

Chapter 28

Appendix to Chapter 14

28.1 A Dynamic Model of Growth and Trade

If we consider the growth-trade relations in a dynamic context, the static equations of the basic neoclassical model treated in Chap. 19 must be supplemented by the following differential equations (a dot indicates the time derivative and the subscript $i = 1, 2$ refers to the countries)

$$\begin{aligned}\dot{L}_i/L_i &= n_i, \\ \dot{K}_i &= S_i, \\ S_i &= s_i I_{iA} = A_i^D, \quad 0 < s_i < 1.\end{aligned}\tag{28.1}$$

The first equation expresses the assumption of an exogenously given growth rate of the labour force.

The second equation defines the increment in the stock of capital (investment) which equals saving, owing to the assumption that all saving gets automatically invested. Also note that we have assumed no technical progress and no capital depreciation; it would not be difficult to allow for factor-augmenting technological change and for “radioactive” depreciation. Factor-augmenting technical progress at constant rates could be taken into account by modifying the first two equations, for example $\dot{L}_i/L_i = n_i + \lambda_i$, $\dot{K}_i = S_i + \lambda'_i K_i$, where λ_i and λ'_i are the relevant rates. As regards radioactive depreciation, this implies depreciation allowances proportional to the existing capital stock, so that a quantity $\delta_i K_i$, $0 < \delta_i < 1$, has to be subtracted from (gross) investment to obtain net investment. However, for simplicity’s sake we ignore them.

The third equation is the saving function, proportional to national income (measured in terms of commodity A). Given the assumption that A is the investment good and that saving is automatically invested, saving coincides with the domestic demand for commodity A . In other words, these assumptions imply that the domestic demand for A , instead of having the general form $A = A_i^D(I_{iA}, p)$,

has the particularly simple form $A_i^D = s_i I_{iA}$. It goes without saying that saving-investment will be higher (lower) than A 's domestic output according to whether country i imports or exports this commodity. In what follows we assume that the structure of international trade is such that A is country 1's importable. To reconcile the assumption of internationally immobile factors of production (the stock of capital among them) with the fact that the investment good, as a produced commodity, is internationally traded, we must introduce the further assumption that this commodity, internationally mobile as a produced commodity subject to international trade, once installed as an addition to the capital stock of a country, becomes internationally immobile. In other words, stocks of existing capital are immobile, flows of investment goods are mobile and traded: to adapt terms used for other purposes in vintage growth theory, we might talk of *ex ante* international mobility, *ex post* immobility.

Before continuing, we note a feature of the steady-state growth path. In steady-state by definition all the physical variables (stock of capital, labour force, output, exports, imports etc.) must grow at the same rate, which coincides with the natural rate of growth, n_i . Now, if we exclude international lending, the usual conditions (19.27) for international equilibrium must hold, that is

$$E_{1A} + E_{2A} = 0, \quad (28.2)$$

whence

$$\dot{E}_{1A} + \dot{E}_{2A} = 0,$$

and so

$$E_{1A} \frac{\dot{E}_{1A}}{E_{1A}} + E_{2A} \frac{\dot{E}_{2A}}{E_{2A}} = 0, \quad (28.3)$$

which can be rewritten—since $E_{2A} = -E_{1A}$ by (28.2)—in the form

$$E_{1A} \left(\frac{\dot{E}_{1A}}{E_{1A}} - \frac{\dot{E}_{2A}}{E_{2A}} \right) = 0. \quad (28.4)$$

Similarly we obtain

$$E_{1B} \left(\frac{\dot{E}_{1B}}{E_{1B}} - \frac{\dot{E}_{2B}}{E_{2B}} \right) = 0. \quad (28.5)$$

Since, as we said above, in steady-state we have $\dot{E}_{1A}/E_{1A} = \dot{E}_{1B}/E_{1B} = n_1$ and $\dot{E}_{2A}/E_{2A} = \dot{E}_{2B}/E_{2B} = n_2$, from Eqs. (28.4) and (28.5) it follows that $n_1 = n_2$. In other words, in the model under consideration there can be no steady-state growth

unless the two countries have the same natural rate of growth. This limitation can be removed if we introduce international lending and investment, but, to keep complications to a minimum, we shall not do so (the reader is referred, for example, to Kemp (1969b) and Bardhan (1970)), i.e. we shall assume that all of a country's stock of capital is owned by residents. Thus we shall assume that $n_1 = n_2 = n$.

As in any growth model we must distinguish between the momentary or short-run equilibrium and the long-run equilibrium. The momentary equilibrium is that occurring at a given point of time. At any moment of time, the labour force and the capital stock are given (the former exogenously, and the latter as a result of past accumulation), so that the equations of the static model described in Chap. 19 determine the equilibrium quantities and prices: this is the momentary equilibrium. The dynamic equations (28.1) determine the path over time of factor endowments and so the steady growth path (the long run equilibrium). Before examining this, it is as well to show further properties of momentary equilibrium not treated in Chap. 19.

28.2 Momentary Equilibrium

If we denote by $q_i \equiv K_i/L_i$ the relative factor endowments in the two countries, the demand and supply functions, and so the excess demand functions, also depend on q_i , as we have seen in Sect. 21.3 (see, in particular, Eqs. (27.46) ff.). We can therefore write $E_{iA} = E_{iA}(p, q_i)$, and so the equation of international equilibrium (19.27) can be expressed as

$$E_{1A}(p, q_1) + E_{2A}(p, q_2) = 0, \quad (28.6)$$

by which—in accordance with the implicit function theorem—we can express p as a function of q_1, q_2 , provided that the condition $\partial E_{1A}/\partial p + \partial E_{2A}/\partial p \neq 0$ is satisfied. We know from Eq. (19.34) that $\partial E_{1A}/\partial p > 0$, similarly we have $\partial E_{2A}/\partial p > 0$; as the condition is satisfied, there exists the differentiable function

$$p = p(q_1, q_2), \quad (28.7)$$

of which we propose to calculate the partial derivatives. For this purpose we define the per-capita excess demands $e_{iA} = E_{iA}/L_i$, so that

$$L_{1A}e_{1A}(p, q_1) + L_{2A}e_{2A}(p, q_2) = 0. \quad (28.8)$$

If we divide through by $L_1 + L_2$ and define $\lambda_i = L_i/(L_1 + L_2)$, we get

$$\lambda_1 e_{1A}(p, q_1) + \lambda_2 e_{2A}(p, q_2) = 0, \quad (28.9)$$

where λ_1, λ_2 are constants owing to the assumption that L_1 and L_2 grow at the same rate n . From Eq. (28.9), account being taken of (28.7), we have

$$\frac{\partial p}{\partial Q_i} = -\frac{\partial e_{iA}/\partial Q_i}{\lambda_1 \frac{\partial e_{1A}}{\partial p} + \lambda_2 \frac{\partial e_{2A}}{\partial p}}. \quad (28.10)$$

Since $E_{iA} = L_i e_{iA}$, we have $L_i(\partial e_{iA}/\partial p) = \partial E_{iA}/\partial p$ and so the denominator in (28.10) is positive. As regards the numerator, we must find an explicit expression for the per-capita excess demand e_{iA} . If we recall that A is the investment good, the demand for which is proportional to income, we have

$$E_{iA} = s_i I_{iA} - L_i a_{iA} g_{iA}(Q_{iA}), \quad (28.11)$$

and so

$$e_{iA} = s_i y_{iA} - l_{iA} g_{iA}(Q_{iA}), \quad y_{iA} \equiv I_{iA}/L_i, \quad l_{iA} \equiv L_i a_{iA}/L_i. \quad (28.12)$$

Since each factor is paid its marginal product and the production functions are homogeneous of the first degree, Euler's theorem tells us that national income equals the sum of factor rewards,

$$I_{iA} = wL_i + rK_i, \quad (28.13)$$

where w is the *MPL* and r the *MPK*, both expressed in terms of commodity A . In per-capita terms,

$$y_{iA} = w + rQ_i, \quad (28.14)$$

and so

$$\frac{\partial y_{iA}}{\partial Q_i} = r = g'_{iA} > 0, \quad (28.15)$$

since g'_{iA} is the *MPK* in sector A .

For future reference, it should be noted that if we use the definition of income as value of output we have

$$y_{iA} = l_{iA} g_{iA}(Q_{iA}) + p l_{iB} g_{iB}(Q_{iB}), \quad (28.16)$$

whence

$$\frac{\partial y_{iA}}{\partial p} = l_{iB} g_{iB}(Q_{iB}) > 0. \quad (28.17)$$

It must be stressed that in (28.16) we assumed a situation of incomplete specialization, so that each country continues to produce both commodities. It is possible to carry out a complete taxonomy of all possible cases (for which see, e.g., Bardhan, 1970; Kemp, 1969b; Oniki & Uzawa, 1965), but we shall consider only the case of incomplete specialization in both countries. We now observe that from the full employment conditions we get

$$\begin{aligned} l_{iA} + l_{iB} &= 1, \\ Q_{iA}l_{iA} + Q_{iB}l_{iB} &= Q_i, \end{aligned} \quad (28.18)$$

from which

$$l_{iA} = \frac{Q_{iB} - Q_i}{Q_{iB} - Q_{iA}}, \quad l_{iB} = \frac{Q_i - Q_{iA}}{Q_{iB} - Q_{iA}}. \quad (28.19)$$

Equipped with these results we can calculate $\partial e_{iA}/\partial Q_i$. From Eq. (28.11) we have

$$\frac{\partial e_{iA}}{\partial Q_i} = s_i \frac{\partial y_{iA}}{\partial Q_i} - \frac{\partial l_{iA}}{\partial Q_i} g_{iA}, \quad (28.20)$$

and so, by substituting (28.15) into Eq. (28.20) and recalling Eqs. (28.19),

$$\frac{\partial e_{iA}}{\partial Q_i} = s_i g'_{iA} + \frac{1}{Q_{iB} - Q_{iA}} g_{iA}. \quad (28.21)$$

This derivative has an unambiguous sign only if $Q_{iB} > Q_{iA}$, that is, if the capital intensity is greater in the consumer good sector than in the capital good sector (this is the capital intensity condition, widely used in growth theory); for simplicity's sake, we assume that this condition holds,¹ so that $\partial e_{iA}/\partial Q_i$. It follows from this result that

$$\frac{\partial p}{\partial Q_i} < 0. \quad (28.22)$$

28.3 Long Run Equilibrium

We can now pass to the long-run equilibrium and consider the derivatives

$$\dot{Q}_i = \frac{\dot{K}_i L_i - \dot{L}_i K_i}{L_i^2} = \frac{\dot{K}_i}{L_i} - \frac{\dot{L}_i}{L_i} Q_i, \quad (28.23)$$

¹In Sect. 14.1 we adopted the opposite assumption, and warned the reader that in that case stability requires other conditions (for example that concerning the elasticity of substitution). Since the formal proof of stability would be further complicated, we prefer here to adopt the traditional capital-intensity condition.

whence, by using (28.1) and remembering that y_{iA} denotes per capita income, we have the differential equation system

$$\begin{aligned}\dot{Q}_1 &= s_1 y_{1A}(p, Q_1) - n Q_1, \\ \dot{Q}_2 &= s_2 y_{2A}(p, Q_2) - n Q_2,\end{aligned}\tag{28.24}$$

the solution of which determines the time paths of the relative factor endowments in the two countries, $Q_i(t)$, and, consequently, of all the other variables. To examine this system it is convenient to rewrite it in the form

$$\begin{aligned}\frac{\dot{Q}_1}{Q_1} &= \frac{s_1 y_{1A}(p, Q_1)}{Q_1} - n = \phi_1(Q_1, Q_2), \\ \frac{\dot{Q}_2}{Q_2} &= \frac{s_2 y_{2A}(p, Q_2)}{Q_2} - n = \phi_2(Q_1, Q_2).\end{aligned}\tag{28.25}$$

Let us now calculate the derivatives of the functions ϕ_i which are

$$\begin{aligned}\frac{\partial \phi_1}{\partial Q_1} &= \frac{s_1}{Q_1^2} \left[\left(\frac{\partial y_{1A}}{\partial p} \frac{\partial p}{\partial Q_1} + \frac{\partial y_{1A}}{\partial Q_1} \right) Q_1 - y_{1A} \right] \\ &= \frac{s_1}{Q_1^2} \left[\frac{\partial y_{1A}}{\partial p} \frac{\partial p}{\partial Q_1} - \left(y_{1A} - \frac{\partial y_{1A}}{\partial Q_1} Q_1 \right) \right] \\ &= \frac{s_1}{Q_1^2} \left[\frac{\partial y_{1A}}{\partial p} \frac{\partial p}{\partial Q_1} - (y_{1A} - g'_{1A} Q_1) \right],\end{aligned}\tag{28.26}$$

$$\frac{\partial \phi_1}{\partial Q_2} = \frac{s_1}{Q_1} \frac{\partial y_{1A}}{\partial p} \frac{\partial p}{\partial Q_2},\tag{28.27}$$

$$\frac{\partial \phi_2}{\partial Q_1} = \frac{s_2}{Q_2} \frac{\partial y_{2A}}{\partial p} \frac{\partial p}{\partial Q_1},\tag{28.28}$$

$$\begin{aligned}\frac{\partial \phi_2}{\partial Q_2} &= \frac{s_2}{Q_2^2} \left[\left(\frac{\partial y_{2A}}{\partial p} \frac{\partial p}{\partial Q_2} + \frac{\partial y_{2A}}{\partial Q_2} \right) Q_2 - y_{2A} \right] \\ &= \frac{s_2}{Q_2^2} \left[\frac{\partial y_{2A}}{\partial p} \frac{\partial p}{\partial Q_2} - (y_{2A} - g'_{2A} Q_2) \right].\end{aligned}\tag{28.29}$$

From (28.17) and (28.22) it immediately follows that $\partial \phi_1 / \partial Q_2$ and $\partial \phi_2 / \partial Q_1$ are both negative. As regards $\partial \phi_1 / \partial Q_1$ and $\partial \phi_2 / \partial Q_2$ —if we note that $y_{iA} - g'_{iA} Q_i$ is positive as it represents (see Eq. (28.14)) the *MPL*—we see that they are negative as well. The reader can also check as an exercise that

$$\frac{\partial \phi_1 / \partial Q_1}{\partial \phi_1 / \partial Q_2} > \frac{\partial \phi_2 / \partial Q_1}{\partial \phi_2 / \partial Q_2}.\tag{28.30}$$

The steady growth solution of system (28.25) corresponds to $\dot{q}_1 = \dot{q}_2 = 0$, that is, to the singular point of the system. Thus we must consider the equations

$$\begin{aligned}\phi_1(q_1, q_2) &= 0, \\ \phi_2(q_1, q_2) &= 0,\end{aligned}\tag{28.31}$$

the solution of which determines the equilibrium values q_1^E, q_2^E . Now, each of the relations in system (28.31) determines a curve in the (q_1, q_2) plane, with slope

$$\left(\frac{dq_2}{dq_1}\right)_{\phi_1=0} = -\frac{\partial\phi_1/\partial q_1}{\partial\phi_1/\partial q_2}, \quad \left(\frac{dq_2}{dq_1}\right)_{\phi_2=0} = -\frac{\partial\phi_2/\partial q_1}{\partial\phi_2/\partial q_2}.\tag{28.32}$$

Since—given the signs of the partial derivatives and given (28.30)—in the relevant interval both curves are monotonically decreasing though with different slopes, they will cross only once in that interval.²

Having thus ascertained the existence and uniqueness of equilibrium, we examine its stability. For this purpose we observe that, as the q_i can take on positive values only, we can introduce the following transformation of variables

$$x_i = \ln q_i, \quad \text{hence } q_i = e^{x_i}, \quad \dot{x}_i = \dot{q}_i/q_i,\tag{28.33}$$

so that system (28.25) can be rewritten as

$$\begin{aligned}\dot{x}_1 &= \frac{s_1 y_{1A}(p, e^{x_1})}{e^{x_1}} - n = \omega_1(x_1, x_2), \\ \dot{x}_2 &= \frac{s_2 y_{2A}(p, e^{x_2})}{e^{x_2}} - n = \omega_2(x_1, x_2).\end{aligned}\tag{28.34}$$

Since the x_i variables are a monotonically increasing transformation of the q_i , the stability of system (28.34) implies the stability of system (28.25). We could examine the stability by way of phase diagrams, but we prefer to use more powerful analytical methods. Let us first observe that

$$\frac{\partial\omega_j}{\partial x_i} = \frac{\partial\phi_j}{\partial q_i} \frac{\partial q_i}{\partial x_i} = \frac{\partial\phi_j}{\partial q_i} q_i, \quad i, j = 1, 2,\tag{28.35}$$

²That is, in the interval of incomplete specialization. For a more detailed analysis of the existence and uniqueness of equilibrium in this and all other possible cases, see, e.g., the already cited works of Kemp (1969b), Oniki and Uzawa (1965), and Bardhan (1970).

so that the Jacobian matrix $[\partial\omega_j/\partial x_i]$ of system (28.34) turns out to be

$$\mathbf{J} = \begin{vmatrix} \frac{\partial\omega_1}{\partial x_1} & \frac{\partial\omega_1}{\partial x_2} \\ \frac{\partial\omega_2}{\partial x_1} & \frac{\partial\omega_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} \frac{\partial\phi_1}{\partial Q_1} \varrho_1 & \frac{\partial\phi_1}{\partial Q_2} \varrho_2 \\ \frac{\partial\phi_2}{\partial Q_1} \varrho_1 & \frac{\partial\phi_2}{\partial Q_2} \varrho_2 \end{vmatrix}. \quad (28.36)$$

Since, as we have seen above, $\partial\phi_j/\partial Q_i < 0$, $i, j = 1, 2$, and given (28.30), it follows that, everywhere in the relevant interval,

$$\begin{aligned} \frac{\partial\omega_1}{\partial x_1} + \frac{\partial\omega_2}{\partial x_2} &< 0, \\ |\mathbf{J}| &> 0, \\ \frac{\partial\omega_1}{\partial x_1} \frac{\partial\omega_2}{\partial x_2} &\neq 0. \end{aligned} \quad (28.37)$$

Therefore, the conditions of Olech's theorem (see, for example, Gandolfo, 2009, p. 376) are satisfied and, consequently, the equilibrium state is globally stable.

Let us note, as a conclusion, that we have examined the problem of the dynamic relations between trade and growth solely in the context of positive economics. Normative problems (including intertemporal welfare maximization) are surveyed in Bhagwati and Srinivasan (1983, chap. 31); see also Smith (1977) and Findlay (1984). For an alternative view to the traditional one, see Parrinello (1979) and Steedman (1979); see also Smith (1984).

References

- Bardhan, P. K. (1970). *Economic growth, development, and foreign trade: A study in pure theory*.
 Bhagwati, J. N., & Srinivasan, T. N. (1983). *Lectures on international trade*.
 Findlay, R. (1984). Growth and development in trade models.
 Gandolfo, G. (2009). *Economic dynamics*.
 Kemp, M. C. (1969b). *The pure theory of international trade and investment*.
 Oniki, H., & Uzawa, H. (1965). Patterns of trade and investment in a dynamic model of international trade.
 Parrinello, S. (1979). Distribution, growth, and international trade.
 Smith, M. A. M. (1977). Capital accumulation in the open two-sector economy.
 Smith, A. (1984). Capital theory and trade theory.
 Steedman, I. (1979). *Trade amongst growing economies*.