LECTURE 16

# Symplectic Lie Algebras

In this lecture we do for the symplectic Lie algebras exactly what we did for the special linear ones in §15.1 and most of §15.2: we will first describe in general the structure of a symplectic Lie algebra (that is, give a Cartan subalgebra, find the roots, describe the Killing form, and so on). We will then work out in some detail the representations of the specific algebra $\mathfrak{sp}_4\mathbb{C}$. As in the case of the corresponding analysis of the special linear Lie algebras, this is completely elementary.



## §16.1. The Structure of $\mathrm{Sp}_{2n}\mathbb{C}$ and $\mathfrak{sp}_{2n}\mathbb{C}$

Let $V$ be a $2n$-dimensional complex vector space, and

$$Q: V \times V \to \mathbb{C},$$

a nondegenerate, skew-symmetric bilinear form on $V$. The symplectic Lie group $\mathrm{Sp}_{2n}\mathbb{C}$ is then defined to be the group of automorphisms $A$ of $V$ preserving $Q$—that is, such that $Q(Av, Aw) = Q(v, w)$ for all $v, w \in V$—and the symplectic Lie algebra $\mathfrak{sp}_{2n}\mathbb{C}$ correspondingly consists of endomorphisms $A: V \to V$ satisfying

$$Q(Av, w) + Q(v, Aw) = 0$$

for all $v$ and $w \in V$. Clearly, the isomorphism classes of the abstract group and Lie algebra do not depend on the particular choice of $Q$; but in order to be able to write down elements of both explicitly we will, for the remainder of our discussion, take $Q$ to be the bilinear form given, in terms of a basis $e_1, \ldots,$

$e_{2n}$ for $V$, by

$$Q(e_i, e_{i+n}) = 1,$$
$$Q(e_{i+n}, e_i) = -1,$$

and

$$Q(e_i, e_j) = 0 \quad \text{if } j \neq i \pm n.$$

The bilinear form $Q$ may be expressed as

$$Q(x, y) = {}^t x \cdot M \cdot y,$$

where $M$ is the $2n \times 2n$ matrix given in block form as

$$M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix};$$

the group $Sp_{2n}\mathbb{C}$ is thus the group of $2n \times 2n$ matrices $A$ satisfying

$$M = {}^t A \cdot M \cdot A$$

and the Lie algebra $\mathfrak{sp}_{2n}\mathbb{C}$ correspondingly the space of matrices $X$ satisfying the relation

$$ {}^t X \cdot M + M \cdot X = 0. \tag{16.1}$$

Writing a $2n \times 2n$ matrix $X$ in block form as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

we have

$$ {}^t X \cdot M = \begin{pmatrix} -{}^t C & {}^t A \\ -{}^t D & {}^t B \end{pmatrix}$$

and

$$M \cdot X = \begin{pmatrix} C & D \\ -A & -B \end{pmatrix}$$

so that this relation is equivalent to saying that *the off-diagonal blocks $B$ and $C$ of $X$ are symmetric, and the diagonal blocks $A$ and $D$ of $X$ are negative transposes of each other.*

With this said, there is certainly an obvious candidate for Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{sp}_{2n}\mathbb{C}$, namely the subalgebra of matrices diagonal in this representation; in fact, this works, as we shall see shortly. The subalgebra $\mathfrak{h}$ is thus spanned by the $n$ $2n \times 2n$ matrices $H_i = E_{i,i} - E_{n+i,n+i}$ whose action on $V$ is to fix $e_i$, send $e_{n+i}$ to its negative, and kill all the remaining basis vectors; we will correspondingly take as basis for the dual vector space $\mathfrak{h}^*$ the dual basis $L_j$, where $\langle L_j, H_i \rangle = \delta_{i,j}$.

We have already seen how the diagonal matrices act on the algebra of all matrices, so that it is easy to describe the action of $\mathfrak{h}$ on $\mathfrak{g}$. For example, for

$1 \le i, j \le n$ the matrix $E_{i,j} \in \mathfrak{gl}_{2n}\mathbb{C}$ is carried into itself under the adjoint action of $H_i$, into minus itself by the action of $H_j$, and to 0 by all the other $H_k$; and the same is true of the matrix $E_{n+j,n+i}$. The element

$$X_{i,j} = E_{i,j} - E_{n+j,n+i} \in \mathfrak{sp}_{2n}\mathbb{C}$$

is thus an eigenvector for the action of $\mathfrak{h}$, with eigenvalue $L_i - L_j$. Similarly, for $i \neq j$ we see that the matrices $E_{i,n+j}$ and $E_{j,n+i}$ are carried into themselves by $H_i$ and $H_j$ and killed by all the other $H_k$; and likewise $E_{n+i,j}$ and $E_{n+j,i}$ are each carried into their negatives by $H_i$ and $H_j$ and killed by the others. Thus, the elements

$$Y_{i,j} = E_{i,n+j} + E_{j,n+i}$$

and

$$Z_{i,j} = E_{n+i,j} + E_{n+j,i}$$

are eigenvectors for the action of $\mathfrak{h}$, with eigenvalues $L_i + L_j$ and $-L_i - L_j$, respectively. Finally, when $i = j$ the same calculation shows that $E_{i,n+i}$ is doubled by $H_i$ and killed by all other $H_j$; and likewise $E_{n+i,i}$ is sent to minus twice itself by $H_i$ and to 0 by the others. Thus, the elements

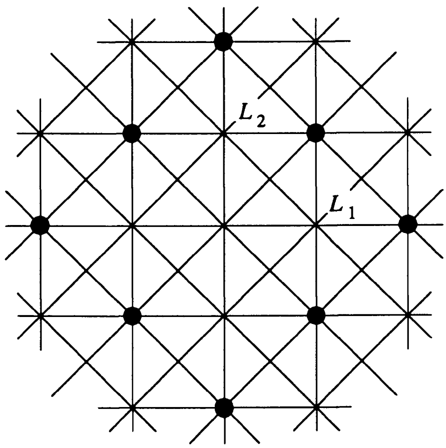
$$U_i = E_{i,n+i}$$

and

$$V_i = E_{n+i,i}$$

are eigenvectors with eigenvalues $2L_i$ and $-2L_i$, respectively. In sum, then, *the roots of the Lie algebra* $\mathfrak{sp}_{2n}\mathbb{C}$ *are the vectors* $\pm L_i \pm L_j \in \mathfrak{h}^*$.

In the first case $n = 1$, of course we just get the root diagram of $\mathfrak{sl}_2\mathbb{C}$, which is the same algebra as $\mathfrak{sp}_2\mathbb{C}$. In case $n = 2$, we have the diagram




(16.2)

As in the case of the special linear Lie algebras, probably the easiest way to determine the Killing form on $\mathfrak{sp}_{2n}\mathbb{C}$ (at least up to scalars) is to use its invariance under the automorphisms of $\mathfrak{sp}_{2n}\mathbb{C}$ preserving $\mathfrak{h}$. For example, we have the automorphisms of $\mathfrak{sp}_{2n}\mathbb{C}$ induced by permutations of the basis vectors $e_i$ of $V$: for any permutation $\sigma$ of $\{1, 2, \dots, n\}$ we can define an automorphism of $V$ preserving $Q$ by sending $e_i$ to $e_{\sigma(i)}$ and $e_{n+i}$ to $e_{n+\sigma(i)}$, and this induces an automorphism of $\mathfrak{sp}_{2n}\mathbb{C}$ preserving $\mathfrak{h}$ and carrying $H_i$ to $H_{\sigma(i)}$. Also, for any $i$ we can define an involution of $V$—and thereby of $\mathfrak{sp}_{2n}\mathbb{C}$—by sending $e_i$ to $e_{n+i}$, $e_{n+i}$ to $-e_i$, and all the other basis vectors to themselves; this will have the effect of sending $H_i$ to $-H_i$ and preserving all the other $H_j$. Now, the Killing form on $\mathfrak{h}$ must be invariant under these automorphisms; from the first batch it follows that for some pair of constants $\alpha$ and $\beta$ we must have

$$B(H_i, H_i) = \alpha$$

and

$$B(H_i, H_j) = \beta \quad \text{for } i \neq j;$$

from the second batch it follows that, in fact, $\beta = 0$. Thus, $B$ is just a multiple of the standard quadratic form $B(H_i, H_j) = \delta_{i,j}$, and the dual form correspondingly a multiple of $B(L_i, L_j) = \delta_{i,j}$; so that the angles in the diagram above are correct.

Also as in the case of $\mathfrak{sl}_n\mathbb{C}$, one can also compute the Killing form directly from the definition: $B(H, H') = \sum \alpha(H)\alpha(H')$, the sum over all roots $\alpha$. For $H = \sum a_i H_i$ and $H' = \sum b_i H_i$, this gives $B(H, H')$ as a sum

$$\sum_{i \neq j} (a_i + a_j)(b_i + b_j) + 2 \sum_i (2a_i)(2bi) + \sum_{i \neq j} (a_i - a_j)(b_i - b_j)$$

which simplifies to

$$B(H, H') = (4n + 4)(\textstyle\sum a_i b_i). \tag{16.3}$$

Our next job is to locate the distinguished copies $\mathfrak{s}_\alpha$ of $\mathfrak{sl}_2\mathbb{C}$, and the corresponding elements $H_\alpha \in \mathfrak{h}$. This is completely straightforward. We start with the eigenvalues $L_i - L_j$ and $L_j - L_i$ corresponding to the elements $X_{i,j}$ and $X_{j,i}$; we have

$$\begin{aligned}[X_{i,j}, X_{j,i}] &= [E_{i,j} - E_{n+j,n+i}, E_{j,i} - E_{n+i,n+j}] \\ &= [E_{i,j}, E_{j,i}] + [E_{n+j,n+i}, E_{n+i,n+j}] \\ &= E_{i,i} - E_{j,j} + E_{n+j,n+j} - E_{n+i,n+i} \\ &= H_i - H_j.\end{aligned}$$

Thus, the distinguished element $H_{L_i - L_j}$ is a multiple of $H_i - H_j$. To see what multiple, recall that $H_{L_i - L_j}$ should act on $X_{i,j}$ by multiplication by 2 and on $X_{j,i}$ by multiplication by $-2$; since we have

$$\begin{aligned}\operatorname{ad}(H_i - H_j)(X_{i,j}) &= ((L_i - L_j)(H_i - H_j)) \cdot X_{i,j} \\ &= 2X_{i,j},\end{aligned}$$

we conclude that

$$H_{L_i - L_j} = H_i - H_j.$$

Next consider the pair of opposite eigenvalues $L_i + L_j$ and $-L_i - L_j$, corresponding to the eigenvectors $Y_{i,j}$ and $Z_{i,j}$. We have

$$\begin{aligned}[Y_{i,j}, Z_{i,j}] &= [E_{i,n+j} + E_{j,n+i}, E_{n+i,j} + E_{n+j,i}] \\ &= [E_{i,n+j}, E_{n+j,i}] + [E_{j,n+i}, E_{n+i,j}] \\ &= E_{i,i} - E_{n+j,n+j} + E_{j,j} - E_{n+i,n+i} \\ &= H_i + H_j.\end{aligned}$$

We calculate then

$$\begin{aligned}\mathrm{ad}(H_i + H_j)(Y_{i,j}) &= ((L_i + L_j)(H_i + H_j)) \cdot Y_{i,j} \\ &= 2 \cdot Y_{i,j},\end{aligned}$$

so we have

$$H_{L_i + L_j} = H_i + H_j$$

and similarly

$$H_{-L_i - L_j} = -H_i - H_j.$$

Finally, we look at the pair of eigenvalues $\pm 2L_i$ coming from the eigenvectors $U_i$ and $V_i$. To complete the span of $U_i$ and $V_i$ to a copy of $\mathfrak{sl}_2\mathbb{C}$ we add

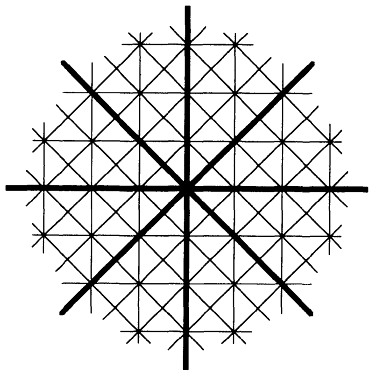
$$\begin{aligned}[U_i, V_i] &= [E_{i,n+i}, E_{n+i,i}] \\ &= E_{i,i} - E_{n+i,n+i} \\ &= H_i.\end{aligned}$$

Since

$$\begin{aligned}\mathrm{ad}(H_i)(U_i) &= (2L_i(H_i)) \cdot U_i \\ &= 2 \cdot U_i,\end{aligned}$$

we conclude that the distinguished element $H_{2L_i}$ is $H_i$, and likewise $H_{-2L_i} = -H_i$. Thus, the distinguished elements $\{H_\alpha\} \subset \mathfrak{h}$ are $\{\pm H_i \pm H_j, \pm H_i\}$; in particular, the weight lattice $\Lambda_W$ of linear forms on $\mathfrak{h}$ integral on all the $H_\alpha$ is exactly the lattice of integral linear combinations of the $L_i$. In Diagram (16.2), for example, this is just the lattice of intersections of the horizontal and vertical lines drawn; observe that for all $n$ the index $[\Lambda_W : \Lambda_R]$ of the root lattice in the weight lattice is just 2.

Next we consider the group of symmetries of the weights of an arbitrary representation of $\mathfrak{sp}_{2n}\mathbb{C}$. For each root $\alpha$ we let $W_\alpha$ be the involution in $\mathfrak{h}^*$ fixing the hyperplane $\Omega_\alpha$ given by $\langle H_\alpha, L \rangle = 0$ and acting as $-I$ on the line spanned by $\alpha$; we observe in this case that, as we claimed will be true in general, the line generated by $\alpha$ is perpendicular to the hyperplane $\Omega_\alpha$, so that the involution is just a reflection in this plane. In the case $n = 2$, for example, we get the dihedral group generated by reflections around the four lines drawn through the origin:



so that the weight diagram of a representation of $\mathfrak{sp}_4\mathbb{C}$ will look like an octagon in general, or (in some cases) a square.

In general, reflection in the plane $\Omega_{2L_i}$ given by $\langle H_i, L\rangle = 0$ will simply reverse the sign of $L_i$ while leaving the other $L_j$ fixed; reflection in the plane $\langle H_i - H_j, L\rangle = 0$ will exchange $L_i$ and $L_j$ and leave the remaining $L_k$ alone. The Weyl group $\mathfrak{W}$ acts as the full automorphism group of the lines spanned by the $L_i$ and fits into a sequence

$$1 \to (\mathbb{Z}/2\mathbb{Z})^n \to \mathfrak{W} \to \mathfrak{S}_n \to 1.$$

Note that the sequence splits: $\mathfrak{W}$ is a semidirect product of $\mathfrak{S}_n$ and $(\mathbb{Z}/2\mathbb{Z})^n$. (This is a special case of a *wreath product*.) In particular the order of $\mathfrak{W}$ is $2^n n!$.

We can choose a positive direction as before:

$$l(\sum a_i L_i) = c_1 a_1 + \cdots + c_n a_n, \qquad c_1 > c_2 > \cdots > c_n > 0.$$

The positive roots are then

$$R^+ = \{L_i + L_j\}_{i \leq j} \cup \{L_i - L_j\}_{i<j}, \tag{16.4}$$

with primitive positive roots $\{L_i - L_{i+1}\}_{i=1,\dots,n-1}$ and $2L_n$. The corresponding (closed) Weyl chamber is

$$\mathscr{W} = \{a_1 L_1 + a_2 L_2 + \cdots + a_n L_n : a_1 \geq a_2 \geq \cdots \geq a_n \geq 0\}; \tag{16.5}$$

note that the walls of this chamber—the cones

$$\{\sum a_i L_i : a_1 > \cdots > a_i = a_{i+1} > \cdots > a_n > 0\}$$

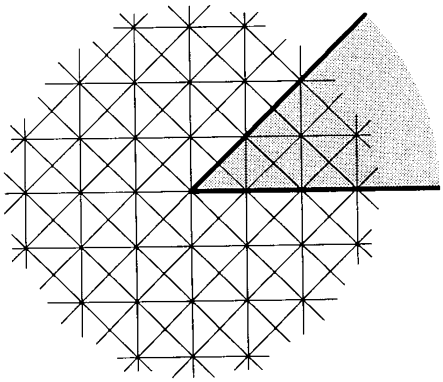
and

$$\{\sum a_i L_i : a_1 > a_2 > \cdots > a_n = 0\}$$

lie in the hyperplanes $\Omega_{L_i - L_{i+1}}$ and $\Omega_{2L_n}$ perpendicular to the primitive positive or negative roots, as expected.

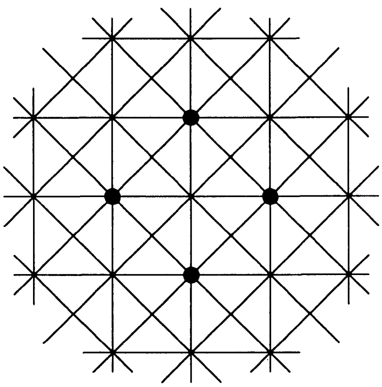
## §16.2. Representations of $\mathfrak{sp}_4\mathbb{C}$

Let us consider now the representations of the algebra $\mathfrak{sp}_4\mathbb{C}$ specifically. Recall that, with the choice of Weyl chamber as above, there is a unique irreducible representation $\Gamma_\alpha$ of $\mathfrak{sp}_4\mathbb{C}$ with highest weight $\alpha$ for any $\alpha$ in the intersection of the closed Weyl chamber $\mathscr{W}$ with the weight lattice: that is, for each lattice vector in the shaded region in the diagram



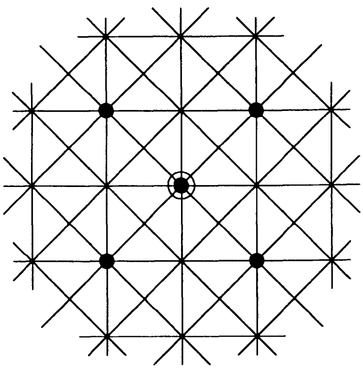
Any such highest weight vector can be written as a non-negative integral linear combination of $L_1$ and $L_1 + L_2$; for simplicity we will just write $\Gamma_{a,b}$ for the irreducible representation $\Gamma_{aL_1+b(L_1+L_2)}$ with highest weight $aL_1 + b(L_1 + L_2) = (a + b)L_1 + bL_2$.

To begin with, we have the standard representation as the algebra of endomorphisms of the four-dimensional vector space $V$; the four standard basis vectors $e_1, e_2, e_3$, and $e_4$ are eigenvectors with eigenvalues $L_1, L_2, -L_1$, and $-L_2$, respectively, so that the weight diagram of $V$ is



$V$ is just the representation $\Gamma_{1,0}$ in the notation above. Note that the dual of this representation is isomorphic to it, which we can see either from the symmetry of the weight diagram, or directly from the fact that the corresponding group representation preserves a bilinear form $V \times V \to \mathbb{C}$ giving an identification of $V$ with $V^*$.

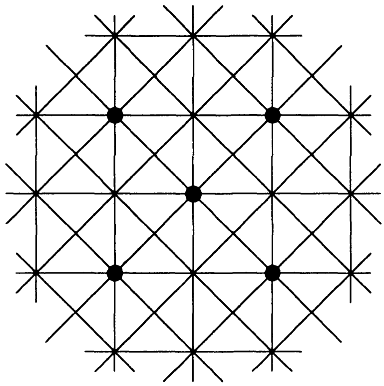
The next representation to consider is the exterior square $\wedge^2 V$. The weights of $\wedge^2 V$, the pairwise sums of distinct weights of $V$, are just the linear forms $\pm L_i \pm L_j$ (each appearing once) and 0 (appearing twice, as $L_1 - L_1$ and $L_2 - L_2$), so that its weight diagram looks like



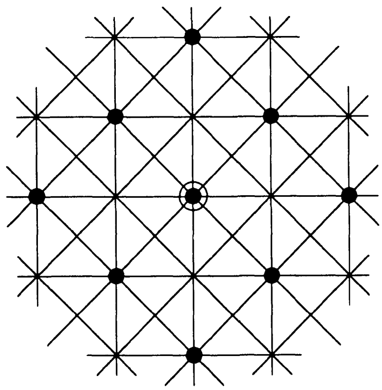
Clearly this representation is not irreducible. We can see this from the weight diagram, using Observation 14.16: there is only one way of getting to the weight space 0 from the highest weight $L_1 + L_2$ by successive applications of the primitive negative root spaces $\mathfrak{g}_{-L_1+L_2}$ (spanned by $X_{2,1} = E_{2,1} - E_{3,4}$) and $\mathfrak{g}_{-2L_2}$ (spanned by $V_2 = E_{4,2}$)—that is, by applying first $V_2$, which takes you to the weight space of $L_1 - L_2$, and then $X_{2,1}$—and so the dimension of the zero weight space in the irreducible representation $\Gamma_{0,1}$ with highest weight $L_1 + L_2$ must be one. Of course, we know in any event that $\wedge^2 V$ cannot be irreducible: the corresponding group action of $\mathrm{Sp}_4\mathbb{C}$ on $V$ by definition preserves the skew form $Q \in \wedge^2 V^* \cong \wedge^2 V$. Either way, we conclude that we have a direct sum decomposition

$$\wedge^2 V = W \oplus \mathbb{C},$$

where $W$ is the irreducible, five-dimensional representation of $\mathfrak{sp}_4\mathbb{C}$ with highest weight $L_1 + L_2$—in our notation, $\Gamma_{0,1}$—and weight diagram



Let us consider next some degree 2 tensors in $V$ and $W$. To begin with, we can write down the weight diagram for the representation $\mathrm{Sym}^2 V$; the weights being just the pairwise sums of the weights of $V$, the diagram is

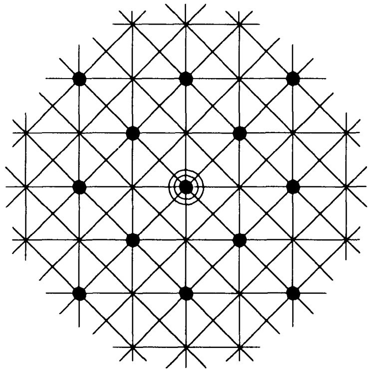


This looks like the weight diagram of the adjoint representation, and indeed that is what it is: in terms of the identification of $V$ and $V^*$ given by the skew form $Q$, the relation (16.1) defining the symplectic Lie algebra says that the subspace

$$\mathfrak{sp}_4\mathbb{C} \subset \mathrm{Hom}(V, V) = V \otimes V^* = V \otimes V$$

is just the subspace $\mathrm{Sym}^2 V \subset V \otimes V$. In particular, $\mathrm{Sym}^2 V$ is the irreducible representation $\Gamma_{2,0}$ with highest weight $2L_1$.

Next, consider the symmetric square $\mathrm{Sym}^2 W$, which has weight diagram



To see if this is irreducible we first look at the weight diagram: this time there are three ways of getting from the weight space with highest weight $2L_1 + 2L_2$ to the space of weight 0 by successively applying $X_{2,1} = E_{2,1} - E_{3,4}$ and $V_2 = E_{4,2}$, so if we want to proceed by this method we are forced to do a little calculation, which we leave as Exercise 16.7.

Alternatively, we can see directly that $\mathrm{Sym}^2 W$ decomposes: the natural map given by wedge product

$$\wedge^2 V \otimes \wedge^2 V \to \wedge^4 V = \mathbb{C}$$

is symmetric, and so factors to give a map

$$\mathrm{Sym}^2(\wedge^2 V)) \to \mathbb{C}.$$

Moreover, since this map is well defined up to scalars—in particular, it does not depend on the choice of skew form $Q$—it cannot contain the subspace $\mathrm{Sym}^2 W \subset \mathrm{Sym}^2(\wedge^2 V))$ in its kernel, so that it restricts to give a surjection

$$\varphi\colon \mathrm{Sym}^2 W \to \mathbb{C}.$$

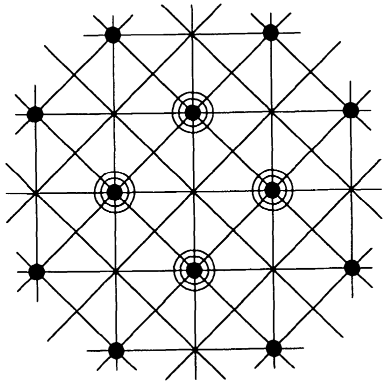
This approach would appear to leave two possibilities open: either the kernel of this map is irreducible, or it is the direct sum of an irreducible representation and a further trivial summand. In fact, however, from the principle that an irreducible representation cannot have two independent invariant bilinear forms, we see that $\mathrm{Sym}^2 W$ can contain at most one trivial summand, and so the former alternative must hold, i.e., we have

$$\mathrm{Sym}^2 W = \Gamma_{0,2} \oplus \mathbb{C}. \tag{16.6}$$

**Exercise 16.7*.** Prove (16.6) directly, by showing that if $v$ is a highest weight vector, then the three vectors $X_{2,1}V_2X_{2,1}V_2v$, $X_{2,1}X_{2,1}V_2V_2v$, and $V_2X_{2,1}X_{2,1}V_2v$ span a two-dimensional subspace of the kernel of $\varphi$.

**Exercise 16.8.** Verify that $\wedge^2 W \cong \mathrm{Sym}^2 V$. The significance of this isomorphism will be developed further in Lecture 18.

Lastly, consider the tensor product $V \otimes W$. First, its weight diagram:



This obviously must contain the irreducible representation $\Gamma_{1,1}$ with highest weight $2L_1 + L_2$; but it cannot be irreducible, for either of two reasons. First, looking at the weight diagram, we see that $\Gamma_{1,1}$ can take on the eigenvalues $\pm L_i$ with multiplicity at most 2, so that $V \otimes W$ must contain at least one copy of the representation $V$. Alternatively, we have a natural map given by wedge product

$$\wedge : V \otimes \wedge^2 V \to \wedge^3 V = V^* = V;$$

and since this map does not depend on the choice of skew form $Q$, it must restrict to give a nonzero (and hence surjective) map

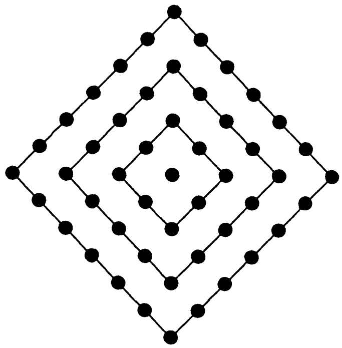
$$\varphi : V \otimes W \to V.$$

**Exercise 16.9.** Show that the kernel of this map is irreducible, and hence that we have

$$V \otimes W = \Gamma_{1,1} \oplus V.$$

What about more general tensors? To begin with, note that we have established the existence half of the standard existence and uniqueness theorem (14.18) in the case of $\mathfrak{sp}_4\mathbb{C}$: the irreducible representation $\Gamma_{a,b}$ may be found somewhere in the tensor product $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b W$. The question that remains is, where? In other words, we would like to be able to say how these tensor products decompose. This will be, as it was in the case of $\mathfrak{sl}_3\mathbb{C}$, nearly tantamount (modulo the combinatorics needed to count the multiplicity with which the tensor product $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b W$ assumes each of its eigenvalues) to specifying the multiplicities of the irreducible representations $\Gamma_{a,b}$.

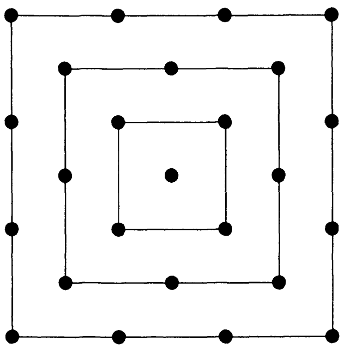
Let us start with the simplest case, namely, the representations $\mathrm{Sym}^a V$. These have weight diagram a sequence of nested diamonds $D_i$ with vertices at $aL_1, (a-2)L_1$, etc.:



Moreover, it is not hard to calculate the multiplicities of $\mathrm{Sym}^a V$: the multiplicity on the outer diamond $D_1$ is one, of course; and then the multiplicities will increase by one on successive rings, so that the multiplicity along the diamond $D_i$ will be $i$.

**Exercise 16.10.** Using the techniques of Lecture 13, show that the representations $\mathrm{Sym}^a V$ are irreducible.

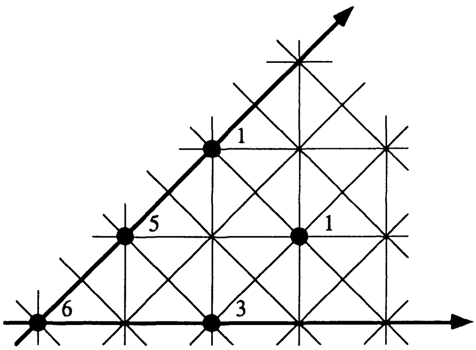
The next simplest representations, naturally enough, are the symmetric powers $\mathrm{Sym}^b W$ of $W$. These have eigenvalue diagrams in the shape of a sequence of squares $S_i$ with vertices at $b(L_1 + L_2)$, $(b - 1)(L_1 + L_2)$, and so on:



Here, however, the multiplicities increase in a rather strange way: they grow quadratically, but only on every other ring. Explicitly, the multiplicity will be one on the outer two rings, then 3 on the next two rings, 6 on the next two; in general, it will be $i(i + 1)/2$ on the $(2i - 1)$st and $(2i)$th squares $S_{2i-1}$ and $S_{2i}$.

**Exercise 16.11.** Show that contraction with the skew form $\varphi \in \mathrm{Sym}^2 W^*$ introduced in the discussion of $\mathrm{Sym}^2 W$ above determines a surjection from $\mathrm{Sym}^b W$ onto $\mathrm{Sym}^{b-2} W$, and that the kernel of this map is the irreducible representation $\Gamma_{0,b}$ with highest weight $b(L_1 + L_2)$. Show that the multiplicities of $\Gamma_{0,b}$ are $i$ on the squares $S_{2i-1}$ and $S_{2i}$ described above.

We will finish by analyzing, naively and in detail, one example of a representation $\Gamma_{a,b}$ with $a$ and $b$ both nonzero, namely, $\Gamma_{2,1}$; one thing we may observe on the basis of this example is that there is not a similarly simple pattern to the multiplicities of the representations $\Gamma_{a,b}$ with general $a$ and $b$. To carry out our analysis, we start of course with the product $\mathrm{Sym}^2 V \otimes W$. We can readily draw the weight diagram for this representation; drawing only one-eighth of the plane and indicating multiplicities by numbers, it is




We know that the representation $\mathrm{Sym}^2 V \otimes W$ contains a copy of the irreducible representation $\Gamma_{2,1}$ with highest weight $2L_1 + (L_1 + L_2)$; and we can see immediately from the diagram that it cannot equal this: for example, $\Gamma_{2,1}$ can take the weight $2L_1$ with multiplicity at most 2 (if $v \in \Gamma_{2,1}$ is its highest weight vector, the corresponding weight space $(\Gamma_{2,1})_{2L_1} \subset \Gamma_{2,1}$ will be spanned by the two vectors $X_{2,1}(Y_2(v))$ and $Y_2(X_{2,1}(v))$); since it cannot contain a copy of the representation $\Gamma_{0,2}$ (the multiplicity of the weight $2(L_1 + L_2)$ being just one) it follows that $\mathrm{Sym}^2 V \otimes W$ must contain a copy of the representation $\Gamma_{2,0} = \mathrm{Sym}^2 V$.

We can, in this way, narrow down the list of possibilities a good deal. For example, $\Gamma_{2,1}$ cannot have multiplicity just one at each of the weights $2L_1$ and $L_1 + L_2$: if it did, $\mathrm{Sym}^2 V \otimes W$ would have to contain two copies of $\mathrm{Sym}^2 V$ and a further two copies of $W$ to make up the multiplicity at $L_1 + L_2$; but since 0 must appear as a weight of $\Gamma_{2,1}$, this would give a total multiplicity of at least 7 for the weight 0 in $\mathrm{Sym}^2 V \otimes W$. Similarly, $\Gamma_{2,1}$ cannot have multiplicity 1 at $2L_1$ and 2 at $L_1 + L_2$: we would then have two copies of $\mathrm{Sym}^2 V$ and one of $W$ in $\mathrm{Sym}^2 V \otimes W$; and since the multiplicity of 0 in $\Gamma_{2,1}$ will in this case be at least 2 (being greater than or equal to the multiplicity of $L_1 + L_2$), this would again imply a multiplicity of at least 7 for the weight 0 in $\text{Sym}^2 V \otimes W$. It follows that $\text{Sym}^2 V \otimes W$ must contain exactly one copy of $\text{Sym}^2 V$; and since the multiplicity of $L_1 + L_2$ in $\Gamma_{2,1}$ is at most 3, it follows that $\text{Sym}^2 V \otimes W$ will contain at least one copy of $\Gamma_{0,1} = W$ as well.

**Exercise 16.12.** Prove, independently of the above analysis, that $\text{Sym}^2 V \otimes W$ must contain a copy of $\text{Sym}^2 V$ and a copy of $W$ by looking at the map

$$\varphi: \text{Sym}^2 V \otimes W \to V \otimes V$$

obtained by sending

$$u \cdot v \otimes (w \wedge z) \mapsto u \otimes \tilde{Q}(v \wedge w \wedge z) + v \otimes \tilde{Q}(u \wedge w \wedge z),$$

where we are identifying $\bigwedge^3 V$ with the dual space $V^*$ and denoting by $\tilde{Q}: V^* \to V$ the isomorphism induced by the skew form $Q$ on $V$. Specifically, show that the image of this map is complementary to the line spanned by the element $Q \in \bigwedge^2 V^* = \bigwedge^2 V \subset V \otimes V$.

The above leaves us with exactly two possibilities for the weights of $\Gamma_{2,1}$: we know that the multiplicity of $2L_1$ in $\Gamma_{2,1}$ is exactly 2; so either the multiplicities of $L_1 + L_2$ and 0 in $\Gamma_{2,1}$ are both 3 and we have

$$\text{Sym}^2 V \otimes W = \Gamma_{2,1} \oplus \text{Sym}^2 V \oplus W;$$

or the multiplicities of $L_1 + L_2$ and 0 in $\Gamma_{2,1}$ are both 2 and we have

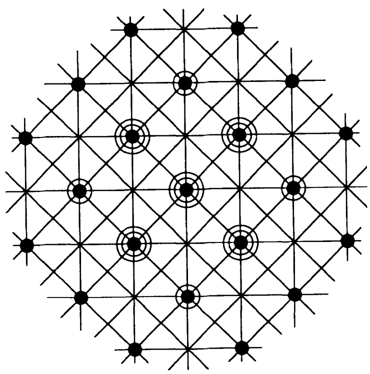
$$\text{Sym}^2 V \otimes W = \Gamma_{2,1} \oplus \text{Sym}^2 V \oplus W^{\oplus 2}.$$

**Exercise 16.13.** Show that the former of these two possibilities actually occurs, by

(a) Showing that if $v$ is the highest weight vector in $\Gamma_{2,1} \subset \text{Sym}^2 V \otimes W$, then the images $(X_{2,1})^2 Y_2(v)$, $X_{2,1} Y_2 X_{2,1}(v)$, and $Y_2(X_{2,1})^2 v$ are independent; and (redundantly)

(b) Showing that the representation $\text{Sym}^2 V \otimes W$ contains only one highest weight vector of weight $L_1 + L_2$.

The weight diagram of $\Gamma_{2,1}$ is therefore



We see from all this that, in particular, the weights of the irreducible representations of $\mathfrak{sp}_4\mathbb{C}$ are not constant on the rings of their weight diagrams.

**Exercise 16.14.** Analyze the representation $V \otimes \mathrm{Sym}^2 W$ of $\mathfrak{sp}_4\mathbb{C}$. Find in particular the multiplicities of the representation $\Gamma_{1,2}$.

**Exercise 16.15.** Analyze the representation $\mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 W$ of $\mathfrak{sp}_4\mathbb{C}$. Find in particular the multiplicities of the representation $\Gamma_{2,2}$.