LECTURE 24

Weyl Character Formula

This lecture is pretty straightforward: we simply state the Weyl character formula in §24.1, then show how it may be worked out in specific examples in §24.2. In particular, we derive in the case of the classical algebras formulas for the character of a given irreducible representation as a polynomial in the characters of certain basic ones (either the alternating or the symmetric powers of the standard representation for $\mathfrak{sl}_n\mathbb{C}$ and their analogues for $\mathfrak{sp}_{2n}\mathbb{C}$ and $\mathfrak{so}_m\mathbb{C}$). The proofs of the formula are deferred to the following two lectures. The techniques involved here are elementary, though the determinantal formulas are fairly complex, involving all the algebra of Appendix A.

§24.1: The Weyl character formula
§24.2: Applications to classical Lie algebras and groups

§24.1. The Weyl Character Formula

We have already seen the Weyl character formula in the case of $\mathfrak{sl}_n\mathbb{C}$, and it is one reason why we were able to calculate so many more representations in that case. We saw in Lectures 6 and 15 that for the representation $\Gamma_\lambda = \mathbb{S}_\lambda\mathbb{C}^n$ of $\mathrm{SL}_n\mathbb{C}$ with highest weight $\lambda = \sum \lambda_i L_i$, the trace of the action of a diagonal matrix $A \in \mathrm{SL}_n\mathbb{C}$ with entries $x_1, \ldots, x_n$ is the symmetric function called the Schur polynomial $S_\lambda(x_1, \ldots, x_n)$. This included a formula for the multiplicities, which are the coefficients of the monomials in these variables.

In order to extend this formula to the other Lie algebras, let us try to rewrite this Schur polynomial in a way that may generalize. The Schur polynomial is defined to be a quotient of two alternating polynomials:

$$S_\lambda(x_1, \ldots, x_n) = \frac{|x_j^{\lambda_i+n-i}|}{|x_j^{n-i}|}.$$

These determinants can be expanded as usual as a sum over the symmetric group $\mathfrak{S}_n$, which is the Weyl group $\mathfrak{W}$. Writing $x_i = e(L_i)$ in $\mathbb{Z}[\Lambda]$ as in the preceding lecture, and writing $(-1)^W$ for $\operatorname{sgn}(W) = \det(W)$ for W in the Weyl group, the numerator may be expanded in the form

$$\sum_{W \in \mathfrak{W}} (-1)^W x_{W(1)}^{\lambda_1+n-1} \cdot \ldots \cdot x_{W(n)}^{\lambda_n} = \sum_{W \in \mathfrak{W}} (-1)^W e(W(\Sigma(\lambda_i + n - i)L_i))$$
$$= \sum_{W \in \mathfrak{W}} (-1)^W e(W(\lambda + \rho)),$$

where we write λ for $\Sigma\lambda_i L_i$ and we set $\rho = \Sigma(n - i)L_i$. Our formula therefore takes the form

$$\operatorname{Char}(\Gamma_\lambda) = \frac{\sum(-1)^W e(W(\lambda + \rho))}{\Sigma(-1)^W e(W(\rho))}.$$

The denominator is the discriminant

$$\Delta(x_1, \ldots, x_n) = \prod_{i<j} (x_i - x_j) = \prod_{i<j} (e(L_i) - e(L_j)).$$

This can be written in terms of the positive roots $L_i - L_j$, $i < j$, as

$$\Delta(x_1, \ldots, x_n) = \prod_{i<j} (e(\tfrac{1}{2}(L_i - L_j) - e(-\tfrac{1}{2}(L_i - L_j))).$$

Note also that

$$\rho = \Sigma(n - i)L_i = L_1 + (L_1 + L_2) + \cdots + (L_1 + \cdots + L_{n-1})$$
$$= \frac{1}{2} \sum_{i<j} (L_i - L_j),$$

which is the *sum of the fundamental weights*, and *half the sum of the positive roots.*

These are the formulas that generalize to the other semisimple Lie algebras: For any weight μ, define $A_\mu \in \mathbb{Z}[\Lambda]$ by

$$A_\mu = \sum_{W \in \mathfrak{W}} (-1)^W e(W(\mu)). \tag{24.1}$$

Note that A_μ is not invariant by the Weyl group, but is *alternating*: $W(A_\mu) = (-1)^W A_\mu$ for $W \in \mathfrak{W}$. The ratio of two alternating polynomials will be invariant.

Theorem 24.2 (Weyl Character Formula). *Let ρ be half the sum of the positive roots. Then ρ is a weight, and $A_\rho \neq 0$. The character of the irreducible representation Γ_λ with highest weight λ is*

$$\operatorname{Char}(\Gamma_\lambda) = \frac{A_{\lambda+\rho}}{A_\rho}. \qquad \text{(WCF)}$$

The assertions about ρ are part of the following lemma and exercise, which will also be useful in the applications:

Lemma 24.3. *The denominator A_ρ of Weyl's formula is*

$$\begin{aligned} A_\rho &= \prod_{\alpha \in R^+} (e(\alpha/2) - e(-\alpha/2)) \\ &= e(\rho) \prod_{\alpha \in R^+} (1 - e(-\alpha)) \\ &= e(-\rho) \prod_{\alpha \in R^+} (e(\alpha) - 1). \end{aligned}$$

Proof. Since $e(\rho) = e(\sum \alpha/2) = \prod e(\alpha/2)$, the equality of the three displayed expressions is evident; denote these expressions temporarily by A. The key point is to see that A is alternating. For this, it suffices to see that A changes sign when a reflection in a hyperplane perpendicular to one of the simple roots is applied to it, since these reflections generate the Weyl group. This follows immediately from the first expression for A and (a) in Exercise 24.4 below.

Now, by the second displayed expression, the highest weight term that appears in A is $e(\rho)$, which is the same as that appearing in A_ρ. Calculating $1/A$ formally as in (24.5) below, we see that A_ρ/A is a formal sum $\sum m_\mu e(\mu)$ that is invariant by the Weyl group, and, using part (c) of the following exercise, it has weight 0. As in Theorem 23.24 it follows that A_ρ/A is constant; and, since A and A_ρ have the same leading term $e(\rho)$, we must have $A_\rho = A$. □

Exercise 24.4*. (a) If $W = W_{\alpha_i}$ is the reflection in the hyperplane perpendicular to a simple root α_i, show that $W(\alpha_i) = -\alpha_i$, and W permutes the other positive roots.

(b) With W as in (a), show that $W(\rho) = \rho - \alpha_i$. Deduce that ρ is the element in $\mathfrak{h}^*$ such that $\rho(H_{\alpha_i}) = 2(\rho, \alpha_i)/(\alpha_i, \alpha_i) = 1$ for each simple root α_i. Equivalently, ρ is the sum of the fundamental weights. In particular, ρ is a weight.

(c) For any $W \neq 1$ in the Weyl group, show that $\rho - W(\rho)$ is a sum of distinct positive roots. Deduce that $W(\rho)$ is not in the closure of the positive Weyl chamber.

Proofs of the character formula will be given in §25.2 and again in §26.2. For now we should at least verify that it is plausible, i.e., that $A_{\lambda+\rho}/A_\rho$ is in $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$ and that the highest weight that occurs is λ. Note that since the numerator and denominator are alternating, the ratio is invariant. The fact that A_ρ is not zero follows from the second expression in the preceding lemma. To see that the ratio is actually in $\mathbb{Z}[\Lambda]$, however, we must verify that it has only a finite number of nonzero coefficients. Write

$$\frac{1}{A_\rho} = e(-\rho) \prod_{\alpha \in R^+} (1 - e(-\alpha))^{-1} = e(-\rho) \prod_{\alpha} \sum_{n=0}^{\infty} e(-n\alpha). \qquad (24.5)$$

When this is multiplied by $A_{\lambda+\rho} = \sum(-1)^W e(W(\lambda + \rho))$, we get a formal sum where the highest weight that occurs is the weight λ. This means in particular that there are only a finite number of nonzero terms corresponding to weights in the fundamental (positive) Weyl chamber $\mathscr{W}$. But since the ratio is invariant by the Weyl group, the same is true for all Weyl chambers, so $A_{\lambda+\rho}/A_\rho$ is in $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$, and has highest weight λ. It follows in particular that the $A_{\lambda+\rho}/A_\rho$, as λ varies over $\mathscr{W} \cap \Lambda$, form an additive basis for $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$.

Before considering the proof or any other special cases, we apply (WCF) to give a formula for the dimension of Γ_λ:

Corollary 24.6. *The dimension of the irreducible representation Γ_λ is*

$$\dim \Gamma_\lambda = \prod_{\alpha \in R^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} = \prod_{\alpha \in R^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)},$$

where $\langle \alpha, \beta \rangle = \alpha(H_\beta) = 2(\alpha, \beta)/(\beta, \beta)$ and (,) is the Killing form.

Proof. The dimension of Γ_λ is obtained by adding the coefficients of all $e(\alpha)$ in $\mathrm{Char}(\Gamma_\lambda)$, i.e., computing the image of $\mathrm{Char}(\Gamma_\lambda)$ by the homomorphism from $\mathbb{Z}[\Lambda]$ to $\mathbb{C}$ which sends each $e(\alpha)$ to 1. However, as in the case of the Schur polynomial, the denominator vanishes if we try to do this directly. To get around this, we factor this homomorphism through the ring of power series:

$$\mathbb{Z}[\Lambda] \xrightarrow{\Psi} \mathbb{C}[[t]] \to \mathbb{C},$$

where the second homomorphism sets the variable t equal to zero, i.e., picks off the constant term of the power series, and the first homomorphism Ψ takes $e(\alpha)$ to $e^{(\rho,\alpha)t}$. More generally, for any weight μ define a homomorphism

$$\Psi_\mu: \mathbb{Z}[\Lambda] \to \mathbb{C}[[t]], \qquad e(\alpha) \mapsto e^{(\mu,\alpha)t}.$$

We claim that $\Psi_\mu(A_\lambda) = \Psi_\lambda(A_\mu)$ for all λ and μ. This is a simple consequence of the invariance of the metric (,) under the Weyl group:

$$\begin{aligned}\Psi_\mu(A_\lambda) &= \sum (-1)^W e^{(\mu, W(\lambda))t} \\ &= \sum (-1)^W e^{(W^{-1}(\mu), \lambda)t} \\ &= \sum (-1)^W e^{(W(\mu), \lambda)t} \\ &= \Psi_\lambda(A_\mu).\end{aligned}$$

Therefore,

$$\begin{aligned}\Psi(A_\lambda) &= \Psi_\rho(A_\lambda) = \Psi_\lambda(A_\rho) \\ &= \prod_{\alpha \in R^+} (e^{(\lambda,\alpha)t/2} - e^{-(\lambda,\alpha)t/2})\end{aligned}$$

$$= \left(\prod_{\alpha \in R^+} (\lambda, \alpha) \right) t^{\#(R^+)} + \text{terms of higher degree in } t.$$

Hence,

$$\Psi(A_{\lambda+\rho}/A_\rho) = \Psi(A_{\lambda+\rho})/\Psi(A_\rho)$$
$$= \frac{\prod (\lambda + \rho, \alpha)}{\prod (\rho, \alpha)} + \text{terms of positive degree in } t,$$

which finishes the proof. □

Exercise 24.7. In the case of $\mathfrak{sl}_n\mathbb{C}$, verify that the above corollary gives the dimension we found in Lecture 6.

Exercise 24.8. Verify directly that the right-hand side of the formula for the dimension is positive.

Since $\chi_\lambda = A_{\lambda+\rho}/A_\rho$ is the character of a virtual representation which takes on a positive value at the identity, as in the case of finite groups, to prove that it is the character of an irreducible representation, it suffices to show that $\int_G \chi_\lambda \overline{\chi}_\lambda = 1$ for an appropriate compact group G. This was the original approach of Weyl, which we will describe in the last lecture. Since the highest weight appearing is λ, we will know then that this irreducible representation must be Γ_λ.

Exercise 24.9. Use Corollary 24.6 to show that if λ is a dominant weight (i.e., in the closure of the positive Weyl chamber), and ω is a fundamental weight, then the dimension of $\Gamma_{\lambda+\omega}$ is greater than the dimension of Γ_λ. Conclude that the nontrivial representations of smallest dimension must be among the n representations Γ_ω with ω a fundamental weight.

§24.2. Applications to Classical Lie Algebras and Groups

In the case of the general linear group $GL_n\mathbb{C}$, the character[1] of the representation Γ_λ is the Schur polynomial

$$S_\lambda(x_1, \ldots, x_n) = \frac{|x_j^{\lambda_i+n-i}|}{|x_j^{n-i}|},$$

[1] We use the representation of $GL_n\mathbb{C}$ instead of its restriction to $SL_n\mathbb{C}$, since the latter would require the product of the variables x_i to be 1.

which has several expressions in terms of simpler symmetric functions. Note that the character of the dth symmetric power of the standard representation is the dth complete symmetric polynomial H_d in n variables (Appendix A.1):

$$H_d = \mathrm{Char}(\mathrm{Sym}^d(\mathbb{C}^n)).$$

The first "Giambelli" or determinantal formula (A.5) of Appendix A gives the character of the representation with highest weight $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$ as an $r \times r$ determinant:

$$\mathrm{Char}(\Gamma_\lambda) = |H_{\lambda_i+j-i}| = \begin{vmatrix} H_{\lambda_1} & H_{\lambda_1+1} \cdots H_{\lambda_1+k-1} \\ H_{\lambda_2-1} H_{\lambda_2} \cdots & \\ \vdots & \\ H_{\lambda_k-k+1} \cdots & H_{\lambda_k} \end{vmatrix}. \tag{24.10}$$

Equivalently, this expresses a general element $\Gamma_\lambda \in R(G)$ of the representation ring as a polynomial in the representations $\mathrm{Sym}^d(\mathbb{C}^n)$. A second determinantal formula, from (A.6), expresses Γ_λ in terms of the basic representations $\bigwedge^d(\mathbb{C}^n)$, whose characters are the elementary symmetric polynomials

$$E_d = \mathrm{Char}(\textstyle\bigwedge^d(\mathbb{C}^n)).$$

This formula is, with μ the conjugate partition to λ,

$$\mathrm{Char}(\Gamma_\lambda) = |E_{\mu_i+j-i}| = \begin{vmatrix} E_{\mu_1} & E_{\mu_1+1} \cdots E_{\mu_1+l-1} \\ E_{\mu_2-1} E_{\mu_2} \cdots & \\ \vdots & \\ E_{\mu_l-l+1} \cdots & E_{\mu_l} \end{vmatrix} \tag{24.11}$$

In this section we work out the character formula for the other classical Lie algebras, including analogues of these determinantal formulas. The analogues of the first determinantal formula (24.10) were given by Weyl, but the analogues of (24.11) were found only recently ([D'H], [Ko-Te]). We also pay, at least by way of exercises, the debts to (WCF) that we owe from earlier lectures.

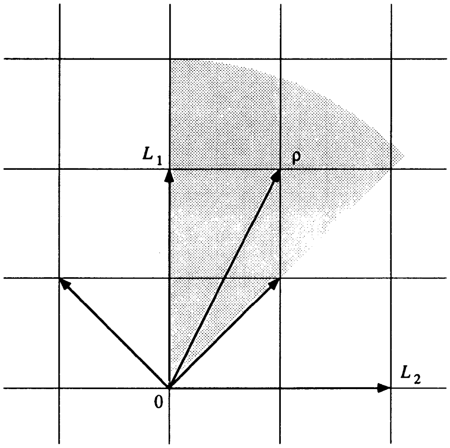
The Symplectic Case

The weights for $\mathfrak{sp}_{2n}\mathbb{C}$ are integral linear combinations of $L_1, \ldots, L_n$. We often write $\mu = (\mu_1, \ldots, \mu_n)$ for the weight $\mu_1 L_1 + \cdots + \mu_n L_n$.

The positive roots are $\{L_i - L_j\}_{i<j}$ and $\{L_i + L_j\}_{i\leq j}$, from which we find

$$\rho = \sum (n + 1 - i)L_i = L_1 + (L_1 + L_2) + \cdots + (L_1 + \cdots + L_n), \tag{24.12}$$

i.e., $\rho = (n, n - 1, \ldots, 1)$.



As we saw in Lecture 16, an element in the Weyl group can be written uniquely as a product $\varepsilon\sigma$, where σ is a permutation of $\{L_1, \ldots, L_n\}$, and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, with $\varepsilon_i = \pm 1$. Hence

$$A_\mu = \sum_\sigma (-1)^\sigma \sum_\varepsilon (-1)^\varepsilon e\left(\sum_{i=1}^n \varepsilon_i \mu_i L_{\sigma(i)}\right); \tag{24.13}$$

here the sign $(-1)^\varepsilon$ is the product of the ε_i. Now with $x_i = e(L_i)$, this can be written

$$A_\mu = \sum_\sigma (-1)^\sigma \prod_{i=1}^n (x_{\sigma(i)}^{\mu_i} - x_{\sigma(i)}^{-\mu_i})$$

or

$$A_\mu = |x_j^{\mu_i} - x_j^{-\mu_i}|, \tag{24.14}$$

where $|a_{i,j}|$ denotes the determinant of the $n \times n$ matrix $(a_{i,j})$. In particular,

$$A_\rho = |x_j^{n-i+1} - x_j^{-(n-i+1)}|. \tag{24.15}$$

From (24.14) or Exercise A.52 we have

$$A_\rho = \Delta(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1}) \cdot (x_1 - x_1^{-1}) \cdot \ldots \cdot (x_n - x_n^{-1}), \tag{24.16}$$

where Δ is the discriminant.

Exercise 24.17. Show that

$$A_\rho = \prod_{i<j} (x_i - x_j)(x_i x_j - 1) \cdot \prod_i (x_i^2 - 1)/(x_1 \cdot \ldots \cdot x_n)^n.$$

The character of the irreducible representation Γ_λ with highest weight $\lambda = \sum \lambda_i L_i$, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, is therefore:

$$\mathrm{Char}(\Gamma_\lambda) = \frac{|x_j^{\lambda_i+n-i+1} - x_j^{-(\lambda_i+n-i+1)}|}{|x_j^{n-i+1} - x_j^{-(n-i+1)}|}. \tag{24.18}$$

The dimension of Γ_λ is easily worked out from Corollary 24.6:

$$\dim(\Gamma_\lambda) = \prod_{i<j} \frac{(l_i - l_j)}{(j-i)} \cdot \prod_{i\le j} \frac{(l_i + l_j)}{(2n+2-i-j)}$$

$$= \prod_{i<j} \frac{(l_i^2 - l_j^2)}{(m_i^2 - m_j^2)} \cdot \prod_i \frac{l_i}{m_i}, \tag{24.19}$$

where $l_i = \lambda_i + n - i + 1$ and $m_i = n - i + 1$.

Exercise 24.20. Show that, setting $l_i' = \lambda_i + n - i$,

$$\dim(\Gamma_\lambda) = \frac{\prod_{i<j} (l_i' - l_j')(l_i' + l_j' + 2) \cdot \prod_i (l_i' + 1)}{(2n-1)! \cdot (2n-3)! \cdot \ldots \cdot 1!}.$$

These formulas give the dimension of the irreducible representation $\Gamma_{a_1,\ldots,a_n}$ with highest weight $a_1\omega_1 + \cdots + a_n\omega_n$, where the ω_i are the fundamental weights, using the relation $\lambda_i = a_i + \cdots + a_n$.

Exercise 24.21. Use Exercise 24.20 to verify that for $\lambda = L_1 + \cdots + L_k$, the dimension of Γ_λ is $2n$ if $k = 1$, and $\binom{2n}{k} - \binom{2n}{k-2}$ if $k \ge 2$. Use this to give another proof that the kernel of the contraction from $\bigwedge^k V$ to $\bigwedge^{k-2} V$ is irreducible.

The first determinantal formula for the symplectic group goes as follows. Let

$$J_d(x_1, \ldots, x_n) = H_d(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}),$$

where H_d is the dth complete symmetric polynomial in $2n$ variables. In other words, J_d is the character of the representation $\mathrm{Sym}^d(\mathbb{C}^{2n})$ of $\mathfrak{sp}_{2n}\mathbb{C}$. From Proposition A.50 of Appendix A we have

Proposition 24.22. *If $\lambda = (\lambda_1 \ge \cdots \ge \lambda_r > 0)$, the character of Γ_λ is the determinant of the $r \times r$ matrix whose ith row is*

$$(J_{\lambda_i-i+1} \quad J_{\lambda_i-i+2} + J_{\lambda_i-i} \quad J_{\lambda_i-i+3} + J_{\lambda_i-i-1} \quad \cdots \quad J_{\lambda_i-i+r} + J_{\lambda_i-i-r+2}).$$

For example, for $\lambda = (d)$, i.e., $\lambda = dL_1$, we have $\mathrm{Char}(\Gamma_{(d)}) = J_d$, which is the character of $\mathrm{Sym}^d(\mathbb{C}^{2n})$. In particular, this verifies that the kth *symmetric powers* $\mathrm{Sym}^k(\mathbb{C}^{2n})$ *of the standard representation are all irreducible.* (This, of course, is a special case of the general description given in §17.3, since all the contraction maps vanish on the symmetric powers.)

Exercise 24.23. (i) Find the character of the representation of $\mathfrak{sp}_4\mathbb{C}$ with highest weight $\omega_1 + \omega_2 = 2L_1 + L_2$, verifying that the multiplicities are as we found in §16.2. (ii) Find the character of the representation of $\mathfrak{sp}_6\mathbb{C}$ with highest weight $\omega_1 + \omega_2$, thus verifying the assertion of Exercise 17.4.

The second Giambelli formula in the symplectic case expresses Γ_λ in terms of the basic representations

$$\Gamma_{\omega_k} = \mathrm{Ker}(\Lambda^k(\mathbb{C}^{2n}) \to \Lambda^{k-2}(\mathbb{C}^{2n}))$$

which are the kernels of the contractions. The character of Γ_{ω_k} is E'_k, where $E'_0 = 1$, $E'_1 = E_1 = x_1 + \cdots + x_n + x_1^{-1} + \cdots + x_n^{-1}$, and

$$E'_k = E_k(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}) - E_{k-2}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1})$$

for $k \geq 2$, where E_k is the kth elementary symmetric polynomial. The formula is

Corollary 24.24. *Let $\mu = (\mu_1, \ldots, \mu_l)$ be the conjugate partition to λ. The character of Γ_λ is equal to the determinant of the $l \times l$ matrix whose* ith *row is*

$$(E'_{\mu_i-i+1} \quad E'_{\mu_i-i+2} + E'_{\mu_i-i} \quad E'_{\mu_i-i+3} + E'_{\mu_i-i-1} \quad \cdots \quad E'_{\mu_i-i+l} + E'_{\mu_i-i-l+2}).$$

PROOF. This follows from the proposition and Proposition A.44, which equates the two determinants before specializing the variables. □

There is also a simple formula for the character in terms of the characters E_k of $\Lambda^k(\mathbb{C}^{2n})$, which also follows from Proposition A.44:

$$\mathrm{Char}(\Gamma_\lambda) = |E_{\mu_i-i+j} - E_{\mu_i-i-j}|. \tag{24.25}$$

Note that $E_{n+k} = E_{n-k}$ (corresponding to the isomorphism $\Lambda^{n+k}\mathbb{C}^{2n} \cong \Lambda^{n-k}\mathbb{C}^{2n}$) and $E'_{n+k} = -E'_{n-k+2}$. In particular, Corollary 24.24 expresses $\mathrm{Char}(\Gamma_\lambda)$ as a polynomial in the characters of the basic representations $\Gamma_{\omega_1}, \ldots, \Gamma_{\omega_n}$.

The Odd Orthogonal Case

For $\mathfrak{so}_{2n+1}\mathbb{C}$ the weights are $\sum \mu_i L_i$, $\mu = (\mu_1, \ldots, \mu_n)$, with all μ_i integers or all half-integers. The positive roots are $\{L_i - L_j\}_{i<j}$, $\{L_i + L_j\}_{i<j}$, and $\{L_i\}$, so ρ is $\frac{1}{2}(L_1 + \cdots + L_n)$ less than in the case for $\mathfrak{sp}_{2n}$:

$$\rho = \sum (n + \tfrac{1}{2} - i)L_i, \tag{24.26}$$

or

$$\rho = (n - \tfrac{1}{2}, n - \tfrac{3}{2}, \ldots, \tfrac{1}{2}).$$

With $x_i^{\pm 1} = e(\pm L_i)$ and $x_i^{\pm 1/2} = e(\pm L_i/2)$, we have the same formula as before [(24.14)] for A_μ.

Exercise 24.27*. Show that

$$A_\rho = |x_j^{n-i+1/2} - x_j^{-(n-i+1/2)}|$$
$$= \Delta(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1})\cdot(x_1^{1/2} - x_1^{-1/2})\cdot\ldots\cdot(x_n^{1/2} - x_n^{-1/2}).$$

If Γ_λ is the irreducible representation with highest weight $\lambda = \sum \lambda_i L_i$, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, then the character formula can be written

$$\mathrm{Char}(\Gamma_\lambda) = \frac{|x_j^{\lambda_i+n-i+1/2} - x_j^{-(\lambda_i+n-i+1/2)}|}{|x_j^{n-i+1/2} - x_j^{-(n-i+1/2)}|}. \tag{24.28}$$

Similarly,

$$\dim(\Gamma_\lambda) = \prod_{i<j} \frac{(l_i - l_j)}{(j - i)} \cdot \prod_{i \leq j} \frac{(l_i + l_j)}{(2n + 1 - i - j)}$$
$$= \prod_{i<j} \frac{(l_i^2 - l_j^2)}{(m_i^2 - m_j^2)} \cdot \prod_i \frac{l_i}{m_i}, \tag{24.29}$$

where $l_i = \lambda_i + n - i + \frac{1}{2}$, and $m_i = n - i + \frac{1}{2}$.

Exercise 24.30. Show that, with $l_i' = \lambda_i + n - i$,

$$\dim(\Gamma_\lambda) = \frac{\prod_{i<j} (l_i' - l_j')(l_i' + l_j' + 1)\cdot \prod_i (2l_i' + 1)}{(2n - 1)!\cdot(2n - 3)!\cdot\ldots\cdot 1!}.$$

These formulas give the dimension of the irreducible representation $\Gamma_{a_1,\ldots,a_n}$ with highest weight $a_1\omega_1 + \cdots + a_n\omega_n$, where the ω_i are the fundamental weights, using the equations

$$\lambda_i = a_i + \cdots + a_{n-1} + \tfrac{1}{2}a_n.$$

Exercise 24.31. Use the dimension formula to verify that for $\lambda = L_1 + \cdots + L_k$, the dimension of Γ_λ is $\binom{2n+1}{k}$. Use this to give another proof that $\wedge^k V$ is irreducible for $1 \leq k \leq n$. Verify that the dimension of the spin representation is 2^n, thus reproving that it is irreducible.

Exercise 24.32. Use the dimension formula to verify that the kernel of the contraction

$$\mathrm{Sym}^d(\mathbb{C}^{2n+1}) \to \mathrm{Sym}^{d-2}(\mathbb{C}^{2n+1})$$

is an irreducible representation with highest weight dL_1.

In case the representation is a representation of $\mathrm{SO}_{2n+1}\mathbb{C}$, i.e., the λ_i are all integral, there is a first determinantal formula that expresses Γ_λ in terms of the kernels of the contractions

$$\mathrm{Ker}(\mathrm{Sym}^d(\mathbb{C}^{2n+1}) \to \mathrm{Sym}^{d-2}(\mathbb{C}^{2n+1})).$$

Let K_d denote the character of this kernel, so $K_0 = 1$, $K_1 = x_1 + \cdots + x_n + x_1^{-1} + \cdots + x_n^{-1} + 1$, and

$$K_d = H_d(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-n}, 1) - H_{d-2}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-n}, 1),$$

where H_d is the dth complete symmetric polynomial. From Proposition A.60 we have

Proposition 24.33. *If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$, with the λ_i integral, then the character of Γ_λ is the determinant of the $r \times r$ matrix whose ith row is*

$$(K_{\lambda_i-i+1} \quad K_{\lambda_i-i+2} + K_{\lambda_i-i} \quad K_{\lambda_i-i+3} + K_{\lambda_i-i-1} \quad \cdots \quad K_{\lambda_i-i+r} + K_{\lambda_i-i-r+2}).$$

In particular, for $\lambda = (d)$, the character is K_d, which verifies that the kernel of $\mathrm{Sym}^d(\mathbb{C}^{2n+1}) \to \mathrm{Sym}^{d-2}(\mathbb{C}^{2n+1})$ is irreducible.

Exercise 24.34. Use the character formula to verify that the multiplicities of the representation $\Gamma_{2L_1+L_2}$ of $\mathfrak{so}_5\mathbb{C}$ are as specified in Exercise 18.9.

The second determinantal formula for $SO_{2n+1}\mathbb{C}$ writes Γ_λ in terms of the representations $\bigwedge^k(\mathbb{C}^{2n+1})$, whose characters are

$$E_k = E_k(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-n}, 1).$$

Applying Proposition 24.33 with Corollary A.46, we have

Corollary 24.35. *Let $\mu = (\mu_1, \ldots, \mu_l)$ be the conjugate partition to λ. The character of Γ_λ is equal to the determinant of the $l \times l$ matrix whose ith row is*

$$(E_{\mu_i-i+1} \quad E_{\mu_i-i+2} + E_{\mu_i-i} \quad \cdots \quad E_{\mu_i-i+l} + E_{\mu_i-i-l+2}).$$

Since $E_{n+k} = E_{n+1-k}$ (corresponding to the isomorphism $\bigwedge^{n+k}\mathbb{C}^{2n+1} \cong \bigwedge^{n+1-k}\mathbb{C}^{2n+1}$), this expresses $\mathrm{Char}(\Gamma_\lambda)$ as a polynomial in $E_1, \ldots, E_n$, with $E_d = \mathrm{Char}(\bigwedge^d\mathbb{C}^{2n+1})$.

The Even Orthogonal Case

For $\mathfrak{so}_{2n}\mathbb{C}$ the weights are the same as in the preceding case. This time the $\{L_i\}$ are not positive roots, however, so ρ is $\frac{1}{2}(L_1 + \cdots + L_n)$ less than in the case of $\mathfrak{so}_{2n+1}\mathbb{C}$, or $L_1 + \cdots + L_n$ less than in the case of $\mathfrak{sp}_{2n}\mathbb{C}$:

$$\rho = \sum (n - i)L_i, \tag{24.36}$$

or

$$\rho = (n - 1, n - 2, \ldots, 0).$$

The calculation of A_μ is similar, but using only those ε of positive sign. This time

$$\sum_{\varepsilon}(-1)^{\varepsilon}e\left(\sum_{i=1}^{n}\varepsilon_i\mu_i L_{\sigma(i)}\right)=\frac{1}{2}\left[\prod_{i=1}^{n}(x_{\sigma(i)}^{\mu_i}+x_{\sigma(i)}^{-\mu_i})+\prod_{i=1}^{n}(x_{\sigma(i)}^{\mu_i}-x_{\sigma(i)}^{-\mu_i})\right].$$

This leads to

$$A_\mu=\tfrac{1}{2}(|x_j^{\mu_i}+x_j^{-\mu_i}|+|x_j^{\mu_i}-x_j^{-\mu_i}|). \tag{24.37}$$

Note that the second determinant term vanishes when any μ_i is zero. In particular,

$$A_\rho=\tfrac{1}{2}|x_j^{n-i}+x_j^{-(n-i)}|. \tag{24.38}$$

From (24.14) or Exercise A.66,

$$A_\rho=\Delta(x_1+x_1^{-1},\dots,x_n+x_n^{-1}). \tag{24.39}$$

This gives, with Γ_λ the irreducible representation with highest weight $\lambda=\sum\lambda_i L_i$, $\lambda_1\geq\cdots\geq|\lambda_n|\geq 0$,

$$\operatorname{Char}(\Gamma_\lambda)=\frac{|x_j^{l_i}+x_j^{-l_i}|+|x_j^{l_i}-x_j^{-l_i}|}{|x_j^{n-i}+x_j^{-(n-i)}|}, \tag{24.40}$$

where $l_i=\lambda_i+n-i$. As before,

$$\begin{aligned}\dim(\Gamma_\lambda)&=\prod_{i<j}\frac{(l_i-l_j)}{(j-i)}\cdot\frac{(l_i+l_j)}{(2n-i-j)}\\&=\prod_{i<j}\frac{(l_i^2-l_j^2)}{(m_i^2-m_j^2)},\end{aligned} \tag{24.41}$$

where $l_i=\lambda_i+n-i$ and $m_i=n-i$. Note that, as expected, the two representations with weights $(\lambda_1,\dots,\lambda_{n-1},\pm\lambda_n)$ have the same dimensions.

Exercise 24.42. Show that

$$\dim(\Gamma_\lambda)=2^{n-1}\frac{\prod_{i<j}(l_i-l_j)(l_i+l_j)}{(2n-2)!\cdot(2n-4)!\cdot\ldots\cdot 2!}.$$

These formulas give the dimension of the irreducible representation $\Gamma_{a_1,\dots,a_n}$ with highest weight $a_1\omega_1+\cdots+a_n\omega_n$, where the ω_i are the fundamental weights, using the equations

$$\lambda_i=a_i+\cdots+a_{n-2}+\tfrac{1}{2}(a_{n-1}+a_n),\quad 1\leq i\leq n-2,$$

$$\lambda_{n-1}=\tfrac{1}{2}(a_{n-1}+a_n),\qquad \lambda_n=\tfrac{1}{2}(-a_{n-1}+a_n).$$

Exercise 24.43. Use the dimension formula to verify that for $\omega=L_1+\cdots+L_k$, $k<n$, the dimension of Γ_ω is $\binom{2n}{k}$, so $\wedge^k(\mathbb{C}^{2n})$ is irreducible. For $\lambda=L_1+\cdots+L_{n-1}\pm L_n$, the dimension is $\frac{1}{2}\binom{2n}{k}$, so $\wedge^n(\mathbb{C}^{2n})$ is the sum of the two corresponding irreducible representations. Verify that the dimension of the two spin representations are 2^{n-1}, proving irreducibility again.

Note that the second term in the numerator in (24.40) changes sign when λ_n is replaced by $-\lambda_n$; in particular, it vanishes when $\lambda_n = 0$. When $\lambda_n = 0$, the representation Γ_λ is a representation of the orthogonal group $O_{2n}\mathbb{C}$. When $\lambda_n \neq 0$, the direct sum of the two representations with highest weights $(\lambda_1, \dots, \pm\lambda_n)$ is an irreducible representation of $O_{2n}\mathbb{C}$. (See Exercises 23.19 and 23.37.)

Let L_d be the character of $\operatorname{Ker}(\operatorname{Sym}^d(\mathbb{C}^{2n}) \to \operatorname{Sym}^{d-2}(\mathbb{C}^{2n}))$, i.e., $L_d = H_d(x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-n}) - H_{d-2}(x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-n})$. In either case, Proposition A.64 applies to give the first determinantal formula:

Proposition 24.44. *Given integers $\lambda_1 \geq \cdots \geq \lambda_r > 0$, the character of the irreducible representation of $O_{2n}\mathbb{C}$ with highest weight $\lambda = (\lambda_1, \dots, \lambda_r)$ is the determinant of the $r \times r$ matrix whose ith row is*

$$(L_{\lambda_i-i+1} \quad L_{\lambda_i-i+2} + L_{\lambda_i-i} \quad \cdots \quad L_{\lambda_i-i+r} + L_{\lambda_i-i-r+2}).$$

Again, for $\lambda = (d)$, this verifies that the kernel of the contraction from $\operatorname{Sym}^d(\mathbb{C}^{2n})$ to $\operatorname{Sym}^{d-2}(\mathbb{C}^{2n})$ is irreducible.

The second determinantal formula is the same as in the odd case, but with $E_k = E_k(x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-n})$:

Corollary 24.45. *Let $\mu = (\mu_1, \dots, \mu_l)$ be the conjugate partition to λ. The character of Γ_λ is equal to the determinant of the $l \times l$ matrix whose ith row is*

$$(E_{\mu_i-i+1} \quad E_{\mu_i-i+2} + E_{\mu_i-i} \quad \cdots \quad E_{\mu_i-i+l} + E_{\mu_i-i-l+2}).$$

Using the fact that $E_{n+k} = E_{n-k}$, this expresses $\operatorname{Char}(\Gamma_\lambda)$ as a polynomial in $E_1, \dots, E_n$, with $E_d = \operatorname{Char}(\wedge^d\mathbb{C}^{2n})$.

Exercise 24.46*. For each of the orthogonal groups $O_m\mathbb{C}$, show that the character of the irreducible representation with highest weight λ can be written in the form

$$\operatorname{Char}(\Gamma_\lambda) = |h_{\lambda_i-i+j} - h_{\lambda_i-i-j}|,$$

where h_k is the character of $\operatorname{Sym}^k(\mathbb{C}^m)$. Another formula for the dimension of Γ_λ is obtained by substituting $\binom{m}{k}$ for h_k in this determinant.

There are other formulas expressing the characters of general representations in terms of simpler ones. Abramsky, Jahn, and King [A-J-K] give one that can be expressed by the *same* formula for the general linear, symplectic, and orthogonal groups. The general irreducible representations are given by partitions λ or Young diagrams, and in their formula the simpler representations are those corresponding to hooks. To express it, let $(a * b)$ denote the hook with horizontal leg of length $a + 1$ and vertical leg of length $b + 1$, i.e., the partition $(a + 1, 1, \dots, 1)$, with b 1's. More generally, given $\mathbf{a} = (a_1 > \cdots > a_r \geq 0)$ and $\mathbf{b} = (b_1 > \cdots > b_r \geq 0)$ with a_r or b_r nonzero, let $(\mathbf{a} * \mathbf{b})$ denote the partition whose Young diagram has legs of these lengths to the right of and below the r diagonal boxes (cf. Frobenius's notation, Exercise 4.17). Let $\chi_{(\mathbf{a}*\mathbf{b})}$ denote the character of the corresponding irreducible representation. Their formula is

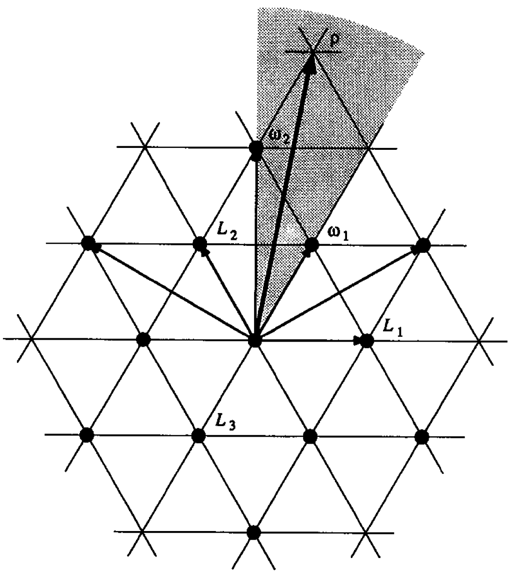
$$\chi_{(\mathbf{a}*\mathbf{b})} = |\chi_{(a_i*b_j)}|_{1\le i,j\le r}. \tag{24.47}$$

Taking the degree of both sides gives new formulas for the dimensions of the irreducible representations. These formulas are particularly useful if the rank r of the partition is small.

Exceptional Cases

We will, as a last example, work out the Weyl character formula for the exceptional Lie algebra $\mathfrak{g}_2$, and thereby verify some of the analysis of its representations given in Lecture 22. The remaining four exceptional Lie algebras we will leave as exercises.

To begin with, the value of ρ is easily seen to be $2L_1 + 3L_2$, in terms of the basis L_1, L_2 for the weight lattice introduced in Lecture 22.



Now, for any weight $\mu = pL_1 + qL_2 + rL_3$, we have

$$A_\mu = \sum_{\sigma\in\mathfrak{S}_3} x_{\sigma(1)}^p\cdot x_{\sigma(2)}^q\cdot x_{\sigma(3)}^r - \sum_{\sigma\in\mathfrak{S}_3} x_{\sigma(1)}^{-p}\cdot x_{\sigma(2)}^{-q}\cdot x_{\sigma(3)}^{-r}$$
$$= \Delta(x)\cdot S_{p,q,r}(x) - \Delta(x^{-1})\cdot S_{p,q,r}(x^{-1}),$$

where we write x for (x_1, x_2, x_3) and x^{-1} for $(x_1^{-1}, x_2^{-1}, x_3^{-1})$, Δ is the discriminant, and $S_{p,q,r}$ the Schur function. Using the relation $\prod x_i = 1$ we can also write this as

$$= \Delta(x)\cdot(S_{p,q,r}(x) - S_{m-p,m-q,m-r}(x))$$

for any $m \geq \max(p, q, r)$. To make this notation agree with the standard notation for Schur polynomials from Appendix (A.4), note that $S_{p,q,r}$ is the Schur polynomial $S_{(s,t)}$ for the partition (s, t), $s \geq t$, where s is two less than the difference between the largest and smallest of p, q, and r, while t is one less than the difference between the second largest and the smallest; if p, q, and r are not distinct, $S_{p,q,r} = 0$. Thus, for example,

$$\begin{aligned} A_\rho &= \Delta(x)\cdot(S_{(1,1)}(x) - S_{(1)}(x)) \\ &= \Delta(x)\cdot(x_1x_2 + x_1x_3 + x_2x_3 - x_1 - x_2 - x_3). \end{aligned}$$

Now, any irreducible representation Γ_λ of $\mathfrak{g}_2$ has highest weight $\lambda = a\omega_1 + b\omega_2$, where $\omega_1 = L_1 + L_2$ and $\omega_2 = L_1 + 2L_2$ are the two fundamental weights, and a and b are non-negative integers. Then $\lambda + \rho = (a + b + 2)L_1 + (a + 2b + 3)L_2$. The Weyl character formula in this case becomes

Proposition 24.48. *The character of the representation of* $\mathfrak{g}_2$ *with highest weight* $a\omega_1 + b\omega_2$ *is*

$$\operatorname{Char}(\Gamma_{a,b}) = \frac{S_{(a+2b+1,a+b+1)} - S_{(a+2b+1,b)}}{S_{(1,1)} - S_{(1)}}.$$

Exercise 24.49. In the case of the standard representation $\Gamma_{1,0}$, the adjoint representation $\Gamma_{0,1}$, and the representation $\Gamma_{2,0}$, use this formula to verify the multiplicities found in Lecture 22.

We can also work out the dimension formula explicitly in this case. The two fundamental weights ω_1 and ω_2 have inner products

$$(\omega_1, \omega_1) = 1, \qquad (\omega_1, \omega_2) = 3/2, \qquad \text{and} \qquad (\omega_2, \omega_2) = 3;$$

ω_1 and ω_2 are among the positive roots of $\mathfrak{g}_2$, and in terms of these the remaining positive roots are $2\omega_1 - \omega_2$, $3\omega_1 - \omega_2$, $\omega_2 - \omega_1$, and $2\omega_2 - 3\omega_1$. The weight ρ is the sum of the fundamental weights ω_1 and ω_2, so that for an arbitrary weight $\lambda = a\omega_1 + b\omega_2$ we have the following table of inner products:

	$(\cdot, \rho)$	$(\cdot, \lambda)$	$(\cdot, \lambda + \rho)$
$2\omega_1 - \omega_2$	$1/2$	$a/2$	$(a + 1)/2$
$3\omega_1 - \omega_2$	3	$3a/2 + 3b/2$	$3(a + b + 2)/2$
ω_1	$5/2$	$a + 3b/2$	$(2a + 3b + 5)/2$
ω_2	$9/2$	$3a/2 + 3b$	$3(a + 2b + 3)/2$
$-\omega_1 + \omega_2$	2	$a/2 + 3b/2$	$(a + 3b + 4)/2$
$-3\omega_1 + 2\omega_2$	$3/2$	$3b/2$	$3(b + 1)/2$

We conclude that *the dimension of the irreducible representation* $\Gamma_{a,b}$ *of* $\mathfrak{g}_2$ *with highest weight* $\lambda = a\omega_1 + b\omega_2$ *is*

$$\dim(\Gamma_{a,b}) = \frac{(a+1)(a+b+2)(2a+3b+5)(a+2b+3)(a+3b+4)(b+1)}{120}.$$

We can check this in the cases $a = 1$, $b = 0$ and $a = 0$, $b = 1$, getting the dimensions 7 and 14 of the standard and adjoint representations, respectively. In case $a = 2$, $b = 0$ we may verify the result of the explicit calculation in Lecture 22, finding that

$$\dim(\Gamma_{2,0}) = 27$$

and, therefore, deducing that $\wedge^3 V = \Gamma_{2,0} \oplus V \oplus \mathbb{C}$ and $\mathrm{Sym}^2 V = \Gamma_{2,0} \oplus \mathbb{C}$.

Exercise 24.50. Show that $\mathrm{Sym}^a V = \bigoplus_{k=0}^{[a/2]} \Gamma_{a-2k,0}$.

We leave the analogous computations for the remaining four Lie algebras as exercises, using the description of the root systems found in Exercise 21.16. Since we have not said much about the Weyl group in the exceptional cases the formula (WCF) cannot be used directly—not to mention the fact that the orders of these Weyl groups are: $2^7 \cdot 3^2 = 1152$ for $\mathfrak{f}_4$; $2^7 \cdot 3^4 \cdot 5 = 51{,}840$ for $\mathfrak{e}_6$, $2^{10} \cdot 3^4 \cdot 5 \cdot 7 = 2{,}903{,}040$ for $\mathfrak{e}_7$, and $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 = 696{,}729{,}600$ for $\mathfrak{e}_8$. However, the dimension formula is available.

Exercise 24.51*. For each of the four remaining exceptional Lie algebras, compute ρ = half the sum of the positive roots. For each of the fundamental weights ω, at least for $\mathfrak{f}_4$, compute the dimension of the irreducible representation with highest weight ω. In particular, find the nontrivial representation of minimal dimension. Use this to verify that (E_6) is not isomorphic to (B_6) or (C_6), i.e., that $\mathfrak{e}_6$ is not isomorphic to $\mathfrak{so}_{13}\mathbb{C}$ or $\mathfrak{sp}_{12}\mathbb{C}$.

Exercise 24.52*. List all irreducible representations V of simple Lie algebras $\mathfrak{g}$ such that $\dim V \le \dim \mathfrak{g}$. Note that these include all cases where the corresponding group representation has a Zariski dense orbit, or a finite number of orbits.