LECTURE 11

Representations of $\mathfrak{sl}_2\mathbb{C}$

This is the first of four lectures—§11–14—that comprise in some sense the heart of the book. In particular, the naive analysis of §11.1, together with the analogous parts of §12 and §13, form the paradigm for the study of finite-dimensional representations of all semisimple Lie algebras and groups. §11.2 is less central; in it we show how the analysis carried out in §11.1 can be used to explicitly describe the tensor products of irreducible representations. §11.3 is least important; it indicates how we can interpret geometrically some of the results of the preceding section. The discussions in §11.1 and §11.2 are completely elementary (we do use the notion of symmetric powers of a vector space, but in a non-threatening way). §11.3 involves a fair amount of classical projective geometry, and can be skimmed or skipped by those not already familiar with the relevant basic notions from algebraic geometry.

§11.1: The irreducible representations
§11.2: A little plethysm
§11.3: A little geometric plethysm

§11.1. The Irreducible Representations

We start our discussion of representations of semisimple Lie algebras with the simplest case, that of $\mathfrak{sl}_2\mathbb{C}$. As we will see, while this case does not exhibit any of the complexity of the more general case, the basic idea that informs the whole approach is clearly illustrated here.

This approach is one already mentioned above, in connection with the representations of the symmetric group on three letters. The idea in that case was that given a representation of our group on a vector space V we first restrict the representation to the abelian subgroup generated by a 3-cycle τ. We obtain a decomposition

W. Fulton et al., *Representation Theory*
© Springer Science+Business Media, Inc. 2004

$$V = \bigoplus V_\alpha$$

of V into eigenspaces for the action of τ; the commutation relations satisfied by the remaining elements σ of the group with respect to τ implied that such σ simply permuted these subspaces V_α, so that the representation was in effect determined by the collection of eigenvalues of τ.

Of course, circumstances in the case of Lie algebra representations are quite different: to name two, it is no longer the case that the action of an abelian object on any vector space admits such a decomposition; and even if such a decomposition exists we certainly cannot expect that the remaining elements of our Lie algebra will simply permute its summands. Nevertheless, the idea remains essentially a good one, as we shall now see.

To begin with, we choose the basis for the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ from the last lecture:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfying

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H. \tag{11.1}$$

These seem like a perfectly natural basis to choose, but in fact the choice is dictated by more than aesthetics; there is, as we will see, a nearly canonical way of choosing a basis of a semisimple Lie algebra (up to conjugation), which will yield this basis in the present circumstance and which will share many of the properties we describe below.

In any event, let V be an irreducible finite-dimensional representation of $\mathfrak{sl}_2\mathbb{C}$. We start by trotting out one of the facts that we quoted in Lecture 9, the preservation of Jordan decomposition; in the present circumstances it implies that

The action of H on V is diagonalizable. (11.2)

We thus have, as indicated, a decomposition

$$V = \bigoplus V_\alpha, \tag{11.3}$$

where the α run over a collection of complex numbers, such that for any vector $v \in V_\alpha$ we have

$$H(v) = \alpha \cdot v.$$

The next question is obviously how X and Y act on the various spaces V_α. We claim that X and Y must each carry the subspaces V_α into other subspaces $V_{\alpha'}$. In fact, we can be more specific: if we want to know where the image of a given vector $v \in V_\alpha$ under the action of X sits in relation to the decomposition (11.3), we have to know how H acts on $X(v)$; this is given by the

Fundamental Calculation (first time):

$$\begin{aligned} H(X(v)) &= X(H(v)) + [H, X](v) \\ &= X(\alpha \cdot v) + 2X(v) \\ &= (\alpha + 2) \cdot X(v); \end{aligned}$$

i.e., *if v is an eigenvector for H with eigenvalue α, then $X(v)$ is also an eigenvector for H, with eigenvalue $\alpha + 2$.* In other words, we have

$$X: V_\alpha \to V_{\alpha+2}.$$

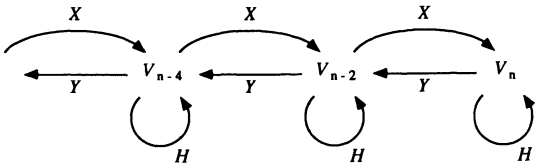
The action of Y on each V_α is similarly calculated; we have $Y(V_\alpha) \subset V_{\alpha-2}$.

Observe that as an immediate consequence of this and the irreducibility of V, *all the complex numbers α that appear in the decomposition* (11.3) *must be congruent to one another* mod 2: for any α_0 that actually occurs, the subspace

$$\bigoplus_{n \in \mathbb{Z}} V_{\alpha_0+2n}$$

would be invariant under $\mathfrak{sl}_2\mathbb{C}$ and hence equal to all of V. Moreover, by the same token, the V_α that appear must form an unbroken string of numbers of the form $\beta, \beta + 2, \ldots, \beta + 2k$. We denote by n the last element in this sequence; at this point we just know n is a complex number, but we will soon see that it must be an integer.

To proceed with our analysis, we have the following picture of the action of $\mathfrak{sl}_2\mathbb{C}$ on the vector space V:



Choose any nonzero vector $v \in V_n$; since $V_{n+2} = (0)$, we must have $X(v) = 0$. We ask now what happens when we apply the map Y to the vector v. To begin with, we have

Claim 11.4. *The vectors $\{v, Y(v), Y^2(v), \ldots\}$ span V.*

Proof. From the irreducibility of V it is enough to show that the subspace $W \subset V$ spanned by these vectors is carried into itself under the action of $\mathfrak{sl}_2\mathbb{C}$. Clearly, Y preserves W, since it simply carries the vector $Y^m(v)$ into $Y^{m+1}(v)$. Likewise, since the vector $Y^m(v)$ is in V_{n-2m}, we have $H(Y^m(v)) = (n - 2m) \cdot Y^m(v)$, so H preserves the subspace W. Thus, it suffices to check that $X(W) \subset W$, i.e., that for each m, X carries $Y^m(v)$ into a linear combination of the $Y^i(v)$. We check this in turn for $m = 0, 1, 2$, etc.

To begin with, we have $X(v) = 0 \in W$. To see what X does to $Y(v)$, we use

the commutation relations for $\mathfrak{sl}_2\mathbb{C}$: we have

$$\begin{aligned} X(Y(v)) &= [X, Y](v) + Y(X(v)) \\ &= H(v) + Y(0) \\ &= n \cdot v. \end{aligned}$$

Next, we see that

$$\begin{aligned} X(Y^2(v)) &= [X, Y](Y(v)) + Y(X(Y(v))) \\ &= H(Y(v)) + Y(n \cdot v) \\ &= (n-2) \cdot Y(v) + n \cdot Y(v). \end{aligned}$$

The pattern now is clear: X carries each vector in the sequence v, $Y(v)$, $Y^2(v)$, ... into a multiple of the previous vector. Explicitly, we have

$$X(Y^m(v)) = (n + (n-2) + (n-4) + \cdots + (n-2m+2)) \cdot Y^{m-1}(v),$$

or

$$X(Y^m(v)) = m(n-m+1) \cdot Y^{m-1}(v), \tag{11.5}$$

as can readily be verified by induction. □

There are a number of corollaries of the calculation in the above Claim. To begin with, we make the observation that

all the eigenspaces V_α of H are one dimensional. (11.6)

Second, since we have in the course of the proof written down a basis for V and said exactly where each of H, X, and Y takes each basis vector, the representation V is completely determined by the one complex number n that we started with; in particular, of course, we have that

V is determined by the collection of α occurring in the decomposition $V = \bigoplus V_\alpha$. (11.7)

To complete our analysis, we have to use one more time the finite dimensionality of V. This tells us that there is a lower bound on the α for which $V_\alpha \neq (0)$ as well as an upper one, so that we must have $Y^k(v) = 0$ for sufficiently large k. But now if m is the smallest power of Y annihilating v, then from the relation (11.5),

$$0 = X(Y^m(v)) = m(n-m+1) \cdot Y^{m-1}(v),$$

and the fact that $Y^{m-1}(v) \neq 0$, we conclude that $n - m + 1 = 0$; in particular, it follows that *n is a non-negative integer*. The picture is thus that the eigenvalues α of H on V form a string of integers differing by 2 and symmetric about the origin in $\mathbb{Z}$. In sum, then, we see that there is a unique representation $V^{(n)}$ for each non-negative integer n; the representation $V^{(n)}$ *is $(n+1)$-dimensional, with H having eigenvalues $n, n-2, \ldots, -n+2, -n$.*

Note that the existence part of this statement may be deduced by checking that the actions of H, X, and Y as given above in terms of the basis v, Yv, $Y^2(v), \ldots, Y^n(v)$ for V do indeed satisfy all the commutation relations for $\mathfrak{sl}_2\mathbb{C}$. Alternatively, we will exhibit them in a moment. Note also that by the symmetry of the eigenvalues we may deduce the useful fact that *any representation V of $\mathfrak{sl}_2\mathbb{C}$ such that the eigenvalues of H all have the same parity and occur with multiplicity one is necessarily irreducible*; more generally, *the number of irreducible factors in an arbitrary representation V of $\mathfrak{sl}_2\mathbb{C}$ is exactly the sum of the multiplicities of* 0 *and* 1 *as eigenvalues of H.*

We can identify in these terms some of the standard representations of $\mathfrak{sl}_2\mathbb{C}$. To begin with, the trivial one-dimensional representation $\mathbb{C}$ is clearly just $V^{(0)}$. As for the standard representation of $\mathfrak{sl}_2\mathbb{C}$ on $V = \mathbb{C}^2$, if x and y are the standard basis for $\mathbb{C}^2$, then we have $H(x) = x$ and $H(y) = -y$, so that $V = \mathbb{C}\cdot x \oplus \mathbb{C}\cdot y = V_{-1} \oplus V_1$ is just the representation $V^{(1)}$ above. Similarly, a basis for the symmetric square $W = \mathrm{Sym}^2 V = \mathrm{Sym}^2\mathbb{C}^2$ is given by $\{x^2, xy, y^2\}$, and we have

$$\begin{aligned} H(x\cdot x) &= x\cdot H(x) + H(x)\cdot x = 2x\cdot x, \\ H(x\cdot y) &= x\cdot H(y) + H(x)\cdot y = 0, \\ H(y\cdot y) &= y\cdot H(y) + H(y)\cdot y = -2y\cdot y, \end{aligned}$$

so the representation $W = \mathbb{C}\cdot x^2 \oplus \mathbb{C}\cdot xy \oplus \mathbb{C}\cdot y^2 = W_{-2} \oplus W_0 \oplus W_2$ is the representation $V^{(2)}$ above. More generally, the nth symmetric power $\mathrm{Sym}^n V$ of V has basis $\{x^n, x^{n-1}y, \ldots, y^n\}$, and we have

$$\begin{aligned} H(x^{n-k}y^k) &= (n-k)\cdot H(x)\cdot x^{n-k-1}y^k + k\cdot H(y)\cdot x^{n-k}y^{k-1} \\ &= (n-2k)\cdot x^{n-k}y^k \end{aligned}$$

so that the eigenvalues of H on $\mathrm{Sym}^n V$ are exactly $n, n-2, \ldots, -n$. By the observation above that a representation for which all eigenvalues of H occur with multiplicity 1 must be irreducible, it follows that $\mathrm{Sym}^n V$ is irreducible, and hence that

$$V^{(n)} = \mathrm{Sym}^n V.$$

In sum then, we can say simply that

> *Any irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ is a symmetric power of the standard representation $V \cong \mathbb{C}^2$.* (11.8)

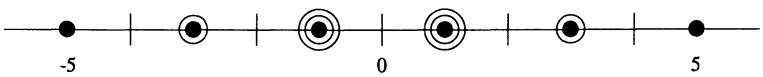
Observe that when we exponentiate the image of $\mathfrak{sl}_2\mathbb{C}$ under the embedding $\mathfrak{sl}_2\mathbb{C} \to \mathfrak{sl}_{n+1}\mathbb{C}$ corresponding to the representation $\mathrm{Sym}^n V$, we arrive at the group $\mathrm{SL}_2\mathbb{C}$ when n is odd and $\mathrm{PGL}_2\mathbb{C}$ when n is even. Thus, *the representations of the group $\mathrm{PGL}_2\mathbb{C}$ are exactly the even powers $\mathrm{Sym}^{2n} V$.*

Exercise 11.9. Use the analysis of the representations of $\mathfrak{sl}_2\mathbb{C}$ to prove the statement made in the previous lecture that the universal cover $\tilde{S}$ of $\mathrm{SL}_2\mathbb{R}$ has no finite-dimensional representations.

§11.2. A Little Plethysm

Clearly, knowing the eigenspace decomposition of given representations tells us the eigenspace decomposition of all their tensor, symmetric, and alternating products and powers: for example, if $V = \bigoplus V_\alpha$ and $W = \bigoplus W_\beta$ then $V \otimes W = \bigoplus (V_\alpha \otimes W_\beta)$ and $V_\alpha \otimes W_\beta$ is an eigenspace for H with eigenvalue $\alpha + \beta$. We can use this to describe the decomposition of these products and powers into irreducible representations of the algebra $\mathfrak{sl}_2\mathbb{C}$.

For example, let $V \cong \mathbb{C}^2$ be the standard representation of $\mathfrak{sl}_2\mathbb{C}$; and suppose we want to study the representation $\mathrm{Sym}^2 V \otimes \mathrm{Sym}^3 V$; we ask in particular whether if it irreducible and, if not, how it decomposes. We have seen that the eigenvalues of $\mathrm{Sym}^2 V$ are 2, 0, and -2, and those of $\mathrm{Sym}^3 V$ are 3, 1, -1, and -3. The 12 eigenvalues of the tensor product $\mathrm{Sym}^2 V \otimes \mathrm{Sym}^3 V$ are thus 5 and -5, 3 and -3 (taken twice), and 1 and -1 (taken three times); we may represent them by the diagram



The eigenvector with eigenvalue 5 will generate a subrepresentation of the tensor product isomorphic to $\mathrm{Sym}^5 V$, which will account for one occurrence of each of the eigenvalues 5, 3, 1, -1, -3, and -5. Similarly, the complement of $\mathrm{Sym}^5 V$ in the tensor product will have eigenvalues 3 and -3, and 1 and -1 (taken twice), and so will contain a copy of the representation $\mathrm{Sym}^3 V$, which will account for one occurrence of the eigenvalues 3, 1, -1 and -3; and the complement of these two subrepresentations will be simply a copy of V. We have, thus,

$$\mathrm{Sym}^2 V \otimes \mathrm{Sym}^3 V \cong \mathrm{Sym}^5 V \oplus \mathrm{Sym}^3 V \oplus V.$$

Note that the projection map

$$\mathrm{Sym}^2 V \otimes \mathrm{Sym}^3 V \to \mathrm{Sym}^5 V$$

on the first factor is just multiplication of polynomials; the other two projections do not admit such obvious interpretations.

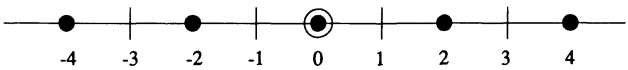
Exercise 11.10. Find, in a similar way, the decomposition of the tensor product $\mathrm{Sym}^2 V \otimes \mathrm{Sym}^5 V$.

Exercise 11.11*. Show, in general, that for $a \geq b$ we have

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V = \mathrm{Sym}^{a+b} V \oplus \mathrm{Sym}^{a+b-2} V \oplus \cdots \oplus \mathrm{Sym}^{a-b} V.$$

As indicated, we can also look at symmetric and exterior powers of given representations; in many ways this is more interesting. For example, let

$V \cong \mathbb{C}^2$ be as above the standard representation of $\mathfrak{sl}_2\mathbb{C}$, and let $W = \mathrm{Sym}^2 V$ be its symmetric square; i.e., in the notation introduced above, take $W = V^{(2)}$. We ask now whether the symmetric square of W is irreducible, and if not what its decomposition is. To answer this, observe that W has eigenvalues -2, 0, and 2, each occurring once, so that the symmetric square of W will have eigenvalues the pairwise sums of these numbers—that is, -4, -2, 0 (occurring twice), 2, and 4. We may represent $\mathrm{Sym}^2 V$ by the diagram:



From this, it is clear that the representation $\mathrm{Sym}^2 W$ must decompose into one copy of the representation $V^{(4)} = \mathrm{Sym}^4 V$, plus one copy of the trivial (one-dimensional) representation:

$$\mathrm{Sym}^2(\mathrm{Sym}^2 V)) = \mathrm{Sym}^4 V \oplus \mathrm{Sym}^0 V. \tag{11.12}$$

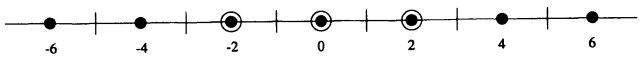
Indeed, we can see this directly: we have a natural map

$$\mathrm{Sym}^2(\mathrm{Sym}^2 V)) \to \mathrm{Sym}^4 V$$

obtained simply by evaluation; this will have a one-dimensional kernel (if x and y are as above the standard basis for V we can write a generator of this kernel as $(x^2)\cdot(y^2) - (x\cdot y)^2$).

Exercise 11.13. Show that the exterior square $\Lambda^2 W$ is isomorphic to W itself. Observe that this, together with the above description of $\mathrm{Sym}^2 W$, agrees with the decomposition of $W \otimes W$ given in Exercise 11.11 above.

We can, in a similar way, describe the decomposition of all the symmetric powers of the representation $W = \mathrm{Sym}^2 V$. For example, the third symmetric power $\mathrm{Sym}^3 W$ has eigenvalues given by the triple sums of the set $\{-2, 0, 2\}$; these are -6, -4, -2 (twice), 0 (twice), 2 (twice), 4, and 6; diagrammatically,



Again, there is no ambiguity about the decomposition; this collection of eigenspaces can only come from the direct sum of $\mathrm{Sym}^6 V$ with $\mathrm{Sym}^2 V$, so we must have

$$\mathrm{Sym}^3(\mathrm{Sym}^2 V) = \mathrm{Sym}^6 V \oplus \mathrm{Sym}^2 V.$$

As before, we can see at least part of this directly: we have a natural evaluation map

$$\mathrm{Sym}^3(\mathrm{Sym}^2 V) \to \mathrm{Sym}^6 V,$$

and the eigenspace decomposition tells us that the kernel is the irreducible representation $\mathrm{Sym}^2 V$.

Exercise 11.14. Use the eigenspace decomposition to establish the formula

$$\mathrm{Sym}^n(\mathrm{Sym}^2 V) = \bigoplus_{\alpha=0}^{[n/2]} \mathrm{Sym}^{2n-4\alpha} V$$

for all n.

§11.3. A Little Geometric Plethysm

We want to give some geometric interpretations of these and similar decompositions of higher tensor powers of representations of $\mathfrak{sl}_2\mathbb{C}$. One big difference is that instead of looking at the action of either the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ or the groups $\mathrm{SL}_2\mathbb{C}$ or $\mathrm{PGL}_2\mathbb{C}$ on a representation W, we look at the action of the group $\mathrm{PGL}_2\mathbb{C}$ on the associated projective space[1] $\mathbb{P}W$. In this context, it is natural to look at various geometric objects associated to the action: for example, we look at closures of orbits of the action, which all turn out to be algebraic varieties, i.e., definable by polynomial equations. In particular, our goal in the following will be to describe the symmetric and exterior powers of W in terms of the action of $\mathrm{PGL}_2\mathbb{C}$ on the projective spaces $\mathbb{P}W$ and various loci in $\mathbb{P}W$.

The main point is that while the action of $\mathrm{PGL}_2\mathbb{C}$ on the projective space $\mathbb{P}V \cong \mathbb{P}^1$ associated to the standard representation V is transitive, its action on the spaces $\mathbb{P}(\mathrm{Sym}^n V) \cong \mathbb{P}^n$ for $n > 1$ is not. Rather, the action will preserve various orbits whose closures are algebraic subvarieties of $\mathbb{P}^n$—for example, the locus of points

$$C = \{[v \cdot v \cdot \ldots \cdot v] : v \in V\} \subset \mathbb{P}(\mathrm{Sym}^n V)$$

corresponding to nth powers in $\mathrm{Sym}^n V$ will be an algebraic curve in $\mathbb{P}(\mathrm{Sym}^n V) \cong \mathbb{P}^n$, called the *rational normal curve*; and this curve will be carried into itself by any element of $\mathrm{PGL}_2\mathbb{C}$ acting on $\mathbb{P}^n$ (more about this in a moment). Thus, a knowledge of the geometry of these subvarieties of $\mathbb{P}W$ may illuminate the representation W, and vice versa. This approach is particularly useful in describing the symmetric powers of W, since these powers can be viewed as the vector spaces of homogeneous polynomials on the projective space $\mathbb{P}(W^*)$ (or, mod scalars, as hypersurfaces in that projective space). Decomposing these symmetric powers should therefore correspond to some interesting projective geometry.

[1] $\mathbb{P}W$ here denotes the projective space of lines through the origin in W, or the quotient space of $W \setminus \{0\}$ by multiplication by nonzero scalars; we write $[w]$ for the point in $\mathbb{P}W$ determined by the nonzero vector w. For $W = \mathbb{C}^{m+1}$, $[z_0, \ldots, z_m]$ is the point in $\mathbb{P}^m = \mathbb{P}W$ determined by a point $(z_0, \ldots, z_m)$ in $\mathbb{C}^{m+1}$.

Digression on Projective Geometry

First, as we have indicated, we want to describe representations of Lie groups in terms of the corresponding actions on projective spaces. The following fact from algebraic geometry is therefore of some moral (if not logical) importance:

Fact 11.15. The group of automorphisms of projective space $\mathbb{P}^n$—either as algebraic variety or as complex manifold—is just the group $\mathrm{PGL}_{n+1}\mathbb{C}$.

For a proof, see [Ha]. (For the Riemann sphere $\mathbb{P}^1$ at least, this should be a familiar fact from complex analysis.)

For any vector space W of dimension $n+1$, $\mathrm{Sym}^k W^*$ is the space of homogeneous polynomials of degree k on the projective space $\mathbb{P}^n = \mathbb{P}W$ of lines in W; dually, $\mathrm{Sym}^k W$ will be the space of homogeneous polynomials of degree k on the projective space $\mathbb{P}^n = \mathbb{P}(W^*)$ of lines in W^*, or of hyperplanes in W. Thus, the projective space $\mathbb{P}(\mathrm{Sym}^k W)$ is the space of hypersurfaces of degree k in $\mathbb{P}^n = \mathbb{P}(W^*)$. (Because of this duality, we usually work with objects in the projective space $\mathbb{P}(W^*)$ rather than the dual space $\mathbb{P}W$ in order to derive results about symmetric powers $\mathrm{Sym}^k W$; this may seem initially more confusing, but we believe it is ultimately less so.)

For any vector space V and any positive integer n, we have a natural map, called the *Veronese embedding*

$$\mathbb{P}V^* \hookrightarrow \mathbb{P}(\mathrm{Sym}^n V^*)$$

that maps the line spanned by $v \in V^*$ to the line spanned by $v^n \in \mathrm{Sym}^n V^*$. We will encounter the Veronese embedding of higher-dimensional vector spaces in later lectures; here we are concerned just with the case where V is two dimensional, so $\mathbb{P}V^* = \mathbb{P}^1$. In this case we have a map

$$\iota_n\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^n = \mathbb{P}(\mathrm{Sym}^n V^*)$$

whose image is called the *rational normal curve* $C = C_n$ *of degree* n. Choosing bases $\{\alpha, \beta\}$ for V^* and $\{\ldots[n!/k!(n-k)!]\alpha^k\beta^{n-k}\ldots\}$ for $\mathrm{Sym}^n V^*$ and expanding out $(x\alpha + y\beta)^n$ we see that in coordinates this map may be given as

$$[x, y] \mapsto [x^n, x^{n-1}y, x^{n-2}y^2, \ldots, xy^{n-1}, y^n].$$

From the definition, the action of $\mathrm{PGL}_2\mathbb{C}$ on $\mathbb{P}^n$ preserves C_n; conversely, since any automorphism of $\mathbb{P}^n$ fixing C_n pointwise is the identity, from Fact 11.15 it follows that *the group G of automorphisms of $\mathbb{P}^n$ that preserve C_n is precisely* $\mathrm{PGL}_2\mathbb{C}$. (Note that conversely if W is any $(n+1)$-dimensional representation of $\mathrm{SL}_2\mathbb{C}$ and $\mathbb{P}W \cong \mathbb{P}^n$ contains a rational normal curve of degree n preserved by the action of $\mathrm{PGL}_2\mathbb{C}$, then we must have $W \cong \mathrm{Sym}^n V$; we leave this as an exercise.[2])

When $n = 2$, C is the *plane conic* defined by the equation

[2] Note that any confusion between $\mathbb{P}W$ and $\mathbb{P}W^*$ is relatively harmless for us here, since the representations $\mathrm{Sym}^n V$ are isomorphic to their duals.

$$F(Z_0, Z_1, Z_2) = Z_0Z_2 - Z_1^2 = 0.$$

For $n = 3$, C is the *twisted cubic curve* in $\mathbb{P}^3$, and is defined by three quadratic polynomials

$$Z_0Z_2 - Z_1^2, \qquad Z_0Z_3 - Z_1Z_2, \qquad \text{and} \qquad Z_1Z_3 - Z_2^2.$$

More generally, the rational normal curve is the common zero locus of the 2×2 minors of the matrix

$$M = \begin{pmatrix} Z_0 Z_1 \dots Z_{n-1} \\ Z_1 Z_2 \dots \ Z_n \end{pmatrix},$$

that is, the locus where the rank of M is 1.

Back to Plethysm

We start with Example (11.12). We can interpret the decomposition given there (or rather the decomposition of the representation of the corresponding Lie group $\mathrm{SL}_2\mathbb{C}$) geometrically via the Veronese embedding $\iota_2\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$. As noted, $\mathrm{SL}_2\mathbb{C}$ acts on $\mathbb{P}^2 = \mathbb{P}(\mathrm{Sym}^2 V^*)$ as the group of motions of $\mathbb{P}^2$ carrying the conic curve C_2 into itself. Its action on the space $\mathrm{Sym}^2(\mathrm{Sym}^2 V))$ of quadratic polynomials on $\mathbb{P}^2$ thus must preserve the one-dimensional subspace $\mathbb{C} \cdot F$ spanned by the polynomial F above that defines the conic C_2. At the same time, we see that pullback via ι_2 defines a map from the space of quadratic polynomials on $\mathbb{P}^2$ to the space of quartic polynomials on $\mathbb{P}^1$, which has kernel $\mathbb{C} \cdot F$; thus, we have an exact sequence

$$0 \to \mathbb{C} = \mathrm{Sym}^0 V \to \mathrm{Sym}^2(\mathrm{Sym}^2 V)) \to \mathrm{Sym}^4 V \to 0,$$

which implies the decomposition of $\mathrm{Sym}^2(\mathrm{Sym}^2 V))$ described above.

Note that what comes to us at first glance is not actually the direct sum decomposition (11.12) of $\mathrm{Sym}^2(\mathrm{Sym}^2 V))$, but just the exact sequence above. The splitting of this sequence of $\mathrm{SL}_2\mathbb{C}$-modules, guaranteed by the general theory, is less obvious. For example, we are saying that given a conic curve C in the plane $\mathbb{P}^2$, there is a subspace U_C of the space of all conics in $\mathbb{P}^2$, complementary to the one-dimensional subspace spanned by C itself and invariant under the action of the group of motions of the plane $\mathbb{P}^2$ carrying C into itself. Is there a geometric description of this space? Yes: the following proposition gives one.

Proposition 11.16. *The subrepresentation* $\mathrm{Sym}^4 V \subset \mathrm{Sym}^2(\mathrm{Sym}^2 V)$ *is the space of conics spanned by the family of double lines tangent to the conic* $C = C_2$.

Proof. One way to prove this is to simply write out this subspace in coordinates: in terms of homogeneous coordinates Z_i on $\mathbb{P}^2$ as above, the tangent line to the conic C at the point $[1, \alpha, \alpha^2]$ is the line

$$L_\alpha = \{Z : \alpha^2 Z_0 - 2\alpha Z_1 + Z_2 = 0\}.$$

The double line $2L_\alpha$ is, thus, the conic with equation

$$\alpha^4 Z_0^2 - 4\alpha^3 Z_0 Z_1 + 2\alpha^2 Z_0 Z_2 + 4\alpha^2 Z_1^2 - 4\alpha Z_1 Z_2 + Z_2^2 = 0.$$

The subspace these conics generate is thus spanned by Z_0^2, $Z_0 Z_1$, $Z_1 Z_2$, Z_2^2, and $Z_0 Z_2 + 2Z_1^2$. By construction, this is invariant under the action of $\mathrm{SL}_2\mathbb{C}$, and it is visibly complementary to the trivial subrepresentation $\mathbb{C} \cdot F = \mathbb{C} \cdot (Z_0 Z_2 - Z_1^2)$.

For those familiar with some algebraic geometry, it may not be necessary to write all this down in coordinates: we could just observe that the map from the conic curve C to the projective space $\mathbb{P}(\mathrm{Sym}^2(\mathrm{Sym}^2 V))$ of conics in $\mathbb{P}^2$ sending each point $p \in C$ to the square of the tangent line to C at p is the restriction to C of the quadratic Veronese map $\mathbb{P}^2 \to \mathbb{P}^5$, and so has image a quartic rational normal curve. This spans a four-dimensional projective subspace of $\mathbb{P}(\mathrm{Sym}^2(\mathrm{Sym}^2 V))$, which must correspond to a subrepresentation isomorphic to $\mathrm{Sym}^4 V$. □

We will return to this notion in Exercise 11.26 below.

We can, in a similar way, describe the decomposition of all the symmetric powers of the representation $W = \mathrm{Sym}^2 V$; in the general setting, the geometric interpretation becomes quite handy. For example, we have seen that the third symmetric power decomposes

$$\mathrm{Sym}^3(\mathrm{Sym}^2 V) = \mathrm{Sym}^6 V \oplus \mathrm{Sym}^2 V.$$

This is immediate from the geometric description: the space of cubics in the plane $\mathbb{P}^2$ naturally decomposes into the space of cubics vanishing on the conic $C = C_2$, plus a complementary space isomorphic (via the pullback map ι_2^*) to the space of sextic polynomials on $\mathbb{P}^1$; moreover, since a cubic vanishing on C_2 factors into the quadratic polynomial F and a linear factor, the space of cubics vanishing on the conic curve $C \subset \mathbb{P}^2$ may be identified with the space of lines in $\mathbb{P}^2$.

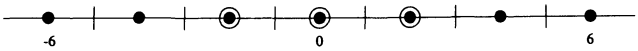
One more special case: from the general formula (11.14), we have

$$\mathrm{Sym}^4(\mathrm{Sym}^2 V) \cong \mathrm{Sym}^8 V \oplus \mathrm{Sym}^4 V \oplus \mathrm{Sym}^0 V.$$

Again, this is easy to see from the geometric picture: the space of quartic polynomials on $\mathbb{P}^2$ consists of the one-dimensional space of quartics spanned by the square of the defining equation F of C itself, plus the space of quartics vanishing on C modulo multiples of F^2, plus the space of quartics modulo those vanishing on C. (We use the word "plus," suggesting a direct sum, but as before only an exact sequence is apparent).

Exercise 11.17. Show that, in general, the order of vanishing on C defines a filtration on the space of polynomials of degree n in $\mathbb{P}^2$, whose successive quotients are the direct sum factors on the right hand side of the decomposition of Exercise 11.14.

We can similarly analyze symmetric powers of the representation $U = \mathrm{Sym}^3 V$. For example, since U has eigenvalues -3, -1, 1, and 3, the symmetric square of U has eigenvalues -6, -4, -2 (twice), 0 (twice), 2 (twice), 4, and 6; diagrammatically, we have



This implies that

$$\mathrm{Sym}^2(\mathrm{Sym}^3 V) \cong \mathrm{Sym}^6 V \oplus \mathrm{Sym}^2 V. \tag{11.18}$$

We can interpret this in terms of the twisted cubic $C = C_3 \subset \mathbb{P}^3$ as follows: the space of quadratic polynomials on $\mathbb{P}^3$ contains, as a subrepresentation, the three-dimensional vector space of quadrics containing the curve C itself; and the quotient is isomorphic, via the pullback map ι_3^*, to the space of sextic polynomials on $\mathbb{P}^1$.

Exercise 11.19*. By the above, the action of $\mathrm{SL}_2\mathbb{C}$ on the space of quadric surfaces containing the twisted cubic curve C is the same as its action on $\mathbb{P}(\mathrm{Sym}^2 V^*) \cong \mathbb{P}^2$. Make this explicit by associating to every quadric containing C a polynomial of degree 2 on $\mathbb{P}^1$, up to scalars.

Exercise 11.20*. The direct sum decomposition (11.18) says that there is a linear space of quadric surfaces in $\mathbb{P}^3$ preserved under the action of $\mathrm{SL}_2\mathbb{C}$ and complementary to the space of quadrics containing C. Describe this space.

Exercise 11.21. The projection map from $\mathrm{Sym}^2(\mathrm{Sym}^3 V)$ to $\mathrm{Sym}^2 V$ given by the decomposition (11.18) above may be viewed as a *quadratic* map from the vector space $\mathrm{Sym}^3 V$ to the vector space $\mathrm{Sym}^2 V$. Show that it may be given in these terms as the *Hessian*, that is, by associating to a homogeneous cubic polynomial in two variables the determinant of the 2×2 matrix of its second partials.

Exercise 11.22. The map in the preceding exercise may be viewed as associating to an unordered triple of points $\{p, q, r\}$ in $\mathbb{P}^1$ an unordered pair of points $\{s, t\} \subset \mathbb{P}^1$. Show that this pair of points is the pair of fixed points of the automorphism of $\mathbb{P}^1$ permuting the three points p, q, and r cyclically.

Exercise 11.23*. Show that

$$\mathrm{Sym}^3(\mathrm{Sym}^3 V) = \mathrm{Sym}^9 V \oplus \mathrm{Sym}^5 V \oplus \mathrm{Sym}^3 V,$$

and interpret this in terms of the geometry of the twisted cubic curve. In particular, show that the space of cubic surfaces containing the curve is the direct sum of the last two factors, and identify the subspace of cubics corresponding to the last factor.

Exercise 11.24. Analyze the representation $\mathrm{Sym}^4(\mathrm{Sym}^3 V)$ similarly. In particular, show that it contains a trivial one-dimensional subrepresentation.

The trivial subrepresentation of $\mathrm{Sym}^4(\mathrm{Sym}^3 V)$ found in the last exercise has an interesting interpretation. To say that $\mathrm{Sym}^4(\mathrm{Sym}^3 V)$ has such an invariant one-dimensional subspace is to say that *there exists a quartic surface in $\mathbb{P}^3$ preserved under all motions of $\mathbb{P}^3$ carrying the rational normal curve $C = C_3$ into itself.* What is this surface? The answer is simple: it is the *tangent developable* to the twisted cubic, that is, the surface given as the union of the tangent lines to C.

Exercise 11.25*. Show that the representation $\mathrm{Sym}^3(\mathrm{Sym}^4 V)$ contains a trivial subrepresentation, and interpret this geometrically.

Problem 11.26. Another way of interpreting the direct sum decomposition of $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ geometrically is to say that given a conic curve $C \subset \mathbb{P}^2$ and given four points on C, we can find a conic $C' = C'(C; p_1, \dots, p_4) \subset \mathbb{P}^2$ intersecting C in exactly these points, in a way that is preserved by the action of the group $\mathrm{PGL}_3\mathbb{C}$ of all motions of $\mathbb{P}^2$ (i.e., for any motion $A\colon \mathbb{P}^2 \to \mathbb{P}^2$ of the plane, we have $A(C'(C; p_1, \dots, p_4)) = C'(AC; Ap_1, \dots, Ap_4)$). What is a description of this process? In particular, show that the cross-ratio of the four points p_i on the curve C' must be a function of the cross-ratio of the p_i on C, and find this function. Observe also that this process gives an endomorphism of the pencil

$$\{C \subset \mathbb{P}^2 : p_1, \dots, p_4 \in C\} \cong \mathbb{P}^1$$

of conics passing through any four points $p_i \in \mathbb{P}^2$. What is the degree of this endomorphism?

The above questions have all dealt with the symmetric powers of $\mathrm{Sym}^n V$. There are also interesting questions about the exterior powers of $\mathrm{Sym}^n V$. To start with, consider the exterior square $\wedge^2(\mathrm{Sym}^3 V)$. The eigenvalues of this representation are just the pairwise sums of distinct elements of $\{3, 1, -1, -3\}$, that is, 4, 2, 0 (twice), -2, and -4; we deduce that

$$\wedge^2(\mathrm{Sym}^3 V) \cong \mathrm{Sym}^4 V \oplus \mathrm{Sym}^0 V. \tag{11.27}$$

Observe in particular that according to this there is a skew-symmetric bilinear form on the space $U = \mathrm{Sym}^3 V$ preserved (up to scalars) by the action of $\mathrm{SL}_2\mathbb{C}$. What is this form? One way of describing it would be in terms of the twisted cubic: the map from C to the dual projective space $(\mathbb{P}^3)^*$ sending each point $p \in C$ to the osculating plane to C at p extends to a skew-symmetric linear isomorphism of $\mathbb{P}^3$ with $(\mathbb{P}^3)^*$.

Exercise 11.28. Show that a line in $\mathbb{P}^3$ is isotropic for this form if and only if, viewed as an element of $\mathbb{P}(\wedge^2 U)$, it lies in the linear span of the locus of tangent lines to the twisted cubic.

Exercise 11.29. Show that the projection on the first factor in the decomposition (11.27) is given explicitly by the map

$$F \wedge G \mapsto F \cdot dG - G \cdot dF$$

and say precisely what this means.

Exercise 11.30. Show that, in general, the representation $\bigwedge^2(\mathrm{Sym}^n V)$ has as a direct sum factor the representation $\mathrm{Sym}^{2n-2} V$, and that the projection on this factor is given as in the preceding exercise. Find the remaining factors of $\bigwedge^2(\mathrm{Sym}^n V)$, and interpret them.

More on Rational Normal Curves

Exercise 11.31. Analyze in general the representations $\mathrm{Sym}^2(\mathrm{Sym}^n V)$; show, using eigenvalues, that we have

$$\mathrm{Sym}^2(\mathrm{Sym}^n V) = \bigoplus_{\alpha \geq 0} \mathrm{Sym}^{2n-4\alpha} V.$$

Exercise 11.32*. Interpret the space $\mathrm{Sym}^2(\mathrm{Sym}^n V)$ of the preceding exercise as the space of quadrics in the projective space $\mathbb{P}^n$, and use the geometry of the rational normal curve $C = C_n \subset \mathbb{P}^n$ to interpret the decomposition of this representation into irreducible factors. In particular, show that direct sum

$$\bigoplus_{\alpha \geq 1} \mathrm{Sym}^{2n-4\alpha} V$$

is the space of quadratic polynomials vanishing on the rational normal curve; and that the direct sum

$$\bigoplus_{\alpha \geq 2} \mathrm{Sym}^{2n-4\alpha} V$$

is the space of quadrics containing the *tangential developable* of the rational normal curve, that is, the union of the tangent lines to C. Can you interpret the sums for $\alpha \geq k$ for $k > 2$?

Exercise 11.33. Note that by Exercise 11.11, the tensor power

$$\mathrm{Sym}^n V \otimes \mathrm{Sym}^n V$$

always contains a copy of the trivial representation; and that by Exercises 11.30 and 11.31, this trivial subrepresentation will lie in $\mathrm{Sym}^2(\mathrm{Sym}^n V)$ if n is even and in $\bigwedge^2(\mathrm{Sym}^n V)$ if n is odd. Show that in either case, the bilinear form on $\mathrm{Sym}^n V$ preserved by $\mathrm{SL}_2\mathbb{C}$ may be described as the isomorphism of $\mathbb{P}^n$ with $(\mathbb{P}^n)^*$ carrying each point p of the rational normal curve $C \subset \mathbb{P}^n$ into the osculating hyperplane to C at p.

Comparing Exercises 11.14 and 11.31, we see that $\mathrm{Sym}^2(\mathrm{Sym}^n V) \cong \mathrm{Sym}^n(\mathrm{Sym}^2 V)$; apparently coincidentally. This is in fact a special case of a more general theorem (cf. Exercise 6.18):

Exercise 11.34. (Hermite Reciprocity). Use the eigenvalues of H to prove the isomorphism

$$\mathrm{Sym}^k(\mathrm{Sym}^n V) \cong \mathrm{Sym}^n(\mathrm{Sym}^k V).$$

Can you exhibit explicitly a map between these two?

Note that in the examples of Hermite reciprocity we have seen, it seems completely coincidental: for example, the fact that the representations $\mathrm{Sym}^3(\mathrm{Sym}^4 V)$ and $\mathrm{Sym}^4(\mathrm{Sym}^3 V)$ both contain a trivial representation corresponds to the facts that the tangential developable of the twisted cubic in $\mathbb{P}^3$ has degree 4, while the chordal variety of the rational normal quartic in $\mathbb{P}^4$ has degree 3.

Exercise 11.35*. Show that $\Lambda^m(\mathrm{Sym}^n V) \cong \mathrm{Sym}^m(\mathrm{Sym}^{n+1-m} V)$.

We will see in Lecture 23 that there is a unique closed orbit in $\mathbb{P}(W)$ for any irreducible representation W. For now, we can do the following special case.

Exercise 11.36. Show that the unique closed orbit of the action of $\mathrm{SL}_2\mathbb{C}$ on the projectivization of any irreducible representation is isomorphic to $\mathbb{P}^1$ (these are the *rational normal curves* introduced above).