LECTURE 6

Weyl's Construction

In this lecture we introduce and study an important collection of functors generalizing the symmetric powers and exterior powers. These are defined simply in terms of the Young symmetrizers c_λ introduced in §4: given a representation V of an arbitrary group G, we consider the dth tensor power of V, on which both G and the symmetric group on d letters act. We then take the image of the action of c_λ on $V^{\otimes d}$; this is again a representation of G, denoted $\mathbb{S}_\lambda(V)$. This gives us a way of generating new representations, whose main application will be to Lie groups: for example, we will generate all representations of $\mathrm{SL}_n\mathbb{C}$ by applying these to the standard representation $\mathbb{C}^n$ of $\mathrm{SL}_n\mathbb{C}$. While it may be easiest to read this material while the definitions of the Young symmetrizers are still fresh in the mind, the construction will not be used again until §15, so that this lecture can be deferred until then.

§6.1. Schur Functors and Their Characters

For any finite-dimensional complex vector space V, we have the canonical decomposition

$$V \otimes V = \mathrm{Sym}^2 V \oplus \wedge^2 V.$$

The group $\mathrm{GL}(V)$ acts on $V \otimes V$, and this is, as we shall soon see, the decomposition of $V \otimes V$ into a direct sum of irreducible $\mathrm{GL}(V)$-representations. For the next tensor power,

$$V \otimes V \otimes V = \mathrm{Sym}^3 V \oplus \wedge^3 V \oplus \text{ another space.}$$

We shall see that this other space is a sum of two copies of an irreducible

GL(V)-representation. Just as $\mathrm{Sym}^d V$ and $\wedge^d V$ are images of symmetrizing operators from $V^{\otimes d} = V \otimes V \otimes \cdots \otimes V$ to itself, so are the other factors. The symmetric group $\mathfrak{S}_d$ acts on $V^{\otimes d}$, say on the right, by permuting the factors

$$(v_1 \otimes \cdots \otimes v_d) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}.$$

This action commutes with the left action of GL(V). For any partition λ of d we have from the last lecture a Young symmetrizer c_λ in $\mathbb{C}\mathfrak{S}_d$. We denote the image of c_λ on $V^{\otimes d}$ by $\mathbb{S}_\lambda V$:

$$\mathbb{S}_\lambda V = \mathrm{Im}(c_\lambda|_{V^{\otimes d}})$$

which is again a representation of GL(V). We call the functor[1] $V \rightsquigarrow \mathbb{S}_\lambda V$ the *Schur functor* or *Weyl module*, or simply *Weyl's construction*, corresponding to λ. It was Schur who made the correspondence between representations of symmetric groups and representations of general linear groups, and Weyl who made the construction we give here.[2] We will give other descriptions later, cf. Exercise 6.14 and §15.5.

For example, the partition $d = d$ corresponds to the functor $V \rightsquigarrow \mathrm{Sym}^d V$, and the partition $d = 1 + \cdots + 1$ to the functor $V \rightsquigarrow \wedge^d V$.

We find something new for the partition $3 = 2 + 1$. The corresponding symmetrizer c_λ is

$$c_{(2,1)} = 1 + e_{(12)} - e_{(13)} - e_{(132)},$$

so the image of c_λ is the subspace of $V^{\otimes 3}$ spanned by all vectors

$$v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_2 \otimes v_1 - v_3 \otimes v_1 \otimes v_2.$$

If $\wedge^2 V \otimes V$ is embedded in $V^{\otimes 3}$ by mapping

$$(v_1 \wedge v_3) \otimes v_2 \mapsto v_1 \otimes v_2 \otimes v_3 - v_3 \otimes v_2 \otimes v_1,$$

then the image of c_λ is the subspace of $\wedge^2 V \otimes V$ spanned by all vectors

$$(v_1 \wedge v_3) \otimes v_2 + (v_2 \wedge v_3) \otimes v_1.$$

It is not hard to verify that these vectors span the kernel of the canonical map from $\wedge^2 V \otimes V$ to $\wedge^3 V$, so we have

$$\mathbb{S}_{(2,1)} V = \mathrm{Ker}(\wedge^2 V \otimes V \to \wedge^3 V).$$

(This gives the missing factor in the decomposition of $V^{\otimes 3}$.)

Note that some of the $\mathbb{S}_\lambda V$ can be zero if V has small dimension. We will see that this is the case precisely when the number of rows in the Young diagram of λ is greater than the dimension of V.

[1] The functoriality means simply that a linear map $\varphi: V \to W$ of vector spaces determines a linear map $\mathbb{S}_\lambda(\varphi): \mathbb{S}_\lambda V \to \mathbb{S}_\lambda W$, with $\mathbb{S}_\lambda(\varphi \circ \psi) = \mathbb{S}_\lambda(\varphi) \circ \mathbb{S}_\lambda(\psi)$ and $\mathbb{S}_\lambda(\mathrm{Id}_V) = \mathrm{Id}_{\mathbb{S}_\lambda V}$

[2] The notion goes by a variety of names and notations in the literature, depending on the context. Constructions differ markedly when not over a field of characteristic zero; and many authors now parametrize them by the conjugate partitions. Our choice of notation is guided by the correspondence between these functors and Schur polynomials, which we will see are their characters.

When $G = \mathrm{GL}(V)$, and for important subgroups $G \subset \mathrm{GL}(V)$, these $\mathbb{S}_\lambda V$ give many of the irreducible representations of G; we will come back to this later in the book. For now we can use our knowledge of symmetric group representations to prove a few facts about them—in particular, we show that they decompose the tensor powers $V^{\otimes d}$, and that they are irreducible representations of $\mathrm{GL}(V)$. We will also compute their characters; this will eventually be seen to be a special case of the Weyl character formula.

Any endomorphism g of V gives rise to an endomorphism of $\mathbb{S}_\lambda V$. In order to tell what representations we get, we will need to compute the trace of this endomorphism on $\mathbb{S}_\lambda V$; we denote this trace by $\chi_{\mathbb{S}_\lambda V}(g)$. For the computation, let $x_1, \ldots, x_k$ be the eigenvalues of g on V, $k = \dim V$. Two cases are easy. For $\lambda = (d)$,

$$\mathbb{S}_{(d)} V = \mathrm{Sym}^d V, \qquad \chi_{\mathbb{S}_{(d)}V}(g) = H_d(x_1, \ldots, x_k), \tag{6.1}$$

where $H_d(x_1, \ldots, x_k)$ is the complete symmetric polynomial of degree d. The definition of these symmetric polynomials is given in (A.1) of Appendix A. The truth of (6.1) is evident when g is a diagonal matrix, and its truth for the dense set of diagonalizable endomorphisms implies it for all endomorphisms; or one can see it directly by using the Jordan canonical form of g. For $\lambda = (1, \ldots, 1)$, we have similarly

$$\mathbb{S}_{(1,\ldots,1)} V = \Lambda^d V, \qquad \chi_{\mathbb{S}_{(1,\ldots,1)}V}(g) = E_d(x_1, \ldots, x_k), \tag{6.2}$$

with $E_d(x_1, \ldots, x_k)$ the elementary symmetric polynomial [see (A.3)]. The polynomials H_d and E_d are special cases of the *Schur polynomials*, which we denote by $S_\lambda = S_\lambda(x_1, \ldots, x_k)$. As λ varies over the partitions of d into at most k parts, these polynomials S_λ form a basis for the symmetric polynomials of degree d in these k variables. Schur polynomials are defined and discussed in Appendix A, especially (A.4)–(A.6). The above two formulas can be written

$$\chi_{\mathbb{S}_\lambda V}(g) = S_\lambda(x_1, \ldots, x_k) \quad \text{for } \lambda = (d) \text{ and } \lambda = (1, \ldots, 1).$$

We will show that this equation is valid for all λ:

Theorem 6.3. (1) *Let $k = \dim V$. Then $\mathbb{S}_\lambda V$ is zero if $\lambda_{k+1} \neq 0$. If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$, then*

$$\dim \mathbb{S}_\lambda V = S_\lambda(1, \ldots, 1) = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

(2) *Let m_λ be the dimension of the irreducible representation V_λ of $\mathfrak{S}_d$ corresponding to λ. Then*

$$V^{\otimes d} \cong \bigoplus_\lambda \mathbb{S}_\lambda V^{\oplus m_\lambda}.$$

(3) *For any $g \in \mathrm{GL}(V)$, the trace of g on $\mathbb{S}_\lambda V$ is the value of the Schur polynomial on the eigenvalues $x_1, \ldots, x_k$ of g on V:*

$$\chi_{\mathbb{S}_\lambda V}(g) = S_\lambda(x_1, \ldots, x_k).$$

(4) *Each* $\mathbb{S}_\lambda V$ *is an irreducible representation of* $\mathrm{GL}(V)$.

This theorem will be proved in the next section. Other formulas for the dimension of $\mathbb{S}_\lambda V$ are given in Exercises A.30 and A.31. The following is another:

Exercise 6.4*. Show that

$$\dim \mathbb{S}_\lambda V = \frac{m_\lambda}{d!} \prod (k - i + j) = \prod \frac{(k - i + j)}{h_{ij}},$$

where the products are over the d pairs (i, j) that number the row and column of boxes for λ, and h_{ij} is the hook number of the corresponding box.

Exercise 6.5. Show that $V^{\otimes 3} \cong \mathrm{Sym}^3 V \oplus \Lambda^3 V \oplus (\mathbb{S}_{(2,1)} V)^{\oplus 2}$, and

$$V^{\otimes 4} \cong \mathrm{Sym}^4 V \oplus \Lambda^4 V \oplus (\mathbb{S}_{(3,1)} V)^{\oplus 3} \oplus (\mathbb{S}_{(2,2)} V)^{\oplus 2} \oplus (\mathbb{S}_{(2,1,1)} V)^{\oplus 3}.$$

Compute the dimensions of each of the irreducible factors.

The proof of the theorem actually gives the following corollary:

Corollary 6.6. *If* $c \in \mathbb{C}\mathfrak{S}_d$, *and* $(\mathbb{C}\mathfrak{S}_d)\cdot c = \bigoplus_\lambda V_\lambda^{\oplus r_\lambda}$ *as representations of* $\mathfrak{S}_d$, *then there is a corresponding decomposition of* $\mathrm{GL}(V)$*-spaces*:

$$V^{\otimes d} \cdot c = \bigoplus_\lambda \mathbb{S}_\lambda V^{\oplus r_\lambda}.$$

If $x_1, \ldots, x_k$ *are the eigenvalues of an endomorphism of* V, *the trace of the induced endomorphism of* $V^{\otimes d} \cdot c$ *is* $\sum r_\lambda S_\lambda(x_1, \ldots, x_k)$.

If λ and μ are different partitions, each with at most $k = \dim V$ parts, the irreducible $\mathrm{GL}(V)$-spaces $\mathbb{S}_\lambda V$ and $\mathbb{S}_\mu V$ are not isomorphic. Indeed, their characters are the Schur polynomials S_λ and S_μ, which are different. More generally, at least for those representations of $\mathrm{GL}(V)$ which can be decomposed into a direct sum of copies of the represenations $\mathbb{S}_\lambda V$'s, *the representations are completely determined by their characters.* This follows immediately from the fact that the Schur polynomials are linearly independent.

Note, however, that we cannot hope to get *all* finite-dimensional irreducible representations of $\mathrm{GL}(V)$ this way, since the duals of these representations are not included. We will see in Lecture 15 that this is essentially the only omission. Note also that although the operation that takes representations of $\mathfrak{S}_d$ to representations of $\mathrm{GL}(V)$ preserves direct sums, the situation with respect to other linear algebra constructions such as tensor products is more complicated.

One important application of Corollary 6.6 is to the decomposition of a tensor product $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V$ of two Weyl modules, with, say, λ a partition of

d and μ a partition of m. The result is

$$\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V \cong \bigoplus_\nu N_{\lambda\mu\nu} \mathbb{S}_\nu V; \tag{6.7}$$

here the sum is over partitions ν of $d + m$, and $N_{\lambda\mu\nu}$ are numbers determined by the *Littlewood–Richardson rule*. This is a rule that gives $N_{\lambda\mu\nu}$ as the number of ways to expand the Young diagram of λ, using μ in an appropriate way, to achieve the Young diagram for ν; see (A.8) for the precise formula. Two important special cases are easier to use and prove since they involve only the simpler Pieri formula (A.7). For $\mu = (m)$, we have

$$\mathbb{S}_\lambda V \otimes \mathrm{Sym}^m V \cong \bigoplus_\nu \mathbb{S}_\nu V, \tag{6.8}$$

the sum over all ν whose Young diagram is obtained by adding m boxes to the Young diagram of λ, with no two in the same column. Similarly for $\mu = (1, \ldots, 1)$,

$$\mathbb{S}_\lambda V \otimes \Lambda^m V = \bigoplus \mathbb{S}_\pi V, \tag{6.9}$$

the sum over all partitions π whose Young diagram is obtained from that of λ by adding m boxes, with no two in the same row.

To prove these formulas, we need only observe that

$$\begin{aligned}\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V &= V^{\otimes n} \cdot c_\lambda \otimes V^{\otimes m} \cdot c_\mu \\ &= V^{\otimes n} \otimes V^{\otimes m} \cdot (c_\lambda \otimes c_\mu) = V^{\otimes (n+m)} \cdot c,\end{aligned}$$

with $c = c_\lambda \otimes c_\mu \in \mathbb{C}\mathfrak{S}_d \otimes \mathbb{C}\mathfrak{S}_m = \mathbb{C}(\mathfrak{S}_d \times \mathfrak{S}_m) \subset \mathbb{C}\mathfrak{S}_{d+m}$. This proves that $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V$ has a decomposition as in Corollary 6.6, and the coefficients are given by knowing the decomposition of the corresponding character. The character of a tensor product is the product of the characters of the factors; so this amounts to writing the product $S_\lambda S_\mu$ of Schur polynomials as a linear combination of Schur polynomials. This is done in Appendix A, and formulas (6.7), (6.8), and (6.9) follow from (A.8), (A.7), and Exercise A.32 (v), respectively.

For example, from $\mathrm{Sym}^d V \otimes V = \mathrm{Sym}^{d+1} V \oplus \mathbb{S}_{(d,1)} V$, it follows that

$$\mathbb{S}_{(d,1)} V = \mathrm{Ker}(\mathrm{Sym}^d V \otimes V \to \mathrm{Sym}^{d+1} V),$$

and similarly for the conjugate partition,

$$\mathbb{S}_{(2,1,\ldots,1)} V = \mathrm{Ker}(\Lambda^d V \otimes V \to \Lambda^{d+1} V).$$

Exercise 6.10*. One can also derive the preceding decompositions of tensor products directly from corresponding decompositions of representations of symmetric groups. Show that, in fact, $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V$ corresponds to the "inner product" representation $V_\lambda \circ V_\mu$ of $\mathfrak{S}_{d+m}$ described in (4.41).

Exercise 6.11*. (a) The Littlewood–Richardson rule also comes into the decomposition of a Schur functor of a direct sum of vector spaces V and W. This generalizes the well-known identities

$$\mathrm{Sym}^n(V \oplus W) = \bigoplus_{a+b=n} (\mathrm{Sym}^a V \otimes \mathrm{Sym}^b W),$$

$$\Lambda^n(V \oplus W) = \bigoplus_{a+b=n} (\Lambda^a V \otimes \Lambda^b W).$$

Prove the general decomposition over $\mathrm{GL}(V) \times \mathrm{GL}(W)$:

$$\mathbb{S}_\nu(V \oplus W) = \bigoplus N_{\lambda\mu\nu}(\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu W),$$

the sum over all partitions λ, μ such that the sum of the numbers partitioned by λ and μ is the number partitioned by ν. (To be consistent with Exercise 2.36 one should use the notation $\boxtimes$ for these "external" tensor products.)

(b) Similarly prove the formula for the Schur functor of a tensor product:

$$\mathbb{S}_\nu(V \otimes W) = \bigoplus C_{\lambda\mu\nu}(\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu W),$$

where the coefficients $C_{\lambda\mu\nu}$ are defined in Exercise 4.51. In particular show that

$$\mathrm{Sym}^d(V \otimes W) = \bigoplus \mathbb{S}_\lambda V \otimes \mathbb{S}_\lambda W,$$

the sum over all partitions λ of d with at most $\dim V$ or $\dim W$ rows. Replacing W by W^*, this gives the decomposition for the space of polynomial functions of degree d on the space $\mathrm{Hom}(V, W)$ over $\mathrm{GL}(V) \times \mathrm{GL}(W)$. For variations on this theme, see [Ho3]. Similarly,

$$\Lambda^d(V \otimes W) = \bigoplus \mathbb{S}_\lambda V \otimes \mathbb{S}_{\lambda'} W,$$

the sum over partitions λ of d with at most $\dim V$ rows and at most $\dim W$ columns.

Exercise 6.12. Regarding

$$\mathrm{GL}_n \mathbb{C} = \mathrm{GL}_n \mathbb{C} \times \{1\} \subset \mathrm{GL}_n \mathbb{C} \times \mathrm{GL}_m \mathbb{C} \subset \mathrm{GL}_{n+m} \mathbb{C},$$

the preceding exercise shows how the restriction of a representation decomposes:

$$\mathrm{Res}(\mathbb{S}_\nu(\mathbb{C}^{n+m})) = \sum (N_{\lambda\mu\nu} \dim \mathbb{S}_\mu(\mathbb{C}^m))\mathbb{S}_\lambda(\mathbb{C}^n).$$

In particular, for $m = 1$, Pieri's formula gives

$$\mathrm{Res}(\mathbb{S}_\nu(\mathbb{C}^{n+1})) = \bigoplus \mathbb{S}_\lambda(\mathbb{C}^n),$$

the sum over all λ obtained from ν by removing any number of boxes from its Young diagram, with no two in any column.

Exercise 6.13*. Show that for any partition $\mu = (\mu_1, \ldots, \mu_r)$ of d,

$$\Lambda^{\mu_1} V \otimes \Lambda^{\mu_2} V \otimes \cdots \otimes \Lambda^{\mu_r} V \cong \bigoplus_\lambda K_{\lambda\mu} \mathbb{S}_{\lambda'} V,$$

where $K_{\lambda\mu}$ is the Kostka number and λ' the conjugate of λ.

Exercise 6.14*. Let $\mu = \lambda'$ be the conjugate partition. Put the factors of the dth tensor power $V^{\otimes d}$ in one-to-one correspondence with the squares of the Young diagram of λ. Show that $\mathbb{S}_\lambda V$ is the image of this composite map:

$$\bigotimes_i (\wedge^{\mu_i} V) \to \bigotimes_i (\otimes^{\mu_i} V) \to V^{\otimes d} \to \bigotimes_j (\otimes^{\lambda_j} V) \to \bigotimes_j (\mathrm{Sym}^{\lambda_j} V),$$

the first map being the tensor product of the obvious inclusions, the second grouping the factors of $V^{\otimes d}$ according to the columns of the Young diagram, the third grouping the factors according to the rows of the Young diagram, and the fourth the obvious quotient map. Alternatively, $\mathbb{S}_\lambda V$ is the image of a composite map

$$\bigotimes_i (\mathrm{Sym}^{\lambda_i} V) \to \bigotimes_i (\otimes^{\lambda_i} V) \to V^{\otimes d} \to \bigotimes_j (\otimes^{\mu_j} V) \to \bigotimes_j (\wedge^{\mu_j} V).$$

In particular, $\mathbb{S}_\lambda V$ can be realized as a subspace of tensors in $V^{\otimes d}$ that are invariant by automorphisms that preserve the rows of a Young tableau of λ, *or* a subspace that is anti-invariant under those that preserve the columns, but not both, cf. Exercise 4.48.

Problem 6.15*. The preceding exercise can be used to describe a basis for the space $\mathbb{S}_\lambda V$. Let $v_1, \ldots, v_k$ be a basis for V. For each semistandard tableau T on λ, one can use it to write down an element v_T in $\bigotimes_i (\wedge^{\mu_i} V)$; v_T is a tensor product of wedge products of basis elements, the ith factor in $\wedge^{\mu_i} V$ being the wedge product (in order) of those basis vectors whose indices occur in the ith column of T. The fact to be proved is that the images of these elements v_T under the first composite map of the preceding exercise form a basis for $\mathbb{S}_\lambda V$.

At the end of Lecture 15, using more representation theory than we have at the moment, we will work out a simple variation of the construction of $\mathbb{S}_\lambda V$ which will give quick proofs of refinements of the preceding exercise and problem.

Exercise 6.16*. The Pieri formula gives a decomposition

$$\mathrm{Sym}^d V \otimes \mathrm{Sym}^d V = \bigoplus \mathbb{S}_{(d+a, d-a)} V,$$

the sum over $0 \le a \le d$. The left-hand side decomposes into a direct sum of $\mathrm{Sym}^2(\mathrm{Sym}^d V)$ and $\wedge^2(\mathrm{Sym}^d V)$. Show that, in fact,

$$\mathrm{Sym}^2(\mathrm{Sym}^d V) = \mathbb{S}_{(2d,0)} V \oplus \mathbb{S}_{(2d-2,2)} V \oplus \mathbb{S}_{(2d-4,4)} V \oplus \cdots,$$

$$\wedge^2(\mathrm{Sym}^d V) = \mathbb{S}_{(2d-1,1)} V \oplus \mathbb{S}_{(2d-3,3)} V \oplus \mathbb{S}_{(2d-5,5)} V \oplus \cdots.$$

Similarly using the dual form of Pieri to decompose $\wedge^d V \otimes \wedge^d V$ into the sum $\bigoplus \mathbb{S}_\lambda V$, the sum over all $\lambda = (2, \ldots, 2, 1, \ldots, 1)$ consisting of $d - a$ 2's and 2a 1's, $0 \le a \le d$, show that $\mathrm{Sym}^2(\wedge^d V)$ is the sum of those factors with a even, and $\wedge^2(\wedge^d V)$ is the sum of those with a odd.

Exercise 6.17*. If λ and μ are any partitions, we can form the composite functor $\mathbb{S}_\mu(\mathbb{S}_\lambda V)$. The original "plethysm" problem—which remains very difficult in general—is to decompose these composites:

$$\mathbb{S}_\mu(\mathbb{S}_\lambda V) = \bigoplus_\nu M_{\lambda\mu\nu}\mathbb{S}_\nu V,$$

the sum over all partitions ν of dm, where λ is a partition of d and μ is a partition of m. The preceding exercise carried out four special cases of this.

(a) Show that there always *exists* such a decomposition for some non-negative integers $M_{\lambda\mu\nu}$ by constructing an element c in $\mathbb{C}\mathfrak{S}_{dm}$, depending on λ and μ, such that $\mathbb{S}_\mu(\mathbb{S}_\lambda V)$ is $V^{\otimes dm}\cdot c$.

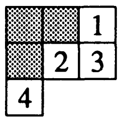
(b) Compute $\mathrm{Sym}^2(\mathbb{S}_{(2,2)}V)$ and $\Lambda^2(\mathbb{S}_{(2,2)}V)$.

Exercise 6.18* "Hermite reciprocity." Show that if $\dim V = 2$ there are isomorphisms

$$\mathrm{Sym}^p(\mathrm{Sym}^q V) \cong \mathrm{Sym}^q(\mathrm{Sym}^p V)$$

of $GL(V)$-representations, for all p and q.

Exercise 6.19*. Much of the story about Young diagrams and representations of symmetric and general linear groups can be generalized to *skew Young diagrams*, which are the differences of two Young diagrams. If λ and μ are partitions with $\mu_i \le \lambda_i$ for all i, λ/μ denotes the complement of the Young diagram for μ in that of λ. For example, if $\lambda = (3, 3, 1)$ and $\mu = (2, 1)$, λ/μ is the numbered part of



To each λ/μ we have a *skew Schur function* $S_{\lambda/\mu}$, which can be defined by any of several generalizations of constructions of ordinary Schur functions. Using the notation of Appendix A, the following definitions are equivalent:

(i) $$S_{\lambda/\mu} = |H_{\lambda_i - \mu_j - i + j}|,$$

(ii) $$S_{\lambda/\mu} = |E_{\lambda'_i - \mu'_j - i + j}|,$$

(iii) $$S_{\lambda/\mu} = \sum m_a x_1^{a_1}\cdot\ldots\cdot x_k^{a_k},$$

where m_a is the number of ways to number the boxes of λ/μ with a_1 1's, a_2 2's, $\ldots$, a_k k's, with nondecreasing rows and strictly increasing columns.

In terms of ordinary Schur polynomials, we have

(iv) $$S_{\lambda/\mu} = \sum N_{\mu\nu\lambda} S_\nu,$$

where $N_{\mu\nu\lambda}$ is the Littlewood–Richardson number.

Each λ/μ determines elements $a_{\lambda/\mu}$, $b_{\lambda/\mu}$, and Young symmetrizers $c_{\lambda/\mu} = a_{\lambda/\mu}b_{\lambda/\mu}$ in $A = \mathbb{C}\mathfrak{S}_d$, $d = \sum \lambda_i - \mu_i$, exactly as in §4.1, and hence a representation denoted $V_{\lambda/\mu} = Ac_{\lambda/\mu}$ of $\mathfrak{S}_d$. Equivalently, $V_{\lambda/\mu}$ is the image of the map $Ab_{\lambda/\mu} \to Aa_{\lambda/\mu}$ given by right multiplication by $a_{\lambda/\mu}$, or the image of the map $Aa_{\lambda/\mu} \to Ab_{\lambda/\mu}$ given by right multiplication by $b_{\lambda/\mu}$. The decomposition of $V_{\lambda/\mu}$ into irreducible representations is

(v) $$V_{\lambda/\mu} = \sum N_{\mu\nu\lambda} V_\nu.$$

Similarly there are *skew Schur functors* $\mathbb{S}_{\lambda/\mu}$, which take a vector space V to the image of $c_{\lambda/\mu}$ on $V^{\otimes d}$; equivalently, $\mathbb{S}_{\lambda/\mu} V$ is the image of a natural map (generalizing that in the Exercise 6.14)

(vi) $$\bigotimes_i (\Lambda^{\lambda'_i - \mu'_i} V) \to V^{\otimes d} \to \bigotimes_j (\mathrm{Sym}^{\lambda_j - \mu_j} V),$$

or

(vii) $$\bigotimes_i (\mathrm{Sym}^{\lambda_i - \mu_i} V) \to V^{\otimes d} \to \bigotimes_j (\Lambda^{\lambda'_j - \mu'_j} V).$$

Given a basis $v_1, \ldots, v_k$ for V and a standard tableau T on λ/μ, one can write down an element v_T in $\bigotimes_i (\Lambda^{\lambda'_j - \mu'_j} V)$; for example, corresponding to the displayed tableau, $v_T = v_4 \otimes v_2 \otimes (v_1 \wedge v_3)$. A key fact, generalizing the result of Exercise 6.15, is that the images of these elements under the map (vi) form a basis for $\mathbb{S}_{\lambda/\mu} V$.

The character of $\mathbb{S}_{\lambda/\mu} V$ is given by the Schur function $S_{\lambda/\mu}$: if g is an endomorphism of V with eigenvalues $x_1, \ldots, x_k$, then

(viii) $$\chi_{\mathbb{S}_{\lambda/\mu} V}(g) = S_{\lambda/\mu}(x_1, \ldots, x_k).$$

In terms of basic Schur functors,

(ix) $$\mathbb{S}_{\lambda/\mu} V \cong \sum N_{\mu\nu\lambda} \mathbb{S}_\nu V.$$

Exercise 6.20*. (a) Show that if $\lambda = (p, q)$, $\mathbb{S}_{(p,q)} V$ is the kernel of the contraction map

$$c_{p,q}\colon \mathrm{Sym}^p V \otimes \mathrm{Sym}^q V \to \mathrm{Sym}^{p+1} V \otimes \mathrm{Sym}^{q-1} V.$$

(b) If $\lambda = (p, q, r)$, show that $\mathbb{S}_{(p,q,r)} V$ is the intersection of the kernels of two contraction maps $c_{p,q} \otimes 1_r$ and $1_p \otimes c_{p,r}$, where 1_i denotes the identity map on $\mathrm{Sym}^i V$.

In general, for $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\mathbb{S}_\lambda V \subset \mathrm{Sym}^{\lambda_1} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_k} V$ is the intersection of the kernels of the $k - 1$ maps

$$\psi_i = 1_{\lambda_1} \otimes \cdots \otimes 1_{\lambda_{i-1}} \otimes c_{\lambda_i, \lambda_{i+1}} \otimes 1_{\lambda_{i+2}} \otimes \cdots \otimes 1_{\lambda_k}, \quad 1 \le i \le k - 1.$$

(c) For $\lambda = (p, 1, \ldots., 1)$, show that $\mathbb{S}_\lambda V$ is the kernel of the contraction map:

$$\mathbb{S}_{(p,1,\ldots,1)} V = \mathrm{Ker}(\mathrm{Sym}^p V \otimes \Lambda^{d-p} V \to \mathrm{Sym}^{p+1} V \otimes \Lambda^{d-p-1} V).$$

In general, for any choice of a between 1 and $k - 1$, the intersection of the kernels of all ψ_i except ψ_a is $\mathbb{S}_\sigma V \otimes \mathbb{S}_\tau V$, where $\sigma = (\lambda_1, \ldots, \lambda_a)$ and $\tau = (\lambda_{a+1}, \ldots, \lambda_k)$; so $\mathbb{S}_\lambda V$ is the kernel of a contraction map defined on $\mathbb{S}_\sigma V \otimes \mathbb{S}_\tau V$. For example, if a is $k - 1$, and we set $r = \lambda_k$, Pieri's formula writes $\mathbb{S}_\sigma V \otimes \mathrm{Sym}^r V$ as a direct sum of $\mathbb{S}_\lambda V$ and other factors $\mathbb{S}_\nu V$; the general assertion in (b) is equivalent to the claim that $\mathbb{S}_\lambda V$ is the only factor that is in the kernel of the contraction, ie.,

$$\mathbb{S}_\lambda V = \mathrm{Ker}(\mathbb{S}_{(\lambda_1, \ldots, \lambda_{k-1})} V \otimes \mathrm{Sym}^r V \to V^{\otimes(d-r+1)} \otimes \mathrm{Sym}^{r-1} V).$$

These results correspond to writing the representations $V_\lambda \subset U_\lambda$ of the symmetric group as the intersection of kernels of maps to $U_{\lambda_1, \ldots, \lambda_i+1, \lambda_{i+1}-1, \ldots, \lambda_k}$.

Exercise 6.21. The functorial nature of Weyl's construction has many consequences, which are not explored in this book. For example, if E_* is a complex of vector spaces, the tensor product $E_*^{\otimes d}$ is also a complex, and the symmetric group $\mathfrak{S}_d$ acts on it; when factors in E_p and E_q are transposed past each other, the usual sign $(-1)^{pq}$ is inserted. The image of the Young symmetrizer c_λ is a complex $\mathbb{S}_\lambda(E_*)$, sometimes called a *Schur complex*. Show that if E_* is the complex $E_{-1} = V \to E_0 = V$, with the boundary map the identity map, and $\lambda = (d)$, then $\mathbb{S}_\lambda(E_*)$ is the Koszul complex

$$0 \to \Lambda^d \to \Lambda^{d-1} \otimes S^1 \to \Lambda^{d-2} \otimes S^2 \to \cdots \to \Lambda^1 \otimes S^{d-1} \to S^d \to 0,$$

where $\Lambda^i = \Lambda^i V$, and $S^j = \mathrm{Sym}^j V$.

§6.2. The Proofs

We need first a small piece of the general story about semisimple algebras, which we work out by hand. For the moment G can be any finite group, although our application is for the symmetric group. If U is a right module over $A = \mathbb{C}G$, let

$$B = \mathrm{Hom}_G(U, U) = \{\varphi: U \to U: \varphi(v \cdot g) = \varphi(v) \cdot g, \forall v \in U, g \in G\}.$$

Note that B acts on U on the left, commuting with the right action of A; B is called the *commutator* algebra. If $U = \bigoplus U_i^{\oplus n_i}$ is an irreducible decomposition with U_i nonisomorphic irreducible right A-modules, then by Schur's Lemma 1.7

$$B = \bigoplus_i \mathrm{Hom}_G(U_i^{\oplus n_i}, U_i^{\oplus n_i}) = \bigoplus_i M_{n_i}(\mathbb{C}),$$

where $M_{n_i}(\mathbb{C})$ is the ring of $n_i \times n_i$ complex matrices.

If W is any left A-module, the tensor product

$$U \otimes_A W = U \otimes_{\mathbb{C}} W/\text{subspace generated by } \{va \otimes w - v \otimes aw\}$$

is a left B-module by acting on the first factor: $b \cdot (v \otimes w) = (b \cdot v) \otimes w$.

Lemma 6.22. *Let U be a finite-dimensional right A-module.*

(i) *For any $c \in A$, the canonical map $U \otimes_A Ac \to Uc$ is an isomorphism of left B-modules.*

(ii) *If $W = Ac$ is an irreducible left A-module, then $U \otimes_A W = Uc$ is an irreducible left B-module.*

(iii) *If $W_i = Ac_i$ are the distinct irreducible left A-modules, with m_i the dimension of W_i, then*

$$U \cong \bigoplus_i (U \otimes_A W_i)^{\oplus m_i} \cong \bigoplus_i (Uc_i)^{\oplus m_i}$$

is the decomposition of U into irreducible left B-modules.

PROOF. Note first that Ac is a direct summand of A as a left A-module; this is a consequence of the semisimplicity of all representations of G (Proposition 1.5). To prove (i), consider the commutative diagram

$$\begin{array}{ccccc} U \otimes_A A & \xrightarrow{\cdot c} & U \otimes_A Ac & \hookrightarrow & U \otimes_A A \\ \downarrow & & \downarrow & & \downarrow \\ U & \xrightarrow{\cdot c} & U \cdot c & \hookrightarrow & U \end{array},$$

where the vertical maps are the maps $v \otimes a \mapsto v \cdot a$; since the left horizontal maps are surjective, the right ones injective, and the outside vertical maps are isomorphisms, the middle vertical map must be an isomorphism.

For (ii), consider first the case where U is an irreducible A-module, so $B = \mathbb{C}$. It suffices to show that $\dim U \otimes_A W \leq 1$. For this we use Proposition 3.29 to identify A with a direct sum $\bigoplus_{i=1}^{r} M_{m_i}\mathbb{C}$ of r matrix algebras. We can identify W with a minimal left ideal of A. Any minimal ideal in the sum of matrix algebras is isomorphic to one which consists of r-tuples of matrices which are zero except in one factor, and in this factor are all zero except for one column. Similarly, U can be identified with the right ideal of r-tuples which are zero except in one factor, and in that factor all are zero except in one row. Then $U \otimes_A W$ will be zero unless the factor is the same for U and W, in which case $U \otimes_A W$ can be identified with the matrices which are zero except in one row and column of that factor. This completes the proof when U is irreducible. For the general case of (ii), decompose $U = \bigoplus_i U_i^{\oplus n_i}$ into a sum of irreducible right A-modules, so $U \otimes_A W = \bigoplus_i (U_i \otimes_A W)^{\oplus n_i} = \mathbb{C}^{\oplus n_k}$ for some k, which is visibly irreducible over $B = \bigoplus M_{n_j}(\mathbb{C})$.

Part (iii) follows, since the isomorphism $A \cong \bigoplus W_i^{\oplus m_i}$ determines an isomorphism

$$U \cong U \otimes_A A \cong U \otimes_A \left(\bigoplus_i W_i^{\oplus m_i}\right) \cong \bigoplus_i (U \otimes_A W_i)^{\oplus m_i}. \qquad \square$$

To prove Theorem 6.3, we will apply Lemma 6.22 to the right $\mathbb{C}\mathfrak{S}_d$-module $U = V^{\otimes d}$. That lemma shows how to decompose U as a B-module, where B

is the algebra of all endomorphisms of U that commute with all permutations of the factors. The endomorphisms of U induced by endomorphisms of V are certainly in this algebra B. Although B is generally much larger than $\mathrm{End}(V)$, we have

Lemma 6.23. *The algebra B is spanned as a linear subspace of* $\mathrm{End}(V^{\otimes d})$ *by* $\mathrm{End}(V)$. *A subspace of $V^{\otimes d}$ is a sub-B-module if and only if it is invariant by* $\mathrm{GL}(V)$.

PROOF. Note that if W is any finite-dimensional vector space, then $\mathrm{Sym}^d W$ is the subspace of $W^{\otimes d}$ spanned by all $w^d = d!w \otimes \cdots \otimes w$ as w runs through W. Applying this to $W = \mathrm{End}(V) = V^* \otimes V$ proves the first statement, since $\mathrm{End}(V^{\otimes d}) = (V^*)^{\otimes d} \otimes V^{\otimes d} = W^{\otimes d}$, with compatible actions of $\mathfrak{S}_d$. The second follows from the fact that $\mathrm{GL}(V)$ is dense in $\mathrm{End}(V)$. □

We turn now to the proof of Theorem 6.3. Note that $\mathbb{S}_\lambda V$ is Uc_λ, so parts (2) and (4) follow from Lemmas 6.22 and 6.23. We use the same methods to give a rather indirect but short proof of part (3); for a direct approach see Exercise 6.28. From Lemma 6.22 we have an isomorphism of $\mathrm{GL}(V)$-modules:

$$\mathbb{S}_\lambda V \cong V^{\otimes d} \otimes_A V_\lambda \tag{6.24}$$

with $V_\lambda = A \cdot c_\lambda$. Similarly for $U_\lambda = A \cdot a_\lambda$, and since the image of right multiplication by a_λ on $V^{\otimes d}$ is the tensor product of symmetric powers, we have

$$\mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_k} V \cong V^{\otimes d} \otimes_A U_\lambda. \tag{6.25}$$

But we have an isomorphism $U_\lambda \cong \bigoplus_\mu K_{\mu\lambda} V_\mu$ of A-modules by Young's rule (4.39), so we deduce an isomorphism of $\mathrm{GL}(V)$-modules

$$\mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_k} V \cong \bigoplus_\mu K_{\mu\lambda} \mathbb{S}_\mu V. \tag{6.26}$$

By what we saw before the statement of the theorem, the trace of g on the left-hand side of (6.26) is the product $H_\lambda(x_1, \ldots, x_k)$ of the complete symmetric polynomials $H_{\lambda_i}(x_1, \ldots, x_k)$. Let $\mathbb{S}_\lambda(g)$ denote the endomorphism of $\mathbb{S}_\lambda V$ determined by an endomorphism g of V. We therefore have

$$H_\lambda(x_1, \ldots, x_k) = \Sigma_\mu K_{\mu\lambda} \,\mathrm{Trace}(\mathbb{S}_\mu(g)).$$

But these are precisely the relations between the functions H_λ and the Schur polynomials S_μ [see formula (A.9)], and these relations are invertible, since the matrix $(K_{\mu\lambda})$ of coefficients is triangular with 1's on the diagonal. It follows that $\mathrm{Trace}(\mathbb{S}_\lambda(g)) = S_\lambda(x_1, \ldots, x_k)$, which proves part (3).

Note that if $\lambda = (\lambda_1, \ldots, \lambda_d)$ with $d > k$ and $\lambda_{k+1} \neq 0$, this same argument shows that the trace is $S_\lambda(x_1, \ldots, x_k, 0, \ldots, 0)$, which is zero, for example by (A.6). For g the identity, this shows that $\mathbb{S}_\lambda V = 0$ in this case. From part (3) we also get

$$\dim \mathbb{S}_\lambda V = S_\lambda(1, \ldots, 1), \tag{6.27}$$

and computing $S_\lambda(1, \ldots, 1)$ via Exercise A.30(ii) yields part (1). □

Exercise 6.28. If you have given an independent proof of Problem 6.15, part (3) of Theorem 6.3 can be seen directly. The basis elements v_T for $\mathbb{S}_\lambda V$ specified in Problem 6.15 are eigenvectors for a diagonal matrix with entries $x_1, \ldots, x_k$, with eigenvalue $X^a = x_1^{a_1} \cdot \ldots \cdot x_k^{a_k}$, where the tableau T has a_1 1's, a_2 2's, ..., a_k k's. The trace is therefore $\sum K_{\lambda a} X^a$, where $K_{\lambda a}$ is the number of ways to number the boxes of the Young diagram of λ with a_1 1's, a_2 2's, ..., a_k k's. This is just the expression for S_λ obtained in Exercise A.31(a).

We conclude this lecture with a few of the standard elaborations of these ideas, in exercise form; they are not needed in these lectures.

Exercise 6.29*. Show that, in the context of Lemma 6.22, if U is a faithful A-module, then A is the commutator of its commutator B:

$$A = \{\psi \colon U \to U \colon \psi(bv) = b\psi(v), \forall v \in U, b \in B\}.$$

If U is not faithful, the canonical map from A to its bicommutator is surjective. Conclude that, in Theorem 6.3, the algebra of endomorphisms of $V^{\otimes d}$ that commute with $\mathrm{GL}(V)$ is spanned by the permutations in $\mathfrak{S}_d$.

Exercise 6.30. Show that, in Lemma 6.22, there is a natural one-to-one correspondence between the irreducible right A-modules U_i that occur in U and the irreducible left B-modules V_i. Show that there is a canonical decomposition

$$U = \bigoplus_i (V_i \otimes_{\mathbb{C}} U_i)$$

as a left B-module and as a right A-module. This shows again that the number of times V_i occurs in U is the dimension of U_i, and dually that the number of times U_i occurs is the dimension of V_i. Deduce the canonical decomposition

$$V^{\otimes d} = \bigoplus \mathbb{S}_\lambda V \otimes_{\mathbb{C}} V_\lambda,$$

the sum over partitions λ of d into at most $k = \dim V$ parts; this decomposition is compatible with the actions of $\mathrm{GL}(V)$ and $\mathfrak{S}_d$. In particular, the number of times V_λ occurs in the representation $V^{\otimes d}$ of $\mathfrak{S}_d$ is the dimension of $\mathbb{S}_\lambda V$.

Exercise 6.31. Let e be an idempotent in the group algebra $A = \mathbb{C}G$, and let $U = eA$ be the corresponding right A-module. Let $E = eAe$, a subalgebra of A. The algebra structure in A makes eA a left E-module. Show that this defines an isomorphism of $\mathbb{C}$-algebras

$$E = eAe \cong \mathrm{Hom}_A(eA, eA) = \mathrm{Hom}_G(U, U) = B.$$

Exercise 6.32. If H is a subgroup of G, and $e \in \mathbb{C}H$ is an idempotent, corresponding to a representation $W = \mathbb{C}H \cdot e$ of H, show that $\mathbb{C}G \cdot e$ is the induced representation $\mathrm{Ind}_H^G(W)$. For example, if $\vartheta\colon H \to \mathbb{C}^*$ is a one-dimensional representation, then

$$\mathrm{Ind}_H^G(\vartheta) = \mathbb{C}G \cdot e_\vartheta, \quad \text{where } e_\vartheta = \frac{1}{|G|} \sum_{g \in G} \overline{\vartheta(g)} e_g.$$