LECTURE 15

$\mathfrak{sl}_4\mathbb{C}$ and $\mathfrak{sl}_n\mathbb{C}$

In this lecture, we will illustrate the general paradigm of the previous lecture by applying it to the Lie algebras $\mathfrak{sl}_n\mathbb{C}$; this is typical of the analyses of specific Lie algebras carried out in this Part. We start in §15.1 by describing the Cartan subalgebra, roots, root spaces, etc., for $\mathfrak{sl}_n\mathbb{C}$ in general. We then give in §15.2 a detailed account of the representations of $\mathfrak{sl}_4\mathbb{C}$, which generalizes directly to $\mathfrak{sl}_n\mathbb{C}$; in particular, we deduce the existence part of Theorem 14.18 for $\mathfrak{sl}_n\mathbb{C}$.

In §15.3 we give an explicit construction of the irreducible representations of $\mathfrak{sl}_n\mathbb{C}$ using the Weyl construction introduced in Lecture 6; analogous constructions of the irreducible representations of the remaining classical Lie algebras will be given in §17.3 and §19.5. This section presupposes familiarity with Lecture 6 and Appendix A, but can be skipped by those willing to forego §17.3 and 19.5 as well. Section 15.4 requires essentially the same degree of knowledge of classical algebraic geometry as §§11.3 and 13.4 (it does not presuppose §15.3), but can also be skipped. Finally, §15.5 describes representations of $GL_n\mathbb{C}$; this appears to involve the Weyl construction but in fact the main statement, Proposition 15.47 (and even its proof) can be understood without the preceding two sections.

§15.1: Analyzing $\mathfrak{sl}_n\mathbb{C}$
§15.2: Representations of $\mathfrak{sl}_4\mathbb{C}$ and $\mathfrak{sl}_n\mathbb{C}$
§15.3: Weyl's construction and tensor products
§15.4: Some more geometry
§15.5: Representations of $GL_n\mathbb{C}$

§15.1. Analyzing $\mathfrak{sl}_n\mathbb{C}$

To begin with, we have to locate a Cartan subalgebra, and this is not hard; as in the case of $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$ the subalgebra of diagonal matrices will work fine. Writing H_i for the diagonal matrix $E_{i,i}$ that takes e_i to itself and kills e_j

for $j \neq i$, we have

$$\mathfrak{h} = \{a_1 H_1 + a_2 H_2 + \cdots + a_n H_n: a_1 + a_2 + \cdots + a_n = 0\};$$

note that H_i is not in $\mathfrak{h}$. We can correspondingly write

$$\mathfrak{h}^* = \mathbb{C}\{L_1, L_2, \ldots, L_n\}/(L_1 + L_2 + \cdots + L_n = 0),$$

where $L_i(H_j) = \delta_{i,j}$. We often write L_i for the image of L_i in $\mathfrak{h}^*$.

We have already seen how the diagonal matrices act on the space of all traceless matrices: if $E_{i,j}$ is the endomorphism of $\mathbb{C}^n$ carrying e_j to e_i and killing e_k for all $k \neq j$, then we have

$$\text{ad}(a_1 H_1 + a_2 H_2 + \cdots + a_n H_n)(E_{i,j}) = (a_i - a_j) \cdot E_{i,j}; \qquad (15.1)$$

or, in other words, $E_{i,j}$ *is an eigenvector for the action of* $\mathfrak{h}$ *with eigenvalue* $L_i - L_j$; in particular, *the roots of* $\mathfrak{sl}_n\mathbb{C}$ *are just the pairwise differences of the* L_i.

Before we try to visualize anything taking place in $\mathfrak{h}$ or $\mathfrak{h}^*$, let us take a moment out and describe the Killing form. To this end, note that the automorphism φ of $\mathbb{C}^n$ sending e_i to e_j, e_j to $-e_i$ and fixing e_k for all $k \neq i, j$ induces an automorphism $\text{Ad}(\varphi)$ of the Lie algebra $\mathfrak{sl}_n\mathbb{C}$ (or even $\mathfrak{gl}_n(\mathbb{C})$) that carries $\mathfrak{h}$ to itself, exchanges H_i and H_j, and fixes all the other H_k. Since the Killing form on $\mathfrak{h}$ must be invariant under all these automorphisms, it must satisfy $B(L_i, L_i) = B(L_j, L_j)$ for all i and j and $B(L_i, L_k) = B(L_j, L_k)$ for all i, j and $k \neq i, j$; it follows that on $\mathfrak{h}$ it must be a linear combination of the forms

$$B'\left(\sum a_i H_i, \sum b_i H_i\right) = \sum a_i b_i$$

and

$$B''\left(\sum a_i H_i, \sum b_i H_i\right) = \sum_{i \neq j} a_i b_j.$$

On the space $\{\sum a_i H_i: \sum a_i = 0\}$, however, we have $0 = (\sum a_i)(\sum b_j) = \sum a_i b_i + \sum a_i b_j$, so in fact these two forms are dependent; and hence we can write the Killing form simply as a multiple of B'. Similarly, the Killing form on $\mathfrak{h}^*$ must be a linear combination of the forms $B'(\sum a_i L_i, \sum b_i L_i) = \sum a_i b_i$ and $B''(\sum a_i L_i, \sum b_i L_i) = \sum_{j \neq i} a_i b_j$; the condition that $B(\sum a_i L_i, \sum b_i L_i) = 0$ whenever $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$ implies that it must be a multiple of

$$B\left(\sum a_i L_i, \sum b_i L_i\right) = \left(\frac{n-1}{n}\right) \sum_i a_i b_i - \frac{1}{n} \sum_{i \neq j} a_i b_j$$

$$= \sum_i a_i b_i - \frac{1}{n} \sum_{i,j} a_i b_j. \qquad (15.2)$$

We may, of course, also calculate the Killing form directly from the definition. By (14.21), since the roots of $\mathfrak{sl}_n\mathbb{C}$ are $\{L_i - L_j\}_{i \neq j}$, we have

$$B\left(\sum a_i H_i, \sum b_i H_i\right) = \sum_{i \neq j} (a_i - a_j)(b_i - b_j)$$

$$= \sum_i \sum_{j \neq i} (a_i b_i + a_j b_j - a_i b_j - a_j b_i).$$

Noting that $\sum_{j \neq i} a_j = -a_i$ and, similarly, $\sum_{j \neq i} b_j = -b_i$, this simplifies to

$$B(\sum a_iH_i, \sum b_iH_i) = 2n\sum a_ib_i. \tag{15.3}$$

It follows with a little calculation that the dual form on $\mathfrak{h}^*$ is

$$B(\sum a_iL_i, \sum b_iL_i) = (1/2n)(\sum_i a_ib_i - (1/n)\sum_{i,j} a_ib_j). \tag{15.4}$$

It is probably simpler just to think of this as the form, unique up to scalars, invariant under the symmetric group $\mathfrak{S}_n$ of permutations of $\{1, 2, \ldots, n\}$. The L_i, therefore, all have the same length, and the angles between all pairs are the same. To picture the roots in $\mathfrak{h}^*$, then, we should think of the points L_i as situated at the vertices of a regular $(n-1)$-simplex Δ, with the origin located at the barycenter of that simplex. This picture is easiest to visualize in the special case $n = 4$, where the L_i will be located at every other vertex of a unit cube centered at the origin:

$$\tag{15.5}$$
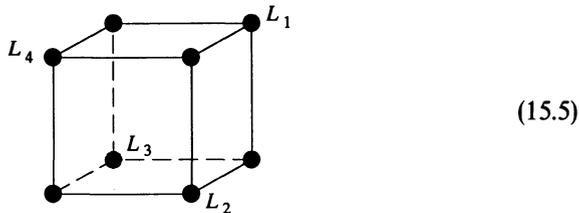
Now, as we said, the roots of $\mathfrak{sl}_n\mathbb{C}$ are now just the pairwise differences of the L_i. The root lattice Λ_R they generate can thus be described as

$$\Lambda_R = \{\sum a_iL_i: a_i \in \mathbb{Z}, \sum a_i = 0\}/(\sum L_i = 0).$$

Both the roots and the root lattice can be drawn in the case of $\mathfrak{sl}_4\mathbb{C}$: if we think of the vectors $L_i \in \mathfrak{h}^*$ as four of the vertices of a cube centered at the origin, the roots will comprise all the midpoints of the edges of a second cube whose linear dimensions are twice the dimensions of the first:

$$\tag{15.6}$$
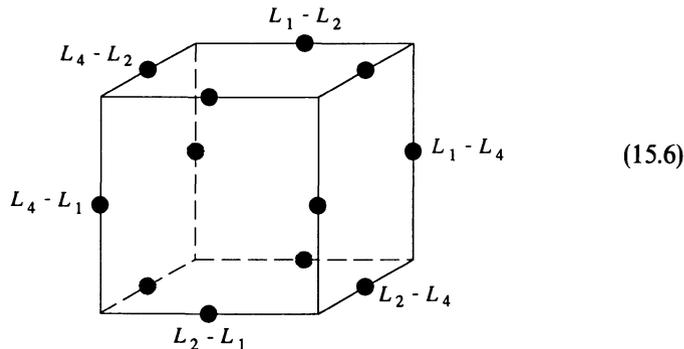
The next step, finding the distinguished subalgebras $\mathfrak{s}_\alpha$, is also very easy. The root space $\mathfrak{g}_{L_i-L_j}$ corresponding to the root $L_i - L_j$ is generated by $E_{i,j}$, so the subalgebra $\mathfrak{s}_{L_i-L_j}$ is generated by

$$E_{i,j}, \quad E_{j,i}, \quad \text{and} \quad [E_{i,j}, E_{j,i}] = H_i - H_j.$$

The eigenvalue of $H_i - H_j$ acting on $E_{i,j}$ is $(L_i - L_j)(H_i - H_j) = 2$, so that the corresponding distinguished element $H_{L_i-L_j}$ in $\mathfrak{h}$ must be just $H_i - H_j$. The annihilator, of course, is the hyperplane $\Omega_{L_i-L_j} = \{\sum a_i L_i : a_i = a_j\}$; note that this is indeed perpendicular to the root $L_i - L_j$ with respect to the Killing form B as described above.

Knowing the H_α we know the weight lattice: in order for a linear functional $\sum a_i L_i \in \mathfrak{h}^*$ to have integral values on all the distinguished elements, it is clearly necessary and sufficient that all the a_i be congruent to one another modulo $\mathbb{Z}$. Since $\sum L_i = 0$ in $\mathfrak{h}^*$, this means that the weight lattice is given as

$$\Lambda_W = \mathbb{Z}\{L_1, \ldots, L_n\}/(\sum L_i = 0).$$

In sum, then, *the weight lattice of* $\mathfrak{sl}_n\mathbb{C}$ *may be realized as the lattice generated by the vertices of a regular* $(n-1)$-*simplex* Δ *centered at the origin; and the roots as the pairwise differences of these vertices.*

While we are at it, having determined Λ_R and Λ_W we might as well compute the quotient Λ_W/Λ_R. This is pretty easy: since the lattice Λ_W can be generated by Λ_R together with any of the vertices L_i of our simplex, the quotient Λ_W/Λ_R will be cyclic, generated by any L_i; since, modulo Λ_R,

$$0 = \sum_j (L_i - L_j) = nL_i - \sum_j L_j = nL_i.$$

we see that L_i has order dividing n in Λ_W/Λ_R.

Exercise 15.7. Show that L_i has order exactly n in Λ_W/Λ_R, so that $\Lambda_W/\Lambda_R \cong \mathbb{Z}/n\mathbb{Z}$.

From the above we can also say what the Weyl group is: the reflection in the hyperplane perpendicular to the root $L_i - L_j$ will exchange L_i and $L_j \in \mathfrak{h}^*$ and leave the other L_k alone, so that *the Weyl group* $\mathfrak{W}$ *is just the group* $\mathfrak{S}_n$, *acting as the symmetric group on the generators* L_i *of* $\mathfrak{h}^*$. Note that we have already verified that these automorphisms of $\mathfrak{h}^*$ do come from automorphisms of the whole Lie algebra $\mathfrak{sl}_n\mathbb{C}$ preserving $\mathfrak{h}$.

To continue, let us choose a direction, and describe the corresponding Weyl chamber. We can write our linear functional l as

$$l(\sum a_i L_i) = \sum c_i a_i$$

with $\sum c_i = 0$; let us suppose that $c_1 > c_2 > \cdots > c_n$. The corresponding ordering of the roots will then be

$$R^+ = \{L_i - L_j : i < j\}$$

and

$$R^- = \{L_i - L_j: j < i\}.$$

The primitive negative roots for this ordering are simply the roots $L_{i+1} - L_i$. (Note that the ordering of the roots depends only on the relative sizes of the c_i, so that the Weyl group acts simply transitively on the set of orderings.) The (closed) Weyl chamber associated to this ordering will then be the set

$$\mathcal{W} = \{\sum a_i L_i: a_1 \geq a_2 \geq \cdots \geq a_n\}.$$

One way to describe this geometrically is to say that if we take the barycentric subdivision of the faces of the simplex Δ, the Weyl chamber will be the cone over one $(n - 2)$-simplex of the barycentric subdivision: e.g., in the case $n = 4$
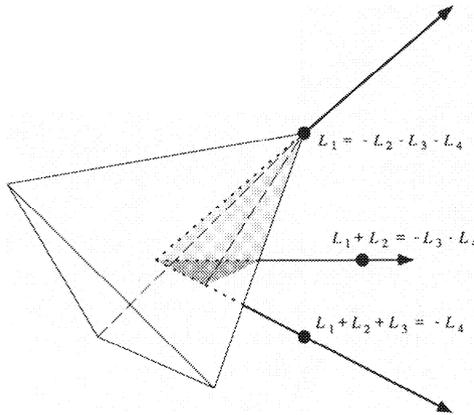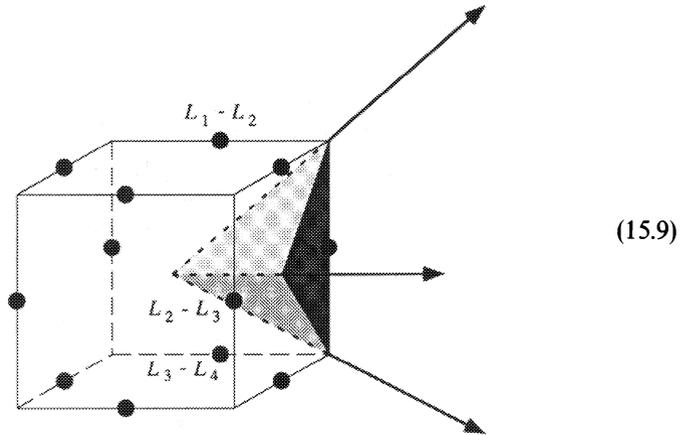

It may be easier to visualize the case $n = 4$ if we introduce the associated cubes: in terms of the cube with vertices at the points $\pm L_i$, we can draw the Weyl chamber as
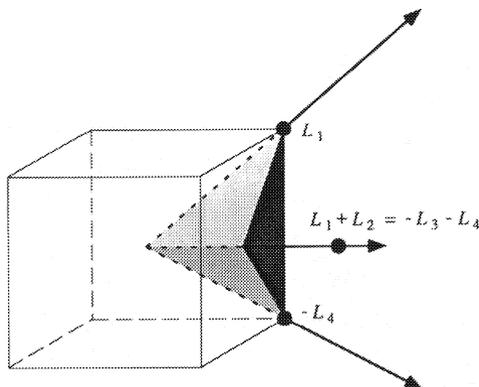
(15.8)

Alternatively, in terms of the slightly larger cube with vertices at the points $\pm 2L_i$, we can draw $\mathcal{W}$ as

$$(15.9)$$

From the first of these pictures we see that the edges of the Weyl chamber are the rays generated by the vectors L_1, $L_1 + L_2$, and $L_1 + L_2 + L_3$; and that the faces of the Weyl chamber are the planes orthogonal to the primitive negative roots $L_2 - L_1$, $L_3 - L_2$, and $L_4 - L_3$. The picture in general is analogous: for $\mathfrak{sl}_n\mathbb{C}$, the Weyl chamber will be the cone over an $(n-2)$-simplex, with edges generated by the vectors

$$L_1, \quad L_1 + L_2, \quad L_1 + L_2 + L_3, \ldots, L_1 + \cdots + L_{n-1} = -L_n.$$

The faces of $\mathcal{W}$ will thus be the hyperplanes

$$\Omega_{L_i - L_{i+1}} = \{\textstyle\sum a_j L_j : a_i = a_{i+1}\}$$

perpendicular to the primitive negative roots $L_{i+1} - L_i$.

Note the important phenomenon: the intersection of the closed Weyl chamber with the lattice Λ_W will be a free semigroup $\mathbb{N}^{n-1}$ generated by the fundamental weights $\omega_i = L_1 + \cdots + L_i$ occurring along the edges of the Weyl chamber. One aspect of its significance that is immediate is that it allows us to index the irreducible representations $\mathfrak{sl}_n\mathbb{C}$ nicely: for an arbitrary $(n-1)$-tuple of natural numbers $(a_1, \ldots, a_{n-1}) \in \mathbb{N}^{n-1}$ we will denote by $\Gamma_{a_1, \ldots, a_{n-1}}$ the irreducible representation of $\mathfrak{sl}_n\mathbb{C}$ with highest weight $a_1 L_1 + a_2(L_1 + L_2) + \cdots + a_{n-1}(L_1 + \cdots + L_{n-1}) = (a_1 + \cdots + a_{n-1})L_1 + (a_2 + \cdots + a_{n-1})L_2 + \cdots + a_{n-1}L_{n-1}$:

$$\Gamma_{a_1, \ldots, a_{n-1}} = \Gamma_{a_1 L_1 + a_2(L_1 + L_2) + \cdots + a_{n-1}(L_1 + \cdots + L_{n-1})}.$$

This also has the nice consequence that once we have located the irreducible representations $V^{(i)}$ with highest weight $L_1 + \cdots + L_i$, the general irreducible

representation $\Gamma_{a_1,\ldots,a_{n-1}}$ with highest weight $\sum a_i(L_1 + \cdots + L_i)$ will occur inside the tensor product of symmetric powers

$$\mathrm{Sym}^{a_1} V^{(1)} \otimes \mathrm{Sym}^{a_2} V^{(2)} \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}} V^{(n-1)}$$

of these representations. Thus, the existence part of the basic Theorem 14.18 is reduced to finding the basic representations $V^{(i)}$; we will do this in due course, though at this point it is probably not too hard an exercise to guess what they are.

§15.2. Representations of $\mathfrak{sl}_4\mathbb{C}$ and $\mathfrak{sl}_n\mathbb{C}$

We begin as usual with the standard representation of $\mathfrak{sl}_4\mathbb{C}$ on $V = \mathbb{C}^4$. The standard basis vectors e_i of $\mathbb{C}^4$ are eigenvectors for the action of $\mathfrak{h}$, with eigenvalues L_i, so that the weight diagram looks like
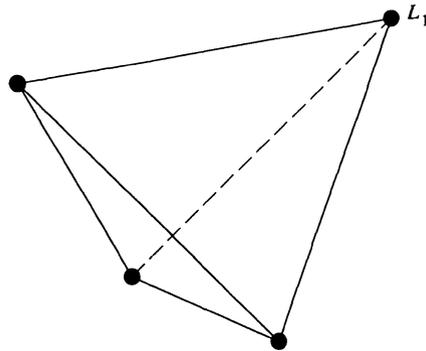
or, with the reference cube drawn as well,
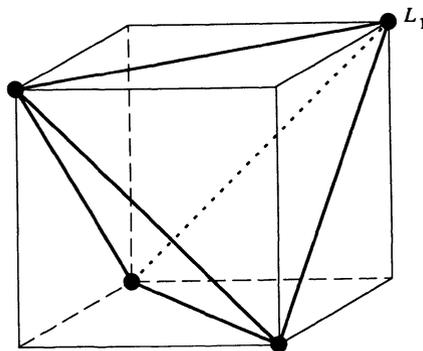
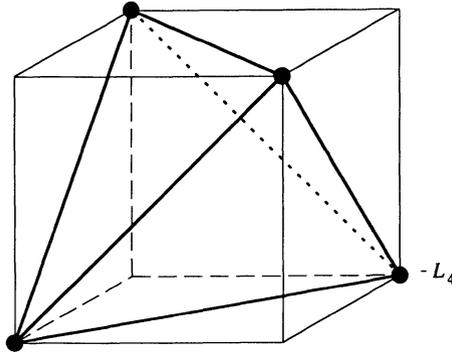The dual representation V^* of course has weights $-L_i$ corresponding to the vectors of the dual basis e_i^* for V^*, so that the weight diagram, with its reference cube, looks like

Note that the highest weight for this representation is $-L_4$, which lies along the bottom edge of the Weyl chamber, as depicted in Diagram (15.8). Note also that the weights of the representation $\wedge^3 V$—the triple sums $L_1 + L_2 + L_3$, $L_1 + L_2 + L_4$, $L_1 + L_3 + L_4$, and $L_2 + L_3 + L_4$ of distinct weights of V—are the same as those of V^*, reflecting the isomorphism of these two representations.

This suggests that we look next at the second exterior power $\wedge^2 V$. This is a six-dimensional representation, with weights $L_i + L_j$ the pairwise sums of distinct weights of V; its weight diagram, in its reference cube, looks like
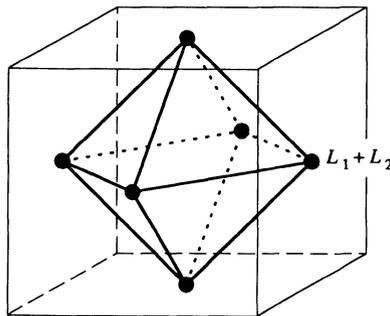
The diagram shows clearly that $\wedge^2 V$ is irreducible since it is not the nontrivial union of two configurations invariant under the Weyl group $\mathfrak{S}_4$ (and all weights occur with multiplicity 1). Note also that the weights are symmetric about the origin, reflecting the isomorphism of $\wedge^2 V$ with $(\wedge^2 V)^* = \wedge^2(V^*)$.

Note that the highest weight $L_1 + L_2$ of the representation $\wedge^2 V$ is the primitive vector along the front edge of the Weyl chamber $\mathcal{W}$ as pictured in Diagram (15.8). Now, we have already seen that the intersection of the closed

Weyl chamber with the weight lattice is a free semigroup generated by the primitive vectors along the three edges of $\mathscr{W}$—that is, every vector in $\mathscr{W} \cap \Lambda_W$ is a non-negative integral linear combination of the three vectors $L_1, L_1 + L_2$, and $L_1 + L_2 + L_3$. As we remarked at the end of the first section of this lecture, it follows that *we have proved the existence half of the general existence and uniqueness theorem* (14.18) *in the case of the Lie algebra* $\mathfrak{sl}_4\mathbb{C}$. Explicitly, since $V, \wedge^2 V$, and $\wedge^3 V = V^*$ have highest weight vectors with weights L_1, $L_1 + L_2$, and $L_1 + L_2 + L_3$, respectively, it follows that *the representation*

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b(\wedge^2 V) \otimes \mathrm{Sym}^c(\wedge^3 V)$$

contains a highest weight vector with weight $aL_1 + b(L_1 + L_2) + c(L_1 + L_2 + L_3)$, *and hence a copy of the irreducible representation* $\Gamma_{a,b,c}$ *with this highest weight.*

Let us continue our examination of representations of $\mathfrak{sl}_4\mathbb{C}$ with a pair of tensor products of the three basic representations: $V \otimes \wedge^2 V$ and $V \otimes \wedge^3 V$. As for the first of these, its weights are easy to find: they consist of the sums $2L_i + L_j$ (which occur once, as the sum of L_i and $L_i + L_j$) and $L_i + L_j + L_k$ (which occur three times). The diagram of these weights looks like
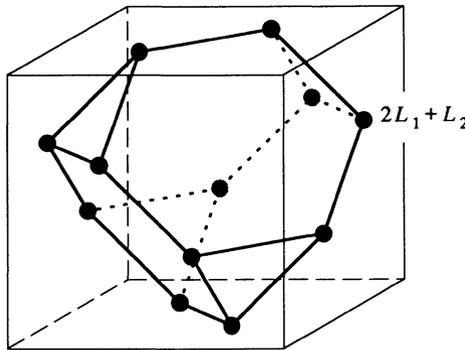
(We have drawn only the vertices of the convex hull of this diagram, thus omitting the weights $L_i + L_j + L_k$; they are located at the centers of the hexagonal faces of this polyhedron.)

Now, the representation $V \otimes \wedge^2 V$ cannot be irreducible, for at least a couple of reasons. First off, just by looking at weights, we see that the irreducible representation $W = \Gamma_{1,1,0}$ with highest weight $2L_1 + L_2$ can have multiplicity at most 2 on the weight $L_1 + L_2 + L_3$: by Observation 14.16, the weight space $W_{L_1+L_2+L_3}$ is generated by the images of the highest weight vector $v \in W_{2L_1+L_2}$ by successive applications of the primitive negative root spaces $\mathfrak{g}_{L_2-L_1}, \mathfrak{g}_{L_3-L_2}$, and $\mathfrak{g}_{L_4-L_3}$. But $L_1 + L_2 + L_3$ is uniquely expressible as a sum of $2L_1 + L_2$ and the primitive negative roots:

$$L_1 + L_2 + L_3 = 2L_1 + L_2 + (L_2 - L_1) + (L_3 - L_2);$$

so that $V_{L_1+L_2+L_3}$ is generated by the subspaces $\mathfrak{g}_{L_2-L_1}(\mathfrak{g}_{L_3-L_2}(v))$ and $\mathfrak{g}_{L_3-L_2}(\mathfrak{g}_{L_2-L_1}(v))$. We can in fact check that the representation $\Gamma_{1,1,0}$ takes on the weight $L_1 + L_2 + L_3$ with multiplicity 2 by writing out these generators explicitly and checking that they are independent: for example, we have

$$\mathfrak{g}_{L_2-L_1}(\mathfrak{g}_{L_3-L_2}(v)) = \mathbb{C} \cdot E_{2,1}(E_{3,2}(e_1 \otimes (e_1 \wedge e_2)))$$

$$= \mathbb{C} \cdot E_{2,1}(e_1 \otimes (e_1 \wedge e_3))$$

$$= \mathbb{C} \cdot (e_2 \otimes (e_1 \wedge e_3) + e_1 \otimes (e_2 \wedge e_3)).$$

This is in fact what is called for in Exercise 15.10.

Alternatively, forgetting weights entirely, we can see from standard multilinear algebra that the representation $V \otimes \wedge^2 V$ cannot be irreducible: we have a natural map of representations

$$\varphi: V \otimes \wedge^2 V \to \wedge^3 V$$

which is obviously surjective. The kernel of this map is a representation with the same set of weights as $V \otimes \wedge^2 V$ (but taking on the weights $L_i + L_j + L_k$ with multiplicity 2 rather than 3), and so must contain the irreducible representation $\Gamma_{1,1,0}$ with highest weight $2L_1 + L_2$.

Exercise 15.10. Prove that the kernel of φ is indeed the irreducible representation $\Gamma_{1,1,0}$.

Finally, consider the tensor product $V \otimes \wedge^3 V$. This has weights $2L_i + L_k + L_l = L_i - L_j$, each occurring once, and 0, occurring four times. Its weight diagrams thus look like
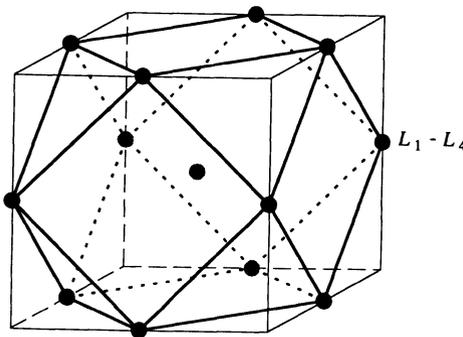
This we may recognize as simply a direct sum of the adjoint representation with a copy of the trivial; this corresponds to the kernel and image of the obvious contraction (or trace) map

$$V \otimes \wedge^3 V = V \otimes V^* = \operatorname{Hom}(V, V) \to \mathbb{C}.$$

(Note that the adjoint representation is the irreducible representation with highest weight $2L_1 + L_2 + L_3$, or in other words the representation $\Gamma_{1,0,1}$.)

Exercise 15.11. Describe the weights of the representations $\operatorname{Sym}^n V$, and deduce that they are all irreducible.

Exercise 15.12. Describe the weights of the representations $\operatorname{Sym}^n(\wedge^2 V)$, and deduce that they are not irreducible. Describe maps

$$\varphi_n \colon \operatorname{Sym}^n(\wedge^2 V) \to \operatorname{Sym}^{n-2}(\wedge^2 V)$$

and show that the kernel of φ_n is the irreducible representation with highest weight $n(L_1 + L_2)$.

Exercise 15.13. The irreducible representation $\Gamma_{1,1,1}$ with highest weight $3L_1 + 2L_2 + L_3$ occurs as a subrepresentation of the tensor product $V \otimes \wedge^2 V \otimes \wedge^3 V$ lying in the kernel of each of the three maps

$$V \otimes \wedge^2 V \otimes \wedge^3 V \to \wedge^3 V \otimes \wedge^3 V$$

$$V \otimes \wedge^2 V \otimes \wedge^3 V \to \wedge^2 V \otimes \wedge^4 V \cong \wedge^2 V$$

$$V \otimes \wedge^2 V \otimes \wedge^3 V \cong V \otimes \wedge^2 V^* \otimes V^* \to V \otimes \wedge^3 V^* \cong V \otimes V$$

obtained by wedging two of the three factors. Is it equal to the intersection of these kernels? To test your graphic abilities, draw a diagram of the weights (ignoring multiplicities) of this representation.

Representations of $\mathfrak{sl}_n\mathbb{C}$

Once the case of $\mathfrak{sl}_4\mathbb{C}$ is digested, the case of the special linear group in general offers no surprises; the main difference in the general case is just the absence of pictures. Of course, the standard representation V of $\mathfrak{sl}_n\mathbb{C}$ has highest weight L_1, and similarly the exterior power $\wedge^k V$ is irreducible with highest weight $L_1 + \cdots + L_k$. It follows that the irreducible representation $\Gamma_{a_1,\dots,a_{n-1}}$ with highest weight $(a_1 + \cdots + a_{n-1})L_1 + \cdots + a_{n-1}L_{n-1}$ will appear inside the tensor product

$$\operatorname{Sym}^{a_1} V \otimes \operatorname{Sym}^{a_2}(\wedge^2 V) \otimes \cdots \otimes \operatorname{Sym}^{a_{n-1}}(\wedge^{n-1} V),$$

demonstrating the existence theorem (14.18) for representations of $\mathfrak{sl}_n\mathbb{C}$.

Exercise 15.14. Verify that the exterior powers of the standard representations of $\mathfrak{sl}_n\mathbb{C}$ are indeed irreducible (though this is not necessary for the truth of the last sentence).

§15.3. Weyl's Construction and Tensor Products

At the end of the preceding section, we saw that the irreducible representation $\Gamma_{a_1,\ldots,a_{n-1}}$ of $\mathfrak{sl}_n\mathbb{C}$ with highest weight $(a_1 + \cdots + a_{n-1})L_1 + \cdots + a_{n-1}L_{n-1}$ will appear as a subspace of the tensor product

$$\mathrm{Sym}^{a_1} V \otimes \mathrm{Sym}^{a_2}(\wedge^2 V) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1} V),$$

or equivalently as a subspace of the dth tensor power $V^{\otimes d}$ of the standard representation V. The natural question is, how can we describe this subspace? We have seen the answer in one case already (two cases, if you count the trivial answer $\Gamma_a = \mathrm{Sym}^a V$ in the case $n = 2$): the representation $\Gamma_{a,b}$ of $\mathfrak{sl}_3\mathbb{C}$ can be realized as the kernel of the contraction map

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b(\wedge^2 V) \to \mathrm{Sym}^{a-1} V \otimes \mathrm{Sym}^{b-1}(\wedge^2 V).$$

This raises the question of whether the representation Γ_a can in general be described as a subspace of the tensor power $\bigotimes(\mathrm{Sym}^{a_i}(\wedge^i V))$ by intersecting kernels of such contraction/wedge product maps. Specifically, for i and j with $i + j \le n$ we can define maps

$$\mathrm{Sym}^{a_1} V \otimes \mathrm{Sym}^{a_2}(\wedge^2 V) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1} V)$$

$$\to \wedge^i V \otimes \wedge^j V \otimes \mathrm{Sym}^{a_1} V \otimes \cdots \otimes \mathrm{Sym}^{a_i-1}(\wedge^i V) \otimes \cdots$$

$$\otimes \mathrm{Sym}^{a_j-1}(\wedge^j V) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1} V)$$

and we have similar maps for $i < j$ with $i + j \ge n$ and i even with $2i \ge n$; there are likewise analogously defined maps in which we split off three or more factors. The representation $\Gamma_{a_1,\ldots,a_{n-1}}$ is in the kernel of all such maps; and we may ask whether the intersection of all such kernels is equal to Γ_a.

The answer, it turns out, is no. (It is a worthwhile exercise to find an example of a representation Γ_a that cannot be realized in this way.) There is, however, another way of describing Γ_a as a subspace of $V^{\otimes d}$: in fact, we have already met these representations in Lecture 6, under the guise of *Schur functors* or *Weyl modules*. In fact, at the end of this lecture we will see how to describe them explicitly as subspaces of the above spaces $\bigotimes(\mathrm{Sym}^{a_i}(\wedge^i V))$. Recall that for $V = \mathbb{C}^n$ an n-dimensional vector space, and any partition

$$\lambda: \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0,$$

we can apply the Schur functor $\mathbb{S}_\lambda$ to V to obtain a representation $\mathbb{S}_\lambda V = \mathbb{S}_\lambda(\mathbb{C}^n)$ of $\mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$. If $d = \sum \lambda_i$, this was realized as

$$\mathbb{S}_\lambda V = V^{\otimes d} \cdot c_\lambda = V^{\otimes d} \otimes_{\mathbb{C}\mathfrak{S}_d} V_\lambda,$$

where c_λ is the Young symmetrizer corresponding to λ, and V_λ is the irreducible representation of $\mathfrak{S}_d$ corresponding to λ.

We saw in Lecture 6 that $\mathbb{S}_\lambda V$ is an irreducible representation of $\mathrm{GL}_n\mathbb{C}$. It follows immediately that $\mathbb{S}_\lambda V$ remains irreducible as a representation of $\mathrm{SL}_n\mathbb{C}$,

since any element of $GL_n\mathbb{C}$ is a scalar multiple of an element of $SL_n\mathbb{C}$. In particular, it determines an irreducible representation of the Lie algebra $\mathfrak{sl}_n\mathbb{C}$.

Proposition 15.15. *The representation* $\mathbb{S}_\lambda(\mathbb{C}^n)$ *is the irreducible representation of* $\mathfrak{sl}_n\mathbb{C}$ *with highest weight* $\lambda_1 L_1 + \lambda_2 L_2 + \cdots + \lambda_n L_n$.

In particular, $\mathbb{S}_\lambda(\mathbb{C}^n)$ and $\mathbb{S}_\mu(\mathbb{C}^n)$ are isomorphic representations of $\mathfrak{sl}_n\mathbb{C}$ if and only if $\lambda_i - \mu_i$ is constant, independent of i. To relate this to our earlier notation, we may say that the irreducible representation $\Gamma_{a_1,\dots,a_{n-1}}$ of $\mathfrak{sl}_n\mathbb{C}$ with highest weight $a_1 L_1 + a_2(L_1 + L_2) + \cdots + a_{n-1}(L_1 + \cdots + L_{n-1})$ is obtained by applying the Schur functor $\mathbb{S}_\lambda$ to the standard representation V, where

$$\lambda = (a_1 + \cdots + a_{n-1}, a_2 + \cdots + a_{n-1}, \dots, a_{n-1}, 0).$$

(If we want a unique Schur functor for each representation, we can restrict to those λ with $\lambda_n = 0$.) In terms of the Young diagram for λ, the coefficients $a_i = \lambda_i - \lambda_{i+1}$ are the differences of lengths of rows. For example, if $n = 6$,
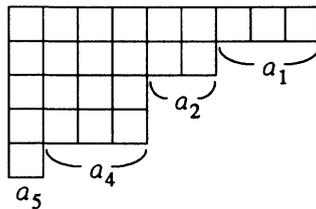
is the Young diagram corresponding to $\Gamma_{3,2,0,3,1}$.

PROOF OF THE PROPOSITION. In Theorem 6.3 we calculated that the trace of a diagonal matrix with entries $x_1, \dots, x_n$ on $\mathbb{S}_\lambda(\mathbb{C}^n)$ is the Schur polynomial $S_\lambda(x_1, \dots, x_n)$. By Equation (A.19), when the Schur polynomial is written out it takes the form

$$S_\lambda(x_1, \dots, x_n) = M_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} M_\mu, \tag{15.16}$$

where M_μ is the sum of the monomial $X^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdot \ldots \cdot x_n^{\mu_n}$ and all distinct monomials obtained from it by permuting the variables, and the $K_{\lambda\mu}$ are certain non-negative integers called Kostka numbers. When $\mathbb{S}_\lambda(\mathbb{C}^n)$ is diagonalized with respect to the group of diagonal matrices in $GL_n(\mathbb{C})$, it is also diagonalized with respect to $\mathfrak{h} \subset \mathfrak{sl}_n(\mathbb{C})$. There is one monomial in the displayed equation for each one-dimensional eigenspace. The weights of $\mathbb{S}_\lambda(\mathbb{C}^n)$ as a representation of $\mathfrak{sl}_n(\mathbb{C})$ therefore consist of all

$$\mu_1 L_1 + \mu_2 L_2 + \cdots + \mu_n L_n,$$

each occurring as often as it does in the monomial X^μ in the polynomial

$S_\lambda(x_1, \ldots, x_n)$. Since the sum is over those partitions μ for which the first nonzero $\lambda_i - \mu_i$ is positive, the highest weight that appears is $\lambda_1 L_1 + \lambda_2 L_2 + \cdots + \lambda_n L_n$, which concludes the proof. [In fact one can describe an explicit basis of eigenvectors for $\mathbb{S}_\lambda(\mathbb{C}^n)$ which correspond to the monomials that appear in (15.16), cf. Problem 6.15 or Proposition 15.55.] $\square$

In particular, we have (by Theorem 6.3) formulas for the dimension of the representation with given highest weight. Explicitly, one formula says that

$$\dim(\Gamma_{a_1, \ldots, a_{n-1}}) = \prod_{1 \le i < j \le n} \frac{(a_i + \cdots + a_{j-1}) + j - i}{j - i}. \tag{15.17}$$

As we saw in the proof, this proposition also gives the multiplicities of all weight spaces as the integers $K_{\lambda\mu}$ that appear in (15.16), which have a simple combinatorial description (p. 456): *the dimension of the weight space with weight μ in the representation $\mathbb{S}_\lambda(\mathbb{C}^n)$ is the number of ways one can fill the Young diagram of λ with μ_1 1's, μ_2 2's, $\ldots, \mu_n$ n's, in such a way that the entries in each row are nondecreasing and those in each column are strictly increasing.*

Exercise 15.18. Use the formula in case $n = 4$ to calculate the dimensions of the irreducible representations $\Gamma_{1,1,0}$ and $\Gamma_{1,1,1}$ of $\mathfrak{sl}_4\mathbb{C}$. In the former case, use this to redo Exercise 15.10; in the latter case, to do Exercise 15.13.

Exercise 15.19*. Use this formula to show that the dimension of the irreducible representation $\Gamma_{a,b}$ of $\mathfrak{sl}_3$ with highest weight $aL_1 + b(L_1 + L_2)$ is $(a + b + 1)(a + 1)(b + 1)/2$. This is the same as the dimension of the kernel of the contraction map

$$\iota_{a,b}: \mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^* \to \mathrm{Sym}^{a-1} V \otimes \mathrm{Sym}^{b-1} V^*.$$

Use this to give another proof of the assertion made in Claim 13.4 that $\Gamma_{a,b}$ is this kernel.

Exercise 15.20*. As an application of the above formula, show that if V is the standard representation of $\mathfrak{sl}_n\mathbb{C}$, then the kernel of the wedge product map

$$V \otimes \textstyle\bigwedge^k V \to \textstyle\bigwedge^{k+1} V$$

is the irreducible representation $\Gamma_{1,0,\ldots,0,1,0,\ldots}$ with highest weight $2L_1 + L_2 + \cdots + L_k$; and that the irreducible representation $\Gamma_{k-1,1,0,\ldots}$ with highest weight $k \cdot L_1 + L_2$ is the kernel of the product map

$$V \otimes \mathrm{Sym}^k V \to \mathrm{Sym}^{k+1} V.$$

Exercise 15.21*. Show that the only nontrivial irreducible representations of $\mathfrak{sl}_n\mathbb{C}$ of dimension less than or equal to n are V and V^*.

One important consequence of the fact that the irreducible representations of $\mathfrak{sl}_n\mathbb{C}$ are obtained by applying Schur functors to the standard representation

is that *identities among the Schur–Weyl functors give rise to identities among representations of* GL_n (and hence SL_n and $\mathfrak{sl}_n$), as we saw in Lecture 6. For example, the representation

$$\operatorname{Sym}^{\lambda_1}(V) \otimes \operatorname{Sym}^{\lambda_2}(V) \otimes \cdots \otimes \operatorname{Sym}^{\lambda_n}(V) \tag{15.22}$$

is a direct sum of representations $\mathbb{S}_\lambda(V) \oplus \bigoplus_\mu K_{\mu\lambda} \mathbb{S}_\mu(V)$, where $K_{\mu\lambda}$ is the coefficient described above. The particular application of this principle that we will use most frequently in the sequel, however, is the consequence that *one knows the decomposition of a tensor product of any two irreducible representations of* $\mathfrak{sl}_n\mathbb{C}$: specifically, the tensor power $\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V)$ decomposes into a direct sum of irreducible representations

$$\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V) = \bigoplus_\nu N_{\lambda\mu\nu} \mathbb{S}_\nu(V), \tag{15.23}$$

where the coefficients $N_{\lambda\mu\nu}$ are given by the *Littlewood–Richardson rule*, which is a formula in terms of the number of ways to fill the Young diagram between λ and ν with μ_1 1's, μ_2 2's, ..., μ_n n's, satisfying a certain combinatorial condition described in (A.8).

Exercise 15.24. Use the Littlewood–Richardson rule to show that the representation $\Gamma_{a_1+b_1,\dots,a_{n-1}+b_{n-1}}$ occurs exactly once in the tensor product $\Gamma_{a_1,\dots,a_{n-1}} \otimes \Gamma_{b_1,\dots,b_{n-1}}$.

A special case of this is the analogue of Pieri's formula, which allows us to decompose the tensor product of an arbitrary irreducible representation with either $\operatorname{Sym}^k V = \Gamma_{k,0,\dots,0}$ or the fundamental representation $\bigwedge^k V = \Gamma_{0,\dots,1,0,\dots,0}$, (where the 1 occurs in the kth place):

Proposition 15.25. (i) *The tensor product of* $\Gamma_{a_1,\dots,a_{n-1}}$ *with* $\operatorname{Sym}^k V = \Gamma_{k,0,\dots,0}$ *decomposes into a direct sum:*

$$\Gamma_{a_1,\dots,a_{n-1}} \otimes \Gamma_{k,\dots,0} = \bigoplus \Gamma_{b_1,\dots,b_{n-1}},$$

the sum over all $(b_1,\dots,b_{n-1})$ *for which there are non-negative integers* $c_1,\dots,c_n$ *whose sum is* k, *with* $c_{i+1} \le a_i$ *for* $1 \le i \le n-1$, *and with* $b_i = a_i + c_i - c_{i+1}$ *for* $1 \le i \le n-1$.
    (ii) *The tensor product of* $\Gamma_{a_1,\dots,a_{n-1}}$ *with* $\bigwedge^k V = \Gamma_{0,\dots,0,1,0,\dots,0}$ *decomposes into a direct sum:*

$$\Gamma_{a_1,\dots,a_{n-1}} \otimes \Gamma_{0,\dots,0,1,0,\dots,0} = \bigoplus \Gamma_{b_1,\dots,b_{n-1}},$$

the sum over all $(b_1,\dots,b_{n-1})$ *for which there is a subset* S *of* $\{1,\dots,n\}$ *of cardinality* k, *such that if* $i \notin S$ *and* $i + 1 \in S$, *then* $a_i > 0$, *with*

$$b_i = \begin{cases} a_i - 1 & \text{if } i \notin S \text{ and } i+1 \in S \\ a_i + 1 & \text{if } i \in S \text{ and } i+1 \notin S \\ a_i & \text{otherwise.} \end{cases}$$

PROOF. This is simply a matter of translating the prescriptions of (6.8) and (6.9), which describe the decompositions in terms of adding boxes to the Young diagrams. In (i), the c_i are the number of boxes added to the ith row, and in (ii), S is the set of rows to which a box is added. $\qquad\square$

Exercise 15.26. Verify the descriptions in Section 2 of this lecture of $V \otimes \wedge^2 V$ and $V \otimes \wedge^3 V$, where V is the standard representation of $\mathfrak{sl}_4\mathbb{C}$.

Exercise 15.27. Use Pieri's formula (with $n = 4$) twice to find the decomposition into irreducibles of $V \otimes \wedge^2 V \otimes \wedge^3 V$, where V is the standard representation of $\mathfrak{sl}_4\mathbb{C}$. Use this to redo Exercise 15.13.

Exercise 15.28. Use Pieri's formula to prove (13.5). You may also want to look around in Lecture 13 to see which other of the decompositions found there by hand may be deduced from these formulas.

Exercise 15.29. Verify that the statement of Exercise 15.20 follows directly from Pieri's formula.

In the following exercises, $V = \mathbb{C}^n$ is the standard representation of $\mathfrak{sl}_n\mathbb{C}$.

Exercise 15.30. Consider now tensor products of the form $\wedge^k V \otimes \wedge^l V$, with, say, $k \geq l$. Show that there is a natural map

$$\wedge^k V \otimes \wedge^l V \to \wedge^{k+1} V \otimes \wedge^{l-1} V$$

given by contraction with the element "trace" (or "identity") in $V \otimes V^* = \text{End}(V)$. Explicitly, this map may be given by

$$(v_1 \wedge \cdots \wedge v_k) \otimes (w_1 \wedge \cdots \wedge w_l)$$
$$\mapsto \sum_{i=1}^{l} (-1)^i (v_1 \wedge \cdots \wedge v_k \wedge w_i) \otimes (w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_l).$$

What is the image of this map? Show that the kernel is the irreducible representation $\Gamma_{0,\ldots,0,1,0,\ldots,0,1,0,\ldots}$ with highest weight $2L_1 + \cdots + 2L_l + L_{l+1} + \cdots + L_k$.

Exercise 15.31*. Carry out an analysis similar to that of the preceding exercise for the maps

$$\text{Sym}^k V \otimes \text{Sym}^l V \to \text{Sym}^{k+1} V \otimes \text{Sym}^{l-1} V$$

defined analogously.

Exercise 15.32*. As a special case of Pieri's formula, we see that if V is the standard representation of $\mathfrak{sl}_n\mathbb{C}$, the tensor product

$$\wedge^k V \otimes \wedge^k V = \bigoplus \mathbb{S}_{(2,\ldots,2,1,\ldots,1,0,\ldots)}(V)$$

$$= \bigoplus \Gamma_{0,\ldots,0,1,0,\ldots,0,1,0,\ldots},$$

where in the ith factor the 1's occur in the $(k-i)$th and $(k+i)$th places. At the same time, of course, we know that

$$\wedge^k V \otimes \wedge^k V = \mathrm{Sym}^2(\wedge^k V) \oplus \wedge^2(\wedge^k V).$$

If we denote the ith term on the right-hand side of the first displayed equation for $\wedge^k V \otimes \wedge^k V$ by Θ_i, show that

$$\mathrm{Sym}^2(\wedge^k V) = \bigoplus \Theta_{2i} \quad \text{and} \quad \wedge^2(\wedge^k V) = \bigoplus \Theta_{2i+1}.$$

Exercise 15.33*. As another special case of Pieri's formula, we see that the tensor product

$$\mathrm{Sym}^k V \otimes \mathrm{Sym}^k V = \bigoplus \mathbb{S}_{(k+i,k-i)}(V)$$

$$= \bigoplus \Gamma_{2i,k-i,0\ldots0}.$$

At the same time, of course, we know that

$$\mathrm{Sym}^k V \otimes \mathrm{Sym}^k V = \mathrm{Sym}^2(\mathrm{Sym}^k V) \oplus \wedge^2(\mathrm{Sym}^k V).$$

Which of the factors appearing in the first decomposition lie in $\mathrm{Sym}^2(\mathrm{Sym}^k V)$, and which in $\wedge^2(\mathrm{Sym}^k V)$?

It follows from the Littlewood–Richardson rule that if λ, μ, and ν all have at most two rows, then the coefficient $N_{\lambda\mu\nu}$ is zero or one (and it is easy to say which occurs). In particular, for the Lie algebras $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$, the decomposition of the tensor product of two irreducible representations is always multiplicity free. Groups whose representations have this property, such as SU(2), SU(3), and SO(3) which are so important in physics, are called "simply reducible," cf. [Mack].

§15.4. Some More Geometry

Let V be an n-dimensional vector space, and $G(k, n) = G(k, V) = \mathrm{Grass}_k V$ the Grassmannian of k-planes in V. $\mathrm{Grass}_k V$ is embedded as a subvariety of the projective space $\mathbb{P}(\wedge^k V)$ by the *Plücker embedding*:

$$\rho: \mathrm{Grass}_k V \hookrightarrow \mathbb{P}(\wedge^k V)$$

sending the plane W spanned by vectors $v_1, \ldots, v_k$ to the alternating tensor $v_1 \wedge \cdots \wedge v_k$. Equivalently, noting that if $W \subset V$ is a k-dimensional subspace, then $\wedge^k W$ is a line in $\wedge^k V$, we may write this simply as

$$\rho: W \mapsto \wedge^k W.$$

This embedding is compatible with the action of the general linear group:

$$\mathrm{PSL}_n\mathbb{C} = \mathrm{Aut}(\mathbb{P}(V)) = \{\sigma \in \mathrm{Aut}(\mathbb{P}(\wedge^k V)) : \sigma(G(k, V)) = G(k, V)\}^\circ.$$

This follows from a fact in algebraic geometry ([Ha]): all automorphisms of the Grassmannian are induced by automorphisms of V, unless $n = 2k$, in which case we can choose an arbitrary isomorphism of V with V^* and compose these with the automorphism that takes W to $(\mathbb{C}^n/W)^*$. Here the superscript $\circ$ denotes the connected component of the identity. As in previous lectures, if we want symmetric powers to correspond to homogeneous polynomials on projective space, we should consider the dual situation: $G = \mathrm{Grass}^k V$ is the Grassmannian of k-dimensional *quotient* spaces of V, and the Plücker embedding embeds G in the projective space $\mathbb{P}(\wedge^k V^*)$ of one-dimensional quotients of $\wedge^k V$.

The space of all homogeneous polynomials of degree m on $\mathbb{P}(\wedge^k V^*)$ is naturally the symmetric power $\mathrm{Sym}^m(\wedge^k V)$. Let $I(G)_m$ denote the subspace of those polynomials of degree m on $\mathbb{P}(\wedge^k V^*)$ that vanish on G. Each $I(G)_m$ is a representation of $\mathfrak{sl}_n\mathbb{C}$:

$$0 \to I(G)_m \to \mathrm{Sym}^m(\wedge^k V) \to W_m \to 0,$$

where W_m denotes the restrictions to G of the polynomials of degree m on the ambient space $\mathbb{P}(\wedge^k V^*)$. We shall see later that W_m is the irreducible representation $\Gamma_{0,\ldots,0,m,0,\ldots}$ with highest weight $m(L_1 + \cdots + L_k)$ (the case $m = 2$ will be dealt with below). In the following discussion, we consider the problem of describing the quadratic part $I(G)_2$ of the ideal as a representation of $\mathfrak{sl}_n\mathbb{C}$.

Exercise 15.34. Consider the first case of a Grassmannian that is not a projective space, that is, $k = 2$. The ideal of the Grassmannian $G(2, V)$ of 2-planes in a vector space is easy to describe: a tensor $\varphi \in \wedge^2 V$ is decomposable if and only if $\varphi \wedge \varphi = 0$ (equivalently, if we think of φ as given by a skew-symmetric $n \times n$ matrix, if and only if the Pfaffians of symmetric 4×4 minors all vanish); and indeed the quadratic relations we get in this way generate the ideal of the Grassmannian. We, thus, have an isomorphism

$$I(G)_2 \cong \wedge^4 V$$

and correspondingly a decomposition into irreducibles

$$\mathrm{Sym}^2(\wedge^2 V) \cong \wedge^4 V \oplus \Gamma_{0,2,0,\ldots,0},$$

where $\Gamma_{0,2,0,\ldots,0}$ is, as above, the irreducible representation with highest weight $2(L_1 + L_2)$, cf. Exercise 15.32.

Exercise 15.35. When $k = 2$ and $n = 4$, G is a quadric hypersurface in $\mathbb{P}^5$, so polynomials vanishing on G are simply those divisible the quadratic polynomial that defines G. Deduce an isomorphism.

$$I(G)_m = \mathrm{Sym}^{m-2}(\wedge^2 V).$$

   The first case of a Grassmannian that is not a projective space or of the form $G(2, V)$ is, of course, $G(3, 6)$, and this yields an interesting example.

Exercise 15.36. Let V be six dimensional. By examining weights, show that the space $I(G)_2$ of quadratic polynomials vanishing on the Grassmannian $G(3, V) \subset \mathbb{P}(\wedge^3 V)$ is isomorphic to the adjoint representation of $\mathfrak{sl}_6\mathbb{C}$, i.e., that we have a map

$$\varphi: \text{Sym}^2(\wedge^3 V) \to V \otimes V^*$$

with image the space of traceless matrices.

Exercise 15.37. Find explicitly the map φ of the preceding exercise.

Exercise 15.38. Again, let V be six dimensional. Show that the representation $\text{Sym}^4(\wedge^3 V)$ has a trivial direct summand, corresponding to the hypersurface in $\mathbb{P}(\wedge^3 V^*)$ dual to the Grassmannian $G = G(3, V) \subset \mathbb{P}(\wedge^3 V)$.

   In general, the ideal $I(G) = \bigoplus I(G)_m$ is generated by the famous *Plücker equations*. These are homogeneous polynomials of degree two, and may be written down explicitly, cf. (15.53), [H-P], or [Ha]. In the following exercises, we will give a more intrinsic description of these relations, which will allow us to identify the space $I(G)_2$ they span as a representation on $\mathfrak{sl}_n\mathbb{C}$ (and to see the general pattern of which the above are special cases).

Exercise 15.39. For a given tensor $\Lambda \in \wedge^k V$, we introduce two associated subspaces:

$$W = \{v \in V: v \wedge \Lambda = 0\} \subset V$$

and

$$W^* = \{v^* \in V^*: v^* \wedge \Lambda^* = 0\} \subset V^*,$$

where, abusing notation slightly, Λ^* is the tensor Λ viewed as an element of $\wedge^k V = \wedge^{n-k} V^*$. Show that the dimensions of W and W^* are at most k and $n - k$, respectively, and that Λ is decomposable if and only if W has dimension k or W^* has dimension $n - k$; and deduce that Λ is decomposable if and only if the annihilator W' of W^* is equal to W.

Exercise 15.40. Now let $\Xi \in \wedge^{k+1} V^* = \wedge^{n-k-1} V$. Wedge product gives a map

$$\iota_\Xi: \wedge^k V \to \wedge^{n-1} V = V^*.$$

Using the preceding exercise, show that Λ is decomposable if and only if

$$\iota_\Xi(\Lambda) \wedge \Lambda = 0 \in \wedge^{k-1} V$$

for all $\Xi \in \wedge^{k+1} V^*$.

Exercise 15.41. Observe that in the preceding exercise we construct a map

$$\textstyle\bigwedge^{k+1} V^* \otimes \mathrm{Sym}^2(\bigwedge^k V) \to \bigwedge^{k-1} V,$$

or, by duality, a map

$$\textstyle\bigwedge^{k+1} V^* \otimes \bigwedge^{k-1} V^* \to \mathrm{Sym}^2(\bigwedge^k V^*) \tag{15.42}$$

whose image is a vector space of quadrics on $\mathbb{P}(\bigwedge^k V)$ whose common zeros are exactly the locus of decomposable vectors, that is, the Grassmannian $G(k, V)$. Show that this image is exactly the span of the Plücker relations above.

Exercise 15.43. Show that the map (15.42) of the preceding exercise is just the dual of the map constructed in Exercise 15.30, with $k = l$ and restricted to the symmetric product. Combining this with the result of Exercise 15.32 (and assuming the statement that the Plücker relations do indeed span $I(G)_2$), deduce that in terms of the description

$$\textstyle\mathrm{Sym}^2(\bigwedge^k V) = \bigoplus \Theta_{2i}$$

of the symmetric square of $\bigwedge^k V$, we have

$$W_2 = \Theta_0 = \Gamma_{0,\dots,0,2,0,\dots}$$

(the irreducible representation with highest weight $2(L_1 + \cdots + L_k)$), and

$$I(G)_2 = \bigoplus_{i \geq 1} \Theta_{2i}.$$

Hard Exercise 15.44. Show that in the last equation the sub-direct sum

$$I(l) = \bigoplus_{i \geq l} \Theta_{2i}$$

is just the quadratic part of the ideal of the *restricted chordal variety* of the Grassmannian: that is, the union of the chords $\overline{LM}$ joining pairs of points in G corresponding to pairs of planes L and M meeting in a subspace of dimension at least $k - 2l + 1$. (Question: What is the actual zero locus of these quadrics?)

Exercise 15.45. Carry out an analysis similar to the above to relate the ideal of a Veronese variety $\mathbb{P}V^* \subset \mathbb{P}(\mathrm{Sym}^k V^*)$ to the decomposition given in Exercise 15.33 of $\mathrm{Sym}^2(\mathrm{Sym}^k V)$. For which k do the quadratic polynomials vanishing the Veronese give an irreducible representation?

Exercise 15.46. (For algebraic geometers and/or commutative algebraists.) Just as the group $\mathrm{PGL}_n\mathbb{C}$ acts on the ring S of polynomials on projective space $\mathbb{P}^N$, preserving the ideal of the Veronese variety, so it acts on that space of relations on the ideal (that is, inasmuch as the ideal is generated by quadrics, the kernel of the multiplication map $I_X(2) \otimes S \to S$), and likewise on the entire minimal resolution of the ideal of X. Show that this resolution has the form

$$\cdots \to R_2 \otimes S \to R_1 \otimes S \to I_X(2) \otimes S,$$

where all the R_i are finite-dimensional representations of PGL$_n$C, and identify the representations R_i in the specific cases of

(i) the rational normal curve in $\mathbb{P}^3$,
(ii) the rational normal curve in $\mathbb{P}^4$, and
(iii) the Veronese surface in $\mathbb{P}^5$.

§15.5. Representations of GL$_n$C

We have said that there is little difference between representations of GL$_n$C and those of the subgroup SL$_n$C of matrices of determinant 1. Our object here is to record the difference, which, naturally enough, comes from the determinant: if $V = \mathbb{C}^n$ is the standard representation, $\wedge^n V$ is trivial for SL$_n$C but not for GL$_n$C. Similarly, V and $\wedge^{n-1} V^*$ are isomorphic for SL$_n$C but not for GL$_n$C.

To relate representations of SL$_n$C and GL$_n$C, we first need to define some representations of GL$_n$C. To begin with, let D_k denote the one-dimensional representation of GL$_n$C given by the kth power of the determinant. When k is non-negative, $D_k = (\wedge^n V)^{\otimes k}$; D_{-k} is the dual $(D_k)^*$ of D_k. Next, note that the irreducible representations of SL$_n$C may be lifted to representations of GL$_n$C in two ways. First, for any index $\mathbf{a} = (a_1, \ldots, a_n)$ of length n we may take $\Phi_{\mathbf{a}}$ to be the subrepresentation of the tensor product

$$\mathrm{Sym}^{a_1} V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1} V) \otimes \mathrm{Sym}^{a_n}(\wedge^n V)$$

spanned by the highest weight vector with weight $a_1 L_1 + a_2(L_1 + L_2) + \cdots + a_{n-1}(L_1 + \cdots + L_{n-1})$—that is, the vector

$$v = (e_1)^{a_1} \cdot (e_1 \wedge e_2)^{a_2} \cdot \ldots \cdot (e_1 \wedge \cdots \wedge e_n)^{a_n}.$$

This restricts to SL$_n$C to give the representation $\Gamma_{a'}$, where $a' = (a_1, \ldots, a_{n-1})$; taking different values of a_n amounts to tensoring the representation with different factors $\mathrm{Sym}^{a_n}(\wedge^n V) = (\wedge^n V)^{\otimes a_n} = D_{a_n}$. In particular, we have

$$\Phi_{a_1, \ldots, a_n + k} = \Phi_{a_1, \ldots, a_n} \otimes D_k,$$

which allows us to extend the definition of $\Phi_{\mathbf{a}}$ to indices a with $a_n < 0$: we simply set

$$\Phi_{a_1, \ldots, a_n} = \Phi_{a_1, \ldots, a_n + k} \otimes D_{-k}$$

for large k.

Alternatively, we may consider the Schur functor $\mathbb{S}_\lambda$ applied to the standard representation V of GL$_n$C, where

$$\lambda = (a_1 + \cdots + a_n, a_2 + \cdots + a_n, \ldots, a_{n-1} + a_n, a_n).$$

We will denote this representation $\mathbb{S}_\lambda V$ of GL$_n$C by Ψ_λ; note that

$$\Psi_{\lambda_1 + k, \ldots, \lambda_n + k} = \Psi_{\lambda_1, \ldots, \lambda_n} \otimes D_k$$

which likewise allows us to define Ψ_λ for any index λ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, even if some of the λ_i are negative: we simply take

$$\Psi_{\lambda_1,\ldots,\lambda_n} = \Psi_{\lambda_1+k,\ldots,\lambda_n+k} \otimes D_{-k}$$

for any sufficiently large k.

As is not hard to see, *the two representations $\Phi_{\mathbf{a}}$ and Ψ_λ are isomorphic as representations of $GL_n\mathbb{C}$*: by §15.3 their restrictions to $SL_n\mathbb{C}$ agree, so it suffices to check their restrictions to the center $\mathbb{C}^* \subset GL_n\mathbb{C}$, where each acts by multiplication by $z^{\sum\lambda_i} = z^{\sum ia_i}$). It is even clearer that there are no coincidences among the $\Phi_{\mathbf{a}}$ (i.e., $\Phi_{\mathbf{a}}$ will be isomorphic to $\Phi_{\mathbf{a}'}$ if and only if $\mathbf{a} = \mathbf{a}'$): if $\Phi_{\mathbf{a}} \cong \Phi_{\mathbf{a}'}$, we must have $a_i = a_i'$ for $i = 1, \ldots, n-1$, so the statement follows from the nontriviality of D_k for $k \neq 0$. Thus, to complete our description of the irreducible finite-dimensional representations of $GL_n\mathbb{C}$, we just have to check that we have found them all. We may then express the completed result as

Proposition 15.47. *Every irreducible complex representation of $GL_n\mathbb{C}$ is isomorphic to Ψ_λ for a unique index $\lambda = \lambda_1, \ldots, \lambda_n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (equivalently, to $\Phi_{\mathbf{a}}$ for a unique index $\mathbf{a} = a_1, \ldots, a_n$ with $a_1, \ldots, a_{n-1} \geq 0$).*

PROOF. We start by going back to the corresponding Lie algebras. The scalar matrices form a one-dimensional ideal $\mathbb{C}$ in $\mathfrak{gl}_n\mathbb{C}$, and in fact $\mathfrak{gl}_n\mathbb{C}$ is a product of Lie algebras:

$$\mathfrak{gl}_n\mathbb{C} = \mathfrak{sl}_n\mathbb{C} \times \mathbb{C}. \qquad (15.48)$$

In particular, $\mathbb{C}$ is the radical of $\mathfrak{gl}_n\mathbb{C}$, and $\mathfrak{sl}_n\mathbb{C}$ is the semisimple part. It follows from Proposition 9.17 that every irreducible representation of $\mathfrak{gl}_n\mathbb{C}$ is a tensor product of an irreducible representation of $\mathfrak{sl}_n\mathbb{C}$ and a one-dimensional representation. More precisely, let $W_\lambda = \mathbb{S}_\lambda(\mathbb{C}^n)$ be the representation of $\mathfrak{sl}_n\mathbb{C}$ determined by the partition λ (extended to $\mathfrak{sl}_n\mathbb{C} \times \mathbb{C}$ by making the second factor act trivially). For $w \in \mathbb{C}$, let $L(w)$ be the one-dimensional representation of $\mathfrak{sl}_n\mathbb{C} \times \mathbb{C}$ which is zero on the first factor and multiplication by w on the second; the proof of Proposition 9.17 shows that any irreducible representation of $\mathfrak{sl}_n\mathbb{C} \times \mathbb{C}$ is isomorphic to a tensor product $W_\lambda \otimes L(w)$. The same is therefore true for the simply connected[1] group $SL_n\mathbb{C} \times \mathbb{C}$ with this Lie algebra.

We write $GL_n\mathbb{C}$ as a quotient modulo a discrete subgroup of the center of $SL_n\mathbb{C} \times \mathbb{C}$:

$$1 \to \mathrm{Ker}(\rho) \to SL_n\mathbb{C} \times \mathbb{C} \xrightarrow{\rho} GL_n\mathbb{C} \to 1, \qquad (15.49)$$

where $\rho(g \times z) = e^z \cdot g$, so the kernel of ρ is generated by $e^s \cdot I \times (-s)$, where $s = 2\pi i/n$.

Our task is simply to see which of the representations $W_\lambda \otimes L(w)$ of $SL_n\mathbb{C} \times \mathbb{C}$ are trivial on the kernel of ρ. Now $e^s \cdot I$ acts on $\mathbb{S}_\lambda\mathbb{C}^n$ by multi-

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[1] For a proof that $SL_n\mathbb{C}$ is simply connected, see §23.1.

plication by e^{sd}, where $d = \sum \lambda_i$; indeed, this is true on the entire representation $(\mathbb{C}^n)^{\otimes d}$ which contains $\mathbb{S}_\lambda \mathbb{C}^n$. And $-s$ acts on $L(w)$ by multiplication by e^{-sw}, so $e^s \cdot I \times (-s)$ acts on the tensor product by multiplication by e^{sd-sw}. The tensor product is, therefore, trivial on the kernel of ρ precisely when $sd - sw \in 2\pi i\mathbb{Z}$, i.e., when

$$w = \sum \lambda_i + kn$$

for some integer k.

We claim finally that any representation $W_\lambda \otimes L(w)$ satisfying this condition is the pullback via ρ of a representation Ψ on $GL_n\mathbb{C}$. In fact, it is not hard to see that it is the pullback of the representation $\Psi_{\lambda_1+k,\ldots,\lambda_n+k}$: the two clearly restrict to the same representation on $SL_n\mathbb{C}$, and their restrictions to $\mathbb{C}$ are just multiplication by $e^{wz} = e^{(\sum \lambda_i + nk)z}$. $\qquad\square$

Exercise 15.50. Show that the dual of the representation Ψ_λ which is isomorphic to $\mathbb{S}_\lambda(V^*)$ is the representation $\Psi_{(-\lambda_n,\ldots,-\lambda_1)}$.

Exercise 15.51*. Show that if $\rho: GL_n\mathbb{C} \to GL(W)$ is a representation (assumed to be holomorphic), then W decomposes into a direct sum of irreducible representations.

Exercise 15.52*. Show that the Hermite reciprocity isomorphism of Exercise 11.34 is an isomorphism over $GL_2\mathbb{C}$, not just over $SL_2\mathbb{C}$.

More Remarks on Weyl's Construction

We close out this lecture by looking once more at the Weyl construction of these representations of $GL(V)$. This will include a realization "by generators and relations," as well as giving a natural basis for each representation. First, it may be illuminating—and it will be useful later—to look more closely at how $\mathbb{S}_\lambda V$ sits in $V^{\otimes d}$. We want to realize $\mathbb{S}_\lambda V$ as a subspace of the subspace

$$\text{Sym}^{a_k}(\wedge^k V) \otimes \text{Sym}^{a_{k-1}}(\wedge^{k-1} V) \otimes \cdots \otimes \text{Sym}^{a_1}(V) \subset V^{\otimes d},$$

where a_i is the number of columns of the Young diagram of λ of length i (and k is the number of rows). This space is embedded in $V^{\otimes d}$ in the natural way: from left to right, a factor $\text{Sym}^a(\wedge^b V)$ is embedded in the corresponding $V^{\otimes ab}$ by mapping a symmetric product of exterior products

$$(v_{1,1} \wedge v_{2,1} \wedge \cdots \wedge v_{b,1}) \cdot (v_{1,2} \wedge v_{2,2} \wedge \cdots \wedge v_{b,2}) \cdot \ldots$$
$$\cdot (v_{1,a} \wedge v_{2,a} \wedge \cdots \wedge v_{b,a})$$

to

$$\sum \text{sgn}(q)(v_{q_1(1),p(1)} \otimes \cdots \otimes v_{q_1(b),p(1)}) \otimes \cdots \otimes (v_{q_a(1),p(a)} \otimes \cdots \otimes v_{q_a(b),p(a)}),$$

the sum over $p \in \mathfrak{S}_a$ and $q = (q_1, \ldots, q_a) \in \mathfrak{S}_b \times \cdots \times \mathfrak{S}_b$. In other words, one

first symmetrizes by permuting columns of the same length, and then performs an alternating symmetrizer on each column.

Letting $\mathbf{a} = (a_1, \ldots, a_k)$, let $A^{\mathbf{a}}(V)$ denote this tensor product of symmetric powers of exterior powers, i.e., set

$$A^{\mathbf{a}}V = \mathrm{Sym}^{a_k}(\textstyle\bigwedge^k V) \otimes \mathrm{Sym}^{a_{k-1}}(\textstyle\bigwedge^{k-1} V) \otimes \cdots \otimes \mathrm{Sym}^{a_1}(V).$$

We want to realize $\mathbb{S}_\lambda V$ as a subspace of $A^{\mathbf{a}}V$. To do this we use the construction of $\mathbb{S}_\lambda V$ as $V^{\otimes d} \cdot c_\lambda$, where c_λ is a Young symmetrizer; to get compatibility with the embedding of $A^{\mathbf{a}}V$ we have just made, we use the tableau which numbers the columns from top to bottom, then left to right.

1	4	6	8	$\lambda_1 = 4$
2	5	7	$a_1 = 1$	$\lambda_2 = 3$
3	$a_2 = 2$			$\lambda_3 = 1$

$a_3 = 1$

$$\begin{array}{cccc} \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \| & \| & \| & \| \\ 3 & 2 & 2 & 1 \end{array}$$

We take $\mu = \lambda' = (\mu_1 \geq \cdots \geq \mu_l > 0)$ to be the conjugate of λ. The symmetrizer c_λ is a product $a_\lambda \cdot b_\lambda$, where $a_\lambda = \sum e_p$, the sum over all p in the subgroup $P = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k}$ of $\mathfrak{S}_d$ preserving the rows, $b_\lambda = \sum \mathrm{sgn}(q)q$, the sum over the subgroup $Q = \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_l}$ preserving the columns, as described in Lecture 4. The symmetrizing by rows can be done in two steps as follows. There is a subgroup

$$R = \mathfrak{S}_{a_k} \times \cdots \times \mathfrak{S}_{a_1}$$

of P, which consists of permutations that move all entries of each column to the same position in some column of the same length; in other words, permutations in R are determined by permuting columns which have the same length. (In the illustration, $R = \{1, (46)(57)\}$.) Set

$$a'_\lambda = \sum_{r \in R} e_r \quad \text{in } \mathbb{C}\mathfrak{S}_d.$$

Now if we define a''_λ to be $\sum e_p$, where the sum is over any set of representatives in P for the left cosets P/R, then the row symmetrizer a_λ is the product of a''_λ and a'_λ. So

$$\mathbb{S}_\lambda(V) = (V^{\otimes d} \cdot a''_\lambda) \cdot a'_\lambda \cdot b_\lambda.$$

The point is that, by what we have just seen,

$$V^{\otimes d} \cdot a'_\lambda \cdot b_\lambda = A^{\mathbf{a}}V.$$

Since $V^{\otimes d} \cdot a''_\lambda$ is a subspace of $V^{\otimes d}$, its image $\mathbb{S}_\lambda(V)$ by $a'_\lambda \cdot b_\lambda$ is a subspace of $A^{\mathbf{a}}(V)$, as we claimed.

There is a simple way to construct all the representations $\mathbb{S}_\lambda V$ of $GL(V)$ at once. In fact, the direct sum of all the representations $\mathbb{S}_\lambda V$, over all (non-negative) partitions λ, can be made into a commutative, graded ring, which we denote by $\mathbb{S}^{\cdot}$ or $\mathbb{S}^{\cdot}(V)$, with simple generators and relations. This is similar to the fact that the symmetric algebra $\mathrm{Sym}^{\cdot}V = \bigoplus \mathrm{Sym}^k V$ and the exterior algebra $\bigwedge^{\cdot}V = \bigoplus \bigwedge^k V$ are easier to describe than the individual graded pieces, and it has some of the similar advantages for studying all the representations at once. This algebra has appeared and reappeared frequently, cf. [H-P]; the construction we give is essentially that of Towber [Tow1].

To construct $\mathbb{S}^{\cdot}(V)$, start with the symmetric algebra on the sum of all the positive exterior products of V: se`

$$A^{\cdot}(V) = \mathrm{Sym}^{\cdot}(V \oplus \wedge^2 V \oplus \wedge^3 V \oplus \cdots \oplus \wedge^n V)$$

$$= \bigoplus_{a_1,\ldots,a_n} \mathrm{Sym}^{a_n}(\wedge^n V) \otimes \cdots \otimes \mathrm{Sym}^{a_2}(\wedge^2 V) \otimes \mathrm{Sym}^{a_1}(V),$$

the sum over all n-tuples $a_1, \ldots, a_n$ of non-negative integers. So $A^{\cdot}(V)$ is the direct sum of the $A^{\mathbf{a}}(V)$ just considered. The ring $\mathbb{S}^{\cdot} = \mathbb{S}^{\cdot}(V)$ is defined to be the quotient of this ring $A^{\cdot}(V)$ modulo the graded, two-sided ideal $I^{\cdot}$ generated by all elements ("Plücker relations") of the form

$$(v_1 \wedge \cdots \wedge v_p) \cdot (w_1 \wedge \cdots \wedge w_q)$$

$$- \sum_{i=1}^{p} (v_1 \wedge \cdots \wedge v_{i-1} \wedge w_1 \wedge v_{i+1} \wedge \cdots \wedge v_p) \cdot (v_i \wedge w_2 \wedge \cdots \wedge w_q)$$

$$\tag{15.53}$$

for all $p \geq q \geq 1$ and all $v_1, \ldots, v_p, w_1, \ldots, w_q \in V$. (If $p = q$, this is an element of $\mathrm{Sym}^2(\wedge^p V)$; if $p > q$, it is in $\wedge^p V \otimes \wedge^q V = \mathrm{Sym}^1(\wedge^p V) \otimes \mathrm{Sym}^1(\wedge^q V)$. Note that the multiplication in $\mathbb{S}^{\cdot}(V)$ comes entirely from its being a symmetric algebra and does not involve the wedge products in $\wedge^{\cdot}V$.)

Exercise 15.54*. Show that $I^{\cdot}$ contains all elements of the form

$$(v_1 \wedge \cdots \wedge v_p) \cdot (w_1 \wedge \cdots \wedge w_q)$$

$$- \sum (v_1 \wedge \cdots \wedge w_1 \wedge \cdots \wedge w_r \wedge \cdots \wedge v_p)$$

$$\cdot (v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_r} \wedge w_{r+1} \wedge \cdots \wedge w_q)$$

for all $p \geq q \geq r \geq 1$ and all $v_1, \ldots, v_p, w_1, \ldots, w_q \in V$, where the sum is over all $1 \leq i_1 < i_2 < \cdots < i_r \leq p$, and the elements $w_1, \ldots, w_r$ are inserted at the corresponding places in $v_1 \wedge \cdots \wedge v_p$.

Remark. You can avoid this exercise by simply taking the elements in the exercise as defining generators for the ideal $I^{\cdot}$. When $p = q = r$, the calcula-

tion of Exercise 15.54 shows that the relation $(v_1 \wedge \cdots \wedge v_p) \cdot (w_1 \wedge \cdots \wedge w_p)$
$= (w_1 \wedge \cdots \wedge w_p) \cdot (v_1 \wedge \cdots \wedge v_p)$ follows from the generating equations for $I^{\cdot}$.
In particular, this commutativity shows that one could define $\mathbb{S}^{\cdot}(V)$ to be the
full tensor algebra on $V \oplus \wedge^2 V \oplus \cdots \oplus \wedge^n V$ modulo the ideal generated by
the same generators.

The algebra $\mathbb{S}^{\cdot}(V)$ is the direct sum of the images $\mathbb{S}^{\mathbf{a}}(V)$ of the summands
$A^{\mathbf{a}}(V)$. Let $e_1, \ldots, e_n$ be a basis for V. We will construct a basis for $\mathbb{S}^{\mathbf{a}}(V)$, with
a basis element e_T for every semistandard tableau T on the partition λ which
corresponds to $\mathbf{a}$. Recall that a semistandard tableau is a numbering of the
boxes of the Young diagram with the integers $1, \ldots, n$, in such a way that the
entries in each row are nondecreasing, and the entries in each column are
strictly increasing. Let $T(i, j)$ be the entry of T in the ith row and the jth
column. Define e_T to be the image in $\mathbb{S}^{\mathbf{a}}(V)$ of the element

$$\prod_{j=1}^{l} e_{T(1,j)} \wedge e_{T(2,j)} \wedge \cdots \wedge e_{T(\mu_j, j)} \in \mathrm{Sym}^{a_n}(\wedge^n V) \otimes \cdots \otimes \mathrm{Sym}^{a_1}(V),$$

i.e., wedge together the basis elements corresponding to the entries in the
columns, and multiply the results in $\mathbb{S}^{\cdot}(V)$.

Proposition 15.55. (1) *The projection from $A^{\mathbf{a}}(V)$ to $\mathbb{S}^{\mathbf{a}}(V)$ maps the subspace
$\mathbb{S}_\lambda(V)$ isomorphically onto $\mathbb{S}^{\mathbf{a}}(V)$.*
   (2) *The e_T for T a semistandard tableau on λ form a basis for $\mathbb{S}^{\mathbf{a}}(V)$.*

PROOF. We show first that the elements e_T span $\mathbb{S}^{\mathbf{a}}(V)$. It is clear that the e_T
span if we allow all tableaux T that number the boxes of λ with integers
between 1 and n with strictly increasing columns, for such elements span before
dividing by the ideal $I^{\cdot}$. We order such tableaux by listing their entries column
by column, from left to right and top to bottom, and using the reverse
lexicographic order: $T' > T$ if the last entry where they differ has a larger entry
for T' than for T. If T is not semistandard, there will be two successive columns
of T, say the jth and $(j+1)$st, in which we have $T(r, j) > T(r, j+1)$ for some
r. It suffices to show how to use relations in $I^{\cdot}$ to write e_T as a linear
combination of elements $e_{T'}$ with $T' > T$. For this we use the relation in
Exercise 15.54, with $v_i = e_{T(i,j)}$ for $1 \le i \le p = \mu_j$, and $w_i = e_{T(i,j+1)}$ for
$1 \le i \le q = \mu_{j+1}$, to interchange the first r of the $\{w_i\}$ with subsets of r of the
$\{v_i\}$. The terms on the right-hand side of the relation will all correspond to
tableaux T' in which the r first entries in the $(j+1)$st column of T are replaced
by r of the entries in the jth column, and are not otherwise changed beyond
the jth column. All of these are larger than T in the ordering, which proves the
assertion.
   It is possible to give a direct proof that the e_T corresponding to semi-
standard tableaux T are linearly independent (see [Tow1]), but we can get by
with less. Among the semistandard tableaux on λ there is a smallest one T_0
whose ith row is filled with the integer i. We need to know that e_{T_0} is not zero

in $\mathbb{S}^{\cdot}$. This is easy to see directly. In fact, the relations among the e_T in $\Gamma \cap A^{\mathbf{a}}(V)$ are spanned by those obtained by substituting r elements from some column of some T to an earlier column, as in the preceding paragraph. Such will never involve the generator e_{T_0} unless the T that is used is T_0, and in this case, the resulting element of Γ is zero. Since e_{T_0} occurs in no nontrivial relation, its image in $\mathbb{S}^{\cdot}$ cannot vanish.

Since e_{T_0} comes from $\mathbb{S}_\lambda(V)$, it follows that the projection from $\mathbb{S}_\lambda(V)$ to $\mathbb{S}^{\mathbf{a}}(V)$ is not zero. Since this projection is a mapping of representations of $SL(V)$, it follows that $\mathbb{S}^{\mathbf{a}}(V)$ must contain a copy of the irreducible representation $\mathbb{S}_\lambda(V)$. We know from Theorem 6.3 and Exercise A.31 that the dimension of $\mathbb{S}_\lambda(V)$ is the number of semistandard tableaux on λ. Since we have proved that the dimension of $\mathbb{S}^{\mathbf{a}}(V)$ is at most this number, the projection from $\mathbb{S}_\lambda(V)$ to $\mathbb{S}^{\mathbf{a}}(V)$ must be surjective, and since $\mathbb{S}_\lambda(V)$ is irreducible, it must be injective as well, and the e_T for T a semistandard tableau on λ must form a basis, as asserted.   □

Note that this proposition gives another description of the representations $\mathbb{S}_\lambda(V)$, as the quotient of the space $A^{\mathbf{a}}(V)$ by the subspace generated by the "Plücker" relations (15.53).

Exercise 15.56. Show that, if the factor $\wedge^n V$ is omitted from the construction, the resulting algebra is the direct sum of all irreducible representations of $SL(V) = SL_n\mathbb{C}$.

It is remarkable that all the representations $\mathbb{S}_\lambda(\mathbb{C}^n)$ of $GL_n\mathbb{C}$ were written down by Deruyts (following Clebsch) a century ago, before representation theory was born, as in the following exercise.

Exercise 15.57*. Let $X = (x_{i,j})$ be an $n \times n$ matrix of indeterminates. The group $G = GL_n\mathbb{C}$ acts on the polynomial ring $\mathbb{C}[x_{i,j}]$ by $g \cdot x_{i,j} = \sum_{k=1}^{n} a_{k,i} x_{k,j}$ for $g = (a_{i,j}) \in GL_n\mathbb{C}$. For any tableau T on the Young diagram of λ consisting of the integers from 1 to n, strictly increasing in the columns, let e_T be the product of minors constructed from X, one for each column, as follows: if the column of T has length μ_j, form the minor using the first μ_j columns, and use the rows that are numbered by the entries of the column of T. Let D_λ be the subspace of $\mathbb{C}[x_{i,j}]$ spanned by these e_T, where d is the number partitioned by λ. Show that: (i) D_λ is preserved by $GL_n\mathbb{C}$; (ii) the e_T, where T is semistandard, form a basis for D_λ; (iii) D_λ is isomorphic to $\mathbb{S}_\lambda(\mathbb{C}^n)$.