LECTURE 20

Spin Representations of $\mathfrak{so}_m\mathbb{C}$

In this lecture we complete the picture of the representations of the orthogonal Lie algebras by constructing the spin representations $S^{\pm}$ of $\mathfrak{so}_m\mathbb{C}$; this also yields a description of the spin groups $\mathrm{Spin}_m\mathbb{C}$. Since the representation-theoretic analysis of the spaces $S^{\pm}$ was carried out in the preceding lecture, we are concerned here primarily with the algebra involved in their construction. Thus, §20.1 and §20.2, while elementary, involve some fairly serious algebra. Section 20.3, where we briefly sketch the notion of triality, may seem mysterious to the reader (this is at least in part because it is so to the authors); if so, it may be skipped. Finally, we should say that the subject of the spin representations of $\mathfrak{so}_m\mathbb{C}$ is a very rich one, and one that accommodates many different points of view; the reader who is interested is encouraged to try some of the other approaches that may be found in the literature.

§20.1. Clifford Algebras and Spin Representations of $\mathfrak{so}_m\mathbb{C}$

We begin this section by trying to motivate the definition of Clifford algebras. We may begin by asking, why were we able to find all the representations of $\mathrm{SL}_n\mathbb{C}$ or $\mathrm{Sp}_{2n}\mathbb{C}$ inside tensor powers of the standard representation, but only half the representations of $\mathrm{SO}_m\mathbb{C}$ arise this way? One difference that points in this direction lies in the topology of these groups: $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{Sp}_{2n}\mathbb{C}$ are simply connected, while $\mathrm{SO}_m\mathbb{C}$ has fundamental group $\mathbb{Z}/2$ for $m > 2$ (for proofs see §23.1). Therefore $\mathrm{SO}_m\mathbb{C}$ has a double covering, the *spin group* $\mathrm{Spin}_m\mathbb{C}$. (For $m \leq 6$, these coverings could also be extracted from our identifications

W. Fulton et al., *Representation Theory*
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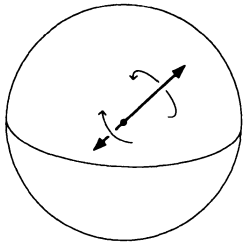
of the adjoint group $PSO_m\mathbb{C}$ with the adjoint group of other simply connected groups; e.g. the double cover of $SO_3\mathbb{C}$ is $SL_2\mathbb{C}$.) We will see that the missing representations are those representations of $Spin_m\mathbb{C}$ that do not come from representations of $SO_m\mathbb{C}$.

This double covering may be most readily visible, and probably familiar, for the case of the real subgroup $SO_3\mathbb{R}$ of rotations: a rotation is specified by an axis to rotate about, given by a unit vector u, and an angle of rotation about u; the two choices $\pm u$ of unit vector give a two-sheeted covering. In other words, if D^3 is the unit ball in $\mathbb{R}^3$, there is a double covering

$$S^3 = D^3/\partial D^3 \to SO_3\mathbb{R},$$

which sends a vector v in D^3 to rotation by the angle $2\pi\|v\|$ about the unit vector $v/\|v\|$ (the origin and the unit sphere ∂D^3 are sent to the identity transformation).



This covering is even easier to see for the entire orthogonal group $O_3\mathbb{R}$, which is generated by reflections R_v in unit vectors v (with $\pm v$ determining the same reflection): we can describe the double cover of $O_3\mathbb{R}$ as the group generated by unit vectors v, with relations

$$v_1 \cdot \ldots \cdot v_n = w_1 \cdot \ldots \cdot w_m$$

whenever the compositions of the corresponding reflections are equal, i.e., whenever

$$R_{v_1} \circ \cdots \circ R_{v_n} = R_{w_1} \circ \cdots \circ R_{w_m};$$

and also relations

$$(-v)\cdot(-w) = v \cdot w$$

for all pairs of unit vectors v and w. (Note that if we restricted ourselves to products of even numbers of the generators $v \in \partial D^3$ we would get back the double cover of the special orthogonal group $SO_3\mathbb{C}$.)

How should we generalize this? The answer is not obvious. For one thing, for various reasons we will not try to construct directly a group that covers the orthogonal group in general. Instead, given a vector space V (real or complex) and a quadratic form Q on V, we will first construct an algebra $\mathrm{Cliff}(V, Q)$, called the *Clifford algebra*. The algebra $\mathrm{Cliff}(V, Q)$ will then turn out to contain in its multiplicative group a subgroup which is a double cover of the orthogonal group $O(V, Q)$ of automorphisms of V preserving Q.

By analogy with the construction of the double cover of $SO_3\mathbb{R}$, the Clifford algebra $\text{Cliff}(V, Q)$ associated to the pair (V, Q) is an associative algebra containing and generated by V. (When we want to describe the spin group inside $\text{Cliff}(V, Q)$ we will restrict ourselves to products of even numbers of elements of V having a fixed norm $Q(v, v)$; if odd products are allowed as well, we get a group called "Pin" which is a double covering of the whole orthogonal group.) To motivate the definition, we would like $\text{Cliff}(V, Q)$ to be the algebra generated by V subject to relations analogous to those above for the double cover of the orthogonal group. In particular, for any vector v with $Q(v, v) = 1$, since the reflection R_v in the hyperplane perpendicular to v is an involution, we want

$$v \cdot v = 1$$

in $\text{Cliff}(V, Q)$. By polarization, this is the same as imposing the relation

$$v \cdot w + w \cdot v = 2Q(v, w)$$

for all v and w in V. In particular, $w \cdot v = -v \cdot w$ if v and w are perpendicular. In fact, the Clifford algebra[1] will be defined below to be the associative algebra generated by V and subject to the equation $v \cdot v = Q(v, v)$.

Looking ahead, we will see later in this section that each complex Clifford algebra contains an orthogonal Lie algebra as a subalgebra. The key theorem is then that $\text{Cliff}(V, Q)$ *is isomorphic either to a matrix algebra or to a sum of two matrix algebras*. This in turn determines either one or two representations of the orthogonal Lie algebras, which turn out to be the representations which were needed to complete the story in the last lecture. Just as in the special linear and symplectic cases, the corresponding Lie groups are not really needed to construct the representations; they can be written down directly from the Lie algebra. In this section we do this, using the Clifford algebras to construct these representations of $\mathfrak{so}_m\mathbb{C}$ directly, and verify that they give the missing spin representations. In the second section of this lecture we will show how the spin groups sit as subgroups in their multiplicative groups.

Clifford Algebras

Given a symmetric bilinear form Q on a vector space V, the *Clifford algebra* $C = C(Q) = \text{Cliff}(V, Q)$ is an associative algebra with unit 1, which contains and is generated by V, with $v \cdot v = Q(v, v) \cdot 1$ for all $v \in V$. Equivalently, we have the equation

$$v \cdot w + w \cdot v = 2Q(v, w), \tag{20.1}$$

[1] The mathematical world seems to be about evenly divided about the choice of signs here, and one must translate from Q to $-Q$ to go from one side to the other.

for all v and w in V. The Clifford algebra can be defined to be the universal algebra with this property: if E is any associative algebra with unit, and a linear mapping $j: V \to E$ is given such that $j(v)^2 = Q(v, v)\cdot 1$ for all $v \in V$, or equivalently

$$j(v)\cdot j(w) + j(w)\cdot j(v) = 2Q(v, w)\cdot 1 \tag{20.2}$$

for all $v, w \in V$, then there should be a unique homomorphism of algebras from $C(Q)$ to E extending j. The Clifford algebra can be constructed quickly by taking the tensor algebra

$$T^{\bullet}(V) = \bigoplus_{n\geq 0} V^{\otimes n} = \mathbb{C} \oplus V \oplus (V\otimes V) \oplus (V\otimes V\otimes V) \oplus \cdots,$$

and setting $C(Q) = T^{\bullet}(V)/I(Q)$, where $I(Q)$ is the two-sided ideal generated by all elements of the form $v \otimes v - Q(v, v)\cdot 1$. It is automatic that this $C(Q)$ satisfies the required universal property.

The facts that the dimension of C is 2^m, where $m = \dim(V)$, and that the canonical mapping from V to C is an embedding, are part of the following lemma:

Lemma 20.3. *If $e_1, \ldots, e_m$ form a basis for V, then the products $e_I = e_{i_1}\cdot e_{i_2}\cdot\ldots\cdot e_{i_k}$, for $I = \{i_1 < i_2 < \cdots < i_k\}$, and with $e_\phi = 1$, form a basis for $C(Q) = \mathrm{Cliff}(V, Q)$.*

PROOF. From the equations $e_i\cdot e_j + e_j\cdot e_i = 2Q(e_i, e_j)$ it follows immediately that the elements e_I generate $C(Q)$. Their independence is not hard to verify directly; it also follows by seeing that the images in the matrix algebras under the mappings constructed below are independent. For another proof, note that when $Q \equiv 0$, the Clifford algebra is just the exterior algebra $\bigwedge^{\bullet} V$. In general, the Clifford algebra can be filtered by subspaces F_k, consisting of those elements which can be written as sums of at most k products of elements in V; one checks that the associated graded space F_k/F_{k+1} is $\bigwedge^k V$. For a third proof, one can verify that the Clifford algebra of the direct sum of two orthogonal spaces is the skew commutative tensor product of the Clifford algebras of the two spaces (cf. Exercise B.9), which reduces one to the trivial case where $\dim V = 1$. □

Since the ideal $I(Q) \subset T(V)$ is generated by elements of even degree, the Clifford algebra inherits a $\mathbb{Z}/2\mathbb{Z}$ grading:

$$C = C^{\text{even}} \oplus C^{\text{odd}} = C^+ \oplus C^-,$$

with $C^+\cdot C^+ \subset C^+, C^+\cdot C^- \subset C^-, C^-\cdot C^+ \subset C^-, C^-\cdot C^- \subset C^+$; C^+ is spanned by products of an even number of elements in V and C^- is spanned by products of an odd number. In particular, C^{even} is a subalgebra of dimension 2^{m-1}.

Since $C(Q)$ is an associative algebra, it determines a Lie algebra, with bracket $[a, b] = a\cdot b - b\cdot a$. From now on we assume Q is nondegenerate. The new representations of $\mathfrak{so}_m\mathbb{C}$ will be found in two steps:

(i) embedding the Lie algebra $\mathfrak{so}(Q) = \mathfrak{so}_m\mathbb{C}$ inside the Lie algebra of the even part of the Clifford algebra $C(Q)$;
(ii) identifying the Clifford algebras with one or two copies of matrix algebras.

To carry out the first step we make explicit the isomorphism of $\bigwedge^2 V$ with $\mathfrak{so}(Q)$ that we have discussed before. Recall that

$$\mathfrak{so}(Q) = \{X \in \operatorname{End}(V)\colon Q(Xv, w) + Q(v, Xw) = 0 \text{ for all } v, w \text{ in } V\}.$$

The isomorphism is given by

$$\textstyle\bigwedge^2 V \stackrel{\cong}{\to} \mathfrak{so}(Q) \subset \operatorname{End}(V), \qquad a \wedge b \mapsto \varphi_{a\wedge b},$$

for a and b in V, where $\varphi_{a\wedge b}$ is defined by

$$\varphi_{a\wedge b}(v) = 2(Q(b, v)a - Q(a, v)b). \tag{20.4}$$

It is a simple verification that $\varphi_{a\wedge b}$ is in $\mathfrak{so}(Q)$. One sees that the natural bases correspond up to scalars, e.g., $e_i \wedge e_{n+j}$ maps to $2(E_{i,j} - E_{n+j,n+i})$, so the map is an isomorphism. (The choice of scalar factor is unimportant here; it was chosen to simplify later formulas.) One calculates what the bracket on $\bigwedge^2 V$ must be to make this an isomorphism of Lie algebras:

$$\begin{aligned}
[\varphi_{a\wedge b}, \varphi_{c\wedge d}](v) &= \varphi_{a\wedge b} \circ \varphi_{c\wedge d}(v) - \varphi_{c\wedge d} \circ \varphi_{a\wedge b}(v) \\
&= 2\varphi_{a\wedge b}(Q(d, v)c - Q(c, v)d) - 2\varphi_{c\wedge d}(Q(b, v)a - Q(a, v)b) \\
&= 4Q(d, v)(Q(b, c)a - Q(a, c)b) \\
&\quad - 4Q(c, v)(Q(b, d)a - Q(a, d)b) \\
&\quad - 4Q(b, v)(Q(d, a)c - Q(c, a)d) \\
&\quad + 4Q(a, v)(Q(d, b)c - Q(c, b)d) \\
&= 2Q(b, c)\varphi_{a\wedge d}(v) - 2Q(b, d)\varphi_{a\wedge c}(v) \\
&\quad - 2Q(a, d)\varphi_{c\wedge b}(v) + 2Q(a, c)\varphi_{d\wedge b}(v).
\end{aligned}$$

This gives an explicit formula for the bracket on $\bigwedge^2 V$:

$$\begin{aligned}
[a \wedge b, c \wedge d] = 2Q(b, c)a \wedge d &- 2Q(b, d)a \wedge c \\
&- 2Q(a, d)c \wedge b + 2Q(a, c)d \wedge b.
\end{aligned} \tag{20.5}$$

On the other hand, the bracket in the Clifford algebra satisfies

$$\begin{aligned}
[a\cdot b, c\cdot d] &= a\cdot b\cdot c\cdot d - c\cdot d\cdot a\cdot b \\
&= (2Q(b, c)a\cdot d - a\cdot c\cdot b\cdot d) - (2Q(a, d)c\cdot b - c\cdot a\cdot d\cdot b) \\
&= 2Q(b, c)a\cdot d - (2Q(b, d)a\cdot c - a\cdot c\cdot d\cdot b) \\
&\quad - 2Q(a, d)c\cdot b + (2Q(a, c)\cdot d\cdot b - a\cdot c\cdot d\cdot b) \\
&= 2Q(b, c)a\cdot d - 2Q(b, d)a\cdot c - 2Q(a, d)c\cdot b + 2Q(a, c)\cdot d\cdot b.
\end{aligned}$$

It follows that the map $\psi\colon \bigwedge^2 V \to \operatorname{Cliff}(V, Q)$ defined by

$$\psi(a \wedge b) = \tfrac{1}{2}(a \cdot b - b \cdot a) = a \cdot b - Q(a, b) \tag{20.6}$$

is a map[2] of Lie algebras, and by looking at basis elements again one sees that it is an embedding. This proves:

Lemma 20.7. *The mapping* $\psi \circ \varphi^{-1}\colon \mathfrak{so}(Q) \to C(Q)^{\text{even}}$ *embeds* $\mathfrak{so}(Q)$ *as a Lie subalgebra of* $C(Q)^{\text{even}}$.

Exercise 20.8. Show that the image of ψ is

$$F_2 \cap C(Q)^{\text{even}} \cap \operatorname{Ker}(\text{trace}),$$

where F_2 is the subspace of $C(Q)$ spanned by products of at most two elements of V, and the trace of an element of $C(Q)$ is the trace of left multiplication by that element on $C(Q)$.

We consider first the *even* case: write $V = W \oplus W'$, where W and W' are n-dimensional isotropic spaces for Q. (Recall that a space is isotropic when Q restricts to the zero form on it.) With our choice of standard Q on $V = \mathbb{C}^{2n}$, W can be taken to be the space spanned by the first n basis vectors, W' by the last n.

Lemma 20.9. *The decomposition* $V = W \oplus W'$ *determines an isomorphism of algebras*

$$C(Q) \cong \operatorname{End}(\textstyle\bigwedge^{\bullet} W),$$

where $\bigwedge^{\bullet} W = \bigwedge^0 W \oplus \cdots \oplus \bigwedge^n W$.

Proof. Mapping $C(Q)$ to the algebra $E = \operatorname{End}(\bigwedge^{\bullet} W)$ is the same as defining a linear mapping from V to E, satisfying (20.2). We must construct maps $l\colon W \to E$ and $l'\colon W' \to E$ such that

$$l(w)^2 = 0, \qquad l'(w')^2 = 0, \tag{20.10}$$

and

$$l(w) \circ l'(w') + l'(w') \circ l(w) = 2Q(w, w')I$$

for any $w \in W$, $w' \in W'$. For each $w \in W$, let $L_w \in E$ be left multiplication by w on the exterior algebra $\bigwedge^{\bullet} W$:

$$L_w(\xi) = w \wedge \xi, \quad \xi \in \textstyle\bigwedge^{\bullet} W.$$

For $\vartheta \in W^*$, let $D_\vartheta \in E$ be the derivation of $\bigwedge^{\bullet} W$ such that $D_\vartheta(1) = 0$, $D_\vartheta(w) = \vartheta(w) \in \bigwedge^0 W = \mathbb{C}$ for $w \in W = \bigwedge^1 W$, and

[2] Note that the bilinear form ψ given by (20.6) is alternating since $\psi(a \wedge a) = 0$, so it defines a linear map on $\bigwedge^2 V$.

$$D_\vartheta(\zeta \wedge \xi) = D_\vartheta(\zeta) \wedge \xi + (-1)^{\deg(\zeta)}\zeta \wedge D_\vartheta(\zeta).$$

Explicitly, $D_\vartheta(w_1 \wedge \cdots \wedge w_r) = \sum(-1)^{i-1}\vartheta(w_i)(w_1 \wedge \cdots \wedge \hat{w}_i \wedge \cdots \wedge w_r)$. Now set

$$l(w) = L_w, \qquad l'(w') = D_\vartheta, \tag{20.11}$$

where $\vartheta \in W^*$ is defined by the identity $\vartheta(w) = 2Q(w, w')$ for all $w \in W$. The required equations (20.10) are straightforward verifications: one checks directly on elements in $W = \bigwedge^1 W$, and then that, if they hold on ζ and ξ, they hold on $\zeta \wedge \xi$. Finally, one may see that the resulting map is an isomorphism by looking at what happens to a basis. □

Exercise 20.12. The left $C(Q)$-module $\bigwedge^\bullet W$ is isomorphic to a left ideal in $C(Q)$. Show that if f is a generator for $\bigwedge^n W'$, then $C(Q)\cdot f = \bigwedge^\bullet W \cdot f$, and the map $\zeta \mapsto \zeta \cdot f$ gives an isomorphism

$$\textstyle\bigwedge^\bullet W \to \bigwedge^\bullet W \cdot f = C(Q)\cdot f$$

of left $C(Q)$-modules.

Now we have a decomposition $\bigwedge^\bullet W = \bigwedge^{\mathrm{even}} W \oplus \bigwedge^{\mathrm{odd}} W$ into the sum of even and odd exterior powers, and $C(W)^{\mathrm{even}}$ respects this splitting. We deduce from Lemma 20.9 an isomorphism

$$C(Q)^{\mathrm{even}} \cong \mathrm{End}(\textstyle\bigwedge^{\mathrm{even}} W) \oplus \mathrm{End}(\bigwedge^{\mathrm{odd}} W). \tag{20.13}$$

Combining with Lemma 20.7, we now have an embedding of Lie algebras:

$$\mathfrak{so}(Q) \subset C(Q)^{\mathrm{even}} \cong \mathfrak{gl}(\textstyle\bigwedge^{\mathrm{even}} W) \oplus \mathfrak{gl}(\bigwedge^{\mathrm{odd}} W), \tag{20.14}$$

and hence we have two representations of $\mathfrak{so}(Q) = \mathfrak{so}_{2n}\mathbb{C}$, which we denote by

$$S^+ = \textstyle\bigwedge^{\mathrm{even}} W \quad \text{and} \quad S^- = \bigwedge^{\mathrm{odd}} W.$$

Proposition 20.15. *The representations $S^\pm$ are the irreducible representations of $\mathfrak{so}_{2n}\mathbb{C}$ with highest weights $\alpha = \frac{1}{2}(L_1 + \cdots + L_n)$ and $\beta = \frac{1}{2}(L_1 + \cdots + L_{n-1} - L_n)$. More precisely,*

$$S^+ = \Gamma_\alpha \quad \textit{and} \quad S^- = \Gamma_\beta \quad \textit{if } n \textit{ is even};$$

$$S^+ = \Gamma_\beta \quad \textit{and} \quad S^- = \Gamma_\alpha \quad \textit{if } n \textit{ is odd}.$$

Proof. We show that the natural basis vectors $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $\bigwedge^\bullet W$ are weight vectors. Tracing through the isomorphisms established above, we see that $H_i = E_{i,i} - E_{n+i,n+i}$ in $\mathfrak{h} \subset \mathfrak{so}_{2n}\mathbb{C}$ corresponds to $\frac{1}{2}(e_i \wedge e_{n+i})$ in $\bigwedge^2 V$, which corresponds to $\frac{1}{2}(e_i \cdot e_{n+i} - 1)$ in $C(Q)$, which maps to

$$\tfrac{1}{2}(L_{e_i} \circ D_{2e_i^*} - I) = L_{e_i} \circ D_{e_i^*} - \tfrac{1}{2}I \in \mathrm{End}(\textstyle\bigwedge^\bullet W).$$

A simple calculation shows that

$$L_{e_i} \circ D_{e_i^*}(e_I) = \begin{cases} e_I & \text{if } i \in I \\ 0 & \text{if } i \notin I. \end{cases}$$

Therefore, e_I spans a weight space with weight $\frac{1}{2}(\sum_{i \in I} L_i - \sum_{j \notin I} L_j)$. All such weights with given $|I|$ mod 2 are congruent by the Weyl group, so each of $S^+ = \bigwedge^{\text{even}} W^+$ and $S^- = \bigwedge^{\text{odd}} W$ must be an irreducible representation. The highest weights are easy to read off. For example, the highest weight for $\bigwedge^{\text{even}} W$ is $\frac{1}{2}\sum L_i = \alpha$ if n is even, while if n is odd, its highest weight is β. □

These two representations S^+ and S^- are usually called the *half-spin representations* of $\mathfrak{so}_{2n}\mathbb{C}$, while their sum $S = S^+ \oplus S^- = \bigwedge^{\bullet} W$ is called the *spin representation.* Frequently, especially when we speak of the even and odd cases together, we call them all simply "spin representations." Elements of S are called *spinors.* For other proofs of the proposition see Exercises 20.34 and 20.35.

For the *odd* case, write $V = W \oplus W' \oplus U$, where W and W' are n-dimensional isotropic subspaces, and U is a one-dimensional space perpendicular to them. For our standard Q on $\mathbb{C}^{2n+1}$, these are spanned by the first n, the second n, and the last basis vector.

Lemma 20.16. *The decomposition $V = W \oplus W' \oplus U$ determines an isomorphism of algebras*

$$C(Q) \cong \operatorname{End}(\textstyle\bigwedge^{\bullet} W) \oplus \operatorname{End}(\bigwedge^{\bullet} W').$$

Proof. Proceeding as in the even case, to map V to $E = \operatorname{End}(\bigwedge^{\bullet} W)$, map $w \in W$ to L_w, $w' \in W'$ to D_ϑ, where $\vartheta(w) = 2Q(w, w')$ as before. Let u_0 be the element in U such that $Q(u_0, u_0) = 1$, and send u_0 to the endomorphism that is the identity on $\bigwedge^{\text{even}} W$, and minus the identity on $\bigwedge^{\text{odd}} W$. Since this involution skew commutes with all L_w and D_ϑ, the resulting map from $V = W \oplus W' \oplus U$ to E determines an algebra homomorphism from $C(Q)$ to E. The map to $\operatorname{End}(\bigwedge^{\bullet} W')$ is defined similarly, reversing the roles of W and W'. Again one checks that the map is an isomorphism by looking at bases. □

Exercise 20.17*. Find a generator for a left ideal of $C(Q)$ that is isomorphic to $\bigwedge^{\bullet} W$.

The subalgebra $C(Q)^{\text{even}}$ of $C(Q)$ is mapped isomorphically onto either of the factors by the isomorphism of the lemma, so we have an isomorphism in the odd case:

$$C(Q)^{\text{even}} \cong \operatorname{End}(\textstyle\bigwedge^{\bullet} W). \tag{20.18}$$

As before, this gives a representation $S = \bigwedge^{\bullet} W$ of Lie algebras:

$$\mathfrak{so}_{2n+1}\mathbb{C} = \mathfrak{so}(Q) \subset C(Q)^{\text{even}} \cong \mathfrak{gl}(\textstyle\bigwedge^{\bullet} W) = \mathfrak{gl}(S). \tag{20.19}$$

Proposition 20.20. *The representation $S = \bigwedge^{\bullet} W$ is the irreducible representation of $\mathfrak{so}_{2n+1}\mathbb{C}$ with highest weight*

$$\alpha = \tfrac{1}{2}(L_1 + \cdots + L_n).$$

PROOF. Exactly as in the even case, each e_I is an eigenvector with with weight $\frac{1}{2}(\sum_{i\in I} L_i - \sum_{j\notin I} L_j)$. This time all such weights are congruent by the Weyl group, so this must be an irreducible representation, and the highest weight is clearly $\frac{1}{2}(L_1 + \cdots + L_n)$. □

As we saw in Lecture 19, the construction of this *spin representation S* finishes the proof of the existence theorem for representations of $\mathfrak{so}_m\mathbb{C}$, and hence for all of the classical complex semisimple Lie algebras.

Exercise 20.21*. Use the above identification of the Clifford algebras with matrix algebras (or direct calculation) to compute their centers. In particular, show that the intersection of the center of C with the even subalgebra C^{even} is always the one-dimensional space of scalars. Show similarly that if x is in C^{odd} and $x\cdot v = -v\cdot x$ for all v in V, then $x = 0$.

Exercise 20.22*. For $X \in \mathfrak{so}(Q)$ and $v \in V$, we have $X\cdot v \in V$ by the standard action of $\mathfrak{so}(Q)$ on V. On the other hand, we have identified $\mathfrak{so}(Q)$ and V as subspaces of the Clifford algebra C, so we can compute the commutator $[X, v]$. Show that these agree:

$$X\cdot v = [X, v] \in V \subset C.$$

Problem 20.23*. Let $C(p, q)$ be the real Clifford algebra corresponding to the quadratic form with p positive and q negative eigenvalues. Lemmas 20.9 and 20.16 actually construct isomorphisms of $C(n, n)$ with a real matrix algebra, and of $C(n + 1, n)$ with a product of two real matrix algebras. Compute $C(p, q)$ for other p and q. All are products of one or two matrix algebras over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

§20.2. The Spin Groups $\mathrm{Spin}_m\mathbb{C}$ and $\mathrm{Spin}_m\mathbb{R}$

The Clifford algebra $C = C(Q)$ is generated by the subspace $V = \mathbb{C}^m$, and C has an anti-involution $x \mapsto x^*$, determined by

$$(v_1\cdot\ldots\cdot v_r)^* = (-1)^r v_r\cdot\ldots\cdot v_1$$

for any $v_1, \ldots, v_r$ in V. This operation $*$, sometimes called the *conjugation*, is the composite of:

the *main antiautomorphism* or reversing map $\tau\colon C \to C$ determined by

$$\tau(v_1 \cdot \ldots \cdot v_r) = v_r \cdot \ldots \cdot v_1 \tag{20.24}$$

for $v_1, \ldots, v_r$ in V, and

the *main involution* α which is the identity on C^{even} and minus the identity on C^{odd}, i.e.,

$$\alpha(v_1 \cdot \ldots \cdot v_r) = (-1)^r v_1 \cdot \ldots \cdot v_r. \tag{20.25}$$

Note that $(x \cdot y)^* = y^* \cdot x^*$, which comes from the identities $\tau(x \cdot y) = \tau(y) \cdot \tau(x)$ and $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$.

Exercise 20.26. Use the universal property for C to verify that these are well defined: show that α is a homomorphism from C to C and τ is a well-defined homomorphism from C to the opposite algebra of C (the algebra with the same vector space structure, but with reversed multiplication: $x \tilde{\cdot} y = y \cdot x$).

Instead of defining the spin group as the set of products of certain elements of V, it will be convenient to start with a more abstract definition. Set

$$\operatorname{Spin}(Q) = \{x \in C(Q)^{\text{even}} : x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\}. \tag{20.27}$$

We see from this definition that $\operatorname{Spin}(Q)$ forms a closed subgroup of the group of units in the (even) Clifford algebra. Any x in $\operatorname{Spin}(Q)$ determines an endomorphism $\rho(x)$ of V by

$$\rho(x)(v) = x \cdot v \cdot x^*, \quad v \in V.$$

Proposition 20.28. *For $x \in \operatorname{Spin}(Q)$, $\rho(x)$ is in $\operatorname{SO}(Q)$. The mapping*

$$\rho \colon \operatorname{Spin}(Q) \to \operatorname{SO}(Q)$$

is a homomorphism, making $\operatorname{Spin}(Q)$ a connected two-sheeted covering of $\operatorname{SO}(Q)$. The kernel of ρ is $\{1, -1\}$.

Proof. We will prove something more. Define a larger subgroup, this time of the multiplicative group of $C(Q)$, by

$$\operatorname{Pin}(Q) = \{x \in C(Q) : x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\}, \tag{20.29}$$

and define a homomorphism

$$\rho \colon \operatorname{Pin}(Q) \to \operatorname{O}(Q), \qquad \rho(x)(v) = \alpha(x) \cdot v \cdot x^*, \tag{20.30}$$

where $\alpha \colon C(Q) \to C(Q)$ is the main involution.

To see that $\rho(x)$ preserves the quadratic form Q, we use the fact that for w in V, $Q(w, w) = w \cdot w = -w \cdot w^*$, and calculate:

$$\begin{aligned} Q(\rho(x)(v), \rho(x)(v)) &= -\alpha(x) \cdot v \cdot x^* \cdot (\alpha(x) \cdot v \cdot x^*)^* \\ &= -\alpha(x) \cdot v \cdot x^* \cdot x \cdot v^* \cdot \alpha(x)^* \end{aligned}$$

$$
\begin{aligned}
&= -\alpha(x)\cdot v\cdot v^*\cdot \alpha(x^*)\\
&= Q(v, v)\alpha(x)\cdot\alpha(x^*)\\
&= Q(v, v)\alpha(x\cdot x^*) = Q(v, v).
\end{aligned}
$$

We claim next that ρ is surjective. This follows from the standard fact (see Exercise 20.32) that the orthogonal group $O(Q)$ is generated by reflections. Indeed, if R_w is the reflection in the hyperplane perpendicular to a vector w, normalized so that $Q(w, w) = -1$, it is easy to see that w is in $\text{Pin}(Q)$ and $\rho(w) = R_w$; in fact,

$$w\cdot w^* = w\cdot(-w) = -Q(w, w) = 1,$$

and so

$$\rho(w)(w) = \alpha(w)\cdot w\cdot w^* = -w\cdot 1 = -w;$$

and if $Q(w, v) = 0$,

$$\rho(w)(v) = \alpha(w)\cdot v\cdot w^* = -w\cdot v\cdot w^* = v\cdot w\cdot w^* = v.$$

The next claim is that the kernel of ρ on the larger group $\text{Pin}(Q)$ is ± 1. Suppose x is in the kernel, and write $x = x_0 + x_1$ with $x_0 \in C^{\text{even}}$ and $x_1 \in C^{\text{odd}}$. Then $x_0\cdot v = v\cdot x_0$ for all $v \in V$, so x_0 is in the center of C. And $x_1\cdot v = -v\cdot x_1$ for all $v \in V$. By Exercise 20.21, x_0 is in $\mathbb{C}\cdot 1$, and $x_1 = 0$. So $x = x_0$ is in $\mathbb{C}$ and $x^2 = 1$; so $x = \pm 1$.

It follows that if $R \in O(Q)$ is written as a product of reflections $R_{w_1}\circ\ldots\circ R_{w_r}$, then the two elements in $\rho^{-1}(R)$ are $\pm w_1\cdot\ldots\cdot w_r$. In particular, we get another description of the spin groups:

$$
\begin{aligned}
\text{Spin}(Q) &= \text{Pin}(Q)\cap C(Q)^{\text{even}} = \rho^{-1}(\text{SO}(Q))\\
&= \{\pm w_1\cdot\ldots\cdot w_{2k}\colon w_i \in V, Q(w_i, w_i) = -1\}. \qquad (20.31)
\end{aligned}
$$

Since $-1 = v\cdot v$ for any v with $Q(v, v) = -1$, we see that the spin group consists of even products of such elements.

To complete the proof, we must check that $\text{Spin}(Q)$ is connected or, equivalently, that the two elements in the kernel of ρ can be connected by a path. We leave this now as an exercise, since much more will be seen shortly. □

Exercise 20.32*. Let Q be a nondegenerate symmetric bilinear form on a real or complex vector space V.

(a) Show that if v and w are vectors in V with $Q(v, v) = Q(w, w) \neq 0$, then there is either a reflection or a product of two reflections that takes v into w.

(b) Deduce that every element of the orthogonal group of Q can be written as the product of at most $2\cdot\dim(V)$ reflections.

Exercise 20.33*. Since $\text{Spin}(Q)$ is a subgroup of the multiplicative group of $C(Q)$, its Lie algebra is a subalgebra of $C(Q)$ with its usual bracket. Verify that this subalgebra is the subalgebra $\mathfrak{so}(Q)$ that was constructed in §20.1.

Exercise 20.34. The fact that $\bigwedge^\bullet W$ (and $\bigwedge^\bullet W'$ in the odd case) is an irreducible module over $C(Q)$ is equivalent to the fact that it is an irreducible representation of the group $\mathrm{Pin}(Q)$ since the linear span of $\mathrm{Pin}(Q)$ is dense in $C(Q)$.

(a) Apply the analysis of §5.1 to the subgroup

$$\mathrm{Spin}(Q) \subset \mathrm{Pin}(Q)$$

of index two. In the odd case, $\bigwedge^\bullet W$ and $\bigwedge^\bullet W'$ are conjugate representations, so their restrictions to $\mathrm{Spin}(Q)$ are isomorphic and irreducible: this is the spin representation. In the even case, $\bigwedge^\bullet W$ is self-conjugate, and its restriction to $\mathrm{Spin}(Q)$ is a sum of two conjugate irreducible representations, which are the two half-spin representations.

(b) Of the representations of $\mathrm{Spin}(Q)$ (i.e., the representations of $\mathfrak{so}_m\mathbb{C}$), which induce irreducible representations of $\mathrm{Pin}(Q)$ and which are restrictions of irreducible representations of $\mathrm{Pin}(Q)$?

Exercise 20.35. Deduce the irreducibility of the spin and half-spin representations from the fact that their restrictions to the 2-groups of Exercise 3.9 are irreducible representations of these finite groups.

Exercise 20.36*. Show that the center of $\mathrm{Spin}_m(\mathbb{C})$ is $\rho^{-1}(1) = \{\pm 1\}$ if m is odd. If m is even show that the center is

$$\rho^{-1}(\pm 1) = \{\pm 1, \pm\omega\},$$

where, in terms of our standard basis,

$$\omega = \frac{ie_1 \cdot e_{n+1} - ie_{n+1} \cdot e_1}{2} \cdot \ldots \cdot \frac{ie_n \cdot e_{2n} - ie_{2n} \cdot e_n}{2}.$$

Exercise 20.37*. Show that the spin representation $\mathrm{Spin}(Q) \to \mathrm{GL}(S)$ maps into the special linear group $\mathrm{SL}(S)$. Show that for $m = 2n$ and n even, the half-spin representations also map into the special linear groups $\mathrm{SL}(S^+)$ and $\mathrm{SL}(S^-)$.

Exercise 20.38*. Construct a nondegenerate bilinear pairing β on the spinor space $S = \bigwedge^\bullet W$ by choosing an isomorphism of $\bigwedge^n W$ with $\mathbb{C}$ and letting $\beta(s, t)$ be the image of $\tau(s) \wedge t \in \bigwedge^\bullet W$ by the projection to $\bigwedge^n W = \mathbb{C}$, where τ is the main antiautomorphism).

(a) When $m = 2n$, show that β can also be defined by the identity $\beta(s, t)f = \tau(s \cdot f) \cdot t \cdot f$ for an appropriate generator f of $\bigwedge^n W'$. Deduce that the action of $\mathrm{Spin}(Q)$ on S respects the bilinear form β.

(b) Show that β is symmetric if n is congruent to 0 or 3 modulo 4, and skew-symmetric otherwise. So the spin representation is a homomorphism

$$\mathrm{Spin}_{2n+1}\mathbb{C} \to \mathrm{SO}_{2^n}\mathbb{C} \quad \text{if } n \equiv 0, 3\ (4),$$

$$\mathrm{Spin}_{2n+1}\mathbb{C} \to \mathrm{Sp}_{2^n}\mathbb{C} \quad \text{if } n \equiv 1, 2\ (4).$$

(c) If $m = 2n$, the restrictions of β to S^+ and S^- are zero if n is odd. For n even, deduce that the half-spin representations are homomorphisms

$$\text{Spin}_{2n}\mathbb{C} \to \text{SO}_{2^{n-1}}\mathbb{C} \quad \text{if } n \equiv 0\ (4),$$

$$\text{Spin}_{2n}\mathbb{C} \to \text{Sp}_{2^{n-1}}\mathbb{C} \quad \text{if } n \equiv 2\ (4).$$

Note in particular that $\text{Spin}_8\mathbb{C}$ has two maps to $\text{SO}_8\mathbb{C}$ in addition to the original covering. "Triality," which we discuss in the next section, describes the relation among these three homomorphisms.

Exercise 20.39. Show that the spin and half-spin representations give the isomorphisms we have seen before:

$$\text{Spin}_2\mathbb{C} \cong \text{GL}(S^+) = \text{GL}_1\mathbb{C} = \mathbb{C}^*,$$

$$\text{Spin}_3\mathbb{C} \cong \text{SL}(S) = \text{SL}_2\mathbb{C},$$

$$\text{Spin}_4\mathbb{C} \cong \text{SL}(S^+) \times \text{SL}(S^-) = \text{SL}_2\mathbb{C} \times \text{SL}_2\mathbb{C},$$

$$\text{Spin}_5\mathbb{C} \cong \text{Sp}(S) = \text{Sp}_4\mathbb{C},$$

$$\text{Spin}_6\mathbb{C} \cong \text{SL}(S^+) = \text{SL}_4\mathbb{C}.$$

Exercise 20.40. Let C_m denote the Clifford algebra of the vector space $\mathbb{C}^m$ with our standard quadratic form Q_m.

(a) The embedding of $\mathbb{C}^{2n} = W \oplus W'$ in $\mathbb{C}^{2n+1} = W \oplus W' \oplus U$ as indicated induces an embedding of C_{2n} in C_{2n+1}, and corresponding embedding of $\text{Spin}_{2n}\mathbb{C}$ in $\text{Spin}_{2n+1}\mathbb{C}$ and of $\text{SO}_{2n}\mathbb{C}$ in $\text{SO}_{2n+1}\mathbb{C}$. Show that the spin representation S of $\text{Spin}_{2n+1}\mathbb{C}$ restricts to the spin representation $S^+ \oplus S^-$ of $\text{Spin}_{2n}\mathbb{C}$.

(b) Similarly there is an embedding of $\text{Spin}_{2n+1}\mathbb{C}$ in $\text{Spin}_{2n+2}\mathbb{C}$ coming from an embedding of $\mathbb{C}^{2n+1} = W \oplus W' \oplus U$ in $\mathbb{C}^{2n+2} = W \oplus W' \oplus U_1 \oplus U_2$; here $U_1 \oplus U_2 = \mathbb{C} \oplus \mathbb{C}$ with the quadratic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $U = \mathbb{C}$ is embedded in $U_1 \oplus U_2$ by sending 1 to $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Show that each of the half-spin representations of $\text{Spin}_{2n+2}\mathbb{C}$ restricts to the spin representation of $\text{Spin}_{2n+1}\mathbb{C}$.

Very little of the above discussion needs to be changed to construct the real spin groups $\text{Spin}_m(\mathbb{R})$, which are double coverings of the real orthogonal groups $\text{SO}_m(\mathbb{R})$. One uses the real Clifford algebra $\text{Cliff}(\mathbb{R}^m, Q)$ associated to the real quadratic form $Q = -Q_m$, where Q_m is the standard positive definite quadratic form on $\mathbb{R}^m$. If v_i are an orthonormal basis, the products in this Clifford algebra are given by

$$v_i \cdot v_j = -v_j \cdot v_i \quad \text{if } i \neq j, \qquad \text{and} \quad v_i \cdot v_i = -1.$$

The same definitions can be given as in the complex case, giving rise to coverings $\mathrm{Pin}_m(\mathbb{R})$ of $\mathrm{O}_m(\mathbb{R})$ and $\mathrm{Spin}_m(\mathbb{R})$ of $\mathrm{SO}_m(\mathbb{R})$.

Exercise 20.41. Show that $\mathrm{Spin}_m\mathbb{R}$ is connected by showing that if v and w are any two perpendicular elements in V with $Q(v, v) = Q(w, w) = -1$, the path

$$t \mapsto (\cos(t)v + \sin(t)w)\cdot(\cos(t)v - \sin(t)w), \quad 0 \le t \le \pi/2$$

connects -1 to 1.

Exercise 20.42. Show that $i \mapsto v_2 \cdot v_3$, $j \mapsto v_3 \cdot v_1$, $k \mapsto v_1 \cdot v_2$ determines an isomorphism of the quaternions $\mathbb{H}$ onto the even part of $\mathrm{Cliff}(\mathbb{R}^3, -Q_3)$, such that conjugation $^-$ in $\mathbb{H}$ corresponds to the conjugation $*$ in the Clifford algebra. Show that this maps $\mathrm{Sp}(2) = \{q \in \mathbb{H} \mid q\bar{q} = 1\}$ isomorphically onto $\mathrm{Spin}_3\mathbb{R}$, and that this isomorphism is compatible with the map to $\mathrm{SO}_3\mathbb{R}$ defined in Exercise 7.15.

More generally, if Q is a quadratic form on $\mathbb{R}^m$ with p positive and q negative eigenvalues, we get a group $\mathrm{Spin}^+(p, q)$ in the Clifford algebra $C(p, q) = \mathrm{Cliff}(\mathbb{R}^m, Q)$, with double coverings

$$\mathrm{Spin}^+(p, q) \to \mathrm{SO}^+(p, q).$$

Exercise 20.43*. Show that $\mathrm{Spin}^+(p, q)$ is connected if p and q are positive, except for the case $p = q = 1$, when it has two components. Show that if in the definition of spin groups one relaxes the condition $x \cdot x^* = 1$ to the condition $x \cdot x^* = \pm 1$, one gets coverings $\mathrm{Spin}(p, q)$ of $\mathrm{SO}(p, q)$.

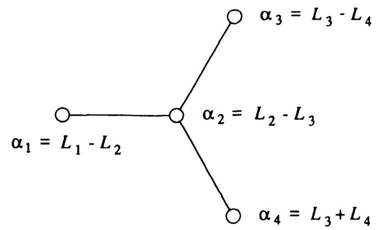
§20.3. $\mathrm{Spin}_8\mathbb{C}$ and Triality

When m is even, there is always an outer automorphism of $\mathrm{Spin}_m(\mathbb{C})$ that interchanges the two spin representations S^+ and S^-, while preserving the basic representation $V = \mathbb{C}^m$ (cf. Exercise 19.9). In case $m = 8$, all three of these representations V, S^+, and S^- are eight dimensional. One basic expression of *triality* is the fact that there are automorphisms of $\mathrm{Spin}_8\mathbb{C}$ or $\mathfrak{so}_8\mathbb{C}$ that permute these three representations arbitrarily. (In fact, the group of outer automorphisms modulo inner automorphisms is the symmetric group on three elements.) We give a brief discussion of this phenomenon in this section, in the form of an extended exercise.

To see where these automorphisms might come from, consider the four simple roots:

$$\alpha_1 = L_1 - L_2, \quad \alpha_2 = L_2 - L_3, \quad \alpha_3 = L_3 - L_4, \quad \alpha_4 = L_3 + L_4.$$

Note that α_1, α_3, and α_4 are mutually perpendicular, and that each makes an angle of 120° with α_2:



Exercise 20.44*. For each of the six permutations of $\{\alpha_1, \alpha_3, \alpha_4\}$ find the orthogonal automorphism of the root space which fixes α_2 and realizes the permutation of α_1, α_3, and α_4.

Each automorphism of this exercise corresponds to an automorphism of the Cartan subalgebra $\mathfrak{h}$. In the next lecture we will see that such automorphisms can be extended (nonuniquely) to automorphisms of the Lie algebra $\mathfrak{so}_8(\mathbb{C})$. (For explicit formulas see [Ca2].)

There is also a purely geometric notion of triality. Recall that an even-dimensional quadric Q can contain linear spaces Λ of at most half the dimension of Q, and that there are two families of linear spaces of this maximal dimension (cf. [G-H], [Ha]). In case Q is six-dimensional, each of these families can themselves be realized as six-dimensional quadrics, which we may denote by Q^+ and Q^- (see below). Moreover, there are correspondences that assign to a point of any one of these quadrics a 3-plane in each of the others:

$$\begin{array}{ccc} \text{Point in } Q & \longrightarrow & \text{3-plane in } Q^+ \\ \swarrow & & \nwarrow \\ \text{3-plane in } Q^- & & \text{Point in } Q^- \\ \nwarrow & & \swarrow \\ \text{Point in } Q^+ & \longrightarrow & \text{3-plane in } Q \end{array} \qquad (20.45)$$

Given $P \in Q$, $\{\Lambda \in Q^+ \colon \Lambda \text{ contains } P\}$ is a 3-plane in Q^+, and $\{\Lambda \in Q^- \colon \Lambda$ contains $P\}$ is a 3-plane in Q^-.

Given $\Lambda \in Q^+$, Λ itself is a 3-plane in Q, and $\{\Gamma \in Q^- \colon \Gamma \cap \Lambda \text{ is a 2-plane}\}$ is a 3-plane in Q^-.

Given $\Lambda \in Q^-$, Λ itself is a 3-plane in Q, and $\{\Gamma \in Q^+ \colon \Gamma \cap \Lambda \text{ is a 2-plane}\}$ is a 3-plane in Q^+.

To relate these two notions of triality, take Q to be our standard quadric in $\mathbb{P}^7 = \mathbb{P}(V)$, with $V = W \oplus W'$ with our usual quadratic space, and let $S^+ = \bigwedge^{\text{even}} W$ and $S^- = \bigwedge^{\text{odd}} W$ be the two spin representations. In Exercise 20.38 we constructed quadratic forms on S^+ and S^-, by choosing an isomorphism of $\bigwedge^4 W$ with $\mathbb{C}$. This gives us two quadrics Q^+ and Q^- in $\mathbb{P}(S^+)$ and $\mathbb{P}(S^-)$.

To identify Q^+ and Q^- with the families of 3-planes in Q, recall the action of V on $S = \bigwedge^{\bullet} W = S^+ \oplus S^-$ which gave rise to the isomorphism of the Clifford algebra with $\mathrm{End}(S)$ (cf. Lemma 20.9). This in fact maps S^+ to S^-

and S^- to S^+; so we have bilinear maps

$$V \times S^+ \dashrightarrow S^- \quad \text{and} \quad V \times S^- \dashrightarrow S^+. \tag{20.46}$$

Exercise 20.47. Show that for each point in Q^+, represented by a vector $s \in S^+$, $\{v \in V: v \cdot s = 0\}$ is an isotropic 4-plane in V, and hence determines a projective 3-plane in Q. Similarly, each point in Q^- determines a 3-plane in Q. Show that every 3-plane in Q arises uniquely in one of these ways.

Let $\langle \ , \ \rangle_V$ denote the symmetric form corresponding to the quadratic form in V, and similarly for S^+ and S^-. Define a product

$$S^+ \times S^- \dashrightarrow V, \qquad s \times t \mapsto s \cdot t, \tag{20.48}$$

by requiring that $\langle v, s \cdot t \rangle_V = \langle v \cdot s, t \rangle_{S^-}$ for all $v \in V$.

Exercise 20.49. Use this product, together with those in (20.46), to show that the other four arrows in the hexagon (20.45) for geometric triality can be described as in the preceding exercise.

This leads to an algebraic version of triality, which we sketch following [Ch2]. The above products determine a commutative but nonassociative product on the direct sum $A = V \oplus S^+ \oplus S^-$. The operation

$$(v, s, t) \mapsto \langle v \cdot s, t \rangle_{S^-}$$

determines a cubic form on A, which by polarization determines a symmetric trilinear form Φ on A.

Exercise 20.50*. One can construct an automorphism J of A of order three that sends V to S^+, S^+ to S^-, and S^- to V, preserving their quadratic forms, and compatible with the cubic form. The definition of J depends on the choice of an element $v_1 \in V$ and $s_1 \in S^+$ with $\langle v_1, v_1 \rangle_V = \langle s_1, s_1 \rangle_{S^+} = 1$; set $t_1 = v_1 \cdot s_1$, so that $\langle t_1, t_1 \rangle_{S^-} = 1$ as well. The map J is defined to be the composite $\mu \circ \nu$ of two involutions μ and ν, which are determined by the following:

(i) μ interchanges S^+ and S^-, and maps V to itself, with $\mu(s) = v_1 \cdot s$ for $s \in S^+$; $\mu(v) = 2\langle v, v_1 \rangle_V v_1 - v$ for $v \in V$.
(ii) ν interchanges V and S^-, maps S^+ to itself, with $\nu(v) = v \cdot s_1$ for $v \in V$; $\nu(s) = 2\langle s, s_1 \rangle_{S^+} s_1 - s$ for $s \in S^+$.

Show that this J satisfies the asserted properties.

Exercise 20.51*. In this algebraic form, triality can be expressed by the assertion that there is an automorphism j of $\mathrm{Spin}_8\mathbb{C}$ of order 3 compatible with J, i.e., such that for all $x \in \mathrm{Spin}_8\mathbb{C}$, the following diagrams commute:

$$\begin{array}{ccccccc}
V & \xrightarrow{J} & S^+ & \xrightarrow{J} & S^- & \xrightarrow{J} & V \\
\downarrow{\scriptstyle \rho(x)} & & \downarrow{\scriptstyle \rho^+(j(x))} & & \downarrow{\scriptstyle \rho^-(j^2(x))} & & \downarrow{\scriptstyle \rho(x)} \\
V & \xrightarrow[J]{} & S^+ & \xrightarrow[J]{} & S^- & \xrightarrow[J]{} & V
\end{array}$$

If $j'\colon \mathfrak{so}_8\mathbb{C} \to \mathfrak{so}_8\mathbb{C}$ is the map induced by j, the fact that j is compatible with the trilinear form Φ (cf. Exercise 20.49) translates to the "local triality" equation

$$\Phi(Xv, s, t) + \Phi(v, Ys, t) + \Phi(v, s, Zt) = 0$$

for $X \in \mathfrak{so}_8\mathbb{C}$, $Y = j'(X)$, $Z = j'(Y)$.