

Chapter 27

Appendix to Chapter 13

27.1 Classification of the Effects of Growth

Let us first consider the consumption effects, which amounts to finding the conditions for the ratio A^D/Y to increase, remain unchanged, decrease, where A is the importable. The derivative of this ratio is

$$\frac{d(A^D/Y)}{dY} = \frac{Y(dA^D/dY) - A^D}{Y^2} = \frac{1}{Y} \left(\frac{dA^D}{dY} - \frac{A^D}{Y} \right), \quad (27.1)$$

which can be written as

$$\frac{d(A^D/Y)}{dY} = \frac{A^D}{Y^2} (\eta_{dY} - 1), \quad (27.2)$$

where

$$\eta_{dY} \equiv \frac{dA^D/A^D}{dY/Y} = \frac{dA^D/dY}{A^D/Y} = \frac{\mu_{dY}}{\alpha_{dY}} \quad (27.3)$$

is the income elasticity of A^D , and μ_{dY}, α_{dY} , are, respectively, the marginal and average propensity to consume commodity A .

We thus have the following conditions for the consumption effects of growth to be pro-trade-biased (P), neutral (N), anti-trade-biased (A):

$$\frac{d(A^D/Y)}{dY} \begin{cases} \geq 0 \\ = 0 \\ \leq 0 \end{cases} \text{ according as } \eta_{dY} \begin{cases} \geq 1 \\ = 1 \\ \leq 1 \end{cases} \text{ i.e. according as } \mu_{dY} \begin{cases} \geq \\ = \\ \leq \end{cases} \alpha_{dY}. \quad (27.4)$$

Growth has ultra-pro-trade biased (UP) or ultra-anti-trade biased (UA) consumption effects when $\mu_{dY} > 1$ or $\mu_{dY} < 0$, respectively.

As regards the production effects, these involve the derivative of the ratio A^S/Y , which is

$$\frac{d(A^S/Y)}{dY} = \frac{1}{Y} \left(\frac{dA^S}{dY} - \frac{A^S}{Y} \right) = \frac{A^S}{Y} (\eta_{sY} - 1), \quad (27.5)$$

where

$$\eta_{sY} \equiv \frac{dA^S/A^S}{dY/Y} = \frac{\mu_{sY}}{\alpha_{sY}}. \quad (27.6)$$

Thus the conditions for the production effects of growth to be pro-trade-biased (P), neutral (N), anti-trade-biased (A) are the following:

$$\frac{d(A^S/Y)}{dY} \begin{cases} \leq 0 \\ \geq 0 \end{cases} \text{ according as } \eta_{sY} \begin{cases} \leq 1 \\ \geq 1 \end{cases} \text{ i.e. according as } \mu_{sY} \begin{cases} \leq \\ \geq \end{cases} \alpha_{sY}. \quad (27.7)$$

Ultra-pro-trade-biased (UP) or ultra-anti-trade-biased (UA) production effects occur when $\mu_{sY} < 0$ or $\mu_{sY} > 1$, respectively. Table 27.1 lists the (intervals of) values for the parameters corresponding to the various cases.

Let us now consider the ratio of the *demand for imports* to income, $(A^D - A^S)/Y$, and calculate its derivative. We have

$$\begin{aligned} \frac{d[(A^D - A^S)/Y]}{dY} &= \frac{d(A^D/Y)}{dY} - \frac{d(A^S/Y)}{dY} \\ &= \frac{1}{Y} \left[\frac{A^D}{Y} (\eta_{dY} - 1) - \frac{A^S}{Y} (\eta_{sY} - 1) \right] \\ &= \frac{1}{Y} [(\mu_{dY} - \alpha_{dY}) - (\mu_{sY} - \alpha_{sY})], \end{aligned} \quad (27.8)$$

where we have used (27.1), (27.2), and (27.5). The definition of the *total* effects of growth, states that growth has pro-trade-biased, neutral, anti-trade-biased total effects according as

$$\frac{d[(A^D - A^S)/Y]}{dY} \begin{cases} \geq 0 \\ \leq 0 \end{cases},$$

and that it has ultra-pro-trade-biased or ultra-anti-trade-biased total effects according as

$$\frac{d(A^D - A^S)}{dY} > 1 \text{ or } < 0, \text{ respectively.}$$

Given this definition, Eq. (27.8), and Table (27.1), it is a simple exercise to derive the results listed in Table 13.1 in the text.

Table 27.1 Consumption and production effects of growth

Effects	Consumption parameters	Production parameters
<i>UP</i>	$\eta_{dY} > 1, \alpha_{dY} < 1 < \mu_{dY}$	$\eta_{sY} < 0, \mu_{sY} < 0 < \alpha_{sY} < 1$
<i>P</i>	$\eta_{dY} > 1, 0 < \alpha_{dY} < \mu_{dY} < 1$	$0 < \eta_{sY} < 1, 0 < \mu_{sY} < \alpha_{sY} < 1$
<i>N</i>	$\eta_{dY} = 1, 0 < \mu_{dY} = \alpha_{dY} < 1$	$\eta_{sY} = 1, 0 < \mu_{sY} = \alpha_{sY} < 1$
<i>A</i>	$0 < \eta_{dY} < 1, 0 < \mu_{dY} < \alpha_{dY} < 1$	$\eta_{sY} > 1, 0 < \alpha_{sY} < \mu_{sY} < 1$
<i>UA</i>	$\eta_{dY} < 0, \mu_{dY} < 0 < \alpha_{dY} < 1$	$\eta_{sY} > 1, \alpha_{sY} < 1 < \mu_{sY}$

We now prove the relation between the various growth rates. If we denote the growth rate of import demand by g_m , the new demand for imports will be $(1 + g_m)(A^D - A^S)$. With a similar notation we can write the new domestic demand for and domestic supply of commodity A (the importable) as $(1 + g_m)A^D$ and $(1 + g_s)A^S$, respectively. Then, by definition,

$$(1 + g_m)(A^D - A^S) = (1 + g_d)A^D - (1 + g_s)A^S, \tag{27.9}$$

whence

$$g_m = \frac{g_d A^D - g_s A^S}{A^D - A^S} = \frac{A^D}{A^D - A^S} g_d - \frac{A^S}{A^D - A^S} g_s. \tag{27.10}$$

By suitably adding and subtracting g_Y we have

$$\begin{aligned} g_m &= \frac{(g_d - g_Y + g_Y) A^D - (g_s - g_Y + g_Y) A^S}{A^D - A^S} \\ &= \frac{g_Y (A^D - A^S) + (g_d - g_Y) A^D - (g_s - g_Y) A^S}{A^D - A^S} \\ &= g_Y + \frac{A^D}{A^D - A^S} (g_d - g_Y) - \frac{A^S}{A^D - A^S} (g_s - g_Y), \end{aligned} \tag{27.11}$$

which is the expression given in Eq. (13.4).

27.2 Comparative Statics of the Effects of Growth in General

Economic growth involves an upward shift of the transformation curve, that is, an increase in the production possibilities of both A and B for any given relative price $p = p_B/p_A$. Since we are not concerned here with the causes of growth, to examine its effects in general it is sufficient to introduce a shift parameter (γ) in the functions

defining the quantities of A and B as a function of p along the transformation curve of the country, which we will assume to be country 1 (see Sect. 19.2)¹

$$A_1 = A_1(p, \gamma), \quad B_1 = B_1(p, \gamma), \quad (27.12)$$

where $\partial A_1/\partial \gamma > 0$ and $\partial B_1/\partial \gamma > 0$ are for the time being considered as exogenously given. To examine the effects of growth (increase in γ) on the terms of trade we start from the international equilibrium condition—see Eq. (19.28)—which becomes

$$E_{1A}(p, \gamma) - pE_{2B}(p) = 0. \quad (27.13)$$

If we differentiate totally with respect to γ , account being taken of the fact that, by the implicit-function theorem, p is a function of γ , we get

$$\frac{\partial E_{1A}}{\partial p} \frac{dp}{d\gamma} + \frac{\partial E_{1A}}{\partial \gamma} - E_{2B} \frac{dp}{d\gamma} - p \frac{dE_{2B}}{dp} \frac{dp}{d\gamma} = 0, \quad (27.14)$$

whence

$$\left(E_{2B} + p \frac{dE_{2B}}{dp} - \frac{\partial E_{1A}}{\partial p} \right) \frac{dp}{d\gamma} = \frac{\partial E_{1A}}{\partial \gamma}. \quad (27.15)$$

We now divide through by E_{2B} (which equals E_{1A}/p , from (27.13) above) and obtain

$$\left(1 + \frac{p}{E_{2B}} \frac{dE_{2B}}{dp} - \frac{p}{E_{1A}} \frac{\partial E_{1A}}{\partial p} \right) \frac{dp}{d\gamma} = \frac{1}{E_{2B}} \frac{\partial E_{1A}}{\partial \gamma}. \quad (27.16)$$

If we solve for $dp/d\gamma$ and use the definitions of the elasticities given in (19.36) and (19.44), we obtain

$$\frac{dp}{d\gamma} = \frac{\partial E_{1A}/\partial \gamma}{E_{2B}(1 + \xi_1 + \xi_2)}, \quad (27.17)$$

where ξ_1 and ξ_2 are the elasticities of the demand for imports of the two countries. To determine the sign of $dp/d\gamma$ we must determine the sign of the fraction on the right-hand side of (27.17). Let us begin by observing that, thanks to Samuelson's correspondence principle (see, for example, [Gandolfo, 2009](#), chap. 20), it is possible

¹For simplicity and in accordance with the notation used in the Chap. 19, we henceforth omit the superscript S , so that A and B with no superscript will indicate the quantities supplied (produced), whilst we maintain the superscript D to denote the quantities demanded. It is as well to inform the reader that in what follows, we shall make ample use of the model explained in Chap. 19.

to determine the sign of the denominator: in fact, if the equilibrium is stable, the stability condition (19.48) tells us that $E_{2B} (1 + \xi_1 + \xi_2) < 0$.²

All that remains is to determine the sign of the numerator. We recall from (19.25) that

$$\begin{aligned} E_{1A}(p, \gamma) &= A_1^D(I_{1A}, p) - A_1(p, \gamma), \\ I_{1A} &= A_1 + pB_1, \end{aligned} \quad (27.18)$$

where I_{1A} is country 1's real income measured in terms of commodity A . Let us note, incidentally, that demand is ultimately a function of γ for any given p , because I_{1A} is a function of γ through the quantities produced. If we differentiate the first equation in (27.18) with respect to γ we get

$$\frac{\partial E_{1A}}{\partial \gamma} = \frac{\partial A_1^D}{\partial I_{1A}} \frac{\partial I_{1A}}{\partial \gamma} - \frac{\partial A_1}{\partial \gamma}, \quad (27.19)$$

from which

$$\frac{\partial E_{1A}}{\partial \gamma} = \frac{\partial I_{1A}}{\partial \gamma} (\mu_{dY} - \mu_{sY}), \quad (27.20)$$

where $\mu_{dY} \equiv \partial A_1^D / \partial I_{1A}$ and $\mu_{sY} \equiv (\partial A_1 / \partial \gamma) / (\partial I_{1A} / \partial \gamma) = \partial A_1 / \partial I_{1A}$ are, respectively, country 1's marginal propensity to consume and marginal propensity to produce commodity A , already met in the previous section.

As $(\partial I_{1A} / \partial \gamma) = \partial A_1 / \partial \gamma + p(\partial B_1 / \partial \gamma)$ is assumed positive, the sign of the numerator will depend on the sign of $(\mu_{dY} - \mu_{sY})$; as the sign of the fraction is the opposite of the sign of the numerator, we finally have

$$\frac{dp}{d\gamma} \gtrless 0 \text{ according as } \mu_{dY} \lesseqgtr \mu_{sY}. \quad (27.21)$$

Relation (27.21), together with Tables 27.1 and 13.1, enables us to immediately obtain the result explained graphically in Sect. 13.3, that is, $dp/d\gamma < 0$ except in the case of globally UA growth. For example, growth with UA consumption effects ($\mu_{dY} < 0$) and P production effects ($0 < \mu_{sY} < \alpha_{sY}$), which has a UA total effect, implies $\mu_{dY} < \mu_{sY}$ and so $dp/d\gamma > 0$. As another example, consider growth with N consumption effects ($\mu_{dY} = \alpha_{dY}$) and P production effects ($0 < \mu_{sY} < \alpha_{sY}$), which has a P total effect. Since A is the importable, the average propensity

²It should be recalled that this condition was derived in Chap. 19 on the basis of the assumption that $E_{2B} > 0$, i.e. that B is country 2's importable (and, therefore, that A is country 1's importable). But the result is unchanged if we assumed the opposite pattern of trade ($E_{2B} < 0$, i.e. B is country 2's exportable, etc.). In this case, in fact, the expression under consideration would become $E_{2B} (1 + \varepsilon_1 + \varepsilon_2)$; given (19.45) and account being taken that $E_{2B} < 0$, this expression is negative if the expression in the text is negative.

to consume is higher than the average propensity to produce, that is, $\alpha_{dY} > \alpha_{sY}$ and so, in our case, $\mu_{dY} > \mu_{sY}$, whence $dp/d\gamma < 0$.

We leave the other cases as an exercise and pass on to the problem of *immiserizing growth*.

27.2.1 Immiserizing Growth

To avoid the problems inherent in the use of social indifference curves we shall use an alternative way of measuring the improvement or impairment in social welfare, that is, we shall consider the situation as better (worse) if the new national income due to growth enables the society, account being taken of the change in prices, to purchase the *same* bundle of commodities as before *plus (minus)* something else. In other words, the situation is better (worse) if the new national income is higher (lower) than the cost (*at the new prices*) of the same bundle of commodities purchased before growth or, equivalently, if the increase in national income is higher (lower) than the increase in the cost of the pre-growth bundle of commodities, where both income and cost are measured in terms of one of the commodities (for example *A*) taken as numéraire.

Let us begin by calculating the increase in income, which is $dI_{1A}/d\gamma$.

It must be stressed that this is a *total* derivative, which takes all the effects of γ on output and p into account, and not the partial derivative previously used. Thus we have

$$\frac{dI_{1A}}{d\gamma} = \frac{\partial I_{1A}}{\partial \gamma} + \frac{\partial I_{1A}}{\partial p} \frac{dp}{d\gamma} = \frac{\partial I_{1A}}{\partial \gamma} + B_1 \frac{dp}{d\gamma}, \quad (27.22)$$

as $\partial I_{1A}/\partial p = B_1$ by (19.22). The pre-growth bundle of goods has a cost, at the pre-growth relative price, of $A_1^D + pB_1^D$; by keeping A_1^D and B_1^D unchanged and letting p vary we get the change in cost, $B_1^D(dp/d\gamma)$, so that the post-growth situation will be better or worse according as

$$\frac{dI_{1A}}{d\gamma} = \frac{\partial I_{1A}}{\partial \gamma} + B_1 \frac{dp}{d\gamma} \geq B_1^D \frac{dp}{d\gamma}, \quad (27.23)$$

that is,

$$\frac{\partial I_{1A}}{\partial \gamma} - (B_1^D - B_1) \frac{dp}{d\gamma} = \frac{\partial I_{1A}}{\partial \gamma} - E_{1B} \frac{dp}{d\gamma} \geq 0. \quad (27.24)$$

If we use the value of $dp/d\gamma$ found above—see Eq. (27.17)—and recall that, by Eqs. (19.27), $-E_{1B} = E_{2B}$ in equilibrium, we get

$$\frac{\partial I_{1A}}{\partial \gamma} + \frac{\partial E_{1A}/\partial \gamma}{1 + \xi_1 + \xi_2} \geq 0, \quad (27.25)$$

whence, by substituting the value of $\partial E_{1A}/\partial\gamma$ from (27.20) and collecting terms, we arrive at

$$\frac{\partial I_{1A}}{\partial\gamma} \left(1 + \frac{\mu_{dY} - \mu_{sY}}{1 + \xi_1 + \xi_2} \right) \geq 0. \quad (27.26)$$

As $\partial I_{1A}/\partial\gamma$ is assumed positive, the condition for a *worsening* (immiserizing growth) is

$$1 + \frac{\mu_{dY} - \mu_{sY}}{1 + \xi_1 + \xi_2} < 0. \quad (27.27)$$

Since we have assumed that the equilibrium is stable, that is $1 + \xi_1 + \xi_2 < 0$ from (19.49), we can rewrite (27.27) as

$$1 + \xi_1 + \xi_2 + \mu_{dY} - \mu_{sY} > 0. \quad (27.28)$$

This condition may, in general, be either realized or not, so that immiserizing growth remains a possibility to be further investigated by an examination of the causes of growth (see Sects. 27.3 and 27.4); this examination will also enable us to obtain exact expressions for the various derivatives $\partial A_1/\partial\gamma$ etc. and in particular for μ_{sY} . However, it is now possible to show the *necessary condition* for immiserizing growth, which is that the terms of trade move against the growing country. In fact, if these were to improve or to remain unchanged, given Eq. (27.21) we have $\mu_{dY} - \mu_{sY} < 0$ and so, as $1 + \xi_1 + \xi_2 < 0$, condition (27.28) is *not* satisfied. This shows that the deterioration in the terms of trade is necessary, though not sufficient, for growth to be immiserizing.

27.3 Changes in Factor Endowments, Rybczynski's Theorem, and the Terms of Trade

Let us assume that the total amount of labour existing in the economy, L , increases. Given the assumption of continuous full employment of all factors of production, the expansion factor γ is identified with L . Thus we have to calculate $\partial I_{1A}/\partial L$ and $\partial A_1/\partial L$ to determine μ_{sY} and then dp/dL , as made clear in (27.20) and (27.21). In what follows we shall drop the subscript 1 for brevity of notation, as it is understood that all magnitudes refer to country 1.

If we consider the relations, derived from Eqs. (19.17)

$$\begin{aligned} L_A + L_B &= L, \\ Q_A L_A + Q_B L_B &= K, \end{aligned} \quad (27.29)$$

we can express L_A and L_B in terms of factor endowments and factor intensities, thus obtaining

$$L_A = \frac{\varrho_B L - K}{\varrho_B - \varrho_A}, \quad L_B = \frac{K - \varrho_A L}{\varrho_B - \varrho_A}. \quad (27.30)$$

The production functions in intensive form—see Eqs. (19.17)—are

$$A = L_A g_A(\varrho_A), \quad B = L_B g_B(\varrho_B), \quad (27.31)$$

where, thanks to the assumption of first-degree homogeneity, ϱ_A and ϱ_B depend solely on the relative price of factors, which is kept constant in computing the partial derivatives $\partial A/\partial L$ and $\partial B/\partial L$.

From (27.31) we therefore obtain

$$\frac{\partial A}{\partial L} = \frac{\partial L_A}{\partial L} g_A, \quad \frac{\partial B}{\partial L} = \frac{\partial L_B}{\partial L} g_B, \quad (27.32)$$

that is, given (27.30),

$$\frac{\partial A}{\partial L} = \frac{\varrho_B}{\varrho_B - \varrho_A} g_A, \quad \frac{\partial B}{\partial L} = \frac{-\varrho_A}{\varrho_B - \varrho_A} g_B, \quad (27.33)$$

so that the two productive levels move in opposite directions. If, for example, one assumes (as in the text) that sector A is labour intensive (so that sector B is capital intensive, whence $\varrho_B > \varrho_A$), it follows that the output of A increases and that of B decreases (the *Rybczynski theorem*). Besides, as $dA/A = (\partial A/\partial L)dL/A = [(\partial A/\partial L)/(A/L)](dL/L)$, and account being taken of Eqs. (27.33), (27.31), and (27.30), we see that

$$\frac{dA}{A} = \frac{\varrho_B}{\varrho_B - \varrho} \frac{dL}{L}, \quad \varrho = K/L, \quad (27.34)$$

whence, as we assumed $\varrho_B > \varrho_A$ (and so $\varrho_B > \varrho > \varrho_A$), it follows that $dA/A > dL/L$. In other words, the output of the expanding sector (in our example, sector A) increases *more than proportionally* to the increase in the factor. Jones (1965) has called this the *magnification effect*.

Let us now go back to the main line and calculate $\partial I_A/\partial L$. Since $I_A = A + pB$, we have

$$\frac{\partial I_A}{\partial L} = \frac{\partial A}{\partial L} + p \frac{\partial B}{\partial L} = \frac{\varrho_B g_A - p \varrho_A g_B}{\varrho_B - \varrho_A}. \quad (27.35)$$

Now, from Eqs. (19.17) we get

$$p g_B = g_A + p \varrho_B g'_B - \varrho_A g'_A = g_A + p \varrho_B g'_B - \varrho_A p g'_B = g_A + p g'_B (\varrho_B - \varrho_A)$$

and so

$$\frac{\partial I_A}{\partial L} = \frac{\varrho_B g_A - \varrho_A g_A - \varrho_A p g'_B (\varrho_B - \varrho_A)}{\varrho_B - \varrho_A} = g_A - \varrho_A p g'_B,$$

whence, as $p g'_B = g'_A$, we get

$$\frac{\partial I_A}{\partial L} = g_A - \varrho_A g'_A, \quad (27.36)$$

which is certainly positive by (19.17).

Given (27.36) and (27.33), we can calculate

$$\mu_{sY} = \frac{\partial A / \partial L}{\partial I_A / \partial L} = \frac{\varrho_B g_A}{(\varrho_B - \varrho_A) (g_A - \varrho_A g'_A)}. \quad (27.37)$$

Since $g_A - \varrho_A g'_A > 0$, the sign of μ_{sY} depends on the sign of $\varrho_B - \varrho_A$ so that

$$\mu_{sY} \gtrless 0 \text{ according as } \varrho_B \gtrless \varrho_A, \quad (27.38)$$

that is, according as sector B 's capital intensity is higher or lower than sector A 's. Besides, it can be shown that, if $\varrho_B > \varrho_A$, then μ_{sY} is not only positive but also greater than one. In fact, we have

$$0 < \varrho_B g_A - \varrho_B \varrho_A g'_A - \varrho_A (g_A - \varrho_A g'_A) < \varrho_B g_A,$$

where the central expression is the denominator of the fraction in (27.37); the left-hand inequality derives from the assumption $\varrho_B > \varrho_A$, and the right-hand one is self-evident. It follows that the denominator under consideration, when positive, is certainly smaller than the numerator in (27.37), so that $\mu_{sY} > 1$. This is an important result because it enables us to determine the direction in which the terms of trade move and to *exclude the possibility of immiserizing growth when no good is inferior*. If we assume—as in the text—that A is the labour-intensive commodity, then $\varrho_A < \varrho_B$ and so $\mu_{sY} > 1$. Now, if no good is inferior, the marginal propensity to consume A must be smaller than one, that is, $\mu_{dY} < 1$. It follows that $\mu_{dY} < \mu_{sY}$ and so, according to (27.21), the relative price $p = p_B / p_A$ increases:

$$\frac{dp}{dL} > 0, \quad (27.39)$$

so that the terms of trade will move in favour of or against the country according as A is the importable or the exportable (*corollary of Rybczynski's theorem*). If we assume that A is the importable, the improvement in the terms of trade *excludes* the possibility of immiserizing growth, as the necessary condition (27.28) is *not* verified.

Immiserizing growth, therefore, requires as a necessary condition that the importable should *not* be intensive in the augmenting factor: only when $\varrho_A > \varrho_B$, in fact, $\mu_{sY} < 0$ and so $dp/dL < 0$, which is the necessary condition for immiserizing growth to occur.

So far we have examined the effects of an increase in the labour force; the same procedure can be used to analyse the effects of an increase in the stock of capital (this includes the case of a transfer from abroad). We only state the results, omitting all the intermediate steps, which are exactly like those detailed above in the case of an increase in L . We begin by

$$\frac{\partial A}{\partial K} = \frac{\partial L_A}{\partial K} g_A, \quad \frac{\partial B}{\partial K} = \frac{\partial L_B}{\partial K} g_B, \quad (27.40)$$

so that, given (27.30),

$$\frac{\partial A}{\partial K} = \frac{-g_A}{\varrho_B - \varrho_A}, \quad \frac{\partial B}{\partial K} = \frac{g_B}{\varrho_B - \varrho_A}, \quad (27.41)$$

and so $\partial A/\partial K \leq 0$ and $\partial B/\partial K \geq 0$ according as $\varrho_B \geq \varrho_A$ (Rybczynski's theorem). We then calculate

$$\frac{\partial I_A}{\partial A} = \frac{\partial A}{\partial K} + p \frac{\partial B}{\partial K} = \frac{-g_A + p g_B}{\varrho_B - \varrho_A}, \quad (27.42)$$

whence, after suitable substitutions from (19.17), we get

$$\frac{\partial I_A}{\partial A} = g'_A > 0, \quad (27.43)$$

and so

$$\mu_{sY} = \frac{\partial A/\partial K}{\partial I_A/\partial K} = \frac{-g_A}{g'_A (\varrho_B - \varrho_A)} \geq 0 \text{ according as } \varrho_B \leq \varrho_A, \quad (27.44)$$

a result symmetrical with that obtained in the case of an increase in L , as can be arrived at intuitively.

Also in this case it is possible to show that μ_{sY} , when positive, is necessarily greater than one. From Eq. (19.17) we recall that $g_A - \varrho_A g'_A > 0$; besides, it is obvious that $g'_A \varrho_A > g'_A (\varrho_A - \varrho_B)$, so that $g_A > g'_A \varrho_A > g'_A (\varrho_A - \varrho_B)$. If we assume $\varrho_A - \varrho_B > 0$ (i.e., μ_{sY} positive), we can divide throughout by $g'_A (\varrho_A - \varrho_B)$ and obtain

$$\frac{g_A}{g'_A (\varrho_A - \varrho_B)} > 1, \quad (27.45)$$

as was to be demonstrated.

Also the results concerning the terms of trade are symmetrical with those obtained above in the case of an increase in L . In fact, if no good is inferior, $dp/dK \geq 0$ according as $\varrho_A \geq \varrho_B$, whence the corollary of Rybczynski's theorem and the usual conclusions on immiserizing growth follow.

27.3.1 Simultaneous Increase in Both Factors

We conclude this section by examining the effects of a *simultaneous increase in both factor endowments*. Let us consider the total differential

$$dA = \frac{\partial A}{\partial L}dL + \frac{\partial A}{\partial K}dK, \tag{27.46}$$

whence, substituting from (27.33) and (27.41), and collecting terms

$$dA = \frac{\varrho_B g_A dL - g_A dK}{\varrho_B - \varrho_A} = \frac{g_A dL [\varrho_B - (dK/dL)]}{\varrho_B - \varrho_A}. \tag{27.47}$$

Similarly we obtain

$$dB = \frac{g_B dL [(dK/dL) - \varrho_A]}{\varrho_B - \varrho_A}. \tag{27.48}$$

In order to be able to analyse the signs of dA and dB we must know the changes in factor endowments. For this purpose we consider their proportional changes and introduce a parameter α such that

$$\frac{dK}{K} = \alpha \frac{dL}{L}, \tag{27.49}$$

where $\alpha \geq 1$ according as the capital stock increases more than proportionally to, in the same proportion as, less than proportionally to the increase in the labour force. We then have

$$\frac{dK}{dL} = \alpha \varrho, \tag{27.50}$$

where $\varrho \equiv K/L$ is the initial factor endowment ratio which, as we recall from Eq.(4.3), is a weighted average of the factor intensities in the two sectors. By substituting (27.50) into (27.47) and (27.48), we obtain

$$dA = \frac{g_A dL (\varrho_B - \alpha \varrho)}{\varrho_B - \varrho_A}, \quad dB = \frac{g_B dL (\alpha \varrho - \varrho_A)}{\varrho_B - \varrho_A}. \tag{27.51}$$

It is possible, when $\alpha = 1$, to reach a definite conclusion, namely that an equiproportional increase in both factor endowments brings about an increase in the output of *both* commodities (conversely, as we have seen above, when only one factor increases, the commodity outputs move in opposite directions). In fact, by definition of average, ϱ is always situated between the minimum and the maximum term, that is

$$\begin{aligned} \varrho_A < \varrho < \varrho_B & \text{ if } \varrho_B > \varrho_A, \\ \varrho_B < \varrho < \varrho_A & \text{ if } \varrho_A > \varrho_B. \end{aligned} \quad (27.52)$$

Thus, letting $\alpha = 1$ in (27.51)—account being taken of (27.52)—we see that when $\varrho_B > \varrho_A$ we have $\varrho_B > \varrho$ and $\varrho > \varrho_A$, so that $dA > 0$, $dB > 0$. And when $\varrho_B < \varrho_A$ we have $\varrho_B < \varrho$ and $\varrho < \varrho_A$ so that $dA > 0$, $dB > 0$.

But when $\alpha \neq 1$ it is no longer possible to reach definite conclusions: the output of one commodity will certainly increase (this will be the commodity intensive in the factor with the higher proportional increase) but the other may either increase or decrease. Let us assume, for example, $\alpha > 1$ (the capital stock increases more than proportionally to the labour force) and $\varrho_B > \varrho_A$ (the capital-intensive commodity is B). Since from (27.52) we have $\varrho_A < \varrho$ it will also be true that $\varrho_A < \alpha\varrho$, hence $dB > 0$. But, although $\varrho < \varrho_B$, $\alpha\varrho$ may in general be smaller than, equal to, or greater than ϱ_B , so that the sign of dA is indeterminate.

We next examine the effects on the terms of trade. For this purpose we must, first, calculate the variation in real income—at unchanged p —and then determine the marginal propensity to produce A . At unchanged p , we have the total differential

$$dI_A = dA + pdB, \quad (27.53)$$

and substituting into it from (27.51) we get

$$dI_A = dL \frac{g_A (\varrho_B - \alpha\varrho) + pg_B (\alpha\varrho - \varrho_A)}{\varrho_B - \varrho_A}. \quad (27.54)$$

In the case in which $\alpha = 1$, we certainly have $dI_A > 0$, whilst the sign of (27.57) is ambiguous when $\alpha \neq 1$. It is however possible to eliminate this ambiguity by rewriting dI_A in the form

$$dI_A = dL (g_A - \varrho_A g'_A + \alpha\varrho g'_A), \quad (27.55)$$

which can be arrived at either by transforming (27.54) by a suitable use of Eqs. (19.17), or by starting from the equivalent definition of the variation in real income

$$dI_A = \frac{\partial I_A}{\partial L} dL + \frac{\partial I_A}{\partial K} dK = dL \left(\frac{\partial I_A}{\partial L} + \alpha\varrho \frac{\partial I_A}{\partial K} \right), \quad (27.56)$$

and substituting from (27.36) and (27.43) into it.

Now, since $g_A - \varrho_A g'_A > 0$ and $\alpha \varrho g'_A > 0$, it follows from (27.55) that dI_A is certainly positive even if $\alpha \neq 1$.

We now calculate, from (27.51) and (27.54),

$$\mu_{sY} = \frac{\partial A}{\partial I_A} = \frac{g_A (\varrho_B - \alpha \varrho)}{g_A (\varrho_B - \alpha \varrho) + p g_B (\alpha \varrho - \varrho_A)}, \quad (27.57)$$

which has a generally indeterminate sign. In the particular case where $\alpha = 1$ it can easily be seen, by using (27.52), that the numerator is smaller in absolute value than the denominator, so that $0 < \mu_{sY} < 1$. If no good is inferior, $0 < \mu_{dY} < 1$ as well, so that—given (27.21)—the terms of trade can move in either direction and the phenomenon of immiserizing growth may appear.

27.4 Technical Progress

A possible way of representing technical progress in a sector, for example in that producing commodity A , is

$$A = f_A (\lambda K_A, \lambda' L_A), \quad (27.58)$$

where λ and λ' are parameters, initially equal to one, which increase when technical progress occurs (if it occurs continuously, λ and λ' will be continuous functions of time). This is called *factor-augmenting* disembodied technological change. Since f_A is, assumed, a first-degree homogeneous production function, we can write it in the intensive form

$$A = \lambda' L_A g_A \left(\frac{\lambda}{\lambda'} \varrho_A \right). \quad (27.59)$$

We then define technical progress as being neutral, capital-saving, or labour-saving according as λ increases in the same proportion as, more than proportionally or less than proportionally to the increase in λ' , that is, according as the ratio λ/λ' remains unchanged, increases or decreases.³

Let us then assume, as in the text, that a technological change occurs in sector A , but not in sector B , so that the production function of the latter, $B = L_B g_B (\varrho_B)$, remains the same. The equilibrium conditions contained in the third and fourth equation of set (19.17) become, account being taken of Eq. (27.59),

³This definition is equivalent to the Hicks classification employed in the text, if the elasticity of substitution between the factors is smaller than one (see, for example, Vanek, 1966; see also Allen, 1967; Hicks, 1932; Johnson, 1955). To avoid unnecessary complications we assume this to be the case.

$$\begin{aligned} \lambda g'_A - p g'_B &= 0, \\ \lambda' g_A - \lambda \varrho_A g'_A - p (g_B - \varrho_B g'_B) &= 0. \end{aligned} \quad (27.60)$$

27.4.1 *Effects of Technical Progress on Factor Intensities and Rewards*

Let us begin by determining the changes in the factor intensities in the two sectors ϱ_A, ϱ_B . For this purpose we can consider Eqs. (27.60) as a set of implicit functions, so that, on the basis of the implicit function theorem, we can express ϱ_A and ϱ_B as differentiable functions of the parameters λ and λ' in a neighbourhood of the equilibrium point, provided that the Jacobian of (27.60) with respect to ϱ_A and ϱ_B , evaluated at the equilibrium point, is not zero. This Jacobian is

$$J = \begin{vmatrix} \lambda g''_A \frac{\lambda}{\lambda'} & -p g''_B \\ \lambda' g'_A \frac{\lambda}{\lambda'} - \lambda g'_A - \lambda \varrho_A g''_A \frac{\lambda}{\lambda'} - p g'_B + p g'_B + p \varrho_B g'_B \end{vmatrix}, \quad (27.61)$$

whence, noting that $\lambda = \lambda' = 1$ in the initial equilibrium point and simplifying,

$$J = -g''_A g''_B p (\varrho_B - \varrho_A), \quad (27.62)$$

which is different from zero since $\varrho_B \neq \varrho_A$. Thus there exist the functions

$$\begin{aligned} \varrho_A &= \varrho_A(\lambda, \lambda'), \\ \varrho_B &= \varrho_B(\lambda, \lambda'). \end{aligned} \quad (27.63)$$

If we differentiate (27.60) with respect to λ , account being taken of (27.63), we have

$$\begin{aligned} g'_A + \lambda g''_A \left(\frac{\varrho_A}{\lambda'} + \frac{\lambda}{\lambda'} \frac{\partial \varrho_A}{\partial \lambda} \right) - p g''_B \frac{\partial \varrho_B}{\partial \lambda} &= 0, \\ \lambda' g'_A \left(\frac{\varrho_A}{\lambda'} + \frac{\lambda}{\lambda'} \frac{\partial \varrho_A}{\partial \lambda} \right) - \varrho_A g'_A - \lambda \frac{\partial \varrho_A}{\partial \lambda} g'_A - \lambda \varrho_A g''_A \left(\frac{\varrho_A}{\lambda'} + \frac{\lambda}{\lambda'} \frac{\partial \varrho_A}{\partial \lambda} \right) \\ - p g'_B \frac{\partial \varrho_B}{\partial \lambda} + p g'_B \frac{\partial \varrho_B}{\partial \lambda} + p \varrho_B g''_B \frac{\partial \varrho_B}{\partial \lambda} &= 0. \end{aligned} \quad (27.64)$$

Again noting that initially $\lambda = \lambda'$, we can simplify and rearrange terms, thus obtaining

$$\begin{aligned} g''_A \frac{\partial \varrho_A}{\partial \lambda} - p g''_B \frac{\partial \varrho_B}{\partial \lambda} &= -g'_A - g''_A \varrho_A, \\ -\varrho_A g''_A \frac{\partial \varrho_A}{\partial \lambda} + p \varrho_B g''_B \frac{\partial \varrho_B}{\partial \lambda} &= g''_A \varrho_A^2, \end{aligned}$$

which, solved for $\partial Q_A/\partial\lambda$ and $\partial Q_B/\partial\lambda$, yields

$$\begin{aligned}\frac{\partial Q_A}{\partial\lambda} &= \frac{-pQ_B g_B'' g_A' - pQ_B g_B'' g_A'' Q_A + p g_B'' g_A'' Q_A^2}{-g_A'' g_B'' p (Q_B - Q_A)} \\ &= -\frac{-pQ_A g_A'' g_B'' (Q_B - Q_A) - pQ_B g_A' g_B''}{-g_A'' g_B'' p (Q_B - Q_A)} \\ &= -\left[Q_A + \frac{Q_B g_A'}{g_A'' (Q_B - Q_A)} \right],\end{aligned}\quad (27.65)$$

and

$$\frac{\partial Q_B}{\partial\lambda} = -\frac{Q_A g_B'}{p g_B'' (Q_B - Q_A)}.\quad (27.66)$$

By the same procedure (differentiate (27.60) with respect to λ' , account being taken of (27.63), and solve the resulting system) one gets

$$\frac{\partial Q_A}{\partial\lambda'} = Q_A - \frac{g_A - Q_A g_A'}{g_A'' (Q_B - Q_A)},\quad (27.67)$$

$$\frac{\partial Q_B}{\partial\lambda'} = -\frac{g_B - Q_B g_B'}{p g_B'' (Q_B - Q_A)}.\quad (27.68)$$

In general, both λ and λ' will increase as a consequence of technical progress; since we have assumed $\lambda = \lambda'$ initially, technical progress will be capital-saving-biased, neutral, labour-saving-biased according as

$$d\lambda \gtrless d\lambda', \text{ that is } d\lambda/d\lambda' \gtrless 1\quad (27.69)$$

If we introduce a parameter β measuring the ratio between the two changes, $\beta = d\lambda/d\lambda'$, we can rewrite (27.69) as

$$\beta \gtrless 1.\quad (27.70)$$

The total effect of technological change on Q_A, Q_B is obtained by calculating the total differentials of these. As regards Q_A we have

$$dQ_A = \frac{\partial Q_A}{\partial\lambda} d\lambda + \frac{\partial Q_A}{\partial\lambda'} d\lambda',\quad (27.71)$$

that is, given the definition of β ,

$$dQ_A = d\lambda' \left(\beta \frac{\partial Q_A}{\partial\lambda} + \frac{\partial Q_A}{\partial\lambda'} \right).\quad (27.72)$$

Since $d\lambda' > 0$, the sign of dQ_A depends solely on the sign of the expression in parentheses; substituting Eqs. (27.65) and (27.67) into it we get

$$\begin{aligned} \beta \frac{\partial Q_A}{\partial \lambda} + \frac{\partial Q_A}{\partial \lambda'} &= -\beta Q_A - \frac{\beta Q_B g'_A}{g''_A (Q_B - Q_A)} + Q_A - \frac{g_A - Q_A g'_A}{g''_A (Q_B - Q_A)} \\ &= Q_A (1 - \beta) - \frac{g_A - Q_A g'_A + \beta Q_B g'_A}{g''_A (Q_B - Q_A)} \\ &= Q_A (1 - \beta) - \frac{p (g_B - Q_B g'_B) + \beta p Q_B g'_B}{g''_A (Q_B - Q_A)}, \end{aligned} \quad (27.73)$$

where the last passage has been made possible by the equilibrium conditions (27.60) for $\lambda = \lambda'$.

Similarly, given

$$dQ_B = d\lambda' \left(\beta \frac{\partial Q_B}{\partial \lambda} + \frac{\partial Q_B}{\partial \lambda'} \right), \quad (27.74)$$

it follows that the sign of dQ_B depends solely on the sign of the following expression

$$\begin{aligned} \beta \frac{\partial Q_B}{\partial \lambda} + \frac{\partial Q_B}{\partial \lambda'} &= -\frac{\beta Q_A g'_B}{p g''_B (Q_B - Q_A)} - \frac{g_B - Q_B g'_B}{p g''_B (Q_B - Q_A)} \\ &= -\frac{g_B - Q_B g'_B + \beta Q_A g'_B}{p g''_B (Q_B - Q_A)}. \end{aligned} \quad (27.75)$$

Let us note for future reference that the numerator of the last fraction in (27.73) and the numerator of the last fraction in (27.75) are in any case positive. In fact, $g_B - Q_B g'_B$ is the marginal productivity of labour in sector B and g'_B is the marginal productivity of capital in the same sector.

We can now consider the various types of technical progress, beginning with the neutral one. Since $\beta = 1$, in Eq. (27.73) only the last fraction remains, which becomes

$$\frac{p g_B}{-g''_A (Q_B - Q_A)}. \quad (27.76)$$

Since $-g''_A > 0$ as $g''_A < 0$ (decreasing marginal productivity), the sign of (27.76) depends exclusively on the sign of $(Q_B - Q_A)$. Therefore

$$dQ_A \geq 0 \text{ according to whether } Q_B \geq Q_A. \quad (27.77)$$

If we now examine (27.75) we can see that, as the numerator is positive and $g''_B < 0$, its sign depends exclusively on $(Q_B - Q_A)$. Therefore

$$dQ_B \geq 0 \text{ according to whether } Q_B \geq Q_A. \quad (27.78)$$

It follows that Q_B and Q_A move in the same direction. If, for example (as assumed in the text), sector A , where technological change occurs, is intensive in capital, that is $Q_B < Q_A$, then factor intensities will decrease in both sectors.

When technical progress has a capital-saving bias, $\beta > 1$ and from Eqs. (27.73) and (27.75) it can easily be seen that both Q_B and Q_A decrease when $Q_A > Q_B$; when, conversely, $Q_B > Q_A$, the sign of dQ_A is ambiguous, whilst dQ_B is certainly negative.

Finally, when technical progress has a labour-saving bias, $\beta < 1$, and from Eqs. (27.73) and (27.75) it can be readily seen that both Q_B and Q_A increase when $Q_B > Q_A$; when, conversely, $Q_B < Q_A$, the sign of dQ_A is ambiguous, whilst Q_B certainly decreases.

Let us now examine *real factor rewards*. Letting $w = p_L/p_A$ and $r = p_K/p_A$, from the equilibrium conditions we have

$$\begin{aligned} w &= \lambda' g_A - \lambda_{Q_A} g'_A = p (g_B - Q_B g'_B), \\ r &= \lambda g'_A = p g'_B. \end{aligned} \quad (27.79)$$

It is clearly simpler to use the last expression in each of these two relations, so that

$$\begin{aligned} dw &= p (dg_B - g'_B dQ_B - Q_B dg'_B) \\ &= p (g'_B dQ_B - g'_B dQ_B - Q_B g''_B dQ_B) = -p Q_B g''_B dQ_B, \\ dr &= p g''_B dQ_B. \end{aligned} \quad (27.80)$$

Since $g''_B < 0$, we see that w (and so p_L , since p_A is fixed by assumption) moves in the same direction as Q_B whilst r (and so p_K) moves in the opposite direction, so that the relative price of factors $w/r = p_L/p_K$ certainly moves in the same direction as Q_B .

27.4.2 Effects of Technical Progress on Output Levels

As regards the *changes in output levels*, we begin with observing that, as technical progress brings about an outward shift of the transformation curve, it is *not* possible for both output levels to decrease: if one decreases, the other must necessarily increase. Since, as we shall see, in most cases it is the output of B which decreases, whilst in the remaining cases the change in the output of B has an ambiguous sign (hence A has too), we shall restrict ourselves to deriving the formulae that give dB , partly because these are relatively simpler. Since $B = L_B g_B(Q_B)$, we have

$$dB = g_B dL_B + L_B g'_B dQ_B. \quad (27.81)$$

We now express dL_B in terms of dQ_A and dQ_B . If we consider the total differentials of Eqs. (27.29) and recall that factor endowments are unchanged, we have

$$\begin{aligned} dL_A + dL_B &= 0, \\ \varrho_A dL_A + L_A d\varrho_A + \varrho_B dL_B + L_B d\varrho_B &= 0, \end{aligned} \quad (27.82)$$

whence, solving for dL_A, dL_B we get

$$dL_A = \frac{L_A d\varrho_A + L_B d\varrho_B}{\varrho_B - \varrho_A}, \quad dL_B = -\frac{L_A d\varrho_A + L_B d\varrho_B}{\varrho_B - \varrho_A}. \quad (27.83)$$

By substituting Eq. (27.83) into (27.81) and rearranging terms we get

$$dB = -\frac{g_B L_A}{\varrho_B - \varrho_A} d\varrho_A - \frac{(g_B - \varrho_B g'_B) L_B + g'_A \varrho_A L_B}{\varrho_B - \varrho_A} d\varrho_B. \quad (27.84)$$

It can be readily checked that the numerators of both fractions are positive: the first is obviously so; as regards the second it is sufficient to remember that $g_B - \varrho_B g'_B$ is the marginal productivity of labour in sector B . We can now examine the sign of dB in the various cases.

1. Neutral technical progress.

- (1a) $\varrho_B > \varrho_A$. We know from the previous analysis—see (27.77) and (27.78)—that $d\varrho_A > 0, d\varrho_B > 0$, and so from (27.84) we get $dB < 0$ (consequently dA will be positive).
- (1b) $\varrho_B < \varrho_A$. In this case $d\varrho_A < 0, d\varrho_B < 0$, so that from (27.84) we again have $dB < 0$ (and so $dA > 0$).

2. Capital-saving technical progress.

- (2a) $\varrho_B > \varrho_A$. In this case $d\varrho_B < 0$, whilst $d\varrho_A$ can have either sign. Therefore the sign of dB (and, consequently, that of dA) is ambiguous.
- (2b) $\varrho_B < \varrho_A$. Here we have $d\varrho_A < 0, d\varrho_B < 0$, so that $dB < 0$ (and, consequently, $dA > 0$).

3. Labour-saving technical progress.

- (3a) $\varrho_B > \varrho_A$. In this case $d\varrho_A > 0, d\varrho_B > 0$, and so $dB < 0$ (thus $dA > 0$).
- (3b) $\varrho_B < \varrho_A$. Here $d\varrho_B < 0$, whilst the sign of $d\varrho_A$ is ambiguous. Therefore dB (and, consequently, dA) have ambiguous signs.

Before passing on to the analysis of the effects of the various types of technological change on the terms of trade it is as well to summarize in tabular form all the results reached up to this point. In Table 27.2 the sign + denotes an increase, the sign – a decrease, the question mark an ambiguous result.

Table 27.2 Effects of the various types of technological change in sector A

Types of progress	$Q_A > Q_B$						$Q_B > Q_A$							
	Q_A	Q_B	p_L	p_K	$\frac{p_L}{p_K}$	A	B	Q_A	Q_B	p_L	p_K	$\frac{p_L}{p_K}$	A	B
Neutral	-	-	-	+	-	+	-	+	+	+	-	+	+	-
Cap.-saving	-	-	-	+	-	+	-	?	+	+	-	+	?	?
Lab.-saving	?	-	-	+	-	?	?	+	+	+	-	+	+	-

27.4.3 Effects of Technical Progress on the Terms of Trade

We can now examine the effects of technical progress on the terms of trade; as usual we must first determine the change in income (at unchanged p) to determine the marginal propensity to produce commodity A . If we consider the total differential (in which p has been kept constant)

$$dI_A = dA + pdB, \tag{27.85}$$

we only have to determine dA , as dB is known from Eqs. (27.29) and (27.84). It is however possible to obtain exact results without having to calculate dA . In fact, if we observe that technological change brings about an outward shift of the transformation curve, we realize that the isoincome which is tangent to the new transformation curve with the same slope as the one tangent to the previous transformation curve, is necessarily higher than the old one, so that $dI_A > 0$. Consider now the relation

$$\mu_{sY} = \left(\frac{dA}{dI_A} \right)_{p=\text{const}} = \left(\frac{dA}{dA + pdB} \right)_{p=\text{const}}, \tag{27.86}$$

from which it follows that, when $dA > 0$ and $dB < 0$, the numerator is certainly greater than the denominator, so that $\mu_{sY} > 1$. Now, as is clear from Table 27.2, all the cases in which technological change has unambiguous effects on productive levels are the ones in which A increases and B decreases. In the cases in which these effects are ambiguous it is, a fortiori, not possible to determine the value of μ_{sY} . Therefore, if we exclude inferior goods (so that $0 < \mu_{dY} < 1$), it follows from (27.21) that in all cases in which technical progress has unambiguous effects on productive levels (neutral technical progress, or factor-saving progress in the sector intensive in the saved factor) the relative price p_B/p_A increases and so p_A/p_B decreases. The terms of trade will therefore move against or in favour of the country according to whether the innovating sector (sector A in our case) is that producing the exportable or the importable. It is also clear that when A is the importable no immiserizing growth is possible, while it is possible when A is the exportable.

In all cases of factor-saving technical progress in the sector intensive in the other factor, the indeterminacy of μ_{sY} does not allow us to find the direction in which p moves, so that any result is possible.

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