

LECTURE 4

Representations of \mathfrak{S}_d : Young Diagrams and Frobenius's Character Formula

In this lecture we get to work. Specifically, we give in §4.1 a complete description of the irreducible representations of the symmetric group, that is, a construction of the representations (via Young symmetrizers) and a formula (Frobenius' formula) for their characters. The proof that the representations constructed in §4.1 are indeed the irreducible representations of the symmetric group is given in §4.2; the proof of Frobenius' formula, as well as a number of others, in §4.3. Apart from their intrinsic interest (and undeniable beauty), these results turn out to be of substantial interest in Lie theory: analogs of the Young symmetrizers will give a construction of the irreducible representations of $SL_n \mathbb{C}$. At the same time, while the techniques of this lecture are completely elementary (we use only a few identities about symmetric polynomials, proved in Appendix A), the level of difficulty is clearly higher than in preceding lectures. The results in the latter half of §4.3 (from Corollary 4.39 on) in particular are quite difficult, and inasmuch as they are not used later in the text may be skipped by readers who are not symmetric group enthusiasts.

§4.1: Statements of the results

§4.2: Irreducible representations of \mathfrak{S}_d

§4.3: Proof of Frobenius's formula

§4.1. Statements of the Results

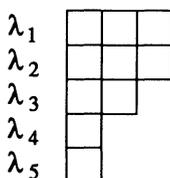
The number of irreducible representations of \mathfrak{S}_d is the number of conjugacy classes, which is the number $p(d)$ of partitions¹ of d : $d = \lambda_1 + \cdots + \lambda_k$, $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$. We have

¹ It is sometimes convenient, and sometimes a nuisance, to have partitions that end in one or more zeros; if convenient, we allow some of the λ_i on the end to be zero. Two sequences define the same partition, of course, if they differ only by zeros at the end.

$$\sum_{d=0}^{\infty} p(d)t^d = \prod_{n=1}^{\infty} \left(\frac{1}{1-t^n} \right) \\ = (1+t+t^2+\dots)(1+t^2+t^4+\dots)(1+t^3+\dots)\dots$$

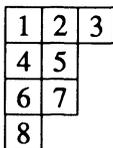
which converges exactly in $|t| < 1$. This partition number is an interesting arithmetic function, whose congruences and growth behavior as a function of d have been much studied (cf. [Har], [And]). For example, $p(d)$ is asymptotically equal to $(1/\alpha d)e^{\beta\sqrt{d}}$, with $\alpha = 4\sqrt{3}$ and $\beta = \pi\sqrt{2/3}$.

To a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ is associated a *Young diagram* (sometimes called a Young frame or Ferrers diagram)



with λ_i boxes in the i th row, the rows of boxes lined up on the left. The *conjugate partition* $\lambda' = (\lambda'_1, \dots, \lambda'_r)$ to the partition λ is defined by interchanging rows and columns in the Young diagram, i.e., reflecting the diagram in the 45° line. For example, the diagram above is that of the partition $(3, 3, 2, 1, 1)$, whose conjugate is $(5, 3, 2)$. (Without reference to the diagram, the conjugate partition to λ can be defined by saying λ'_i is the number of terms in the partition λ that are greater than or equal to i .)

Young diagrams can be used to describe projection operators for the regular representation, which will then give the irreducible representations of \mathfrak{S}_d . For a given Young diagram, number the boxes, say consecutively as shown:



More generally, define a *tableau* on a given Young diagram to be a numbering of the boxes by the integers $1, \dots, d$. Given a tableau, say the canonical one shown, define two subgroups² of the symmetric group

² If a tableau other than the canonical one were chosen, one would get different groups in place of P and Q , and different elements in the group ring, but the representations constructed this way will be isomorphic.

$$P = P_\lambda = \{g \in \mathfrak{S}_d: g \text{ preserves each row}\}$$

and

$$Q = Q_\lambda = \{g \in \mathfrak{S}_d: g \text{ preserves each column}\}.$$

In the group algebra $\mathbb{C}\mathfrak{S}_d$, we introduce two elements corresponding to these subgroups: we set

$$a_\lambda = \sum_{g \in P} e_g \quad \text{and} \quad b_\lambda = \sum_{g \in Q} \text{sgn}(g) \cdot e_g. \quad (4.1)$$

To see what a_λ and b_λ do, observe that if V is any vector space and \mathfrak{S}_d acts on the d th tensor power $V^{\otimes d}$ by permuting factors, the image of the element $a_\lambda \in \mathbb{C}\mathfrak{S}_d \rightarrow \text{End}(V^{\otimes d})$ is just the subspace

$$\text{Im}(a_\lambda) = \text{Sym}^{\lambda_1} V \otimes \text{Sym}^{\lambda_2} V \otimes \cdots \otimes \text{Sym}^{\lambda_k} V \subset V^{\otimes d},$$

where the inclusion on the right is obtained by grouping the factors of $V^{\otimes d}$ according to the rows of the Young tableaux. Similarly, the image of b_λ on this tensor power is

$$\text{Im}(b_\lambda) = \wedge^{\mu_1} V \otimes \wedge^{\mu_2} V \otimes \cdots \otimes \wedge^{\mu_l} V \subset V^{\otimes d},$$

where μ is the conjugate partition to λ .

Finally, we set

$$c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}\mathfrak{S}_d; \quad (4.2)$$

this is called a *Young symmetrizer*. For example, when $\lambda = (d)$, $c_{(d)} = a_{(d)} = \sum_{g \in \mathfrak{S}_d} e_g$, and the image of $c_{(d)}$ on $V^{\otimes d}$ is $\text{Sym}^d V$. When $\lambda = (1, \dots, 1)$, $c_{(1, \dots, 1)} = b_{(1, \dots, 1)} = \sum_{g \in \mathfrak{S}_d} \text{sgn}(g) e_g$, and the image of $c_{(1, \dots, 1)}$ on $V^{\otimes d}$ is $\wedge^d V$. We will eventually see that the image of the symmetrizers c_λ in $V^{\otimes d}$ provide essentially all the finite-dimensional irreducible representations of $\text{GL}(V)$. Here we state the corresponding fact for representations of \mathfrak{S}_d :

Theorem 4.3. *Some scalar multiple of c_λ is idempotent, i.e., $c_\lambda^2 = n_\lambda c_\lambda$, and the image of c_λ (by right multiplication on $\mathbb{C}\mathfrak{S}_d$) is an irreducible representation V_λ of \mathfrak{S}_d . Every irreducible representation of \mathfrak{S}_d can be obtained in this way for a unique partition.*

We will prove this theorem in the next section. Note that, as a corollary, each irreducible representation of \mathfrak{S}_d can be defined over the rational numbers since c_λ is in the rational group algebra $\mathbb{Q}\mathfrak{S}_d$. Note also that the theorem gives a direct correspondence between conjugacy classes in \mathfrak{S}_d and irreducible representations of \mathfrak{S}_d , something which has never been achieved for general groups.

For example, for $\lambda = (d)$,

$$V_{(d)} = \mathbb{C}\mathfrak{S}_d \cdot \sum_{g \in \mathfrak{S}_d} e_g = \mathbb{C} \cdot \sum_{g \in \mathfrak{S}_d} e_g$$

is the trivial representation U , and when $\lambda = (1, \dots, 1)$,

$$V_{(1, \dots, 1)} = \mathbb{C}\mathfrak{S}_d \cdot \sum_{g \in \mathfrak{S}_d} \text{sgn}(g)e_g = \mathbb{C} \cdot \sum_{g \in \mathfrak{S}_d} \text{sgn}(g)e_g$$

is the alternating representation U' . For $\lambda = (2, 1)$,

$$c_{(2,1)} = (e_1 + e_{(12)}) \cdot (e_1 - e_{(13)}) = 1 + e_{(12)} - e_{(13)} - e_{(132)}$$

in $\mathbb{C}\mathfrak{S}_3$, and $V_{(2,1)}$ is spanned by $c_{(2,1)}$ and $(13) \cdot c_{(2,1)}$, so $V_{(2,1)}$ is the standard representation of \mathfrak{S}_3 .

Exercise 4.4*. Set $A = \mathbb{C}\mathfrak{S}_d$, so $V_\lambda = Ac_\lambda = Aa_\lambda b_\lambda$.

(a) Show that $V_\lambda \cong Ab_\lambda a_\lambda$.

(b) Show that V_λ is the image of the map from Aa_λ to Ab_λ given by right multiplication by b_λ . By (a), this is isomorphic to the image of $Ab_\lambda \rightarrow Aa_\lambda$ given by right multiplication by a_λ .

(c) Using (a) and the description of V_λ in the theorem show that

$$V_{\lambda'} = V_\lambda \otimes U',$$

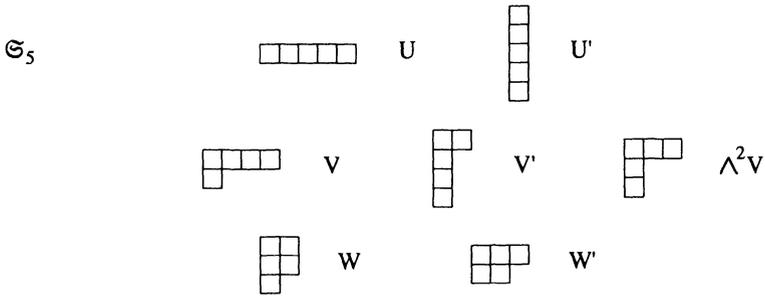
where λ' is the conjugate partition to λ and U' is the alternating representation.

Examples 4.5. In earlier lectures we described the irreducible representations of \mathfrak{S}_d for $d \leq 5$. From the construction of the representation corresponding to a Young diagram it is not hard to work out which representations come from which diagrams:

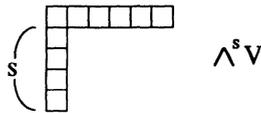
\mathfrak{S}_2 trivial alternating

\mathfrak{S}_3 U trivial U' alternating
 V standard

\mathfrak{S}_4 U U'
 V V' W



Exercise 4.6*. Show that for general d , the standard representation V corresponds to the partition $d = (d - 1) + 1$. As a challenge, you can try to prove that the exterior powers of the standard representation V are represented by a “hook”:



Note that this recovers our theorem that the $\wedge^s V$ are irreducible.

Next we turn to Frobenius's formula for the character χ_λ of V_λ , which includes a formula for its dimension. Let C_i denote the conjugacy class in \mathfrak{S}_d determined by a sequence

$$\mathbf{i} = (i_1, i_2, \dots, i_d) \quad \text{with} \quad \sum \alpha i_x = d:$$

C_i consists of those permutations that have i_1 1-cycles, i_2 2-cycles, \dots , and i_d d -cycles.

Introduce independent variables x_1, \dots, x_k , with k at least as large as the number of rows in the Young diagram of λ . Define the *power sums* $P_j(x)$, $1 \leq j \leq d$, and the *discriminant* $\Delta(x)$ by

$$\begin{aligned} P_j(x) &= x_1^j + x_2^j + \dots + x_k^j, \\ \Delta(x) &= \prod_{i < j} (x_i - x_j). \end{aligned} \tag{4.7}$$

If $f(x) = f(x_1, \dots, x_k)$ is a formal power series, and (l_1, \dots, l_k) is a k -tuple of non-negative integers, let

$$[f(x)]_{(l_1, \dots, l_k)} = \text{coefficient of } x_1^{l_1} \dots x_k^{l_k} \text{ in } f. \tag{4.8}$$

Given a partition $\lambda: \lambda_1 \geq \dots \geq \lambda_k \geq 0$ of d , set

$$l_1 = \lambda_1 + k - 1, \quad l_2 = \lambda_2 + k - 2, \dots, l_k = \lambda_k, \tag{4.9}$$

a strictly decreasing sequence of k non-negative integers. The character of V_λ evaluated on $g \in C_i$ is given by the remarkable

Frobenius Formula 4.10

$$\chi_\lambda(C_i) = \left[\Delta(x) \cdot \prod_j P_j(x)^{i_j} \right]_{(i_1, \dots, i_k)}.$$

For example, if $d = 5$, $\lambda = (3, 2)$, and C_i is the conjugacy class of $(12)(345)$, i.e., $i_1 = 0$, $i_2 = 1$, $i_3 = 1$, then

$$\chi_{(3,2)}(C_i) = [(x_1 - x_2) \cdot (x_1^2 + x_2^2)(x_1^3 + x_2^3)]_{(4,2)} = 1.$$

Other entries in our character tables for \mathfrak{S}_3 , \mathfrak{S}_4 , and \mathfrak{S}_5 can be verified as easily, verifying the assertions of Examples 4.5.

In terms of certain symmetric functions S_λ called *Schur polynomials*, Frobenius's formula can be expressed by

$$\prod_j P_j(x)^{i_j} = \sum \chi_\lambda(C_i) S_\lambda,$$

the sum over all partitions λ of d in at most k parts (cf. Proposition 4.37 and (A.27)). Although we do not use Schur polynomials explicitly in this lecture, they play the central role in the algebraic background developed in Appendix A.

Let us use the Frobenius formula to compute the dimension of V_λ . The conjugacy class of the identity corresponds to $\mathbf{i} = (d)$, so

$$\dim V_\lambda = \chi_\lambda(C_{(d)}) = [\Delta(x) \cdot (x_1 + \dots + x_k)^d]_{(d_1, \dots, d_k)}.$$

Now $\Delta(x)$ is the Vandermonde determinant:

$$\begin{vmatrix} 1 & x_k & \dots & x_k^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_1 & \dots & x_1^{k-1} \end{vmatrix} = \sum_{\sigma \in \mathfrak{S}_k} (\text{sgn } \sigma) x_k^{\sigma(1)-1} \dots x_1^{\sigma(k)-1}.$$

The other term is

$$(x_1 + \dots + x_k)^d = \sum \frac{d!}{r_1! \dots r_k!} x_1^{r_1} x_2^{r_2} \dots x_k^{r_k},$$

the sum over k -tuples (r_1, \dots, r_k) that sum to d . To find the coefficient of $x_1^{l_1} \dots x_k^{l_k}$ in the product, we pair off corresponding terms in these two sums, getting

$$\sum \text{sgn}(\sigma) \cdot \frac{d!}{(l_1 - \sigma(k) + 1)! \dots (l_k - \sigma(1) + 1)!},$$

the sum over those σ in \mathfrak{S}_k such that $l_{k-i+1} - \sigma(i) + 1 \geq 0$ for all $1 \leq i \leq k$. This sum can be written as

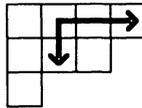
$$\begin{aligned} & \frac{d!}{l_1! \cdots l_k!} \sum_{\sigma \in \mathfrak{S}_k} \text{sgn}(\sigma) \prod_{j=1}^k l_j(l_j - 1) \cdots (l_j - \sigma(k - j + 1) + 2) \\ &= \frac{d!}{l_1! \cdots l_k!} \begin{vmatrix} 1 & l_k & l_k(l_k - 1) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & l_1 & l_1(l_1 - 1) & \cdots \end{vmatrix}. \end{aligned}$$

By column reduction this determinant reduces to the Vandermonde determinant, so

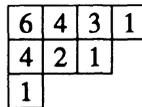
$$\dim V_\lambda = \frac{d!}{l_1! \cdots l_k!} \prod_{i < j} (l_i - l_j), \tag{4.11}$$

with $l_i = \lambda_i + k - i$.

There is another way of expressing the dimensions of the V_λ . The *hook length* of a box in a Young diagram is the number of squares directly below or directly to the right of the box, including the box once.



In the following diagram, each box is labeled by its hook length:



Hook Length Formula 4.12.

$$\dim V_\lambda = \frac{d!}{\prod (\text{Hook lengths})}.$$

For the above partition $4 + 3 + 1$ of 8, the dimension of the corresponding representation of \mathfrak{S}_8 is therefore $8!/6 \cdot 4 \cdot 4 \cdot 2 \cdot 3 = 70$.

Exercise 4.13*. Deduce the hook length formula from the Frobenius formula (4.11).

Exercise 4.14*. Use the hook length formula to show that the only irreducible representations of \mathfrak{S}_d of dimension less than d are the trivial and alternating representations U and U' of dimension 1, the standard representation V and $V' = V \otimes U'$ of dimension $d - 1$, and three other examples: the two-dimensional representation of \mathfrak{S}_4 corresponding to the partition $4 = 2 + 2$, and the two five-dimensional representations of \mathfrak{S}_6 corresponding to the partitions $6 = 3 + 3$ and $6 = 2 + 2 + 2$.

Exercise 4.15*. Using Frobenius's formula or otherwise, show that:

$$\chi_{(d-1,1)}(C_i) = i_1 - 1;$$

$$\chi_{(d-2,1,1)}(C_i) = \frac{1}{2}(i_1 - 1)(i_1 - 2) - i_2;$$

$$\chi_{(d-2,2)}(C_i) = \frac{1}{2}(i_1 - 1)(i_1 - 2) + i_2 - 1.$$

Can you continue this list?

Exercise 4.16*. If g is a cycle of length d in \mathfrak{S}_d , show that $\chi_\lambda(g)$ is ± 1 if λ is a hook, and zero if λ is not a hook:

$$\chi_\lambda(g) = \begin{cases} (-1)^s & \text{if } \lambda = (d - s, 1, \dots, 1), 0 \leq s \leq d - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 4.17. Frobenius [Fro1] used his formula to compute the value of χ_λ on a cycle of length $m \leq d$.

(a) Following the procedure that led to (4.11)—which was the case $m = 1$ —show that

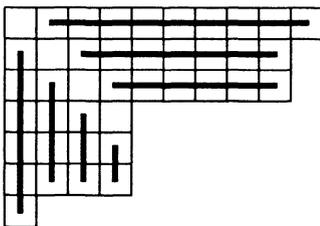
$$\chi_\lambda((12 \dots m)) = \frac{\dim V_\lambda}{-m^2 h_m} \sum_{p=1}^k \frac{\psi(l_p)}{\varphi'(l_p)}, \tag{4.18}$$

where $h_m = d!/(d - m)!m$ is the number of cycles of length m (if $m > 1$), and

$$\varphi(x) = \prod_{i=1}^k (x - l_i), \quad \psi(x) = \varphi(x - m) \prod_{j=1}^m (x - j + 1).$$

The sum in (4.18) can be realized as the coefficient of x^{-1} in the Laurent expansion of $\psi(x)/\varphi(x)$ at $x = \infty$.

Define the *rank* r of a partition to be the length of the diagonal of its Young diagram, and let a_i and b_i be the number of boxes below and to the right of the i th box of the diagonal, reading from lower right to upper left. Frobenius called $\begin{pmatrix} a_1 a_2 \dots a_r \\ b_1 b_2 \dots b_r \end{pmatrix}$ the *characteristics* of the partition. (Many writers now use a reverse notation for the characteristics, writing $(b_r, \dots, b_1 | a_r, \dots, a_1)$ instead.) For the partition $(10, 9, 9, 4, 4, 1)$:



$$r = 4$$

$$\text{characteristics} = \begin{pmatrix} 2 & 3 & 4 & 6 \\ 0 & 6 & 7 & 9 \end{pmatrix}$$

Algebraically, r and the characteristics $a_1 < \dots < a_r$ and $b_1 < \dots < b_r$ are determined by requiring the equality of the two sets

$$\{l_1, \dots, l_k, k-1-a_1, \dots, k-1-a_r\} \quad \text{and} \\ \{0, 1, \dots, k-1, k+b_1, \dots, k+b_r\}.$$

(b) Show that $\psi(x)/\varphi(x) = g(y)/f(y)$, where $y = x - d$ and

$$f(y) = \frac{\prod_{i=1}^r (y - b_i)}{\prod_{i=1}^r (y + a_i + 1)}, \quad g(y) = f(y - m) \prod_{j=1}^m (y - j + 1).$$

Deduce that the sum in (4.18) is the coefficient of x^{-1} in $g(x)/f(x)$.

(c) When $m = 2$, use this to prove the formula

$$\chi_\lambda((12)) = \frac{\dim V_\lambda}{d(d-1)} \sum_{i=1}^r (b_i(b_i+1) - a_i(a_i+1)).$$

Hurwitz [Hur] used this formula of Frobenius to calculate the number of ways to write a given permutation as a product of transpositions. From this he gave a formula for the number of branched coverings of the Riemann sphere with a given number of sheets and given simple branch points. Ingram [In] has given other formulas for $\chi_\lambda(g)$, when g is a somewhat more complicated conjugacy class.

Exercise 4.19*. If V is the standard representation of \mathfrak{S}_d , prove the decompositions into irreducible representations:

$$\text{Sym}^2 V \cong U \oplus V \oplus V_{(d-2, 2)}, \\ V \otimes V = \text{Sym}^2 V \oplus \wedge^2 V \cong U \oplus V \oplus V_{(d-2, 2)} \oplus V_{(d-2, 1, 1)}.$$

Exercise 4.20*. Suppose λ is symmetric, i.e., $\lambda = \lambda'$, and let $q_1 > q_2 > \dots > q_r > 0$ be the lengths of the symmetric hooks that form the diagram of λ ; thus, $q_1 = 2\lambda_1 - 1, q_2 = 2\lambda_2 - 3, \dots$. Show that if g is a product of disjoint cycles of lengths q_1, q_2, \dots, q_r , then

$$\chi_\lambda(g) = (-1)^{(d-r)/2}.$$

§4.2. Irreducible Representations of \mathfrak{S}_d

We show next that the representations V_λ constructed in the first section are exactly the irreducible representations of \mathfrak{S}_d . This proof appears in many standard texts (e.g. [C-R], [Ja-Ke], [N-S], [We1]), so we will be a little concise.

Let $A = \mathbb{C}\mathfrak{S}_d$ be the group ring of \mathfrak{S}_d . For a partition λ of d , let P and Q be the corresponding subgroups preserving the rows and columns of a Young tableau T corresponding to λ , let $a = a_\lambda, b = b_\lambda$, and let $c = c_\lambda = ab$ be

the corresponding Young symmetrizer, so $V_\lambda = Ac_\lambda$ is the corresponding representation. (These groups and elements should really be subscripted by T to denote dependence on the tableau chosen, but the assertions made depend only on the partition, so we usually omit reference to T .)

Note that $P \cap Q = \{1\}$, so an element of \mathfrak{S}_d can be written in at most one way as a product $p \cdot q$, $p \in P$, $q \in Q$. Thus, c is the sum $\sum \pm e_g$, the sum over all g that can be written as $p \cdot q$, with coefficient ± 1 being $\text{sgn}(q)$; in particular, the coefficient of e_1 in c is 1.

Lemma 4.21. (1) For $p \in P$, $p \cdot a = a \cdot p = a$.

(2) For $q \in Q$, $(\text{sgn}(q)q) \cdot b = b \cdot (\text{sgn}(q)q) = b$.

(3) For all $p \in P$, $q \in Q$, $p \cdot c \cdot (\text{sgn}(q)q) = c$, and, up to multiplication by a scalar, c is the only such element in A .

PROOF. Only the last assertion is not obvious. If $\sum n_g e_g$ satisfies the condition in (3), then $n_{pqg} = \text{sgn}(q)n_g$ for all g, p, q ; in particular, $n_{pq} = \text{sgn}(q)n_1$. Thus, it suffices to verify that $n_g = 0$ if $g \notin PQ$. For such g it suffices to find a transposition t such that $p = t \in P$ and $q = g^{-1}tg \in Q$; for then $g = pqg$, so $n_g = -n_g$. If $T' = gT$ is the tableau obtained by replacing each entry i of T by $g(i)$, the claim is that there are two distinct integers that appear in the same row of T and in the same column of T' ; t is then the transposition of these two integers. We must verify that if there were no such pair of integers, then one could write $g = p \cdot q$ for some $p \in P$, $q \in Q$. To do this, first take $p_1 \in P$ and $q'_1 \in Q' = gQg^{-1}$ so that $p_1 T$ and $q'_1 T'$ have the same first row; repeating on the rest of the tableau, one gets $p \in P$ and $q' \in Q'$ so that $pT = q'T'$. Then $pT = q'gT$, so $p = q'g$, and therefore $g = pq$, where $q = g^{-1}(q')^{-1}g \in Q$, as required. \square

We order partitions *lexicographically*:

$$\lambda > \mu \quad \text{if the first nonvanishing } \lambda_i - \mu_i \text{ is positive.} \quad (4.22)$$

Lemma 4.23. (1) If $\lambda > \mu$, then for all $x \in A$, $a_\lambda \cdot x \cdot b_\mu = 0$. In particular, if $\lambda > \mu$, then $c_\lambda \cdot c_\mu = 0$.

(2) For all $x \in A$, $c_\lambda \cdot x \cdot c_\lambda$ is a scalar multiple of c_λ . In particular, $c_\lambda \cdot c_\lambda = n_\lambda c_\lambda$ for some $n_\lambda \in \mathbb{C}$.

PROOF. For (1), we may take $x = g \in \mathfrak{S}_d$. Since $g \cdot b_\mu \cdot g^{-1}$ is the element constructed from gT' , where T' is the tableau used to construct b_μ , it suffices to show that $a_\lambda \cdot b_\mu = 0$. One verifies that $\lambda > \mu$ implies that there are two integers in the same row of T and the same column of T' . If t is the transposition of these integers, then $a_\lambda \cdot t = a_\lambda$, $t \cdot b_\mu = -b_\mu$, so $a_\lambda \cdot b_\mu = a_\lambda \cdot t \cdot t \cdot b_\mu = -a_\lambda \cdot b_\mu$, as required. Part (2) follows from Lemma 4.21 (3). \square

Exercise 4.24*. Show that if $\lambda \neq \mu$, then $c_\lambda \cdot A \cdot c_\mu = 0$; in particular, $c_\lambda \cdot c_\mu = 0$.

Lemma 4.25. (1) Each V_λ is an irreducible representation of \mathfrak{S}_d .

(2) If $\lambda \neq \mu$, then V_λ and V_μ are not isomorphic.

PROOF. For (1) note that $c_\lambda V_\lambda \subset \mathbb{C}c_\lambda$ by Lemma 4.23. If $W \subset V_\lambda$ is a subrepresentation, then $c_\lambda W$ is either $\mathbb{C}c_\lambda$ or 0. If the first is true, then $V_\lambda = A \cdot c_\lambda \subset W$. Otherwise $W \cdot W \subset A \cdot c_\lambda W = 0$, but this implies $W = 0$. Indeed, a projection from A onto W is given by right multiplication by an element $\varphi \in A$ with $\varphi = \varphi^2 \in W \cdot W = 0$. This argument also shows that $c_\lambda V_\lambda \neq 0$.

For (2), we may assume $\lambda > \mu$. Then $c_\lambda V_\lambda = \mathbb{C}c_\lambda \neq 0$, but $c_\lambda V_\mu = c_\lambda \cdot A c_\mu = 0$, so they cannot be isomorphic A -modules. \square

Lemma 4.26. For any λ , $c_\lambda \cdot c_\lambda = n_\lambda c_\lambda$, with $n_\lambda = d!/\dim V_\lambda$.

PROOF. Let F be right multiplication by c_λ on A . Since F is multiplication by n_λ on V_λ , and zero on $\text{Ker}(c_\lambda)$, the trace of F is n_λ times the dimension of V_λ . But the coefficient of e_g in $e_g \cdot c_\lambda$ is 1, so $\text{trace}(F) = |\mathfrak{S}_d| = d!$. \square

Since there are as many irreducible representations V_λ as conjugacy classes of \mathfrak{S}_d , these must form a complete set of isomorphism classes of irreducible representations, which completes the proof of Theorem 4.3. In the next section we will prove Frobenius's formula for the character of V_λ , and, in a series of exercises, discuss a little of what else is known about them: how to decompose tensor products or induced or restricted representations, how to find a basis for V_λ , etc.

§4.3. Proof of Frobenius's Formula

For any partition λ of d , we have a subgroup, often called a *Young subgroup*,

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k} \hookrightarrow \mathfrak{S}_d. \tag{4.27}$$

Let U_λ be the representation of \mathfrak{S}_d induced from the trivial representation of \mathfrak{S}_λ . Equivalently, $U_\lambda = A \cdot a_\lambda$, with a_λ as in the preceding section. Let

$$\psi_\lambda = \chi_{U_\lambda} = \text{character of } U_\lambda. \tag{4.28}$$

Key to this investigation is the relation between U_λ and V_λ , i.e., between ψ_λ and the character χ_λ of V_λ . Note first that V_λ appears in U_λ , since there is a surjection

$$U_\lambda = A a_\lambda \twoheadrightarrow V_\lambda = A a_\lambda b_\lambda, \quad x \mapsto x \cdot b_\lambda. \tag{4.29}$$

Alternatively,

$$V_\lambda = A a_\lambda b_\lambda \cong A b_\lambda a_\lambda \subset A a_\lambda = U_\lambda,$$

by Exercise 4.4. For example, we have

$$U_{(d-1, 1)} \cong V_{(d-1, 1)} \oplus V_{(d)}$$

which expresses the fact that the permutation representation \mathbb{C}^d of \mathfrak{S}_d is the sum of the standard representation and the trivial representation. Eventually we will see that every U_λ contains V_λ with multiplicity one, and contains only other V_μ for $\mu > \lambda$.

The character of U_λ is easy to compute directly since U_λ is an induced representation, and we do this next.

For $\mathbf{i} = (i_1, \dots, i_d)$ a d -tuple of non-negative integers with $\sum ai_\alpha = d$, denote by

$$C_{\mathbf{i}} \subset \mathfrak{S}_d$$

the conjugacy class consisting of elements made up of i_1 1-cycles, i_2 2-cycles, \dots , i_d d -cycles. The number of elements in $C_{\mathbf{i}}$ is easily counted to be

$$|C_{\mathbf{i}}| = \frac{d!}{1^{i_1}i_1!2^{i_2}i_2!\cdots d^{i_d}i_d!}. \tag{4.30}$$

By the formula for characters of induced representations (Exercise 3.19),

$$\begin{aligned} \psi_\lambda(C_{\mathbf{i}}) &= \frac{1}{|C_{\mathbf{i}}|} [\mathfrak{S}_d : \mathfrak{S}_\lambda] \cdot |C_{\mathbf{i}} \cap \mathfrak{S}_\lambda| \\ &= \frac{1^{i_1}i_1! \cdots d^{i_d}i_d!}{d!} \cdot \frac{d!}{\lambda_1! \cdots \lambda_k!} \cdot \sum \prod_{p=1}^k \frac{\lambda_p!}{1^{r_{p1}}r_{p1}! \cdots d^{r_{pd}}r_{pd}!}, \end{aligned}$$

where the sum is over all collections $\{r_{pq}; 1 \leq p \leq k, 1 \leq q \leq d\}$ of non-negative integers satisfying

$$\begin{aligned} i_q &= r_{1q} + r_{2q} + \cdots + r_{kq}, \\ \lambda_p &= r_{p1} + 2r_{p2} + \cdots + dr_{pd}. \end{aligned}$$

(To count $C_{\mathbf{i}} \cap \mathfrak{S}_\lambda$, write the p th component of an element of \mathfrak{S}_λ as a product of r_{p1} 1-cycles, r_{p2} 2-cycles, \dots) Simplifying,

$$\psi_\lambda(C_{\mathbf{i}}) = \sum \prod_{q=1}^d \frac{i_q!}{r_{1q}!r_{2q}! \cdots r_{kq}!}, \tag{4.31}$$

the sum over the same collections of integers $\{r_{pq}\}$.

This sum is exactly the coefficient of the monomial $X^\lambda = x_1^{\lambda_1} \cdots x_k^{\lambda_k}$ in the power sum symmetric polynomial

$$P^{(\mathbf{i})} = (x_1 + \cdots + x_k)^{i_1} \cdot (x_1^2 + \cdots + x_k^2)^{i_2} \cdots (x_1^d + \cdots + x_k^d)^{i_d}. \tag{4.32}$$

So we have the formula

$$\psi_\lambda(C_{\mathbf{i}}) = [P^{(\mathbf{i})}]_\lambda = \text{coefficient of } X^\lambda \text{ in } P^{(\mathbf{i})}. \tag{4.33}$$

To prove Frobenius's formula, we need to compare these coefficients with the coefficients $\omega_\lambda(\mathbf{i})$ defined by

$$\omega_\lambda(\mathbf{i}) = [\Delta \cdot P^{(\mathbf{i})}]_l, \quad l = (\lambda_1 + k - 1, \lambda_2 + k - 2, \dots, \lambda_k). \quad (4.34)$$

Our goal, Frobenius's formula, is the assertion that $\chi_\lambda(C_i) = \omega_\lambda(\mathbf{i})$.

There is a general identity, valid for any symmetric polynomial P , relating such coefficients:

$$[P]_\lambda = \sum_{\mu} K_{\mu\lambda} [\Delta \cdot P]_{(\mu_1+k-1, \mu_2+k-2, \dots, \mu_k)},$$

where the coefficients $K_{\mu\lambda}$ are certain universally defined integers, called *Kostka numbers*. For any partitions λ and μ of d , the integer $K_{\mu\lambda}$ may be defined combinatorially as the number of ways to fill the boxes of the Young diagram for μ with λ_1 1's, λ_2 2's, up to λ_k k 's, in such a way that the entries in each row are nondecreasing, and those in each column are strictly increasing; such are called *semistandard tableaux on μ of type λ* . In particular,

$$K_{\lambda\lambda} = 1, \quad \text{and } K_{\mu\lambda} = 0 \text{ for } \mu < \lambda.$$

The integer $K_{\mu\lambda}$ may be also be defined to be the coefficient of the monomial $X^\lambda = x_1^{\lambda_1} \cdots x_k^{\lambda_k}$ in the Schur polynomial S_μ corresponding to μ . For the proof that these are equivalent definitions, see (A.9) and (A.19) of Appendix A. In the present case, applying Lemma A.26 to the polynomial $P = P^{(\mathbf{i})}$, we deduce

$$\psi_\lambda(C_i) = \sum_{\mu} K_{\mu\lambda} \omega_\mu(\mathbf{i}) = \omega_\lambda(\mathbf{i}) + \sum_{\mu > \lambda} K_{\mu\lambda} \omega_\mu(\mathbf{i}). \quad (4.35)$$

The result of Lemma A.28 can be written, using (4.30), in the form

$$\frac{1}{d!} \sum_{\mathbf{i}} |C_i| \omega_\lambda(\mathbf{i}) \omega_\mu(\mathbf{i}) = \delta_{\lambda\mu}. \quad (4.36)$$

This indicates that the functions ω_λ , regarded as functions on the conjugacy classes of \mathfrak{S}_d , satisfy the same orthogonality relations as the irreducible characters of \mathfrak{S}_d . In fact, one can deduce formally from these equations that the ω_λ must be the irreducible characters of \mathfrak{S}_d , which is what Frobenius proved. A little more work is needed to see that ω_λ is actually the character of the representation V_λ , that is, to prove

Proposition 4.37. *Let $\chi_\lambda = \chi_{V_\lambda}$ be the character of V_λ . Then for any conjugacy class C_i of \mathfrak{S}_d ,*

$$\chi_\lambda(C_i) = \omega_\lambda(\mathbf{i}).$$

PROOF. We have seen in (4.29) that the representation U_λ , whose character is ψ_λ , contains the irreducible representation V_λ . In fact, this is all that we need to know about the relation between U_λ and V_λ . It implies that we have

$$\psi_\lambda = \sum_{\mu} n_{\lambda\mu} \chi_\mu, \quad n_{\lambda\lambda} \geq 1, \text{ all } n_{\lambda\mu} \geq 0. \quad (4.38)$$

Consider this equation together with (4.35). We deduce first that each ω_λ is a

virtual character: we can write

$$\omega_\lambda = \sum m_{\lambda\mu} \chi_\mu, \quad m_{\lambda\mu} \in \mathbb{Z}.$$

But the ω_λ , like the χ_λ , are orthonormal by (4.36), so

$$1 = (\omega_\lambda, \omega_\lambda) = \sum_\mu m_{\lambda\mu}^2,$$

and hence ω_λ is $\pm \chi$ for some irreducible character χ . (It follows from the hook length formula that the plus sign holds here, but we do not need to assume this.)

Fix λ , and assume inductively that $\chi_\mu = \omega_\mu$ for all $\mu > \lambda$, so by (4.35)

$$\psi_\lambda = \omega_\lambda + \sum_{\mu > \lambda} K_{\mu\lambda} \chi_\mu.$$

Comparing this with (4.38), and using the linear independence of characters, the only possibility is that $\omega_\lambda = \chi_\lambda$. \square

Corollary 4.39 (Young's rule). *The integer $K_{\mu\lambda}$ is the multiplicity of the irreducible representation V_μ in the induced representation U_λ :*

$$U_\lambda \cong V_\lambda \oplus \bigoplus_{\mu > \lambda} K_{\mu\lambda} V_\mu, \quad \psi_\lambda = \chi_\lambda + \sum_{\mu > \lambda} K_{\mu\lambda} \chi_\mu.$$

Note that when $\lambda = (1, \dots, 1)$, U_λ is just the regular representation, so $K_{\mu(1, \dots, 1)} = \dim V_\mu$. This shows that *the dimension of V_λ is the number of standard tableaux on λ* , i.e., the number of ways to fill the Young diagram of λ with the numbers from 1 to d , such that all rows and columns are increasing. The hook length formula gives another combinatorial formula for this dimension. Frame, Robinson, and Thrall proved that these two numbers are equal. For a short and purely combinatorial proof, see [G-N-W]. For another proof that the dimension of V_λ is the number of standard tableaux, see [Jam]. The latter leads to a canonical decomposition of the group ring $A = \mathbb{C}\mathfrak{S}_d$ as the direct sum of left ideals Ae_T , summing over all standard tableaux, with $e_T = (\dim V_\lambda/d!) \cdot c_T$, and c_T the Young symmetrizer corresponding to T , cf. Exercises 4.47 and 4.50. This, in turn, leads to explicit calculation of matrices of the representations V_λ with integer coefficients.

For another example of Young's rule, we have a decomposition

$$U_{(d-a, a)} = \bigoplus_{i=0}^a V_{(d-i, i)}.$$

In fact, the only μ whose diagrams can be filled with $d - a$ 1's and a 2's, nondecreasing in rows and strictly increasing in columns, are those with at most two rows, with the second row no longer than a ; and such a diagram has only one such tableau, so there are no multiplicities.

Exercise 4.40*. The characters ψ_λ of \mathfrak{S}_d have been defined only when λ is a partition of d . Extend the definition to any k -tuple $a = (a_1, \dots, a_k)$ of integers

that add up to d by setting $\psi_a = 0$ if any of the a_i are negative, and otherwise $\psi_a = \psi_\lambda$, where λ is the reordering of a_1, \dots, a_k in descending order. In this case ψ_a is the character of the representation induced from the trivial representation by the inclusion of $\mathfrak{S}_{a_1} \times \dots \times \mathfrak{S}_{a_k}$ in \mathfrak{S}_d . Use (A.5) and (A.9) of Appendix A to prove the *determinantal formula* for the irreducible characters χ_λ in terms of the induced characters ψ_μ :

$$\chi_\lambda = \sum_{\tau \in \mathfrak{S}_k} \operatorname{sgn}(\tau) \psi_{(\lambda_1 + \tau(1) - 1, \lambda_2 + \tau(2) - 2, \dots, \lambda_k + \tau(k) - k)}.$$

If one writes ψ_a as a formal product $\psi_{a_1} \cdot \psi_{a_2} \cdot \dots \cdot \psi_{a_k}$, the preceding formula can be written

$$\chi_\lambda = |\psi_{\lambda_i + j - i}| = \begin{vmatrix} \psi_{\lambda_1} & \psi_{\lambda_1+1} & \psi_{\lambda_1+k-1} \\ \psi_{\lambda_2-1} & \psi_{\lambda_2} \cdots & \vdots \\ \vdots & \vdots & \vdots \\ \psi_{\lambda_k-k+1} \cdots & \psi_{\lambda_k} & \vdots \end{vmatrix}.$$

The formal product of the preceding exercise is the character version of an “outer product” of representations. Given any non-negative integers d_1, \dots, d_k , and representations V_i of \mathfrak{S}_{d_i} , denote by $V_1 \circ \dots \circ V_k$ the (isomorphism class of the) representation of \mathfrak{S}_d , $d = \sum d_i$, induced from the tensor product representation $V_1 \boxtimes \dots \boxtimes V_k$ of $\mathfrak{S}_{d_1} \times \dots \times \mathfrak{S}_{d_k}$ by the inclusion of $\mathfrak{S}_{d_1} \times \dots \times \mathfrak{S}_{d_k}$ in \mathfrak{S}_d (see Exercise 2.36). This product is commutative and associative. It will turn out to be useful to have a procedure for decomposing such a representation into its irreducible pieces. For this it is enough to do the case of two factors, and with the individual representations V_i irreducible. In this case, one has, for V_λ the representation of \mathfrak{S}_d corresponding to the partition λ of d and V_μ the representation of \mathfrak{S}_m corresponding to the partition μ of m ,

$$V_\lambda \circ V_\mu = \sum N_{\lambda\mu\nu} V_\nu, \quad (4.41)$$

the sum over all partitions ν of $d + m$, with $N_{\lambda\mu\nu}$ the coefficients given by the *Littlewood–Richardson rule* (A.8) of Appendix A. Indeed, by the exercise, the character of $V_\lambda \circ V_\mu$ is the product of the corresponding determinants, and, by (A.8), that is the sum of the characters $N_{\lambda\mu\nu} \chi_\nu$.

When $m = 1$ and $\mu = (m)$, V_μ is trivial; this gives

$$\operatorname{Ind}_{\mathfrak{S}_d}^{\mathfrak{S}_{d+1}} V_\lambda = \sum V_\nu, \quad (4.42)$$

the sum over all ν whose Young diagram is obtained from that of λ by adding one box. This formula uses only a simpler form of the Littlewood–Richardson rule known as *Pieri's formula*, which is proved in (A.7).

Exercise 4.43*. Show that the Littlewood–Richardson number $N_{\lambda\mu\nu}$ is the multiplicity of the irreducible representation $V_\lambda \boxtimes V_\mu$ in the restriction of V_ν from \mathfrak{S}_{d+m} to $\mathfrak{S}_d \times \mathfrak{S}_m$. In particular, taking $m = 1$, $\mu = (1)$, Pieri's formula (A.7) gives

$$\operatorname{Res}_{\mathfrak{S}_d}^{\mathfrak{S}_{d+1}} V_\nu = \sum V_\lambda,$$

the sum over all λ obtained from ν by removing one box. This is known as the “branching theorem,” and is useful for inductive proofs and constructions, particularly because the decomposition is multiplicity free. For example, you can use it to reprove the fact that the multiplicity of V_λ in U_μ is the number of semistandard tableaux on μ of type λ . It can also be used to prove the assertion made in Exercise 4.6 that the representations corresponding to hooks are exterior powers of the standard representation.

Exercise 4.44* (Pieri's rule). Regard \mathfrak{S}_d as a subgroup of \mathfrak{S}_{d+m} as usual. Let λ be a partition of d and ν a partition of $d + m$. Use Exercise 4.40 to show that the multiplicity of V_ν in the induced representation $\text{Ind}(V_\lambda)$ is zero unless the Young diagram of λ is contained in that of ν , and then it is the number of ways to number the skew diagram lying between them with the numbers from 1 to m , increasing in both row and column. By Frobenius reciprocity, this is the same as the multiplicity of V_λ in $\text{Res}(V_\nu)$.

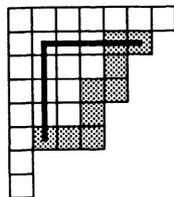
When applied to $d = 0$ (or 1), this implies again that the dimension of V_ν is the number of standard tableaux on the Young diagram of ν .

For a sampling of the many applications of these rules, see [Dia §7, §8].

Problem 4.45*. The *Murnaghan–Nakayama rule* gives an efficient inductive method for computing character values: If λ is a partition of d , and $g \in \mathfrak{S}_d$ is written as a product of an m -cycle and a disjoint permutation $h \in \mathfrak{S}_{d-m}$, then

$$\chi_\lambda(g) = \sum (-1)^{r(\mu)} \chi_\mu(h),$$

where the sum is over all partitions μ of $d - m$ that are obtained from λ by removing a skew hook of length m , and $r(\mu)$ is the number of vertical steps in the skew hook, i.e., one less than the number of rows in the hook. A *skew hook* for λ is a connected region of boundary boxes for its Young diagram such that removing them leaves a smaller Young diagram; there is a one-to-one correspondence between skew hooks and ordinary hooks of the same size, as indicated:



$$\lambda = (7, 6, 5, 5, 4, 4, 1, 1)$$

$$\mu = (7, 4, 4, 3, 3, 1, 1, 1)$$

$$\text{hook length} = 9, r = 4$$

For example, if λ has no hooks of length m , then $\chi_\lambda(g) = 0$.

The Murnaghan–Nakayama rule may be written inductively as follows: If g is written as a product of disjoint cycles of lengths m_1, m_2, \dots, m_p , with the lengths m_i taken in any order, then $\chi_\lambda(g)$ is the sum $\sum (-1)^{r(s)}$, where the sum is over all ways s to decompose the Young diagram of λ by successively

removing p skew hooks of lengths m_1, \dots, m_p , and $r(s)$ is the total number of vertical steps in the hooks of s .

(a) Deduce the Murnaghan–Nakayama rule from (4.41) and Exercise 4.16, using the Littlewood–Richardson rule. Or:

(b) With the notation of Exercise 4.40, show that

$$\psi_{a_1} \psi_{a_2} \cdots \psi_{a_k}(g) = \sum_{i=1}^k \psi_{a_1} \psi_{a_2} \cdots \psi_{a_i-m} \psi_{a_{i+1}} \cdots \psi_{a_k}(h).$$

Exercise 4.46*. Show that Corollary 4.39 implies the “Snapper conjecture”: the irreducible representation V_μ occurs in the induced representation U_λ if and only if

$$\sum_{i=1}^j \lambda_i \leq \sum_{i=1}^j \mu_i \quad \text{for all } j \geq 1.$$

Problem 4.47*. There is a more intrinsic construction of the irreducible representation V_λ , called a *Specht module*, which does not involve the choice of a tableau; it is also useful for studying representations of \mathfrak{S}_d in positive characteristic. Define a *tabloid* $\{T\}$ to be an equivalence class of tableaux (numberings by the integers 1 to d) on λ , two being equivalent if the rows are the same up to order. Then \mathfrak{S}_d acts by permutations on the tabloids, and the corresponding representation, with basis the tabloids, is isomorphic to U_λ . For each tableau T , define an element E_T in this representation space, by

$$E_T = b_T \{T\} = \sum \operatorname{sgn}(q) \{qT\},$$

the sum over the q that preserve the columns of T . The span of all E_T 's is isomorphic to V_λ , and the E_T 's, where T varies over the standard tableaux, form a basis.

Another construction of V_λ is to take the subspace of the polynomial ring $\mathbb{C}[x_1, \dots, x_d]$ spanned by all polynomials F_T , where $F_T = \prod (x_i - x_j)$, the product over all pairs $i < j$ which occur in the same column in the tableau T .

Exercise 4.48*. Let U'_λ be the representation $A \cdot b_\lambda$, which is the representation of \mathfrak{S}_d induced from the tensor product of the alternating representations on the subgroup $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_r}$, where $\mu = \lambda'$ is the conjugate partition. Show that the decomposition of U'_λ is

$$U'_\lambda = \sum_{\mu} K_{\mu' \lambda} V_\mu.$$

Deduce that V_λ is the only irreducible representation that occurs in both U_λ and U'_λ , and it occurs in each with multiplicity one.

Note, however, that in general $A \cdot c_\lambda \neq A \cdot a_\lambda \cap A \cdot b_\lambda$ since $A \cdot c_\lambda$ may not be contained in $A \cdot a_\lambda$.

Exercise 4.49*. With notation as in (4.41), if $U' = V_{(1, \dots, 1)}$ is the alternating representation of \mathfrak{S}_m , show that $V_\lambda \circ V_{(1, \dots, 1)}$ decomposes into a direct sum $\bigoplus V_\pi$, the sum over all π whose Young diagram can be obtained from that of λ by adding m boxes, with no two in the same row.

Exercise 4.50. We have seen that $A = \mathbb{C}\mathfrak{S}_d$ is isomorphic to a direct sum of m_λ copies of $V_\lambda = A c_\lambda$, where $m_\lambda = \dim V_\lambda$ is the number of standard tableaux on λ . This can be seen explicitly as follows. For each standard tableau T on each λ , let c_T be the element of $\mathbb{C}\mathfrak{S}_d$ constructed from T . Then $A = \bigoplus A \cdot c_T$. Indeed, an argument like that in Lemma 4.23 shows that $c_T \cdot c_{T'} = 0$ whenever T and T' are tableaux on the same diagram and $T > T'$, i.e., the first entry (reading from left to right, then top to bottom) where the tableaux differ has the entry of T larger than that of T' . From this it follows that the sum $\Sigma A \cdot c_T$ is direct. A dimension count concludes the proof. (This also gives another proof that the dimension of V_λ is the number of standard tableaux on λ , provided one verifies that the sum of the squares of the latter numbers is $d!$, cf. [Boe] or [Ke].)

Exercise 4.51*. There are several methods for decomposing a tensor product of two representations of \mathfrak{S}_d , which amounts to finding the coefficients $C_{\lambda\mu\nu}$ in the decomposition

$$V_\lambda \otimes V_\mu \cong \Sigma_\nu C_{\lambda\mu\nu} V_\nu,$$

for $\lambda, \mu,$ and ν partitions of d . Since one knows how to express V_μ in terms of the induced representations U_ν , it suffices to compute $V_\lambda \otimes U_\nu$, which is isomorphic to $\text{Ind}(\text{Res}(V_\lambda))$, restricting and inducing from the subgroup $\mathfrak{S}_\nu = \mathfrak{S}_{\nu_1} \times \mathfrak{S}_{\nu_2} \times \dots$; this restriction and induction can be computed by the Littlewood–Richardson rule. For $d \leq 5$, you can work out these coefficients using only restriction to \mathfrak{S}_{d-1} and Pieri's formula.

(a) Prove the following closed-form formula for the coefficients, which shows in particular that they are independent of the ordering of the subscripts $\lambda, \mu,$ and ν :

$$C_{\lambda\mu\nu} = \sum_{\mathbf{i}} \frac{1}{z(\mathbf{i})} \omega_\lambda(\mathbf{i}) \omega_\mu(\mathbf{i}) \omega_\nu(\mathbf{i}),$$

the sum over all $\mathbf{i} = (i_1, \dots, i_d)$ with $\Sigma \alpha i_\alpha = d$, and with $\omega_\lambda(\mathbf{i}) = \chi_\lambda(C_{\mathbf{i}})$ and $z(\mathbf{i}) = i_1! 1^{i_1} \cdot i_2! 2^{i_2} \cdot \dots \cdot i_d! d^{i_d}$.

(b) Show that

$$C_{\lambda\mu(d)} = \begin{cases} 1 & \text{if } \mu = \lambda \\ 0 & \text{otherwise,} \end{cases} \quad C_{\lambda\mu(1, \dots, 1)} = \begin{cases} 1 & \text{if } \mu = \lambda' \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 4.52*. Let $R_d = R(\mathfrak{S}_d)$ denote the representation ring, and set $R = \bigoplus_{d=0}^\infty R_d$. The outer product of (4.41) determines maps

$$R_n \otimes R_m \rightarrow R_{n+m},$$

which makes R into a commutative, graded \mathbb{Z} -algebra. Restriction determines maps

$$R_{n+m} = R(\mathfrak{S}_{n+m}) \rightarrow R(\mathfrak{S}_n \times \mathfrak{S}_m) = R_n \otimes R_m,$$

which defines a *co-product* $\delta: R \rightarrow R \otimes R$. Together, these make R into a (graded) Hopf algebra. (This assertion implies many of the formulas we have proved in this lecture, as well as some we have not.)

(a) Show that, as an algebra,

$$R \cong \mathbb{Z}[H_1, \dots, H_d, \dots],$$

where H_d is an indeterminate of degree d ; H_d corresponds to the trivial representation of \mathfrak{S}_d . Show that the co-product δ is determined by

$$\delta(H_n) = H_n \otimes 1 + H_{n-1} \otimes H_1 + \dots + 1 \otimes H_n.$$

If we set $\Lambda = \mathbb{Z}[H_1, \dots, H_d, \dots] = \bigoplus \Lambda_d$, we can identify Λ_d with the symmetric polynomials of degree d in $k \geq d$ variables. The basic symmetric polynomials in Λ_d defined in Appendix A therefore correspond to virtual representations of \mathfrak{S}_d .

(b) Show that E_d corresponds to the alternating representation U' , and

$$H_\lambda \leftrightarrow U_\lambda, \quad S_\lambda \leftrightarrow V_\lambda, \quad E_\lambda \leftrightarrow U'_\lambda.$$

(c) Show that the scalar product $\langle \cdot, \cdot \rangle$ defined on Λ_d in (A.16) corresponds to the scalar product defined on class functions in (2.11).

(d) Show that the involution \mathfrak{g} of Exercise A.32 corresponds to tensoring a representation with the alternating representation U' .

(e) Show that the inverse map from R_d to Λ_d takes a representation W to

$$\sum_{\mathbf{i}} \frac{1}{z(\mathbf{i})} \chi_W(C_{(\mathbf{i})}) P^{(\mathbf{i})},$$

where $z(\mathbf{i}) = i_1! 1^{i_1} \cdot i_2! 2^{i_2} \cdot \dots \cdot i_d! d^{i_d}$.

The (inner) tensor product of representations of \mathfrak{S}_d gives a map $R_d \otimes R_d \rightarrow R_d$ which corresponds to an "inner product" on symmetric functions, sometimes denoted $*$.

(f) Show that

$$P^{(\mathbf{i})} * P^{(\mathbf{j})} = \begin{cases} 0 & \text{for } \mathbf{j} \neq \mathbf{i} \\ z(\mathbf{i}) P^{(\mathbf{i})} & \text{if } \mathbf{j} = \mathbf{i}. \end{cases}$$

Since these $P^{(\mathbf{i})}$ form a basis for $\Lambda_d \otimes \mathbb{Q}$, this formula determines the inner product.