LECTURE 5

Representations of $\mathfrak{A}_d$ and $GL_2(\mathbb{F}_q)$

In this lecture we analyze the representation of two more types of groups: the alternating groups $\mathfrak{A}_d$ and the linear groups $GL_2(\mathbb{F}_q)$ and $SL_2(\mathbb{F}_q)$ over finite fields. In the former case, we prove some general results relating the representations of a group to the representations of a subgroup of index two, and use what we know about the symmetric group; this should be completely straightforward given just the basic ideas of the preceding lecture. In the latter case we start essentially from scratch. The two sections can be read (or not) independently; neither is logically necessary for the remainder of the book.

§5.1. Representations of $\mathfrak{A}_d$

The alternating groups $\mathfrak{A}_d$, $d \geq 5$, form one of the infinite families of simple groups. In this section, continuing the discussion of §3.1, we describe their irreducible representations. The basic method for analyzing representations of $\mathfrak{A}_d$ is by restricting the representations we know from $\mathfrak{S}_d$.

In general when H is a subgroup of index two in a group G, there is a close relationship between their representations. We will see this phenomenon again in Lie theory for the subgroups SO_n of the orthogonal groups O_n.

Let U and U' denote the trivial and nontrivial representation of G obtained from the two representations of G/H. For any representation V of G, let $V' = V \otimes U'$; the character of V' is the same as the character of V on elements of H, but takes opposite values on elements not in H. In particular, $\mathrm{Res}_H^G V' = \mathrm{Res}_H^G V$.

If W is any representation of H, there is a *conjugate* representation defined by conjugating by any element t of G that is not in H; if ψ is the character of W, the character of the conjugate is $h \mapsto \psi(tht^{-1})$. Since t is unique up to multiplication by an element of H, the conjugate representation is unique up to isomorphism.

Proposition 5.1. *Let V be an irreducible representation of G, and let $W = \mathrm{Res}_H^G V$ be the restriction of V to H. Then exactly one of the following holds:*

(1) *V is not isomorphic to V'; W is irreducible and isomorphic to its conjugate;* $\mathrm{Ind}_H^G W \cong V \oplus V'$.

(2) *$V \cong V'$; $W = W' \oplus W''$, where W' and W'' are irreducible and conjugate but not isomorphic;* $\mathrm{Ind}_H^G W' \cong \mathrm{Ind}_H^G W'' \cong V$.

Each irreducible representation of H arises uniquely in this way, noting that in case (1) *V' and V determine the same representation.*

Proof. Let χ be the character of V. We have

$$|G| = 2|H| = \sum_{h \in H} |\chi(h)|^2 + \sum_{t \notin H} |\chi(t)|^2.$$

Since the first sum is an integral multiple of $|H|$, this multiple must be 1 or 2, which are the two cases of the proposition. This shows that W is either irreducible or the sum of two distinct irreducible representations W' and W''. Note that the second case happens when $\chi(t) = 0$ for all $t \notin H$, which is the case when V' is isomorphic to V. In the second case, W' and W'' must be conjugate since W is self-conjugate, and if W' and W'' were self-conjugate V would not be irreducible. The other assertions in (1) and (2) follow from the isomorphism $\mathrm{Ind}(\mathrm{Res}\ V) = V \otimes (U \oplus U')$ of Exercise 3.16. Similarly, for any representation W of H, $\mathrm{Res}(\mathrm{Ind}\ W)$ is the direct sum of W and its conjugate—as follows say from Exercise 3.19—from which the last statement follows readily. □

Most of this discussion extends with little change to the case where H is a normal subgroup of arbitrary prime index in G, cf. [B-tD, pp. 293–296]. Clifford has extended much of this proposition to arbitrary normal subgroups of finite index, cf. [Dor, §14].

There are two types of conjugacy classes c in H: those that are also conjugacy classes in G, and those such that $c \cup c'$ is a conjugacy class in G, where $c' = tct^{-1}$, $t \notin H$; the latter are called *split*. When W is irreducible, its character assumes the same values—those of the character of the representation V of G that restricts to W—on pairs of split conjugacy classes, whereas in the other case the characters of W' and W'' agree on nonsplit classes, but they must disagree on some split classes. If $\chi_{W'}(c) = \chi_{W''}(c') = x$, and $\chi_{W'}(c') = \chi_{W''}(c) = y$, we know the sum $x + y$, since it is the value of the character of the representation V that gives rise to W' and W'' on $c \cup c'$. Often the exact values of x and y can be determined from orthogonality considerations.

Exercise 5.2*. Show that the number of split conjugacy classes is equal to the number of irreducible representations V of G that are isomorphic to V', or to the number of irreducible representations of H that are not isomorphic to their conjugates. Equivalently, the number of nonsplit classes in H is same as the number of conjugacy classes of G that are not in H.

We apply these considerations to the alternating subgroup of the symmetric group. Consider restrictions of the representations V_λ from $\mathfrak{S}_d$ to $\mathfrak{A}_d$. Recall that if λ' is the conjugate partition to λ, then

$$V_{\lambda'} = V_\lambda \otimes U',$$

with U' the alternating representation. The two cases of the proposition correspond to the cases (1) $\lambda' \neq \lambda$ and (2) $\lambda' = \lambda$. If $\lambda' \neq \lambda$, let W_λ be the restriction of V_λ to $\mathfrak{A}_d$. If $\lambda' = \lambda$, let W'_λ and W''_λ be the two representations whose sum is the restriction of V_λ. We have

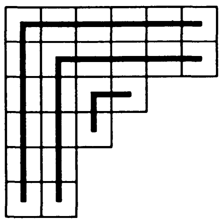
$$\operatorname{Ind} W_\lambda = V_\lambda \oplus V_{\lambda'}, \qquad \operatorname{Res} V_\lambda = \operatorname{Res} V_{\lambda'} = W_\lambda \quad \text{when } \lambda' \neq \lambda,$$

$$\operatorname{Ind} W'_\lambda = \operatorname{Ind} W''_\lambda = V_\lambda, \qquad \operatorname{Res} V_\lambda = W'_\lambda \oplus W''_\lambda \quad \text{when } \lambda' = \lambda.$$

Note that

$$\begin{aligned} &\#\{\text{self-conjugate representations of } \mathfrak{S}_d\} \\ &= \#\{\text{symmetric Young diagrams}\} \\ &= \#\{\text{split pairs of conjugacy classes in } \mathfrak{A}_d\} \\ &= \#\{\text{conjugacy classes in } \mathfrak{S}_d \text{ breaking into two classes in } \mathfrak{A}_d\}. \end{aligned}$$

Now a conjugacy class of an element written as a product of disjoint cycles is split if and only if there is no odd permutation commuting with it, which is equivalent to all the cycles having odd length, and no two cycles having the same length. So the number of self-conjugate representations is the number of partitions of d as a sum of distinct odd numbers. In fact, there is a natural correspondence between these two sets: any such partition corresponds to a symmetric Young diagram, assembling hooks as indicated:



If λ is the partition, the lengths of the cycles in the corresponding split conjugacy classes are $q_1 = 2\lambda_1 - 1$, $q_2 = 2\lambda_2 - 3$, $q_3 = 2\lambda_3 - 5, \ldots$.

For a self-conjugate partition λ, let χ'_λ and χ''_λ denote the characters of W'_λ and W''_λ, and let c and c' be a pair of split conjugacy classes, consisting of cycles of odd lengths $q_1 > q_2 > \cdots > q_r$. The following proposition of Frobenius completes the description of the character table of $\mathfrak{A}_d$.

Proposition 5.3. (1) *If c and c′ do not correspond to the partition λ, then*

$$\chi'_\lambda(c) = \chi'_\lambda(c') = \chi''_\lambda(c) = \chi''_\lambda(c') = \tfrac{1}{2}\chi_\lambda(c \cup c').$$

(2) *If c and c′ correspond to λ, then*

$$\chi'_\lambda(c) = \chi''_\lambda(c') = x, \qquad \chi'_\lambda(c') = \chi''_\lambda(c) = y,$$

with x and y the two numbers

$$\tfrac{1}{2}\left((-1)^m \pm \sqrt{(-1)^m q_1 \cdot \ldots \cdot q_r}\right),$$

and $m = \frac{1}{2}(d - r) = \frac{1}{2}\sum (q_i - 1) \equiv \frac{1}{2}(\prod q_i - 1) \pmod 2$.

For example, if $d = 4$ and $\lambda = (2, 2)$, we have $r = 2$, $q_1 = 3$, $q_2 = 1$, and x and y are the cube roots of unity; the representations W'_λ and W''_λ are the representations labeled U' and U'' in the table in §2.3. For $d = 5$, $\lambda = (3, 1, 1)$, $r = 1$, $q_1 = 5$, and we find the representations called Y and Z in §3.1. For $d \leq 7$, there is at most one split pair, so the character table can be derived from orthogonality alone.

Note that since only one pair of character values is not taken care of by the first case of Frobenius's formula, the choice of which representation is W'_λ and which W''_λ is equivalent to choosing the plus and minus sign in (2). Note also that the integer m occurring in (2) is the number of squares above the diagonal in the Young diagram of λ.

We outline a proof of the proposition as an exercise:

Exercise 5.4*. *Step 1.* Let $q = (q_1 > \cdots > q_r)$ be a sequence of positive odd integers adding to d, and let $c' = c'(q)$ and $c'' = c''(q)$ be the corresponding conjugacy classes in $\mathfrak{A}_d$. Let λ be a self-conjugate partition of d, and let χ'_λ and χ''_λ be the corresponding characters of $\mathfrak{A}_d$. Assume that χ'_λ and χ''_λ take on the same values on each element of $\mathfrak{A}_d$ that is not in c' or c''. Let $u = \chi'_\lambda(c') = \chi''_\lambda(c'')$ and $v = \chi'_\lambda(c'') = \chi''_\lambda(c')$.

(i) Show that u and v are real when $m = \frac{1}{2}\Sigma(q_i - 1)$ is even, and $\bar{u} = v$ when m is odd.

(ii) Let $\vartheta = \chi'_\lambda - \chi''_\lambda$. Deduce from the equation $(\vartheta, \vartheta) = 2$ that $|u - v|^2 = q_1 \cdot \ldots \cdot q_r$.

(iii) Show that λ is the partition that corresponds to q and that $u + v = (-1)^m$, and deduce that u and v are the numbers specified in (2) of the proposition.

Step 2. Prove the proposition by induction on d, and for fixed d, look at that q which has smallest q_1, and for which some character has values on the classes $c'(q)$ and $c''(q)$ other than those prescribed by the proposition.

(i) If $r = 1$, so $q_1 = d = 2m + 1$, the corresponding self-conjugate partition is $\lambda = (m + 1, 1, \ldots, 1)$. By induction, Step 1 applies to χ'_λ and χ''_λ.

(ii) If $r > 1$, consider the imbedding $H = \mathfrak{A}_{q_1} \times \mathfrak{A}_{d-q_1} \subset G = \mathfrak{A}_d$, and let X' and X'' be the representations of G induced from the representations $W'_1 \boxtimes W'_2$ and $W''_1 \boxtimes W'_2$, where W'_1 and W''_1 are the representations of $\mathfrak{A}_{q_1}$ corresponding to q_1, i.e., to the self-conjugate partition $(\frac{1}{2}(q_1 - 1), 1, \ldots, 1)$ of q_1; W'_2 is one of the representations of $\mathfrak{A}_{d-q_1}$ corresponding to $(q_2, \ldots, q_r)$; and $\boxtimes$ denotes the external tensor product (see Exercise 2.36). Show that X' and X'' are conjugate representations of $\mathfrak{A}_d$, and their characters χ' and χ'' take equal values on each pair of split conjugacy classes, with the exception of $c'(q)$ and $c''(q)$, and compute the values of these characters on $c'(q)$ and $c''(q)$.

(iii) Let $\vartheta = \chi' - \chi''$, and show that $(\vartheta, \vartheta) = 2$. Decomposing X' and X'' into their irreducible pieces, deduce that $X' = Y \oplus W'_\lambda$ and $X'' = Y \oplus W''_\lambda$ for some self-conjugate representation Y and some self-conjugate partition λ of d.

(iv) Apply Step 1 to the characters χ'_λ and χ''_λ, and conclude the proof.

Exercise 5.5*. Show that if $d > 6$, the only irreducible representations of $\mathfrak{A}_d$ of dimension less than d are the trivial representation and the $(n - 1)$-dimensional restriction of the standard representation of $\mathfrak{S}_d$. Find the exceptions for $d \le 6$.

We have worked out the character tables for all $\mathfrak{S}_d$ and $\mathfrak{A}_d$ for $d \le 5$. With the formulas of Frobenius, an interested reader can construct the tables for a few more d—until the number of partitions of d becomes large.

§5.2. Representations of $GL_2(\mathbb{F}_q)$ and $SL_2(\mathbb{F}_q)$

The groups $GL_2(\mathbb{F}_q)$ of invertible 2×2 matrices with entries in the finite field $\mathbb{F}_q$ with q elements, where q is a prime power, form another important series of finite groups, as do their subgroups $SL_2(\mathbb{F}_q)$ consisting of matrices of determinant one. The quotient $PGL_2(\mathbb{F}_q) = GL_2(\mathbb{F}_q)/\mathbb{F}_q^*$ is the automorphism group of the finite projective line $\mathbb{P}^1(\mathbb{F}_q)$. The quotients $PSL_2(\mathbb{F}_q) = SL_2(\mathbb{F}_q)/\{\pm 1\}$ are simple groups if $q \neq 2, 3$ (Exercise 5.9). In this section we sketch the character theory of these groups.

We begin with $G = GL_2(\mathbb{F}_q)$. There are several key subgroups:

$$G \supset B = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right\} \supset N = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right\}.$$

(This "Borel subgroup" B and the group of upper triangular unipotent matrices N will reappear when we look at Lie groups.) Since G acts transitively on the projective line $\mathbb{P}^1(\mathbb{F}_q)$, with B the isotropy group of the point (1:0), we have

$$|G| = |B| \cdot |\mathbb{P}^1(\mathbb{F}_q)| = (q - 1)^2 q(q + 1).$$

We will also need the diagonal subgroup

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\} = \mathbb{F}^* \times \mathbb{F}^*,$$

where we write $\mathbb{F}$ for $\mathbb{F}_q$. Let $\mathbb{F}' = \mathbb{F}_{q^2}$ be the extension of $\mathbb{F}$ of degree two, unique up to isomorphism. We can identify $GL_2(\mathbb{F}_q)$ as the group of all $\mathbb{F}$-linear invertible endomorphisms of $\mathbb{F}'$. This makes evident a large cyclic subgroup $K = (\mathbb{F}')^*$ of G. At least if q is odd, we may make this isomorphism explicit by choosing a generator ε for the cyclic group $\mathbb{F}^*$ and choosing a square root $\sqrt{\varepsilon}$ in $\mathbb{F}'$. Then 1 and $\sqrt{\varepsilon}$ form a basis for $\mathbb{F}'$ as a vector space over $\mathbb{F}$, so we can make the identification:

$$K = \left\{ \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} \right\} \cong (\mathbb{F}')^*, \qquad \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} \leftrightarrow \zeta = x + y\sqrt{\varepsilon};$$

K is a cyclic subgroup of G of order $q^2 - 1$. We often make this identification, leaving it as an exercise to make the necessary modifications in case q is even.

The conjugacy classes in G are easily found:

Representative	No. Elements in Class	No. Classes
$a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	1	$q - 1$
$b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$	$q^2 - 1$	$q - 1$
$c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, x \neq y$	$q^2 + q$	$\dfrac{(q-1)(q-2)}{2}$
$d_{x,y} = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}, y \neq 0$	$q^2 - q$	$\dfrac{q(q-1)}{2}$

Here $c_{x,y}$ and $c_{y,x}$ are conjugate by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $d_{x,y}$ and $d_{x,-y}$ are conjugate by any $\begin{pmatrix} a & -\varepsilon c \\ c & -a \end{pmatrix}$. To count the number of elements in the conjugacy class of b_x, look at the action of G on this class by conjugation; the isotropy group is $\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right\}$, so the number of elements in the class is the index of this group in G, which is $q^2 - 1$. Similarly the isotropy group for $c_{x,y}$ is D, and the isotropy group for $d_{x,y}$ is K. To see that the classes are disjoint, consider the eigenvalues and the Jordan canonical forms. Since they account for $|G|$ elements, the list is complete.

There are $q^2 - 1$ conjugacy classes, so we must find the same number of irreducible representations. Consider first the permutation representation of G on $\mathbb{P}^1(\mathbb{F})$, which has dimension $q + 1$. It contains the trivial representation;

let V be the complementary q-dimensional representation. The values of the character χ of V on the four types of conjugacy classes are $\chi(a_x) = q$, $\chi(b_x) = 0$, $\chi(c_{x,y}) = 1$, $\chi(d_{x,y}) = -1$, which we display as the table:

$$V: \quad q \quad 0 \quad 1 \quad -1$$

Since $(\chi, \chi) = 1$, V is irreducible.

For each of the $q - 1$ characters $\alpha\colon \mathbb{F}^* \to \mathbb{C}^*$ of $\mathbb{F}^*$, we have a one-dimensional representation U_α of G defined by $U_\alpha(g) = \alpha(\det(g))$. We also have the representations $V_\alpha = V \otimes U_\alpha$. The values of the characters of these representations are

U_α:	$\alpha(x)^2$	$\alpha(x)^2$	$\alpha(x)\alpha(y)$	$\alpha(x^2 - \varepsilon y^2)$
V_α:	$q\alpha(x)^2$	0	$\alpha(x)\alpha(y)$	$-\alpha(x^2 - \varepsilon y^2)$

Note that if we identify $\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$ with $\zeta = x + y\sqrt{\varepsilon}$ in $\mathbb{F}'$, then

$$x^2 - \varepsilon y^2 = \det\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} = \mathrm{Norm}_{\mathbb{F}'/\mathbb{F}}(\zeta) = \zeta \cdot \zeta^q = \zeta^{q+1}.$$

The next place to look for representations is at those that are induced from large subgroups. For each pair α, β of characters of $\mathbb{F}^*$, there is a character of the subgroup B:

$$B \to B/N = D = \mathbb{F}^* \times \mathbb{F}^* \to \mathbb{C}^* \times \mathbb{C}^* \xrightarrow{\cdot} \mathbb{C}^*,$$

which takes $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ to $\alpha(a)\beta(d)$. Let $W_{\alpha,\beta}$ be the representation induced from B to G by this representation; this is a representation of dimension $[G:B] = q + 1$. By Exercise 3.19 its character values are found to be:

$$W_{\alpha,\beta}: \quad (q+1)\alpha(x)\beta(x) \quad \alpha(x)\beta(x) \quad \alpha(x)\beta(y) + \alpha(y)\beta(x) \quad 0$$

We see from this that $W_{\alpha,\beta} \cong W_{\beta,\alpha}$, that $W_{\alpha,\alpha} \cong U_\alpha \oplus V_\alpha$, and that for $\alpha \neq \beta$ the representation is irreducible. This gives $\frac{1}{2}(q - 1)(q - 2)$ more irreducible representations, of dimension $q + 1$.

Comparing with the list of conjugacy classes, we see that there are $\frac{1}{2}q(q - 1)$ irreducible characters left to be found. A natural way to find new characters is to induce characters from the cyclic subgroup K. For a representation

$$\varphi\colon K = (\mathbb{F}')^* \to \mathbb{C}^*,$$

the character values of the induced representation of dimension $[G:K] = q^2 - 1$ are

$$\mathrm{Ind}(\varphi): \quad q(q-1)\varphi(x) \quad 0 \quad 0 \quad \varphi(\zeta) + \varphi(\zeta)^q$$

Here again $\zeta = x + y\sqrt{\varepsilon} \in K = (\mathbb{F}')^*$. Note that $\mathrm{Ind}(\varphi^q) \cong \mathrm{Ind}(\varphi)$, so the representations $\mathrm{Ind}(\varphi)$ for $\varphi^q \neq \varphi$ give $\frac{1}{2}q(q - 1)$ different representations.

However, these represenations are not irreducible: the character χ of $\mathrm{Ind}(\varphi)$ satisfies $(\chi, \chi) = q - 1$ if $\varphi^q \neq \varphi$, and otherwise $(\chi, \chi) = q$. We will have to work a little harder to get irreducible representations from these $\mathrm{Ind}(\varphi)$.

Another attempt to find more representations is to look inside tensor products of representations we know. We have $V_\alpha \otimes U_\gamma = V_{\alpha\gamma}$, and $W_{\alpha,\beta} \otimes U_\gamma \cong W_{\alpha\gamma,\beta\gamma}$, so there are no new ones to be found this way. But tensor products of the V_α's and $W_{\alpha,\beta}$'s are more promising. For example, $V \otimes W_{\alpha,1}$ has character values:

$$V \otimes W_{\alpha,1}: \qquad q(q+1)\alpha(x) \qquad 0 \qquad \alpha(x) + \alpha(y) \qquad 0$$

We can calculate some inner products of these characters with each other to estimate how many irreducible representations each contains, and how many they have in common. For example,

$$(\chi_{V \otimes W_{\alpha,1}}, \chi_{W_{\alpha,1}}) = 2,$$
$$(\chi_{\mathrm{Ind}(\varphi)}, \chi_{W_{\alpha,1}}) = 1 \quad \text{if } \varphi|_{\mathbb{F}^*} = \alpha,$$
$$(\chi_{V \otimes W_{\alpha,1}}, \chi_{V \otimes W_{\alpha,1}}) = q + 3,$$
$$(\chi_{V \otimes W_{\alpha,1}}, \chi_{\mathrm{Ind}(\varphi)}) = q \quad \text{if } \varphi|_{\mathbb{F}^*} = \alpha,$$

Comparing with the formula $(\chi_{\mathrm{Ind}(\varphi)}, \chi_{\mathrm{Ind}(\varphi)}) = q - 1$, one deduces that $V \otimes W_{\alpha,1}$ and $\mathrm{Ind}(\varphi)$ contain many of the same representations. With any luck, $\mathrm{Ind}(\varphi)$ and $W_{\alpha,1}$ should both be contained in $V \otimes W_{\alpha,1}$. This guess is easily confirmed; the virtual character

$$\chi_\varphi = \chi_{V \otimes W_{\alpha,1}} - \chi_{W_{\alpha,1}} - \chi_{\mathrm{Ind}(\varphi)}$$

takes values $(q-1)\alpha(x)$, $-\alpha(x)$, 0, and $-(\varphi(\zeta) + \varphi(\zeta)^q)$ on the four types of conjugacy classes. Therefore, $(\chi_\varphi, \chi_\varphi) = 1$, and $\chi_\varphi(1) = q - 1 > 0$, so χ_φ is, in fact, the character of an irreducible subrepresentation of $V \otimes W_{\alpha,1}$ of dimension $q - 1$. We denote this representation by X_φ. These $\frac{1}{2}q(q-1)$ representations, for $\varphi \neq \varphi^q$, and with $X_\varphi = X_{\varphi^q}$, therefore complete the list of irreducible representations for $GL_2(\mathbb{F})$. The character table is

	1	$q^2 - 1$	$q^2 + q$	$q^2 - q$
$GL_2(\mathbb{F}_q)$	$a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	$b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$	$c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	$d_{x,y} = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} = \zeta$
U_α	$\alpha(x^2)$	$\alpha(x^2)$	$\alpha(xy)$	$\alpha(\zeta^q)$
V_α	$q\alpha(x^2)$	0	$\alpha(xy)$	$-\alpha(\zeta^q)$
$W_{\alpha,\beta}$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y) + \alpha(y)\beta(x)$	0
X_φ	$(q-1)\varphi(x)$	$-\varphi(x)$	0	$-(\varphi(\zeta) + \varphi(\zeta^q))$

Exercise 5.6. Find the multiplicity of each irreducible representation in the representations $V \otimes W_{\alpha,1}$ and $\mathrm{Ind}(\varphi)$.

Exercise 5.7. Find the character table of $PGL_2(\mathbb{F}) = GL_2(\mathbb{F})/\mathbb{F}^*$. Note that its characters are just the characters of $GL_2(\mathbb{F})$ that take the same values on elements equivalent mod $\mathbb{F}^*$.

We turn next to the subgroup $SL_2(\mathbb{F}_q)$ of 2×2 matrices of determinant one, with q odd. The conjugacy classes, together with the number of elements in each conjugacy class, and the number of conjugacy classes of each type, are

	Representative	No. Elements in Class	No. Classes
(1)	$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	1
(2)	$-e = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	1	1
(3)	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\frac{q^2-1}{2}$	1
(4)	$\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$	$\frac{q^2-1}{2}$	1
(5)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$	$\frac{q^2-1}{2}$	1
(6)	$\begin{pmatrix} -1 & \varepsilon \\ 0 & -1 \end{pmatrix}$	$\frac{q^2-1}{2}$	1
(7)	$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, x \neq \pm 1$	$q(q+1)$	$\frac{q-3}{2}$
(8)	$\begin{pmatrix} x & y \\ \varepsilon y & x \end{pmatrix}, x \neq \pm 1$	$q(q-1)$	$\frac{q-1}{2}$

The verifications are very much as we did for $GL_2(\mathbb{F}_q)$. In (7), the classes of $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ and $\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}$ are the same. In (8), the classes for (x, y) and $(x, -y)$ are the same; as before, a better labeling is by the element ζ in the cyclic group

$$C = \{\zeta \in (\mathbb{F}')^* : \zeta^{q+1} = 1\};$$

the elements ± 1 are not used, and the classes of ζ and ζ^{-1} are the same.

The total number of conjugacy classes is $q + 4$, so we turn to the task of finding $q + 4$ irreducible representations. We first see what we get by restricting representations from $GL_2(\mathbb{F}_q)$. Since we know the characters, there is no problem working this out, and we simply state the results:

(1) The U_α all restrict to the trivial representation U. Hence, if we restrict any representation, we will get the same for all tensor products by U_α's.

(2) The restriction V of the V_α's is irreducible.
(3) The restriction W_α of $W_{\alpha,1}$ is irreducible if $\alpha^2 \neq 1$, and $W_\alpha \cong W_\beta$ when $\beta = \alpha$ or $\beta = \alpha^{-1}$. These give $\frac{1}{2}(q-3)$ irreducible representations of dimension $q+1$.
(3′) Let τ denote the character of $\mathbb{F}^*$ with $\tau^2 = 1, \tau \neq 1$. The restriction of $W_{\tau,1}$ is the sum of two distinct irreducible representations, which we denote W' and W''.
(4) The restriction of X_φ depends only on the restriction of φ to the subgroup C, and φ and φ^{-1} determine the same representation. The representation is irreducible if $\varphi^2 \neq 1$. This gives $\frac{1}{2}(q-1)$ irreducible representations of dimension $q-1$.
(4′) If ψ denotes the character of C with $\psi^2 = 1, \psi \neq 1$, the restriction of X_ψ is the sum of two distinct irreducible representations, which we denote X' and X''.

Altogether this list gives $q+4$ distinct irreducible representations, and it is therefore the complete list. To finish the character table, the problem is to describe the four representations W', W'', X', and X''. Since we know the sum of the squares of the dimensions of all representations, we can deduce that the sum of the squares of these four representations is q^2+1, which is only possible if the first two have dimension $\frac{1}{2}(q+1)$ and the other two $\frac{1}{2}(q-1)$. This is similar to what we saw happens for restrictions of representations to subgroups of index two. Although the index here is larger, we can use what we know about index two subgroups by finding a subgroup H of index two in $GL_2(\mathbb{F}_q)$ that contains $SL_2(\mathbb{F}_q)$, and analyzing the restrictions of these four representations to H.

For H we take the matrices in $GL_2(\mathbb{F}_q)$ whose determinant is a square. The representatives of the conjugacy classes are the same as those for $GL_2(\mathbb{F}_q)$, including, of course, only those representatives whose determinant is a square, but we must add classes represented by the elements $\begin{pmatrix} x & \varepsilon \\ 0 & x \end{pmatrix}$, $x \in \mathbb{F}^*$. These are conjugate to the elements $\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$ in $GL_2(\mathbb{F}_q)$, but not in H. These are the $q-1$ split conjugacy classes. The procedure of the preceding section can be used to work out all the representations of H, but we need only a little of this.

Note that the sign representation U' from G/H is U_τ, so that $W_{\tau,1} \cong W_{\tau,1} \otimes U'$ and $X_\psi \cong X_\psi \otimes U'$; their restrictions to H split into sums of conjugate irreducible representations of half their dimensions. This shows these representations stay irreducible on restriction from H to $SL_2(\mathbb{F}_q)$, so that W' and W'' are conjugate representations of dimension $\frac{1}{2}(q+1)$, and X' and X'' are conjugate representations of dimension $\frac{1}{2}(q-1)$. In addition, we know that their character values on all nonsplit conjugacy classes are the same as half the characters of the representations $W_{\tau,1}$ and X_ψ, respectively. This is all the information we need to finish the character table. Indeed, the only values not covered by this discussion are

	$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}1&\varepsilon\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&1\\0&-1\end{pmatrix}$	$\begin{pmatrix}-1&\varepsilon\\0&-1\end{pmatrix}$
W'	s	t	s'	t'
W''	t	s	t'	s'
X'	u	v	u'	v'
X''	v	u	v'	u'

The first two rows are determined as follows. We know that $s + t = \chi_{W_{\tau,1}}\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right) = 1$. In addition, since $\begin{pmatrix}1&1\\0&1\end{pmatrix}^{-1} = \begin{pmatrix}1&-1\\0&1\end{pmatrix}$ is conjugate to $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ if q is congruent to 1 modulo 4, and to $\begin{pmatrix}1&\varepsilon\\0&1\end{pmatrix}$ otherwise, and since $\chi(g^{-1}) = \overline{\chi(g)}$ for any character, we conclude that s and t are real if $q \equiv 1 \bmod(4)$, and $s = \bar{t}$ if $q \equiv 3 \bmod(4)$. In addition, since $-e$ acts as the identity or minus the identity for any irreducible representation (Schur's lemma),

$$\chi(-g) = \chi(g)\cdot\chi(1)/\chi(-e)$$

for any irreducible character χ. This gives the relations $s' = \tau(-1)s$ and $t' = \tau(-1)t$. Finally, applying the equation $(\chi, \chi) = 1$ to the character of W' gives a formula for $s\bar{t} + t\bar{s}$. Solving these equations gives $s, t = \frac{1}{2} \pm \frac{1}{2}\sqrt{\omega q}$, where $\omega = \tau(-1)$ is 1 or -1 according as $q \equiv 1$ or $3 \bmod(4)$. Similarly one computes that u and v are $-\frac{1}{2} \pm \frac{1}{2}\sqrt{\omega q}$. This concludes the computations needed to write out the character table.

Exercise 5.8. By considering the action of $SL_2(\mathbb{F}_q)$ on the set $\mathbb{P}^1(\mathbb{F}_q)$, show that $SL_2(\mathbb{F}_2) \cong \mathfrak{S}_3$, $PSL_2(\mathbb{F}_3) \cong \mathfrak{A}_4$, and $SL_2(\mathbb{F}_4) \cong \mathfrak{A}_5$.

Exercise 5.9*. Use the character table for $SL_2(\mathbb{F}_q)$ to show that $PSL_2(\mathbb{F}_q)$ is a simple group if q is odd and greater than 3.

Exercise 5.10. Compute the character table of $PSL_2(\mathbb{F}_q)$, either by regarding it as a quotient of $SL_2(\mathbb{F}_q)$, or as a subgroup of index two in $PGL_2(\mathbb{F}_q)$.

Exercise 5.11*. Find the conjugacy classes of $GL_3(\mathbb{F}_q)$, and compute the characters of the permutation representations obtained by the action of $GL_3(\mathbb{F}_q)$ on (i) the projective plane $\mathbb{P}^2(\mathbb{F}_q)$ and (ii) the "flag variety" consisting of a point on a line in $\mathbb{P}^2(\mathbb{F}_q)$. Show that the first is irreducible and that the second is a sum of the trivial representation, two copies of the first representation, and an irreducible representation.

Although the characters of the above groups were found by the early pioneers in representation theory, actually producing the representations in a natural way is more difficult. There has been a great deal of work extending this story to $GL_n(\mathbb{F}_q)$ and $SL_n(\mathbb{F}_q)$ for $n > 2$ (cf. [Gr]), and for corresponding groups, called finite Chevalley groups, related to other Lie groups. For some hints in this direction see [Hu3], as well as [Ti2]. Since all but a finite number of finite simple groups are now known to arise this way (or are cyclic or alternating groups, whose characters we already know), such representations play a fundamental role in group theory. In recent work their Lie-theoretic origins have been exploited to produce their representations, but to tell this story would go far beyond the scope of these lecture(r)s.