Representations of $\mathfrak{sl}_3\mathbb{C}$, Part II: Mainly Lots of Examples

In this lecture we complete the analysis of the irreducible representations of $\mathfrak{sl}_3\mathbb{C}$, culminating in §13.2 with the answers to all three of the questions raised at the end of the last lecture: we explicitly construct the unique irreducible representation with given highest weight, and in particular determine its multiplicities. The latter two sections correspond to §11.2 and §11.3 in the lecture on $\mathfrak{sl}_2\mathbb{C}$. In particular, §13.4, like §11.3, involves some projective algebraic geometry and may be skipped by those to whom this is unfamiliar.

§13.1. Examples

This lecture will be largely concerned with studying examples, giving constructions and analyzing tensor products of representations of $\mathfrak{sl}_3\mathbb{C}$. We start, however, by at least stating the basic existence and uniqueness theorem that provides the context for this analysis.

To state this, recall from the previous lecture than any irreducible, finite-dimensional representation of $\mathfrak{sl}_3\mathbb{C}$ has a vector, unique up to scalars, that is simultaneously an eigenvector for the subalgebra $\mathfrak{h}$ and killed by the three subspaces $\mathfrak{g}_{L_1-L_2}$, $\mathfrak{g}_{L_1-L_3}$, and $\mathfrak{g}_{L_2-L_3}$. We called such a vector a *highest weight vector* of the representation V; its associated eigenvalue will, of course, be called the *highest weight* of V. More generally, in any finite-dimensional representation W of $\mathfrak{sl}_3\mathbb{C}$, any vector $v \in W$ with these properties will be called a highest weight vector; we saw that it will generate an irreducible sub-

representation V of W. Finally, from the description given in the last lecture of the possible configurations of eigenvalues for a representation of $\mathfrak{sl}_3\mathbb{C}$, we see that any highest weight vector must lie in the $(\frac{1}{6})$-plane described by the inequalities $\langle H_{1,2}, L \rangle \geq 0$ and $\langle H_{2,3}, L \rangle \geq 0$, i.e., it must be of the form $(a + b)L_1 + bL_2 = aL_1 - bL_3$ for some pair of non-negative integers a and b. We can now state

Theorem 13.1. *For any pair of natural numbers a, b there exists a unique irreducible, finite-dimensional representation $\Gamma_{a,b}$ of $\mathfrak{sl}_3\mathbb{C}$ with highest weight $aL_1 - bL_3$.*

We will defer the proof of this theorem until the second section of this lecture, not so much because it is in any way difficult but simply because it is time to get to some examples. We will remark, however, that whereas in the case of $\mathfrak{sl}_2\mathbb{C}$ the analysis that led to the concept of highest weight vector immediately gave the uniqueness part of the analogous theorem, here to establish uniqueness we will be forced to resort to a more indirect trick. The proof of existence, by contrast, will be very much like that of the corresponding statement for $\mathfrak{sl}_2\mathbb{C}$: we will construct the representations $\Gamma_{a,b}$ out of the standard representation by multilinear algebra.

For the time being, though, we would like to apply the analysis of the previous lecture to some of the obvious representations of $\mathfrak{sl}_3\mathbb{C}$, partly to gain some familiarity with what goes on and partly in the hopes of seeing a general multilinear-algebraic construction.

We begin with the standard representation of $\mathfrak{sl}_3\mathbb{C}$ on $V \cong \mathbb{C}^3$. Of course, the eigenvectors for the action of $\mathfrak{h}$ are just the standard basis vectors e_1, e_2, and e_3; they have eigenvalues L_1, L_2, and L_3, respectively. The weight diagram for V is thus
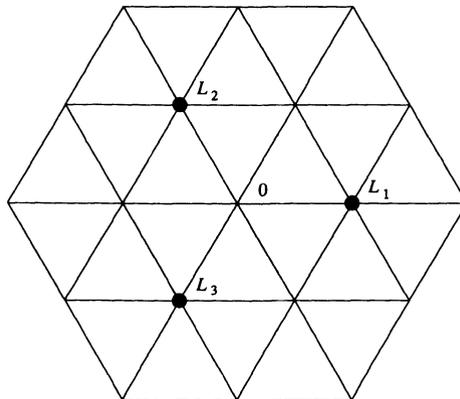

Next, consider the dual representation V^*. The eigenvalues of the dual of a representation of a Lie algebra are just the negatives of the eigenvalues of the original, so the diagram of V^* is
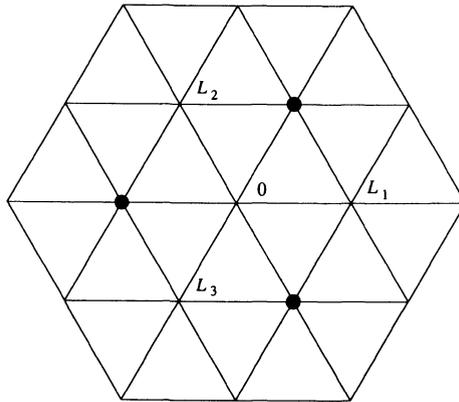
Alternatively, of course, we can just observe that the dual basis vectors e_i^* are eigenvectors with eigenvalues $-L_i$.

Note that while in the case of $\mathfrak{sl}_2\mathbb{C}$ the weights of any representation were symmetric about the origin, and correspondingly each representation was isomorphic to its dual, the same is not true here (that the diagrams for V and V^* look the same is a reflection of the fact that the two representations are carried into one another by an automorphism of $\mathfrak{sl}_3\mathbb{C}$, namely, the automorphism $X \mapsto -{}^tX$). Observe also that V^* is also isomorphic to the representation $\wedge^2 V$, whose weights are the pairwise sums of the distinct weights of V; and that likewise V is isomorphic as representation to $\wedge^2 V^*$.

Next, consider the degree 2 tensor products of V and V^*. Since the weights of the symmetric square of a representation are the pairwise sums of the weights of the original, the weight diagram of $\mathrm{Sym}^2 V$ will look like
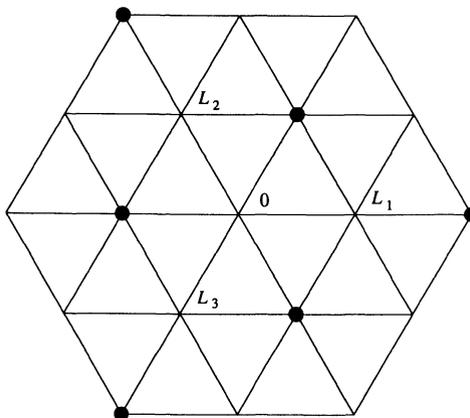

and likewise the symmetric square $\mathrm{Sym}^2 V^*$ has weights $\{-2L_i, \; -L_i - L_j\} = \{-2L_i - 2L_j, \; L_k\}$:
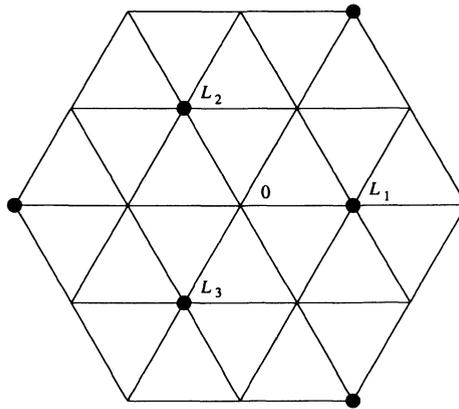
We see immediately from these diagrams that $\text{Sym}^2 V$ and $\text{Sym}^2 V^*$ are irreducible, since neither collection of weights is the union of two collections arising from representations of $\mathfrak{sl}_3\mathbb{C}$.

As for the tensor product $V \otimes V^*$, its weights are just the sums of the weights $\{L_i\}$ of V with those $\{-L_i\}$ of V^*, that is, the linear functionals $L_i - L_j$ (each occurring once, with weight vector $e_i \otimes e_j^*$) and 0 (occurring with multiplicity three, with weight vectors $e_i \otimes e_i^*$). We can represent these weights by the diagram
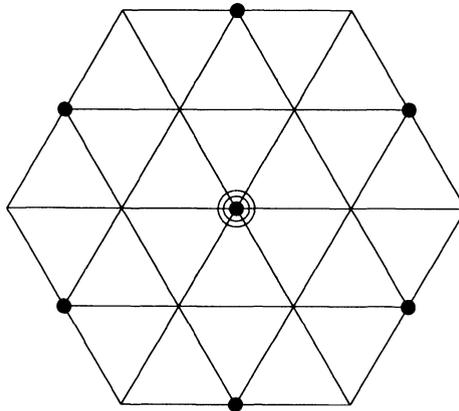
where the triple circle is intended to convey the fact that the weight space V_0 is three dimensional. By contrast with the last two examples, this representation is not irreducible: there is a linear map

$$V \otimes V^* \to \mathbb{C}$$

given simply by the contraction

$$v \otimes u^* \mapsto \langle v, u^* \rangle = u^*(v)$$

(or, in terms of the identification $V \otimes V^* \cong \operatorname{Hom}(V, V)$, by the *trace*) that is a map of $\mathfrak{sl}_3\mathbb{C}$-modules (with $\mathbb{C}$ the trivial representation, of course). The kernel of this map is then the subspace of $V \otimes V^*$ of traceless matrices, which is just the adjoint representation of the Lie algebra $\mathfrak{sl}_3\mathbb{C}$ and is irreducible (we can see this either from our explicit description of the adjoint representation—for example, $E_{1,3}$ is the unique weight vector for $\mathfrak{h}$ killed by $g_{L_1-L_2}$, $g_{L_1-L_3}$, and $g_{L_2-L_3}$—or, if we take as known the fact that $SL_3\mathbb{C}$ is simple, from the fact that a subrepresentation of the adjoint representation is an *ideal* in a Lie algebra, and exponentiates to a normal subgroup, cf. Exercise 8.43.)

(Physicists call this adjoint representation of $\mathfrak{sl}_3\mathbb{C}$ (or $SU(3)$) the "eightfold way," and relate its decomposition to mesons and baryons. The standard representation V is related to "quarks" and V^* to "antiquarks." See [S-W], [Mack].)

(We note that, in general, if V is any faithful representation of a Lie algebra, the adjoint representation will appear as a subrepresentation of the tensor $V \otimes V^*$.)

Let us continue now with some of the triple tensor products of V and V^*, which will be the last specific cases we look at. To begin with, we have the symmetric cubes $\operatorname{Sym}^3 V$ and $\operatorname{Sym}^3 V^*$, with weight diagrams
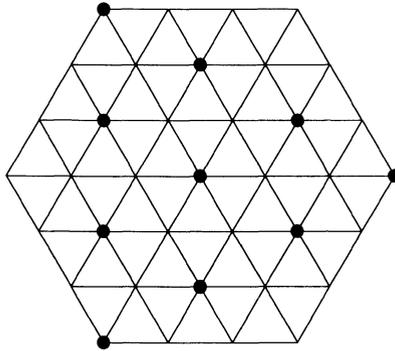
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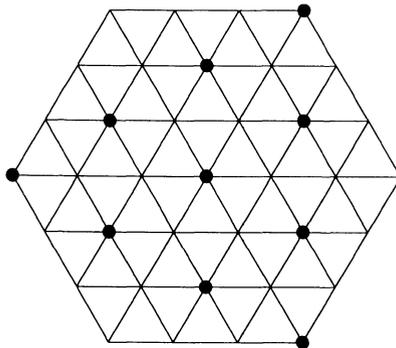
respectively. In general it is clear that, in terms of the description given in the preceding lecture of the possible weight diagrams of irreducible representations of $\mathfrak{sl}_3\mathbb{C}$, the symmetric powers of V and V^* will be exactly the representations with triangular, as opposed to hexagonal, diagrams.

It also follows from the above description and the fact that the weights of the symmetric powers $\text{Sym}^n V$ occur with multiplicity 1 that $\text{Sym}^n V$ and $\text{Sym}^n V^*$ are all irreducible, i.e., we have, in the notation of Theorem 13.1,

$$\text{Sym}^n V = \Gamma_{n,0} \quad \text{and} \quad \text{Sym}^n V^* = \Gamma_{0,n}.$$

By way of notation, we will often write $\text{Sym}^n V$ in place of $\Gamma_{n,0}$.

Consider now the mixed tensor $\text{Sym}^2 V \otimes V^*$. Its weights are the sums of the weights of $\text{Sym}^2 V$—that is, the pairwise sums of the L_i—with the weights of V^*; explicitly, these are $L_i + L_j - L_k$ and $2L_i - L_j$ (each occurring once) and the L_i themselves (each occurring three times, as $L_i + L_j - L_j$). Diagrammatically, the representation looks like
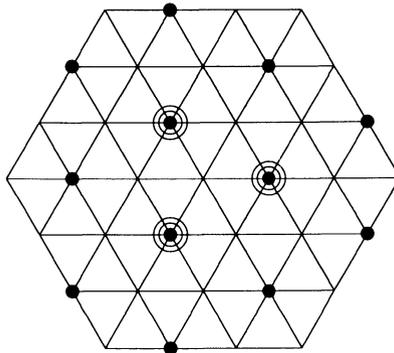
Now, we know right off the bat that this is not irreducible: we have a natural map

$$\iota \colon \text{Sym}^2 V \otimes V^* \to V$$

given again by contraction, that is, by the map

$$vw \otimes u^* \mapsto \langle v, u^* \rangle \cdot w + \langle w, u^* \rangle \cdot v,$$

which is a map of $\mathfrak{sl}_3\mathbb{C}$-modules.[1] What does the kernel of this map look like? Of course, its weight diagram is

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[1] Another way to see that $\text{Sym}^2 V \otimes V^*$ is not irreducible is to observe that if a representation W is generated by a highest weight vector v of weight $2L_1 - L_3$, as $\text{Sym}^2 V \otimes V^*$ must be if it is irreducible, the eigenvalue L_1 can be taken with multiplicity at most 2, the corresponding eigenspace being generated by $E_{2,1} \circ E_{3,2}(v)$ and $E_{3,2} \circ E_{2,1}(v)$.
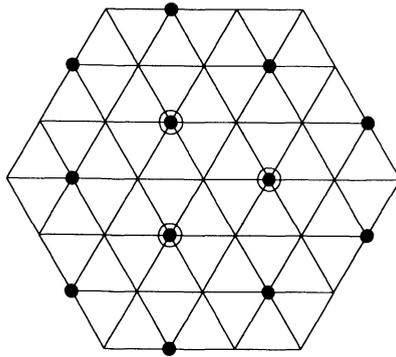
and we know one other thing: certainly any vector in the weight space of $2L_1 - L_3$—that is to say, of course, any multiple of the vector $e_1^2 \otimes e_3^*$—is killed by $\mathfrak{g}_{L_1-L_2}$, $\mathfrak{g}_{L_1-L_3}$, and $\mathfrak{g}_{L_2-L_3}$, so that the kernel of ι will contain an irreducible representation $\Gamma = \Gamma_{2,1}$ with $2L_1 - L_3$ as its highest weight. Since Γ must then assume every weight of $\mathrm{Ker}(\iota)$, there are exactly two possibilities: either $\mathrm{Ker}(\iota) = \Gamma$, which assumes the weights L_i with multiplicity 2; or all the weights of Γ occur with multiplicity one and $\mathrm{Ker}(\iota) \cong \Gamma \oplus V$.

How do we settle this issue? There are at least three ways. To begin with, we can try to analyze directly the structure of the kernel of ι. An alternative approach would be to determine a priori with what multiplicities the weights of $\Gamma_{a,b}$ are taken. Certainly it is clear that a formula giving us the latter information will be tremendously valuable—it would for one thing clear up the present confusion instantly—and indeed there exist several such, one of which, the *Weyl character formula*, we will prove later in the book. (We will also prove the *Kostant multiplicity formula*, which can be applied to deduce directly the independence statement we arrive at below.) As a third possibility, we can identify the representations $\Gamma_{a,b}$ as Weyl modules and appeal to Lecture 6. Rather than invoke such general formulas at present, however, we will take the first approach here. This is straightforward: in terms of the notation we have been using, the highest weight vector for the representation $\Gamma \subset \mathrm{Sym}^2 V \otimes V^*$ is the vector $e_1^2 \otimes e_3^*$, and so the eigenspace $\Gamma_{L_1} \subset \Gamma$ with eigenvalue L_1 will be spanned by the images of this vector under the two compositions $E_{2,1} \circ E_{3,2}$ and $E_{3,2} \circ E_{2,1}$. These are, respectively,

$$E_{2,1} \circ E_{3,2}(e_1^2 \otimes e_3^*) = E_{2,1}(E_{3,2}(e_1^2) \otimes e_3^* + e_1^2 \otimes E_{3,2}(e_3^*))$$

$$= E_{2,1}(-e_1^2 \otimes e_2^*)$$

$$= -2(e_1 \cdot e_2) \otimes e_2^* + e_1^2 \otimes e_1^*$$

and

$$E_{3,2} \circ E_{2,1}(e_1^2 \otimes e_3^*) = E_{3,2}(E_{2,1}(e_1^2) \otimes e_3^* + e_1^2 \otimes E_{2,1}(e_3^*))$$

$$= E_{3,2}((2e_1 \cdot e_2) \otimes e_3^*)$$

$$= 2(e_1 \cdot e_3) \otimes e_3^* - 2(e_1 \cdot e_2) \otimes e_2^*.$$

Since these are independent, we conclude that the weight L_1 does occur in Γ with multiplicity 2, and hence that the kernel of ι is irreducible, i.e.,

$$\mathrm{Sym}^2 V \otimes V^* \cong \Gamma_{2,1} \oplus V.$$

§13.2. Description of the Irreducible Representations

At this point, rather than go on with more examples we should state some of the general principles that have emerged so far. The first and most important (though pretty obvious) is the basic

Observation 13.2. *If the representations V and W have highest weight vectors v and w with weights α and β, respectively, then the vector $v \otimes w \in V \otimes W$ is a highest weight vector of weight $\alpha + \beta$.*

Of course, there are numerous generalizations of this: the vector $v^n \in \mathrm{Sym}^n V$ is a highest weight vector of weight $n\alpha$, etc.[2] Just the basic statement above, however, enables us to give the

PROOF OF THEOREM 13.1. First, the existence statement follows immediately from the observation: the representation $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ will contain an irreducible subrepresentation $\Gamma_{a,b}$ with highest weight $aL_1 - bL_3$.

The uniqueness part is only slightly harder (if less explicit): Given irreducible representations V and W with highest weight α, let $v \in V$ and $w \in W$ be highest weight vectors with weight α. Then (v, w) is again a highest weight vector in the representation $V \oplus W$ with highest weight α; let $U \subset V \oplus W$ be the irreducible subrepresentation generated by (v, w). The projection maps $\pi_1 \colon U \to V$ and $\pi_2 \colon U \to W$, being nonzero maps between irreducible representations of $\mathfrak{sl}_3\mathbb{C}$, must be isomorphisms, and we deduce that $V \cong W$. □

Exercise 13.3*. Let $\mathbb{S}_\lambda$ be the Schur functor introduced in Lecture 6. What can you say about the highest weight vectors in the representation $\mathbb{S}_\lambda(V)$ obtained by applying it to a given representation V?

To continue our discussion of tensor products like $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ in general, as we indicated we would like to make more explicit the construction of the representation $\Gamma_{a,b}$, which we know to be lying in $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$. To begin with, we have in general a contraction map

$$\iota_{a,b} \colon \mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^* \to \mathrm{Sym}^{a-1} V \otimes \mathrm{Sym}^{b-1} V^*$$

analogous to the map ι introduced above; we can describe this map either (in fancy language) as the dual of the map from $\mathrm{Sym}^{a-1} V \otimes \mathrm{Sym}^{b-1} V^*$ to $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ given by multiplication by the identity element in

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[2] One slightly less obvious statement is this: if the weights of V are $\alpha_1, \alpha_2, \alpha_3 \ldots$ with $l(\alpha_1) > l(\alpha_2) > \ldots$, then $\bigwedge^n V$ possesses a highest weight vector weight $\alpha_1 + \cdots + \alpha_n$. Note that since the ordering of the α_i may in fact depend on the choice of l (even with the restriction $a > b > c$ on the coefficients of l as above), this may in some cases imply the existence of several subrepresentations of $\bigwedge^n V$.

$V \otimes V^* = \mathrm{Hom}(V, V)$; or, concretely, by sending

$$(v_1 \cdot \ldots \cdot v_a) \otimes (v_1^* \cdot \ldots \cdot v_b^*)$$

$$\mapsto \sum \langle v_i, v_j^* \rangle (v_1 \cdot \ldots \cdot \hat{v}_i \cdot \ldots \cdot v_a) \otimes (v_1^* \cdot \ldots \cdot \hat{v}_j^* \cdot \ldots \cdot v_b^*).$$

Clearly this map is surjective, and, since the target does not have eigenvalue $aL_1 - bL_3$, the subrepresentation $\Gamma_{a,b} \subset \mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ must lie in the kernel. In fact, we have, just as in the case of $\mathrm{Sym}^2 V \otimes V^*$ above,

Claim 13.4. *The kernel of the map $\iota_{a,b}$ is the irreducible representation $\Gamma_{a,b}$.*

We will defer the proof of this for a moment and consider some of its consequences. To begin with, we can deduce from this assertion the complete decomposition of $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$: we must have (if, say, $b \leq a$)

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^* = \bigoplus_{i=0}^{b} \Gamma_{a-i, b-i}. \tag{13.5}$$

Since we know, a priori, all the multiplicities of the eigenvalues of the tensor product $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$, this will, in turn, determine (inductively at least) all the multiplicities of the representations $\Gamma_{a,b}$. In fact, the answer turns out to be very nice. To express it, observe first that if $a \geq b$, the weight diagram of either $\Gamma_{a,b}$ or $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ looks like a sequence of b shrinking concentric (not in general regular) hexagons H_i with vertices at the points $(a - i)L_1 - (b - i)L_3$ for $i = 0, 1, \ldots, b - 1$, followed (after the shorter three sides of the hexagon have shrunk to points) by a sequence of $[(a - b)/3] + 1$ triangles T_j with vertices at the points $(a - b - 3j)L_1$ for $j = 0, 1, \ldots,$ $[(a - b)/3]$ (it will be convenient notationally to refer to T_0 as H_b occasionally). Diagram (13.6) shows the picture of the weights of $\mathrm{Sym}^6 V \otimes \mathrm{Sym}^2 V^*$:

$$\tag{13.6}$$
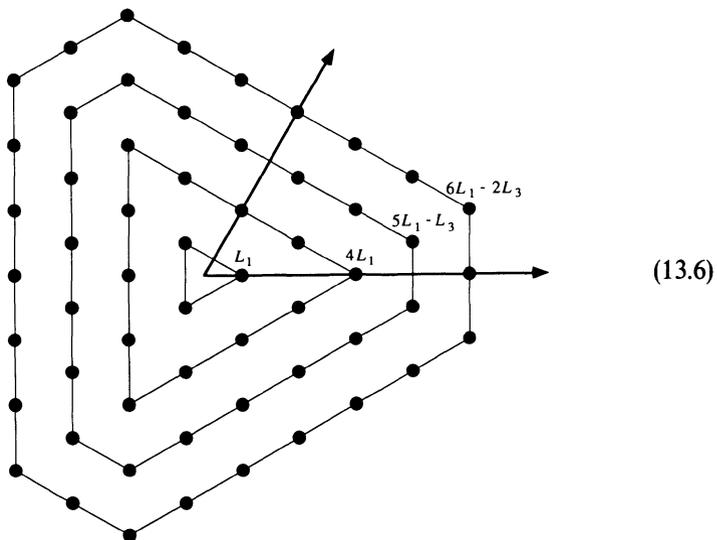
(Note that by the decomposition (13.5), the weights of the highest weight vectors in $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ will be $aL_1 - bL_3$, $(a-1)L_1 - (b-1)L_3$, ..., $(a-b)L_1$, as shown in the diagram.)

An examination of the representation $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ shows that it has multiplicity $(i+1)(i+2)/2$ on the hexagon H_i, and then a constant multiplicity $(b+1)(b+2)/2$ on all the triangles T_j; and it follows from the decomposition (13.5), in general, that the representation $\Gamma_{a,b}$ has multiplicity $(i+1)$ on H_i and $b+1$ on T_j. In English, *the multiplicities of $\Gamma_{a,b}$ increase by one on each of the concentric hexagons of the eigenvalue diagram and are constant on the triangles*. Note in particular that the description of $\Gamma_{2,1}$ in the preceding section is a special case of this.

PROOF OF CLAIM 13.4. We remark first that the claim will be implied by the Weyl character formula or by the description via Weyl's construction in Lecture 15; so the reader who wishes to can skip the following without dire consequences to the logical structure of the book. Otherwise, observe first that the claim is equivalent to asserting the decomposition (13.5); this, in turn, is equivalent to the statement that the representation $W = \mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ has exactly $b+1$ irreducible components (still assuming $a \geq b$). The irreducible factors in a representation correspond to the highest weight vectors in the representation up to scalars; so in sum the claim is equivalent to the assertion that *the eigenspace W_α of $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ contains a unique highest weight vector (up to scalars) if α is of the form $(a-i)L_1 - (b-i)L_3$ for $i \leq b$, and none otherwise*; this is what we shall prove.

To begin with, the "none otherwise" part of the statement follows (given the other) just from looking at the diagram: if, for example, any of the eigenspaces W_α corresponding to a point α on a hexagon H_i (other than the vertex $(a-i)L_1 - (b-i)L_3$ of H_i) possessed a highest weight vector, the multiplicity of α in W would be strictly greater than of $(a-i)L_1 - (b-i)L_3$, which we know is not the case; similarly, the fact that the multiplicities of W in the triangular part of the eigenvalue diagram are constant implies that there can be no highest weight vectors with eigenvalue on a T_j for $j \geq 1$. Thus, we just have to check that the weight spaces W_α for $\alpha = (a-i)L_1 - (b-i)L_3$ contain only the one highest weight vector we know is there; and we do this by explicit calculation.

To start, for any monomial index $I = (i_1, i_2, i_3)$ of degree $\sum i_\gamma = i$, we denote by $e^I \in \mathrm{Sym}^i V$ the corresponding monomial $\prod (e_\gamma^{i_\gamma})$ and define $(e^*)^I \in \mathrm{Sym}^i V^*$ similarly. We can then write any element of the weight space $W_{(a-i)L_1 - (b-i)L_3}$ of $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ as

$$v = \sum c_I \cdot (e_1^{a-i} \cdot e^I) \otimes ((e_3^*)^{b-i} \cdot (e^*)^I).$$

In these terms, it is easy to write down the action of the two operators $E_{1,2}$

and $E_{2,3}$. First, $E_{1,2}$ kills both $e_1 \in V$ and $e_3^* \in V^*$, so that we have

$$E_{1,2}((e_1^{a-i} \cdot e^I) \otimes ((e_3^*)^{b-i} \cdot (e^*)^I))$$

$$= i_2(e_1^{a-i} \cdot e^{I'}) \otimes ((e_3^*)^{b-i} \cdot (e^*)^I)$$

$$- i_1(e_1^{a-i} \cdot e^I) \otimes ((e_3^*)^{b-i} \cdot (e^*)^{I''}),$$

where $I' = (i_1 + 1, i_2 - 1, i_3)$ and $I'' = (i_1 - 1, i_2 + 1, i_3)$ (and we adopt the convention that $e^I = 0$ if $i_\gamma < 0$ for any γ). It follows that *the vector v above is in the kernel of $E_{1,2}$ if and only if the coefficients c_I satisfy $i_2 c_I = (i_1 + 1)c_{I'}$;* and by the analogous calculation that v is in the kernel of $E_{2,3}$ if and only if $i_3 c_I = (i_2 + 1)c_J$ whenever the indices I and J are related by $j_1 = i_1$, $j_2 = i_2 + 1$, and $j_3 = i_3 - 1$. These conditions are equivalent to saying that the numbers $i_1! i_2! i_3! c_I$ are independent of I. We see, in other words, that v is a highest weight vector if and only if all the coefficients c_I are equal to $c/i_1! i_2! i_3!$ for some constant c. $\square$

§13.3. A Little More Plethysm

We would like to consider here, as we did in the case of $\mathfrak{sl}_2 \mathbb{C}$ in Lecture 11, how the tensor products and powers of the representations we have described decompose. We start with one general remark: given our knowledge of the eigenvalue diagrams of the irreducible representations of $\mathfrak{sl}_3 \mathbb{C}$ (*with* multiplicities), there can be no possible ambiguity about the decomposition of any representation U given as the tensor product of representations whose eigenvalue diagrams are known. Indeed, we have an algorithm for determining the components of that decomposition, as follows:

1. Write down the eigenvalue decomposition of U.
2. Find the eigenvalue $\alpha = aL_1 - bL_3$ appearing in this diagram for which the value of $l(\alpha)$ is maximal.
3. We now know that U will contain a copy of the irreducible representation $\Gamma_\alpha = \Gamma_{a,b}$, i.e., $U \cong \Gamma_\alpha \oplus U'$ for some U'. Since we also know the eigenvalue diagram of Γ_α, we can thus write down the eigenvalue diagram of U' as well.
4. Repeat this process for U'.

To see how this goes in practice, consider some examples of tensor products of the basic irreducible representations described so far. We have already seen how the tensor products of the symmetric powers of the standard representation V of $\mathfrak{sl}_3 \mathbb{C}$ and symmetric powers of its dual decompose; let us look now at an example of a more general tensor product of irreducible representations: say V itself and the representation $\Gamma_{2,1}$. We start by writing down the weights of the tensor product: since $\Gamma_{2,1}$ has weights $2L_i - L_j$, $L_i + L_j - L_k$, and L_i

(taken twice) and V has weights L_i, the tensor product will have weights $3L_i - L_j$, $2L_i + L_j - L_k$ (taken twice), $2L_i$ (taken four times), and $L_i + L_j$ (taken five times). The diagram is thus
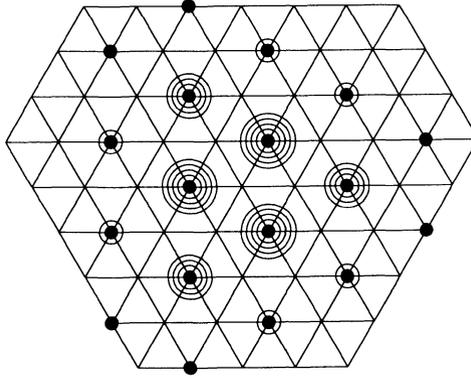

(One thing we may deduce from this diagram is that we are soon going to need a better system for presenting the data of the weights of a representation. In the future, we may simply draw one sector of the plane, and label weights with numbers to indicate multiplicities.)

We know right off the bat that the tensor product $V \otimes \Gamma_{2,1}$ contains a copy of the irreducible representation $\Gamma_{3,1}$ with highest weight $3L_1 - L_3$. By what we have said, the weight diagram of $\Gamma_{3,1}$ is
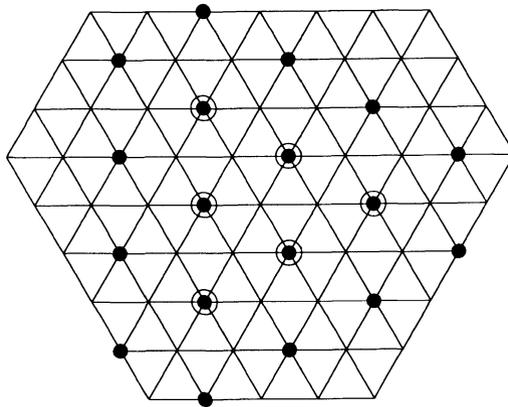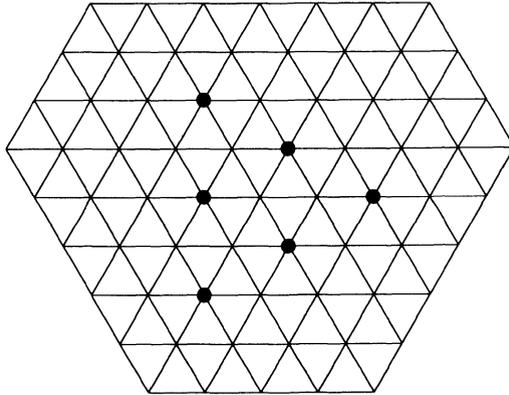

so the complement of $\Gamma_{3,1}$ in the tensor product $V \otimes \Gamma_{2,1}$ will look like
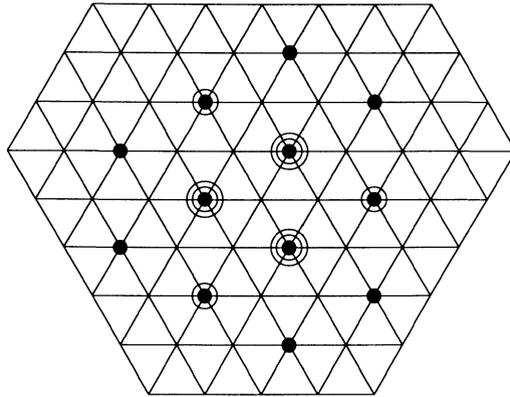
One obvious highest weight in this representation is $2L_1 + L_2 - L_3 = L_1 - 2L_3$, so that the tensor product will contain a copy of the irreducible representation $\Gamma_{1,2}$ as well; since this has weight diagram
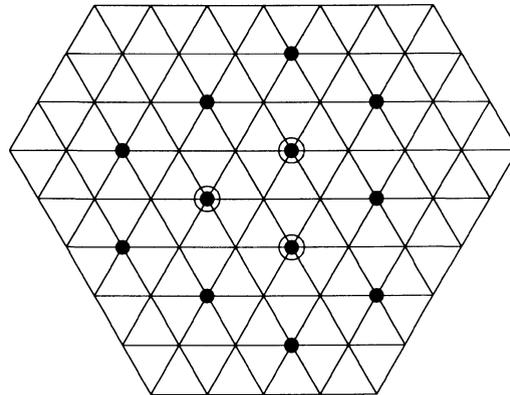
the remaining part of the tensor product will have weight diagram

which we recognize as the weight diagram of the symmetric square $\mathrm{Sym}^2 V = \Gamma_{2,0}$ of the standard representation. We have, thus,

$$V \otimes \Gamma_{2,1} = \Gamma_{3,1} \oplus \Gamma_{1,2} \oplus \Gamma_{2,0}. \tag{13.7}$$

Exercise 13.8*. Find the decomposition into irreducible representations of the tensor products $V \otimes \Gamma_{1,1}$, $V \otimes \Gamma_{1,2}$ and $V \otimes \Gamma_{3,1}$. Can you find a general pattern to the outcomes?

As in the case of $\mathfrak{sl}_2\mathbb{C}$, the next thing to look at are the tensor powers—symmetric and exterior—of representations other than the standard; we look first at tensors of the symmetric square $W = \mathrm{Sym}^2 V$. First, consider the symmetric square $\mathrm{Sym}^2 W = \mathrm{Sym}^2(\mathrm{Sym}^2 V)$. We know the diagram for $\mathrm{Sym}^2 W$; it is
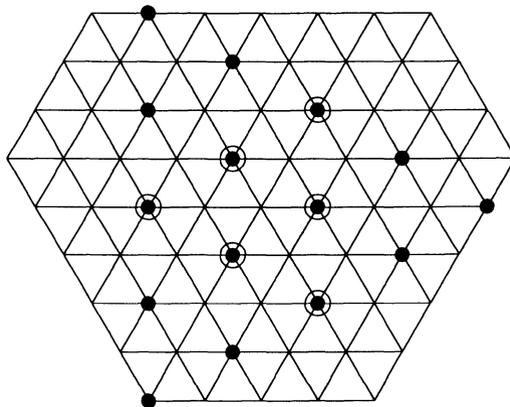
Now, there is only one possible decomposition of a representation whose eigenvalue diagram looks like this: we must have

$$\text{Sym}^2(\text{Sym}^2 V)) \cong \text{Sym}^4 V \oplus \text{Sym}^2 V^*.$$

Indeed, the presence of the $\text{Sym}^4 V$ factor is clear: there is an obvious map

$$\varphi: \text{Sym}^2(\text{Sym}^2 V)) \to \text{Sym}^4 V$$

obtained simply by multiplying out. The identification of the kernel of this map with the representation $\text{Sym}^2 V^*$ is certainly less obvious, but can still be made explicit. We can identify V^* with $\wedge^2 V$ as we saw, and then define a map

$$\tau: \text{Sym}^2(\wedge^2 V) \to \text{Sym}^2(\text{Sym}^2 V))$$

by sending the generator $(u \wedge v) \cdot (w \wedge z) \in \text{Sym}^2(\wedge^2 V)$ to the element $(u \cdot w) \cdot (v \cdot z) - (u \cdot z) \cdot (v \cdot w) \in \text{Sym}^2(\text{Sym}^2 V)$, which is clearly in the kernel of φ.

Exercise 13.9. Verify that this map is well defined and that it extends linearly to an isomorphism of $\text{Sym}^2(\wedge^2 V)$ with $\text{Ker}(\varphi)$.

Exercise 13.10. Apply the techniques above to show that the representation $\wedge^2(\text{Sym}^2 V)$ is isomorphic to $\Gamma_{2,1}$.

Exercise 13.11. Apply the same techniques to determine the irreducible factors of the representation $\wedge^3(\text{Sym}^2 V)$. Note: we will return to this example in Exercise 13.22.

Exercise 13.12. Find the decomposition into irreducibles of the representations $\text{Sym}^2(\text{Sym}^3 V)$ and $\text{Sym}^3(\text{Sym}^2 V)$ (observe in particular that Hermite reciprocity has bitten the dust). Describe the projection maps to the various factors. Note: we will describe these examples further in the following section.

§13.4. A Little More Geometric Plethysm

Just as in the case of $\mathfrak{sl}_2 \mathbb{C}$, some of these identifications can also be seen in geometric terms. To do this, recall from §11.3 the definition of the *Veronese embedding*: if $\mathbb{P}^2 = \mathbb{P}V^*$ is the projective space of one-dimensional subspaces of V^*, there is then a natural embedding of $\mathbb{P}^2$ in the projective space $\mathbb{P}^5 = \mathbb{P}(\text{Sym}^2 V^*)$, obtained simply by sending the point $[v^*] \in \mathbb{P}^2$ corresponding to the vector $v^* \in V^*$ to the point $[v^{*2}] \in \mathbb{P}(\text{Sym}^2 V^*)$ associated to the vector $v^{*2} = v^* \cdot v^* \in \text{Sym}^2 V^*$. The image $S \subset \mathbb{P}^5$ is called the *Veronese surface*. As in the case of the rational normal curves discussed in Lecture 11, it is not hard to see that the group of automorphisms of $\mathbb{P}^5$ carrying S into itself is exactly the group $\text{PGL}_3 \mathbb{C}$ of automorphisms of $S = \mathbb{P}^2$.

Now, a quadratic polynomial in the homogeneous coordinates of the space $\mathbb{P}(\text{Sym}^2 V^*) \cong \mathbb{P}^5$ will restrict to a quartic polynomial on the Veronese surface $S = \mathbb{P}V^*$, which corresponds to the natural evaluation map φ of the preceding section; the kernel of this map is thus the vector space of quadratic poly-

nomials in $\mathbb{P}^5$ vanishing on the Veronese surface S, on which the group of automorphisms of $\mathbb{P}^5$ carrying S to itself obviously acts. Now, for any pair of points $P = [u^*]$, $Q = [v^*] \in S$, it is not hard to see that the cone over the Veronese surface with vertex the line $\overline{PQ} \subset \mathbb{P}^5$ (that is, the union of the 2-planes $\overline{PQR}$ as R varies over the surface S) will be a quadric hypersurface in $\mathbb{P}^5$ containing the Veronese surface; sending the generator $u^* \cdot v^* \in \mathrm{Sym}^2 V^*$ to this quadric hypersurface will then define an isomorphism of the space of such quadrics with the projective space associated to $\mathrm{Sym}^2 V^*$.

Exercise 13.13. Verify the statements made in the last paragraph: that the union of the $\overline{PQR}$ is a quadric hypersurface and that this extends to a linear isomorphism $\mathbb{P}(\mathrm{Sym}^2 V^*) \cong \mathbb{P}(\mathrm{Ker}(\varphi))$. Verify also that this isomorphism coincides with the one given in Exercise 13.9.

There is another way of representing the Veronese surface that will shed some light on this kernel. If, in terms of some coordinates e_i on V^*, we think of $\mathrm{Sym}^2 V^*$ as the vector space of symmetric 3×3 matrices, then the Veronese surface is just the locus, in the associated projective space, of rank 1 matrices up to scalars, i.e., in terms of homogeneous coordinates $Z_{i,j} = e_i \cdot e_j$ on $\mathbb{P}^5$,

$$
S = \left\{ [Z] : \mathrm{rank} \begin{pmatrix} Z_{1,1} & Z_{1,2} & Z_{1,3} \\ Z_{1,2} & Z_{2,2} & Z_{2,3} \\ Z_{1,3} & Z_{2,3} & Z_{3,3} \end{pmatrix} = 1 \right\}.
$$

The vector space of quadratic polynomials vanishing on S is then generated by the 2×2 minors of the matrix $(Z_{i,j})$; in particular, for any pair of linear combinations of the rows and pair of linear combinations of the columns we get a 2×2 matrix whose determinant vanishes on S.

Exercise 13.14. Show that this is exactly the isomorphism $\mathrm{Sym}^2(\wedge^2 V) \cong \mathrm{Ker}(\varphi)$ described above.

We note in passing that if indeed the space of quadrics containing the Veronese surface, with the action of the group $\mathrm{PGL}_3\mathbb{C}$ of motions of $\mathbb{P}^5$ preserving S, is the projectivization of the representation $\mathrm{Sym}^2 V^*$, then it must contain its own Veronese surface, i.e., there must be a surface $T = \mathbb{P}(V^*) \subset \mathbb{P}(\mathrm{Ker}(\varphi))$ invariant under this action. This turns out to be just the set of *quadrics of rank 3* containing the Veronese, that is, the quadrics whose singular locus is a 2-plane. In fact, the 2-plane will be the tangent plane to S at a point, giving the identification $T = S$.

Let us consider one more example of this type, namely, the symmetric cube $\mathrm{Sym}^3(\mathrm{Sym}^2 V))$. (We promise we will stop after this one.) As before, it is easy to write down the eigenvalues of this representation; they are just the triple sums of the eigenvalues $\{2L_i, L_i + L_j\}$ of $\mathrm{Sym}^2 V$. The diagram (we will draw here only one-sixth of the plane and indicate multiplicities with numbers rather than rings) thus looks like
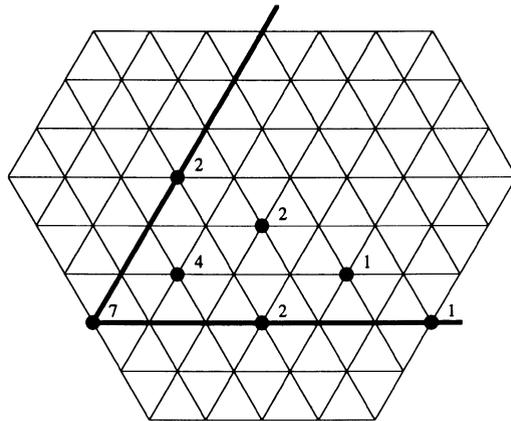
from which we see what the decomposition must be: as representations we have

$$\text{Sym}^3(\text{Sym}^2 V)) \cong \text{Sym}^6 V \oplus \Gamma_{2,2} \oplus \mathbb{C}. \qquad (13.15)$$

As before, the map to the first factor is just the obvious one; it is the identification of the kernel that is intriguing, and especially the identification of the last factor.

To see what is going on here, we should look again at the geometry of the Veronese surface $S \subset \mathbb{P}^5 = \mathbb{P}(\text{Sym}^2 V^*)$. The space $\text{Sym}^3(\text{Sym}^2 V))$ is just the space of homogeneous cubic polynomials on the ambient space $\mathbb{P}^5$, and as before the map to the first factor of the right-hand side of (13.15) is just the restriction, so that the last two factors of (13.15) represent the vector space $I(S)_3$ of cubic polynomials vanishing on S. Note that we could in fact prove (13.15) without recourse to eigenvalue diagrams from this: since the ideal of the Veronese surface is generated by the vector space $I(S)_2$ of quadratic polynomials vanishing on it, we have a surjective map

$$I(S)_2 \otimes W = \text{Sym}^2 V^* \otimes \text{Sym}^2 V \to I(S)_3.$$

But we already know how the left hand side decomposes: we have

$$\text{Sym}^2 V^* \otimes \text{Sym}^2 V = \Gamma_{2,2} \oplus \Gamma_{1,1} \oplus \mathbb{C}, \qquad (13.16)$$

so that $I(S)_3$ must be a partial direct sum of these three irreducible representations; by dimension considerations it can only be $\Gamma_{2,2} \oplus \mathbb{C}$.

This, in turn, tells us how to make the isomorphism (13.15) explicit (assuming we want to): we can define a map

$$\text{Sym}^2(\wedge^2 V) \otimes \text{Sym}^2 V \to \text{Sym}^3(\text{Sym}^2 V)$$

by sending

$$(u \wedge v) \cdot (w \wedge z) \otimes (s \cdot t) \mapsto ((u \cdot w) \cdot (v \cdot z) - (u \cdot z) \cdot (v \cdot w)) \cdot (s \cdot t)$$

and then just check that this gives an isomorphism of $\Gamma_{2,2} \oplus \mathbb{C} \subset$ $\text{Sym}^2 V^* \otimes \text{Sym}^2 V$ with the kernel of projection on the first factor of the right-hand side of (13.15).

What is really most interesting in this whole situation, though, is the trivial summand in the expression (13.15). To say that there is such a summand is to say that *there exists a cubic hypersurface X in $\mathbb{P}^5$ preserved under all automorphisms of $\mathbb{P}^5$ carrying S to itself*. Of course, we have already run into this one: it is the determinant of the 3×3 matrix $(Z_{i,j})$ introduced above. To express this more intrinsically, if we think of the Veronese as the set of rank 1 tensors in $\text{Sym}^2 V^*$, it is just the set of tensors of rank 2 or less. This, in turn, yields another description of X: since a rank 2 tensor is just one that can be expressed as a linear combination of two rank 1 tensors, we see that X is the famous *chordal variety* of the Veronese surface: it is the union of the chords to S, and at the same time the union of all the tangent planes to S.

Exercise 13.17. Show that the only symmetric powers of $\text{Sym}^2 V$ that possess trivial summands are the powers $\text{Sym}^{3k}(\text{Sym}^2 V))$ divisible by 3, and that the unique trivial summand in this is just the kth power of the trivial summand of $\text{Sym}^3(\text{Sym}^2 V))$.

Exercise 13.18. Given the isomorphism of the projectivization of the vector space $I(S)_2$—that is, the projective space of quadric hypersurfaces containing the Veronese surface—with $\mathbb{P}(\text{Sym}^2 V^*)$, find the unique cubic hypersurface in $I(S)_2$ invariant under the action of $\text{PGL}_3\mathbb{C}$.

Exercise 13.19. Analyze the representation $\text{Sym}^2(\text{Sym}^3 V))$ of $\mathfrak{sl}_3\mathbb{C}$. Interpret the direct sum factors in terms of the geometry of the Veronese embedding of $\mathbb{P}V^* = \mathbb{P}^2$ in $\mathbb{P}(\text{Sym}^3 V^*) = \mathbb{P}^9$.

Exercise 13.20*. Show that the representations $\text{Sym}^4(\text{Sym}^3 V))$ and $\text{Sym}^6(\text{Sym}^3 V))$ contain trivial summands, and that the representation $\text{Sym}^{12}(\text{Sym}^3 V))$ contains two. Interpret these.

Exercise 13.21. Apply the techniques above to show that the representation $\wedge^2(\text{Sym}^2 V)$ is isomorphic to $\Gamma_{2,1}$.

Exercise 13.22*. Apply the techniques above to analyze the representation $\wedge^3(\text{Sym}^2 V)$, and in particular to interpret its decomposition into irreducible representations.

Exercise 13.23. If $\mathbb{P}^5 = \mathbb{P}(\text{Sym}^2 V^*)$ is the ambient space of the Veronese surface, the Grassmannian $\mathbb{G}(2, 5)$ of 2-planes in $\mathbb{P}^5$ naturally embeds in the projective space $\mathbb{P}(\wedge^3(\text{Sym}^2 V))$. Describe, in terms of the decomposition in the preceding exercise, the span of the locus of tangent 2-planes to the

Veronese, and the span of the locus of 2-planes in $\mathbb{P}^5$ spanned by the images in S of lines in $\mathbb{P}V^*$.

Exercise 13.24*. Show that the unique closed orbit of the action of $SL_3\mathbb{C}$ on the representation $\Gamma_{a,b}$ is either isomorphic to $\mathbb{P}^2$ (embedded as the Veronese surface) if either a or b is zero, or to the incidence correspondence

$$\Sigma = \{(p, l): p \in l\} \subset \mathbb{P}^2 \times \mathbb{P}^{2*}$$

if neither a or b is zero.