LECTURE 12

Representations of $\mathfrak{sl}_3\mathbb{C}$, Part I

This lecture develops results for $\mathfrak{sl}_3\mathbb{C}$ analogous to those of §11.1 (though not in exactly the same order). This involves generalizing some of the basic terms of §11 (e.g., the notions of eigenvalue and eigenvector have to be redefined), but the basic ideas are in some sense already in §11. Certainly no techniques are involved beyond those of §11.1.

We come now to a second important stage in the development of the theory: in the following, we will take our analysis of the representations of $\mathfrak{sl}_2\mathbb{C}$ and see how it goes over in the next case, the algebra $\mathfrak{sl}_3\mathbb{C}$. As we will see, a number of the basic constructions need to be modified, or at least rethought. There are, however, two pieces of good news that should be borne in mind. First, we will arrive, by the end of the following lecture, at a classification of the representations of $\mathfrak{sl}_3\mathbb{C}$ that is every bit as detailed and explicit as the classification we arrived at previously for $\mathfrak{sl}_2\mathbb{C}$. Second, once we have redone our analysis in this context, *we will need to introduce no further concepts to carry out the classification of the finite-dimensional representations of all remaining semisimple Lie algebras.*

We will proceed by analogy with the previous lecture. To begin with, we started out our analysis of $\mathfrak{sl}_2\mathbb{C}$ with the basis $\{H, X, Y\}$ for the Lie algebra; we then proceeded to decompose an arbitrary representation V of $\mathfrak{sl}_2\mathbb{C}$ into a direct sum of eigenspaces for the action of H. What element of $\mathfrak{sl}_3\mathbb{C}$ in particular will play the role of H? The answer—and this is the first and perhaps most wrenching change from the previous case—is that no one element really allows us to see what is going on.[1] Instead, we have to replace

[1] This is not literally true: as we will see from the following analysis, if H is any diagonal matrix whose entries are independent over $\mathbb{Q}$, then the action of H on any representation V of $\mathfrak{sl}_3\mathbb{C}$ determines the representation (i.e., if we know the eigenvalues of H we know V). But (as we will also see) trying to carry this out in practice would be sheer perversity.

the single element $H \in \mathfrak{sl}_2\mathbb{C}$ with a *subspace* $\mathfrak{h} \subset \mathfrak{sl}_3\mathbb{C}$, namely, the two-dimensional subspace of all diagonal matrices. The idea is a basic one; it comes down to the observation that *commuting diagonalizable matrices are simultaneously diagonalizable*. This translates in the present circumstances to the statement that any finite-dimensional representation V of $\mathfrak{sl}_3\mathbb{C}$ admits a decomposition $V = \bigoplus V_\alpha$, where every vector $v \in V_\alpha$ is an eigenvector for every element $H \in \mathfrak{h}$.

At this point some terminology is clearly in order, since we will be dealing with the action not of a single matrix H but rather a vector space $\mathfrak{h}$ of them. To begin with, by an *eigenvector* for $\mathfrak{h}$ we will mean, reasonably enough, a vector $v \in V$ that is an eigenvector for every $H \in \mathfrak{h}$. For such a vector v we can write

$$H(v) = \alpha(H)\cdot v, \tag{12.1}$$

where $\alpha(H)$ is a scalar depending linearly on H, i.e., $\alpha \in \mathfrak{h}^*$. This leads to our second notion: by an *eigenvalue* for the action of $\mathfrak{h}$ we will mean an element $\alpha \in \mathfrak{h}^*$ such that there exists a nonzero element $v \in V$ satisfying (12.1); and by the *eigenspace* associated to the eigenvalue α we will mean the subspace of all vectors $v \in V$ satisfying (12.1). Thus we may phrase the statement above as

(12.2) *Any finite-dimensional representation V of $\mathfrak{sl}_3\mathbb{C}$ has a decomposition*

$$V = \bigoplus V_\alpha,$$

where V_α is an eigenspace for $\mathfrak{h}$ and α ranges over a finite subset of $\mathfrak{h}^*$.

This is, in fact, a special case of a more general statement: for any semisimple Lie algebra $\mathfrak{g}$, we will be able to find an abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$, such that the action of $\mathfrak{h}$ on any $\mathfrak{g}$-module V will be diagonalizable, i.e., we will have a direct sum decomposition of V into eigenspaces V_α for $\mathfrak{h}$.

Having decided what the analogue for $\mathfrak{sl}_3\mathbb{C}$ of $H \in \mathfrak{sl}_2\mathbb{C}$ is, let us now consider what will play the role of X and Y. The key here is to look at the commutation relations

$$[H, X] = 2X \quad \text{and} \quad [H, Y] = -2Y$$

in $\mathfrak{sl}_2\mathbb{C}$. The correct way to interpret these is as saying that *X and Y are eigenvectors for the adjoint action of H on* $\mathfrak{sl}_2\mathbb{C}$. In our present circumstances, then, we want to look for eigenvectors (in the new sense) for the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}_3\mathbb{C}$. In other words, we apply (12.2) to the adjoint representation of $\mathfrak{sl}_3\mathbb{C}$ to obtain a decomposition

$$\mathfrak{sl}_3\mathbb{C} = \mathfrak{h} \oplus \left(\bigoplus \mathfrak{g}_\alpha\right), \tag{12.3}$$

where α ranges over a finite subset of $\mathfrak{h}^*$ and $\mathfrak{h}$ acts on each space $\mathfrak{g}_\alpha$ by scalar multiplication, i.e., for any $H \in \mathfrak{h}$ and $Y \in \mathfrak{g}_\alpha$,

$$[H, Y] = \mathrm{ad}(H)(Y) = \alpha(H)\cdot Y.$$

This is probably easier to carry out in practice than it is to say; we are being longwinded here because once this process is understood it will be straightforward to apply it to the other Lie algebras. In any case, to do it in the present circumstances, we just observe that multiplication of a matrix M on the left by a diagonal matrix D with entries a_i multiplies the ith row of M by a_i, while multiplication on the right multiplies the ith column by a_i; if the entries of M are $m_{i,j}$, the entries of the commutator $[D, M]$ are thus $(a_i - a_j)m_{i,j}$. We see then that the commutator $[D, M]$ will be a multiple of M for all D if and only if all but one entry of M are zero. Thus, if we let $E_{i,j}$ be the 3×3 matrix whose (i, j)th entry is 1 and all of whose other entries are 0, we see that the $E_{i,j}$ exactly generate the eigenspaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$.

Explicitly, we have

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} : a_1 + a_2 + a_3 = 0 \right\}$$

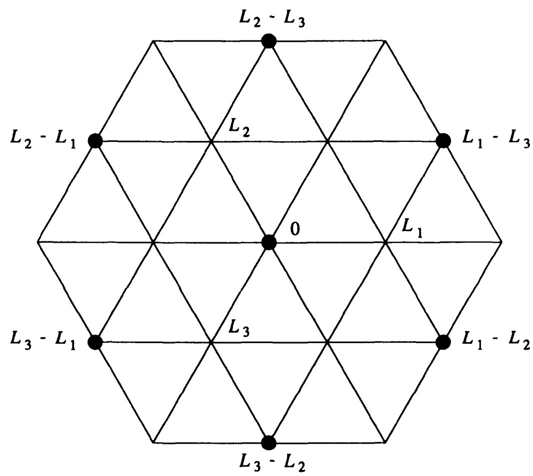
and so we can write

$$\mathfrak{h}^* = \mathbb{C}\{L_1, L_2, L_3\}/(L_1 + L_2 + L_3 = 0)\},$$

where

$$L_i \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i.$$

The linear functionals $\alpha \in \mathfrak{h}^*$ appearing in the direct sum decomposition (12.3) are thus the six functionals $L_i - L_j$; the space $\mathfrak{g}_{L_i - L_j}$ will be generated by the element $E_{i,j}$. To draw a picture



(12.4)

The virtue of this decomposition and the corresponding picture is that we can read off from it pretty much the entire structure of the Lie algebra. Of course, the action of $\mathfrak{h}$ on $\mathfrak{g}$ is clear from the picture: $\mathfrak{h}$ carries each of the subspaces $\mathfrak{g}_\alpha$ into itself, acting on each $\mathfrak{g}_\alpha$ by scalar multiplication by the linear functional represented by the corresponding dot. Beyond that, though, we can also see, much as in the case of representations of $\mathfrak{sl}_2\mathbb{C}$, how the rest of the Lie algebra acts. Basically, we let X be any element of $\mathfrak{g}_\alpha$ and ask where ad(X) sends a given vector $Y \in \mathfrak{g}_\beta$; the answer as before comes from knowing how $\mathfrak{h}$ acts on ad$(X)(Y)$. Explicitly, we let H be an arbitrary element of $\mathfrak{h}$ and as on page 148 we make the

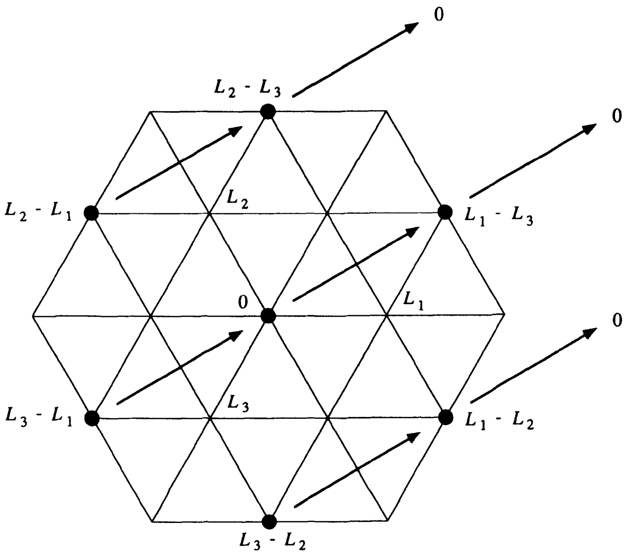
Fundamental Calculation (second time):

$$\begin{aligned}[H, [X, Y]] &= [X, [H, Y]] + [[H, X], Y] \\ &= [X, \beta(H)\cdot Y] + [\alpha(H)\cdot X, Y] \\ &= (\alpha(H) + \beta(H))\cdot[X, Y].\end{aligned}$$

In other words, $[X, Y] = \mathrm{ad}(X)(Y)$ *is again an eigenvector for* $\mathfrak{h}$, *with eigenvalue* $\alpha + \beta$. Thus,

$$\mathrm{ad}(\mathfrak{g}_\alpha)\colon \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta};$$

in particular, the action of ad$(\mathfrak{g}_\alpha)$ preserves the decomposition (12.3) in the sense that it carries each eigenspace $\mathfrak{g}_\beta$ into another. We can interpret this in terms of the diagram (12.4) of eigenspaces by saying that each $\mathfrak{g}_\alpha$ acts, so to speak, by "translation"; that is, it carries each space $\mathfrak{g}_\beta$ corresponding to a dot in the diagram into the subspace $\mathfrak{g}_{\alpha+\beta}$ corresponding to that dot translated by α. For example, the action of $\mathfrak{g}_{L_1-L_3}$ may be pictured as



(12.5)

i.e., it carries $\mathfrak{g}_{L_2-L_1}$ into $\mathfrak{g}_{L_2-L_3}$; $\mathfrak{g}_{L_3-L_1}$ into $\mathfrak{h}$; $\mathfrak{h}$ into $\mathfrak{g}_{L_1-L_3}$, $\mathfrak{g}_{L_3-L_2}$ into $\mathfrak{g}_{L_1-L_2}$, and kills $\mathfrak{g}_{L_2-L_3}$, $\mathfrak{g}_{L_1-L_3}$, and $\mathfrak{g}_{L_1-L_2}$. Of course, not all the data can be read off of the diagram, at least on the basis on what we have said so far. For example, we do not at present see from the diagram the kernel of $\mathrm{ad}(\mathfrak{g}_{L_1-L_3})$ on $\mathfrak{h}$, though we will see later how to read this off as well. We do, however, have at least a pretty good idea of who is doing what to whom.

Pretty much the same picture applies to any representation V of $\mathfrak{sl}_3\mathbb{C}$: we start from the eigenspace decomposition $V = \bigoplus V_\alpha$ for the action of $\mathfrak{h}$ that we saw in (12.2). Next, the commutation relations for $\mathfrak{sl}_3\mathbb{C}$ tell us exactly how the remaining summands of the decomposition (12.3) of $\mathfrak{sl}_3\mathbb{C}$ act on the space V, and again we will see that each of the spaces $\mathfrak{g}_\alpha$ acts by carrying one eigenspace V_β into another. As usual, for any $X \in \mathfrak{g}_\alpha$ and $v \in V_\beta$ we can tell where X will send v if we know how an arbitrary element $H \in \mathfrak{h}$ will act on $X(v)$. This we can determine by making the

Fundamental Calculation (*third time*):

$$\begin{aligned} H(X(v)) &= X(H(v)) + [H, X](v) \\ &= X(\beta(H)\cdot v) + (\alpha(H)\cdot X)(v) \\ &= (\alpha(H) + \beta(H))\cdot X(v). \end{aligned}$$

We see from this that $X(v)$ *is again an eigenvector for the action of* $\mathfrak{h}$, *with eigenvalue* $\alpha + \beta$; in other words, the action of $\mathfrak{g}_\alpha$ carries V_β to $V_{\alpha+\beta}$. We can thus represent the eigenspaces V_α of V by dots in a plane diagram so that each $\mathfrak{g}_\alpha$ acts again "by translation," as we did for representations of $\mathfrak{sl}_2\mathbb{C}$ in the preceding lecture and the adjoint representation of $\mathfrak{sl}_3\mathbb{C}$ above. Just as in the case of $\mathfrak{sl}_2\mathbb{C}$ (page 148), we have

Observation 12.6. *The eigenvalues α occurring in an irreducible representation of $\mathfrak{sl}_3\mathbb{C}$ differ from one other by integral linear combinations of the vectors $L_i - L_j \in \mathfrak{h}^*$.*

Note that these vectors $L_i - L_j$ generate a lattice in $\mathfrak{h}^*$, which we will denote by Λ_R, and that all the α lie in some translate of this lattice.

At this point, we should begin to introduce some of the terminology that appears in this subject. The basic object here, the eigenvalue $\alpha \in \mathfrak{h}^*$ of the action of $\mathfrak{h}$ on a representation V of $\mathfrak{g}$, is called a *weight* of the representation; the corresponding eigenvectors in V_α are called, naturally enough, *weight vectors* and the spaces V_α themselves *weight spaces*. Clearly, the weights that occur in the adjoint representation are special; these are called the *roots* of the Lie algebra and the corresponding subspaces $\mathfrak{g}_\alpha \subset \mathfrak{g}$ *root spaces*; by convention, zero is not a root. The lattice $\Lambda_R \subset \mathfrak{h}^*$ generated by the roots α is called the *root lattice*.

To see what the next step should be, we go back to the analysis of representations of $\mathfrak{sl}_2\mathbb{C}$. There, at this stage we continued our analysis by going to an extremal eigenspace V_α and taking a vector $v \in V_\alpha$. The point was that since V_α was extremal, the operator X, which would carry V_α to $V_{\alpha+2}$, would have to kill v; so that v would be then both an eigenvector for H and in the kernel of X. We then saw that these two facts allowed us to completely describe the representation V in terms of images of v.

What would be the appropriately analogous setup in the case of $\mathfrak{sl}_3\mathbb{C}$? To start at the beginning, there is the question of what we mean by *extremal*: in the case of $\mathfrak{sl}_2\mathbb{C}$, since we knew that all the eigenvalues were scalars differing by integral multiples of 2, there was not much ambiguity about what we meant by this. In the present circumstance this does involve a priori a choice (though as we shall see the choice does not affect the outcome): we have to choose a direction, and look for the farthest α in that direction appearing in the decomposition (12.3). What this means is that we should choose a linear functional

$$l\colon \Lambda_R \to \mathbb{R},$$

extend it by linearity to a linear functional $l\colon \mathfrak{h}^* \to \mathbb{C}$, and then for any representation V we should go to the eigenspace V_α for which the real part of $l(\alpha)$ is maximal.[2] Of course, to avoid ambiguity we should choose l to be irrational with respect to the lattice Λ_R, that is, to have no kernel.

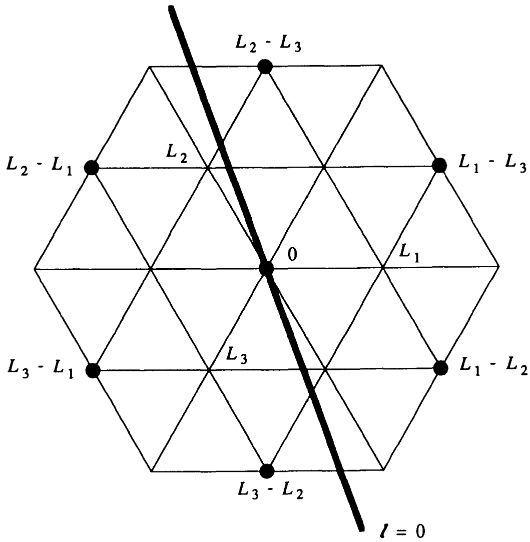
What is the point of this? The answer is that, just as in the case of a representation V of $\mathfrak{sl}_2\mathbb{C}$ we found in this way a vector $v \in V$ that was simultaneously in the kernel of the operator X and an eigenvector for H, in the present case what we will find is a vector $v \in V_\alpha$ that is an eigenvector for $\mathfrak{h}$, *and at the same time in the kernel of the action of* $\mathfrak{g}_\beta$ *for every* β *such that* $l(\beta) > 0$—that is, *that is killed by half the root spaces* $\mathfrak{g}_\beta$ (specifically, the root spaces corresponding to dots in the diagram (12.4) lying in a half plane). This will likewise give us a nearly complete description of the representation V.

To carry this out explicitly, choose our functional l to be given by

$$l(a_1L_1 + a_2L_2 + a_3L_3) = aa_1 + ba_2 + ca_3,$$

where $a + b + c = 0$ and $a > b > c$, so that the spaces $\mathfrak{g}_\alpha \subset \mathfrak{g}$ for which we have $l(\alpha) > 0$ are then exactly $\mathfrak{g}_{L_1-L_3}$, $\mathfrak{g}_{L_2-L_3}$, and $\mathfrak{g}_{L_1-L_2}$; they correspond to matrices with one nonzero entry above the diagonal.

[2] The real-versus-complex business is a red herring since (it will turn out very shortly) all the eigenvalues α actually occurring in any representation will in fact be in the real (in fact, the rational) linear span of Λ_R.



(12.7)

Thus, for $i < j$, the matrices $E_{i,j}$ *generate the positive root spaces,* and the $E_{j,i}$ *generate the negative root spaces.* We set

$$H_{i,j} = [E_{i,j}, E_{j,i}] = E_{i,i} - E_{j,j}. \tag{12.8}$$

Now let V be any irreducible, finite-dimensional representation of $\mathfrak{sl}_3\mathbb{C}$. The upshot of all the above is the

Lemma 12.9. *There is a vector $v \in V$ with the properties that*

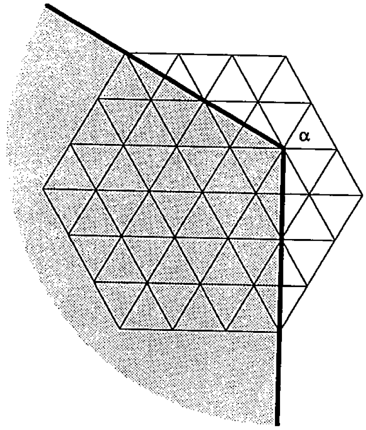
(i) *v is an eigenvector for $\mathfrak{h}$, i.e. $v \in V_\alpha$ for some α; and*
(ii) *v is killed by $E_{1,2}$, $E_{1,3}$, and $E_{2,3}$.*

For any representation V of $\mathfrak{sl}_3\mathbb{C}$, a vector $v \in V$ with these properties is called a *highest weight vector.*

In the case of $\mathfrak{sl}_2\mathbb{C}$, having found an eigenvector v for H killed by X, we argued that the images of v under successive applications of Y generated the representation. The situation here is the same: analogous to Claim 11.4 we have

Claim 12.10. *Let V be an irreducible representation of $\mathfrak{sl}_3\mathbb{C}$, and $v \in V$ a highest weight vector. Then V is generated by the images of v under successive applications of the three operators $E_{2,1}$, $E_{3,1}$, and $E_{3,2}$.*

Before we check the claim, we note three immediate consequences. First, it says that all the eigenvalues $\beta \in \mathfrak{h}^*$ occurring in V lie in a sort of $\frac{1}{3}$-plane with corner at α:



Second, we see that the dimension of V_α itself is 1, so that v is the unique eigenvector with this eigenvalue (up to scalars, of course). (We will see below that in fact v is the unique highest weight vector of V up to scalars; see Proposition 12.11.) Lastly, it says that the spaces $V_{\alpha+n(L_2-L_1)}$ and $V_{\alpha+n(L_3-L_2)}$ are all at most one dimensional, since they must be spanned by $(E_{2,1})^n(v)$ and $(E_{3,2})^n(v)$, respectively.

Proof of Claim 12.10. This is formally the same as the proof of the corresponding statement for $\mathfrak{sl}_2\mathbb{C}$: we argue that the subspace W of V spanned by images of v under the subalgebra of $\mathfrak{sl}_3\mathbb{C}$ generated by $E_{2,1}$, $E_{3,1}$, and $E_{3,2}$ is, in fact, preserved by all of $\mathfrak{sl}_3\mathbb{C}$ and hence must be all of V. To do this we just have to check that $E_{1,2}$, $E_{2,3}$, and $E_{1,3}$ carry W into itself (in fact it is enough to do this for the first two, the third being their commutator), and this is straightforward. To begin with, v itself is in the kernel of $E_{1,2}$, $E_{2,3}$, and $E_{1,3}$, so there is no problem there. Next we check that $E_{2,1}(v)$ is kept in W: we have

$$\begin{aligned} E_{1,2}(E_{2,1}(v)) &= (E_{2,1}(E_{1,2}(v)) + [E_{1,2}, E_{2,1}](v) \\ &= \alpha([E_{1,2}, E_{2,1}])\cdot v \end{aligned}$$

since $E_{1,2}(v) = 0$ and $[E_{1,2}, E_{2,1}] \in \mathfrak{h}$; and

$$\begin{aligned} E_{2,3}(E_{2,1}(v)) &= (E_{2,1}(E_{2,3}(v)) + [E_{2,3}, E_{2,1}](v) \\ &= 0 \end{aligned}$$

since $E_{2,3}(v) = 0$ and $[E_{2,3}, E_{2,1}] = 0$. A similar computation shows that $E_{3,2}(v)$ is also carried into V by $E_{1,2}$ and $E_{2,3}$.

More generally, we may argue the claim by a sort of induction: we let w_n denote any word of length n or less in the letters $E_{2,1}$ and $E_{3,2}$ and take W_n to be the vector space spanned by the vectors $w_n(v)$ for all such words; note that W is the union of the spaces W_n, since $E_{3,1}$ is the commutator of $E_{3,2}$ and $E_{2,1}$. We claim that $E_{1,2}$ and $E_{2,3}$ carry W_n into W_{n-1}. To see this, we can write w_n as either $E_{2,1} \circ w_{n-1}$ or $E_{3,2} \circ w_{n-1}$; in either case $w_{n-1}(v)$ will be an eigenvector for $\mathfrak{h}$ with eigenvalue β for some β. In the former case we have

$$\begin{aligned} E_{1,2}(w_n(v)) &= E_{1,2}(E_{2,1}(w_{n-1}(v))) \\ &= E_{2,1}(E_{1,2}(w_{n-1}(v))) + [E_{1,2}, E_{2,1}](w_{n-1}(v)) \\ &\in E_{2,1}(W_{n-2}) + \beta([E_{1,2}, E_{2,1}])\cdot w_{n-1}(v) \\ &\subset W_{n-1} \end{aligned}$$

since $[E_{1,2}, E_{2,1}] \in \mathfrak{h}$; and

$$\begin{aligned} E_{2,3}(w_n(v)) &= E_{2,3}(E_{2,1}(w_{n-1}(v))) \\ &= E_{2,1}(E_{2,3}(w_{n-1}(v))) + [E_{2,3}, E_{2,1}](w_{n-1}(v)) \\ &\in E_{2,1}(W_{n-2}) \\ &\subset W_{n-1} \end{aligned}$$

since $[E_{2,3}, E_{2,1}] = 0$. Essentially the same calculation covers the latter case $w_n = E_{3,2} \circ w_{n-1}$, establishing the claim. □

This argument shows a little more; in fact, it proves

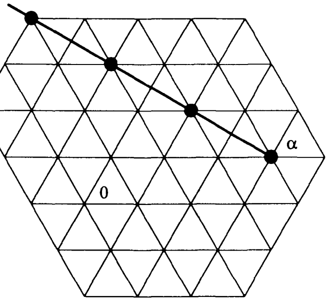
Proposition 12.11. *If V is any representation of $\mathfrak{sl}_3\mathbb{C}$ and $v \in V$ is a highest weight vector, then the subrepresentation W of V generated by the images of v by successive applications of the three operators $E_{2,1}$, $E_{3,1}$, and $E_{3,2}$ is irreducible.*

PROOF. Let α be the weight of v. The above shows that W is a subrepresentation, and it is clear that W_α is one dimensional. If W were not irreducible, we would have $W = W' \oplus W''$ for some representations W' and W''. But since projection to W' and W'' commute with the action of $\mathfrak{h}$, we have $W_\alpha = W'_\alpha \oplus W''_\alpha$. This shows that one of these spaces is zero, which implies that v belongs to W' or W'', and hence that W is W' or W''. □

As a corollary of this proposition we see that any irreducible representation of $\mathfrak{sl}_3\mathbb{C}$ has a unique highest weight vector, up to scalars; more generally, the set of highest weight vectors in V forms a union of linear subspaces Ψ_W corresponding to the irreducible subrepresentations W of V, with the dimension of Ψ_W equal to the number of times W appears in the direct sum decomposition of V into irreducibles.

What do we do next? Well, let us continue to look at the border vectors $(E_{2,1})^k(v)$. We call these border vectors because they live in (and, as we saw, span) a collection of eigenspaces $\mathfrak{g}_\alpha$, $\mathfrak{g}_{\alpha+L_2-L_1}$, $\mathfrak{g}_{\alpha+2(L_2-L_1)}$, ... that correspond to points on the boundary of the diagram above of possible eigenvalues of V. We also know that they span an uninterrupted string of nonzero eigenspaces $\mathfrak{g}_{\alpha+k(L_2-L_1)} \cong \mathbb{C}$, $k = 0, 1, \ldots,$ until we get to the first m such that

$(E_{2,1})^m(v) = 0$; after that we have $\mathfrak{g}_{\alpha+k(L_2-L_1)} = (0)$ for all $k \geq m$. The picture is thus:



(12.12)

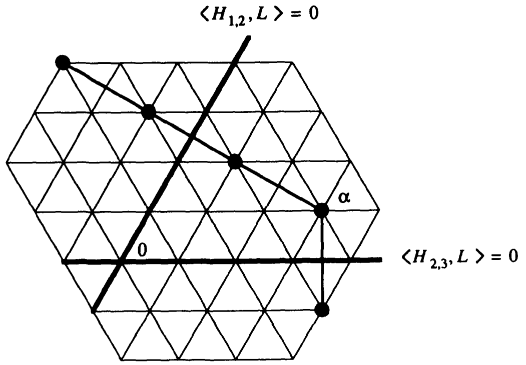
where we have no dots above/to the right of the bold line, and no dots on that line other than the ones marked.

The obvious question now is how long the string of dots along this line is. One way to answer this would be to make a calculation analogous to the one in the preceding lecture: use the computation made above to say explicitly for any k what multiple of $(E_{2,1})^{k-1}(v)$ the image of $(E_{2,1})^k(v)$ under the map $E_{1,2}$ is, and use the fact that $(E_{2,1})^m(v) = 0$ to determine m. It will be simpler—and more useful in general—if instead we just use what we have already learned about representations of $\mathfrak{sl}_2\mathbb{C}$. The point is, *the elements $E_{1,2}$ and $E_{2,1}$, together with their commutator $[E_{1,2}, E_{2,1}] = H_{1,2}$, span a subalgebra of $\mathfrak{sl}_3\mathbb{C}$ isomorphic to $\mathfrak{sl}_2\mathbb{C}$ via an isomorphism carrying $E_{1,2}$, $E_{2,1}$ and $H_{1,2}$ to the elements X, Y and H.* We will denote this subalgebra by $\mathfrak{s}_{L_1-L_2}$ (the notation may appear awkward, but this is a special case of a general construction). By the description we have already given of the action of $\mathfrak{sl}_3\mathbb{C}$ on the representation V in terms of the decomposition $V = \bigoplus V_\alpha$, we see that the subalgebra $\mathfrak{s}_{L_1-L_2}$ will shift eigenspaces V_α only in the direction of $L_2 - L_1$; in particular, the direct sum of the eigenspaces in question, namely the subspace

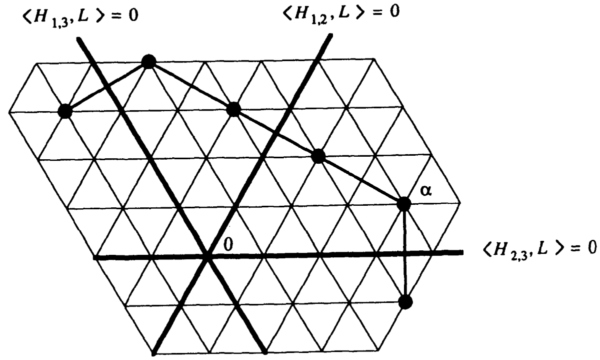
$$W = \bigoplus_k \mathfrak{g}_{\alpha+k(L_2-L_1)} \tag{12.13}$$

of V will be preserved by the action of $\mathfrak{s}_{L_1-L_2}$. In other words, W is a representation of $\mathfrak{s}_{L_1-L_2} \cong \mathfrak{sl}_2\mathbb{C}$ and we may deduce from this that *the eigenvalues of $H_{1,2}$ on W are integral, and symmetric with respect to zero.* Leaving aside the integrality for the moment, this says that the string of dots in diagram (12.12) must be symmetric with respect to the line $\langle H_{1,2}, L\rangle = 0$ in the plane $\mathfrak{h}^*$. Happily (though by no means coincidentally, as we shall see), this line is perpendicular to the line spanned by $L_1 - L_2$ in the picture we have drawn; so we can say simply that *the string of dots occurring in diagram* (12.12) *is preserved under reflection in the line $\langle H_{1,2}, L\rangle = 0$.*

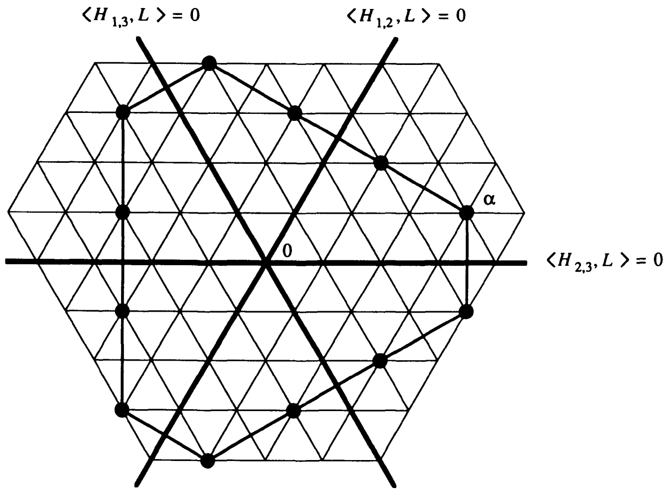
In general, for any $i \neq j$ the elements $E_{i,j}$ and $E_{j,i}$, together with their commutator $[E_{i,j}, E_{j,i}] = H_{i,j}$, span a subalgebra $\mathfrak{s}_{L_i-L_j}$ of $\mathfrak{sl}_3\mathbb{C}$ isomorphic to $\mathfrak{sl}_2\mathbb{C}$ via an isomorphism carrying $E_{i,j}$, $E_{j,i}$, and $H_{i,j}$ to the elements X, Y, and H. (Note that $H_{i,j} = -H_{j,i}$.) Analyzing the action of the subalgebra $\mathfrak{s}_{L_2-L_3}$ in particular then shows that the string of dots corresponding to the eigenspaces $\mathfrak{g}_{\alpha+k}(L_3 - L_2)$ is likewise preserved under reflection in the line $\langle H_{2,3}, L\rangle = 0$ in $\mathfrak{h}^*$. The picture is thus



Let us now take a look at the last eigenspace in the first string, that is, V_β where m is as before the smallest integer such that $(E_{2,1})^m(v) = 0$ and $\beta = \alpha + (m-1)(L_2 - L_1)$. If $v' \in V_\beta$ is any vector, then, by definition, we have $E_{2,1}(v') = 0$; and since there are no eigenspaces V_γ corresponding to γ above the bold line in diagram (12.12), we have as well that $E_{2,3}(v') = E_{1,3}(v') = 0$. Thus, v', like v itself, satisfies the statement of Lemma 12.9, except for the exchange of the indices 2 and 1; or in other words, if we had chosen the linear functional l above differently—precisely, with coefficients $b > a > c$—then the vector whose existence is implied by Lemma 12.9 would have turned out to be v' rather than v. If, indeed, we had carried out the above analysis with respect to the vector v' instead of v, we would find that all eigenvalues of V occur below or to the right of the lines through β in the directions of $L_1 - L_2$ and $L_3 - L_1$, and that the strings of eigenvalues occurring on these two lines were symmetric about the lines $\langle H_{1,2}, L\rangle = 0$ and $\langle H_{1,3}, L\rangle = 0$, respectively. The picture now is



Needless to say, we can continue to play the same game all the way around: at the end of the string of eigenvalues $\{\beta + k(L_3 - L_1)\}$ we will arrive at a vector v'' that is an eigenvector for $\mathfrak{h}$ and killed by $E_{3,1}$ and $E_{2,1}$, and to which therefore the same analysis applies. In sum, then, we see that the set of eigenvalues in V will be bounded by a hexagon symmetric with respect to the lines $\langle H_{i,j}, L\rangle = 0$ and with one vertex at α; indeed, this characterizes the hexagon as the convex hull of the union of the images of α under the group of isometries of the plane generated by reflections in these three lines.



(12.14)

We will see in a moment that the set of eigenvalues will include all the points congruent to α modulo the lattice Λ_R generated by the $L_i - L_j$ lying on the boundary of this hexagon, and that each of these eigenvalues will occur with multiplicity one.

The use of the subalgebras $\mathfrak{s}_{L_i - L_j}$ does not stop here. For one thing, observe that as an immediate consequence of our analysis of $\mathfrak{sl}_2\mathbb{C}$, all the eigenvalues of the elements $H_{i,j}$ must be integers; it is not hard to see that this means that all the eigenvalues occurring in (12.2) must be integral linear combinations of the L_i, i.e., in terms of the diagrams above, all dots must lie in the lattice Λ_W of interstices (as indeed we have been drawing them). Thus, we have

Proposition 12.15. *All the eigenvalues of any irreducible finite-dimensional representation of* $\mathfrak{sl}_3\mathbb{C}$ *must lie in the lattice* $\Lambda_W \subset \mathfrak{h}^*$ *generated by the* L_i *and be congruent modulo the lattice* $\Lambda_R \subset \mathfrak{h}^*$ *generated by the* $L_i - L_j$.

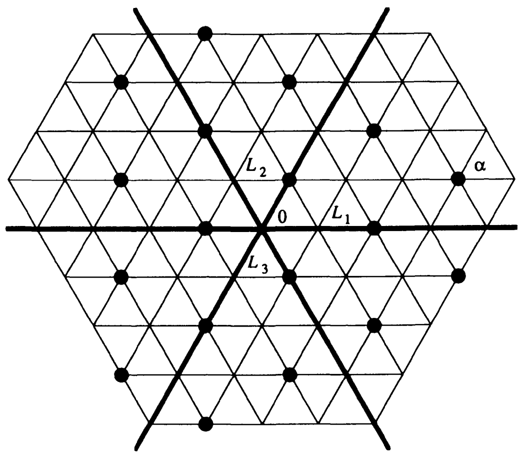
This is exactly analogous to he situation of the previous lecture: there we saw that the eigenvalues of H in any irreducible, finite-dimensional representation of $\mathfrak{sl}_2\mathbb{C}$ lay in the lattice $\Lambda_W \cong \mathbb{Z}$ of linear forms on $\mathbb{C}H$ integral on H, and were congruent to one another modulo the sublattice $\Lambda_R = 2 \cdot \mathbb{Z}$ generated by the eigenvalues of H under the adjoint representation. Note that in the case of $\mathfrak{sl}_2\mathbb{C}$ we have $\Lambda_W/\Lambda_R \cong \mathbb{Z}/2$, while in the present case we have $\Lambda_W/\Lambda_R \cong \mathbb{Z}/3$; we will see later how this reflects a general pattern. The lattice Λ_W is called the *weight lattice.*

Exercise 12.16. Show that the two conditions that the eigenvalues of V are congruent to one another modulo Λ_R and are preserved under reflection in the three lines $\langle H_{i,j}, L\rangle = 0$ imply that they all lie in Λ_W, and that, in fact, this characterizes Λ_W.

To continue, we can go into the interior of the diagram (12.14) of eigenvalues of V by observing that the direct sums (12.13) are not the only visible subspaces of V preserved under the action of the subalgebras $\mathfrak{s}_{L_i-L_j}$; more generally, for any $\beta \in \mathfrak{h}^*$ appearing in the decomposition (12.2) and any i, j the direct sum

$$W = \bigoplus_k \mathfrak{g}_{\beta+k(L_i-L_j)}$$

will be a representation of $\mathfrak{s}_{L_i-L_j}$ (not necessarily irreducible, of course); in particular it follows that the values of k for which $V_{\beta+k(L_i-L_j)} \neq (0)$ form an unbroken string of integers. Observing that if β is any of the "extremal" eigenvalues pictured in diagram (12.14), then this string will include another; so that all eigenvalues congruent to the dots pictured in diagram (12.14) and lying in their convex hull must also occur. Thus, the complete diagram of eigenvalues will look like



(12.17)

We can summarize this description in

Proposition 12.18. *Let V be any irreducible, finite-dimensional representation of $\mathfrak{sl}_3\mathbb{C}$. Then for some $\alpha \in \Lambda_W \subset \mathfrak{h}^*$, the set of eigenvalues occurring in V is* *exactly the set of linear functionals congruent to* α *modulo the lattice* Λ_R *and lying in the hexagon with vertices the images of* α *under the group generated by reflections in the lines* $\langle H_{i,j}, L\rangle = 0$.

Remark. We did, in the analysis thus far, make one apparently arbitrary choice when we defined the notion of "extremal" eigenvalue by choosing a linear functional l on $\mathfrak{h}^*$. We remark here that, in fact, the choice was not as broad as might at first have appeared. Indeed, given the fact that the configuration of eigenvalues occurring in any irreducible finite-dimensional representation of $\mathfrak{sl}_3\mathbb{C}$ is always either a triangle or a hexagon, the "extremal" eigenvalue picked out by l will always turn out to be one of the three or six vertices of this figure; in other words, if we define the linear functional l to take $a_1L_1 + a_2L_2 + a_3L_3$ to $aa_1 + ba_2 + ca_3$, then only the ordering of the three real numbers a, b, and c matters. Indeed, in hindsight this choice was completely analogous to the choice we made (implicitly) in the case of $\mathfrak{sl}_2\mathbb{C}$ in choosing one of the two directions along the real line.

We said at the outset of this lecture that our goal was to arrive at a description of representations of $\mathfrak{sl}_3\mathbb{C}$ as complete as that for $\mathfrak{sl}_2\mathbb{C}$. We have now, certainly, as complete a description of the possible configurations of eigenvalues; but clearly much more is needed. Specifically, we should have

an existence and uniqueness theorem;

an explicit construction of each representations, analogous to the statement that every representation of $\mathfrak{sl}_2\mathbb{C}$ is a symmetric power of the standard; and

for the purpose of analyzing tensor products of representations of $\mathfrak{sl}_3\mathbb{C}$, we need a description not just of the set of eigenvalues, but of the multiplicities with which they occur.

(Note that the last question is one that has no analogue in the case of $\mathfrak{sl}_2\mathbb{C}$: in both cases, any irreducible representation is generated by taking a single eigenvector $v \in V_\alpha$ and pushing it around by elements of $\mathfrak{g}_\alpha$; but whereas in the previous case there was only one way to get from V_α to V_β—that is, by applying Y over and over again—in the present circumstance there will be more than one way of getting, for example, from V_α to $V_{\alpha+L_3-L_1}$; and these may yield independent eigenvectors.) This has been, however, already too long a lecture, and so we will defer these questions, along with all examples, to the next.