Chapter 29
Appendix to Chapter 15

29.1 Endogenous Growth and Traditional Trade Theory

The constant-returns-to scale production functions for the tradable goods A, B and for the R&D services Z can be written in the intensive form (see Sect. 19.2.1)

$$\begin{aligned} A(t) &= \lambda(t) L_A g_A(\rho_A). \\ B(t) &= \lambda(t) L_B g_B(\rho_B), \\ Z &= L_Z g_Z(\rho_Z), \end{aligned} \tag{29.1}$$

where $\rho_i \equiv K_i/L_i, i = A, B, Z$ are the factor intensities in the three sectors, and $\lambda(t)$ is an index of technological efficiency, assumed uniform for both tradables. The crucial assumption (Findlay, 1995) is that $\lambda(t)$ is endogenously determined by the per-capita output of R&D services. More precisely, the proportional rate of change of λ is assumed to depend on $z = Z/L = l_Z g_Z(\rho_Z)$, where $L = L_A + L_B + L_Z$ is the given and constant amount of labour existing in the economy and $l_Z = L_Z/L$ is the share of the labour force employed in the R&D sector:

$$\dot{\lambda}/\lambda = \phi(z), \quad \phi'(z) > 0, \quad \phi''(z) < 0. \tag{29.2}$$

A greater output of R&D services enhances the rate of technical progress, but this enhancement is subject to diminishing returns ($\phi''(z) < 0$).

Due to the small open economy assumption, the terms of trade or relative price of tradables $p = p_B/p_A$, is given from the outside. Let us take good A as the numéraire, so that $p = p_B$. At the initial time $t = 0$ we can set $A(0) = 1$ and, assuming that all three goods are produced, knowledge of the relative price p_B and of the production functions is sufficient to determine all factor intensities and rewards as well as the price of the non-traded good p_Z in terms of the numéraire (see Sects. 6.6 and 22.5) independently of demand conditions.

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To solve the optimal resource allocation problem between tradables and R&D (since K and L are given and constant, more current output of tradables means less technical progress and vice versa) Findlay first defines the value of the per capita output of tradables. This can easily be done thanks to the assumption of a given p. Denoting such a value by $\nu(t)$, we have

$$\begin{aligned}\nu(t) &= [A(t) + p_B B(t)]/L = \lambda(t)\,[L_A g_A(\rho_A) + p_B L_B g_B(\rho_B)]/L \\ &= \lambda(t)\,[l_A g_A(\rho_A) + p_B l_B g_B(\rho_B)],\end{aligned} \tag{29.3}$$

where $l_A = L_A/L, l_B = L_B/L$ are the shares of the total labour force allocated to tradables. For any given amount of labour L_Z allocated to the R&D sector, the remaining amount $L-L_Z$ will be optimally allocated to A and B so that (the optimal values of) l_A, l_B will be determined. Hence ν will change through time only because of technical progress λ, since the expression in square brackets is constant as long as L_Z is constant.

Equation (29.3) clearly shows the trade-off between technical progress and current output. If more resources are allocated to R&D, namely if more z is produced, less resources will be allocated to A and B. Hence at any given point in time per capita output $l_A g_A(\rho_A) + p_B l_B g_B(\rho_B)$ will negatively depend on z, so that we can write

$$l_A g_A(\rho_A) + p_B l_B g_B(\rho_B) = \upsilon(z), \quad \upsilon'(z) < 0, \quad \upsilon''(z) = 0. \tag{29.4}$$

Let us observe that $\upsilon''(z) = 0$ is not an assumption, for υ is a negative linear function of z. To show this, let us first observe that, since the relative price p_B is fixed, we can apply Hicks' theorem (1939, 1946) according to which, if the relative prices of a group of goods remain constant as the quantity of the goods themselves varies, the different goods in the group can be considered as a single whole, that is, as if they were a single good. We next observe that, at any given point in time, in our neoclassical perfectly competitive setting resources are optimally allocated, which implies that $(-\mathrm{d}\upsilon/\mathrm{d}z)$, namely the marginal rate of transformation between the composite commodity υ (tradables) and z, equals the price ratio p_B/p_Z (see Chap. 3). Since, as we have seen above, not only p_B but also p_Z is given from the outside, it follows that $\upsilon'(z) = \mathrm{d}\upsilon/\mathrm{d}z = -p_B/p_Z$ is a constant.

To ascertain the effect of R&D expenditure on the future value of the output of tradable goods we begin by differentiating $\nu(t)$ with respect to time. This gives

$$\begin{aligned}\dot{\nu} &= \dot{\lambda}\,[l_A g_A(\rho_A) + p_B l_B g_B(\rho_B)] \\ &= \lambda(t)\phi(z)\,[l_A g_A(\rho_A) + p_B l_B g_B(\rho_B)] \\ &= \lambda(t)\phi(z)\upsilon(z),\end{aligned} \tag{29.5}$$

where we have used (29.2) and (29.4). Setting $\lambda(t) = 1$ at $t = 0$ (see above) we can ascertain the marginal benefit from R&D by differentiating Eq. (29.5) with respect to z:

$$\left(\frac{\partial \dot{\nu}}{\partial z}\right)_{t=0} = \phi'(z)\upsilon(z) + \phi(z)\upsilon'(z) = \frac{\phi(z)\upsilon(z)}{z}\left[\frac{z}{\phi}\frac{\mathrm{d}\phi(z)}{\mathrm{d}z} + \frac{z}{\upsilon}\frac{\mathrm{d}\upsilon(z)}{\mathrm{d}z}\right], \tag{29.6}$$

where the two expressions in square brackets are the elasticities of $\phi(z)$ and $\upsilon(z)$ with respect to z.

Since this marginal benefit will accrue from now to infinity, its present value is simply

$$\frac{1}{\delta}\left(\frac{\partial \dot{\nu}}{\partial z}\right)_{t=0}. \tag{29.7}$$

Under perfect competition, the marginal cost of a unit of R&D output is simply p_Z. Hence marginal benefit and marginal cost are equated when

$$\frac{1}{\delta}\left(\frac{\partial \dot{\nu}}{\partial z}\right)_{t=0} = p_Z$$

or

$$\frac{\phi(z)\upsilon(z)}{z}\left[\frac{z}{\phi}\frac{\mathrm{d}\phi(z)}{\mathrm{d}z} + \frac{z}{\upsilon}\frac{\mathrm{d}\upsilon(z)}{\mathrm{d}z}\right] = \delta p_Z. \tag{29.8}$$

The second order condition for a maximum is

$$\left(\frac{\partial^2 \dot{\nu}}{\partial z^2}\right)_{t=0} = \left[\phi''(z)\upsilon(z) + 2\phi'(z)\upsilon'(z) + \phi(z)\upsilon''(z)\right] < 0, \tag{29.9}$$

which implies the concavity of the $\dot{\nu}$ function (curve *OF* shown in Fig. 15.1 in the text). Condition (29.9) is certainly satisfied given the signs of the various derivatives.

29.2 Endogenous Growth and Trade: Innovation, Imitation, and Product Cycles

The model described here is due to Grossman and Helpman (1991a, 1991b, 1991c).

29.2.1 Demand

There is a continuum of different products, indexed by $j \in [0, 1]$. At any moment only a subset of these products is available, represented by the number of brands that have been so far developed. The number of available brands is represented by the set $[0, n(t)]$, where $n(t)$ can be considered as the measure of products developed before time t. In each product line j there are several qualities ranked in increasing order of quality from 0 to m; the highest quality available is the state-of-the-art. The quality $q_i(j)$ is represented by a parameter $\lambda > 1$, and units of quality are chosen so that the lowest quality existing (quality 0) offers one unit of service, so that $q_0(j) = \lambda^0 = 1$, and $q_m(j) = \lambda^m$.

The representative household is assumed to maximize utility over an infinite horizon

$$U_t = \int_t^\infty e^{-\rho(\tau - t)} \log D(\tau) d\tau, \tag{29.10}$$

where $D(\tau)$ is an index of consumption at time τ and ρ is the subjective discount rate. The natural logarithm of $D(\tau)$ measures instantaneous utility at time τ.

To simplify the problem we assume functional separability, so that the household can solve its intertemporal maximization problem in two stages. In one stage it chooses the composition of any given level of expenditure so as to maximize instantaneous utility. In the other stage it optimizes the time path of spending. The two stages can be taken in any order. Let us begin with the intertemporal maximization problem.

The household is endowed with one unit of labour and possesses an amount of wealth $W(t)$. It can freely lend or borrow at the instantaneous interest rate $r(t)$. The household's intertemporal maximization problem subject to its intertemporal budget constraint can thus be written as

$$\begin{aligned} &\max U_t = \int_t^\infty e^{-\rho(\tau - t)} \log D(\tau) d\tau \\ &\text{sub} \\ &\int_t^\infty e^{-[R(\tau) - R(t)]} P_D(\tau) D(\tau) d\tau \leq \int_t^\infty e^{-[R(\tau) - R(t)]} w(\tau) d\tau + W(t), \end{aligned} \tag{29.11}$$

where $R(\tau) = \int_0^\tau r(s) ds$ is the discount factor from time τ to time zero, $P_D(\tau)$ is the aggregate price index corresponding to the quantity index $D(\tau)$ in the instantaneous budget constraint

$$D(\tau) = \frac{E(\tau)}{P_D(\tau)}, \tag{29.12}$$

where $E(\tau)$ is the given value of spending, and $w(\tau)$ is the wage rate. The intertemporal budget constraint requires the present value of spending not to exceed the present value of (labour) income plus the initial wealth. index corresponding to the quantity index $D(\tau)$.

The maximization problem (29.37) is a simple calculus-of-variations problem with an integral constraint. It can be solved (see, for example, Gandolfo, 2009, chap. 27, sect. 27.2.2; Kamien & Schwartz, 1991, chap. 7) by forming the Lagrangian

$$\Lambda = \mathrm{e}^{-\rho(\tau-t)} \log D(\tau) - \zeta(t) \left\{ \mathrm{e}^{-[R(\tau)-R(t)]} \left[P_D(\tau) D(\tau) - w(\tau) \right] \right\}, \tag{29.13}$$

where ζ is a Lagrange multiplier, and then calculating the first-order condition

$$\frac{\partial \Lambda}{\partial D(\tau)} = \frac{\mathrm{e}^{-\rho(\tau-t)}}{D(\tau)} - \zeta(t) \mathrm{e}^{-[R(\tau)-R(t)]} P_D(\tau) = 0, \quad \forall \tau \geq t, \tag{29.14}$$

whence

$$\frac{\mathrm{e}^{-\rho(\tau-t)}}{D(\tau)} = \zeta(t) \mathrm{e}^{-[R(\tau)-R(t)]} P_D(\tau). \tag{29.15}$$

Let us now consider household spending as defined in (29.12), and differentiate it logarithmically with respect to time (τ). We have

$$\frac{\dot{E}}{E} = \frac{\dot{D}}{D} + \frac{\dot{P}_D}{P_D}.$$

Logarithmic differentiation of (29.15) with respect to τ yields

$$-\rho - \frac{\dot{D}}{D} = -\frac{\mathrm{d}}{\mathrm{d}t} R(t) + \frac{\dot{P}_D}{P_D}.$$

Now,

$$\frac{\mathrm{d}}{\mathrm{d}t} R(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^\tau r(s) \mathrm{d}s = r(\tau),$$

where we have used the rule for the differentiation of an integral with respect to a parameter that occurs in the limit(s) of integration.

Substituting these results into the expression for $\dot{E}/E$ we obtain

$$\frac{\dot{E}}{E} = r - \rho, \tag{29.16}$$

namely in each instant aggregate expenditure must grow at a rate given by the difference between the interest rate and the subjective discount rate.

For convenience Grossman and Helpman (1991a) use as a normalization condition the condition that nominal spending remains constant,

$$E = 1. \tag{29.17}$$

From (29.16) and (29.17) we get

$$r = \rho, \tag{29.18}$$

namely the nominal interest rate is always equal to the subjective discount rate.

Let us now consider the stage in which the household chooses the composition of the given level of expenditure so as to maximize instantaneous utility.

Preferences embodied in instantaneous utility are assumed to be of the type

$$\log D(t) = \int_0^1 \log \left[\sum_m q_m(j) x_{mt}(j) \right] \mathrm{d}j, \tag{29.19}$$

where the summation extends over the set of qualities of product j that is available at time t. These preferences have the convenient property that vertically differentiated products in any industry are perfect substitutes for one another, once quantities are adjusted for quality differences (for example, one $1/\lambda^m$ units of the state-of-the-art product are equivalent to one unit of the lowest quality product). Further, products of different industries enter the utility index symmetrically, and the elasticity of substitution between every pair of product lines is constant and equal to one. We know that with this sort of preferences (see Sect. 23.2.1) the household's income will be equally divided between all available product lines; besides, in each line the household will purchase only one quality, namely the quality $\tilde{m}_t(j)$ that has the lowest price per unit of quality. Thus we have the demand functions

$$x_{mt}(j) = \begin{cases} \dfrac{E(t)}{p_{mt}(j)} & \text{for } m = \tilde{m}_t(j), \\ \\ 0 & \text{otherwise,} \end{cases}$$

where $p_{mt}(j)$ is the price of quality m at time t, and $E(t)$ denotes spending at time t.

When we consider a two-country world, the representative household is taken to be the same in both countries as is the interest rate, and the normalization condition on $E(t)$ also refers to world expenditure.

29.2.2 *Supply: Product Quality, Innovation, and Imitation*

Labour is the only factor of production in each region (North and South), and can be used for manufacturing or research.

29.2.2.1 The Manufacturing Sector

In manufacturing, production takes place under fixed coefficients, and units are chosen so that one unit of labour is needed to produce one unit of output, whatever the quality (the development of a higher-quality commodity requires R&D expenditure, on which more below, but when this higher-quality commodity is manufactured, its labour requirement is the same as that for lower qualities). Thus marginal and average cost is simply the wage rate. Letting w^N, w^S respectively denote the wage rate in North and South, and assuming that oligopolistic competition takes the form of price competition, the possibility of imitation by South requires $w^N > w^S$.

The three categories of firms have been described in the text (see Sect. 15.3), and their measure is denoted by n^{NN}, n^{NS}, n^S, normalized in such a way that

$$n^{NN} + n^{NS} + n^S = 1. \tag{29.20}$$

Let us now consider the profit rate for each category of firms and show that each good in the continuum has only one producer, beginning with a category (iii) firm. This is a Southern firm that has mastered via imitation the technique for producing some state-of-the-art product, and competes with the Northern firm that has developed this product. Assuming a Bertrand duopoly (hence price competition: on Bertrand duopoly in general see, for example, Mas-Colell, Whinston, & Green, 1995, chap. 12, sect. 12.C), the Southern firm perceives a perfectly elastic demand when it charges a price of w^N (the unit production cost of its Northern rival), a unit elastic demand at lower prices, and zero demand at higher prices. It follows that this firm maximizes profit by setting a limit price exactly equal to w^N, the unit cost of production of its Northern rival. The latter will of course have zero profit and hence leave the market. Also, no other Southern firm is willing to invest resources just to become the second regional producer of the commodity and earn zero profit (in fact, if two Southern firms learn to imitate the same state-of-the-art product, in their Bertrand competition each sets a price equal to its marginal cost w^S and earns zero profit).

Thus there will be only one producer of the commodity, a Southern firm, whose sales at the price w^N are $E/w^N = 1/w^N$ (given the normalization condition $E = 1$, see Sect. 29.2.1). Its profit rate is

$$\pi^S = \frac{w^N - w^S}{w^N}. \tag{29.21}$$

We next consider a category (ii) firm, namely a Northern firm that has just innovated and can produce the next generation commodity, while the Southern firm that is producing the previous state-of-the-art commodity finds that its product is now second-to-top. The Northern leader can charge a quality premium λ over the price of the second-to-top commodity. Since the Southern firm cannot profitably set a price below w^S, the Northern leader can capture the entire market by setting a price marginally below λw^S and hence selling $1/\lambda w^S$ units of output. This outcome will of course only be possible if the quality premium is sufficiently great so that the Northern leader can charge a price higher than its unit cost of production, namely it is required that $\lambda w^S > w^N$. This is assumed to be the case, for otherwise there would be no innovation at all.

Hence the profit rate of the Northern leader competing with a Southern firm will be

$$\pi^{NS} = \frac{\lambda w^S - w^N}{\lambda w^S}. \tag{29.22}$$

We finally consider a category (i) firm, namely a Northern leader that has just innovated and is competing with a Northern follower that can produce the second highest quality. In this case the limit price that maximizes the leader's profit and captures the entire market is λw^N, which entails an output of $1/\lambda w^N$. Hence the leader's profit rate will be

$$\pi^{NN} = \frac{\lambda w^N - w^N}{\lambda w^N}. \tag{29.23}$$

When a second Northern firm gains the ability to produce the same top-quality product, price competition between these two firms drives both to charge a price equal to w^N, hence zero profits. Since innovation and imitation are costly activities, there can be only one producer in this case as well.

29.2.2.2 Innovation

Innovation is assumed to take place only in North. New higher-quality products can be developed only through costly R&D activity. Broadly following Aghion and Howitt (1990, 1992), Grossman and Helpman model this activity as a risky process, in the sense that an entrepreneur who devotes resources to R&D has a probability of success (i.e., of actually developing a new higher-quality variety of the product) that is proportional to the scale of his efforts but smaller than unity. More precisely, a Northern entrepreneur can obtain a probability $\iota \mathrm{d}t$ of success in the time interval $\mathrm{d}t$ by devoting $a\iota$ units of labour to research during $\mathrm{d}t$.

We must however distinguish between leaders and followers. A Northern leader, through the development of the current top-quality product, has accumulated a stock of knowledge and product-specific information that can help to achieve a further technological advance. This knowledge is not available to other firms, hence a Northern follower will have to invest a greater amount of resources to try to upgrade the state-of-the-art product (i.e., to develop the next generation product). Let $a_L \iota < a_F \iota$ (where a_L, a_F are constant coefficients) be the amount of labour that a leader and a follower must respectively allocate to research in the time interval $\mathrm{d}t$ to achieve the probability $\iota \mathrm{d}t$ of success.

This research advantage entails that only leaders will invest resources in R&D to try and recapture the market when a Southern firm has successfully imitated the state-of-the-art product.

To show this, let us recall that successful imitation by a Southern firm crowds the Northern leader out of the market. When this occurs, neither the Northern leader nor Northern followers are producing in the product line under consideration. Hence a Northern firm that succeeds in upgrading the (South-produced) state-of-the-art product earns the same profit independently of whether it is a leader or a follower, as it makes the same two-step profit jump (from zero to the profit corresponding to the production of the next-generation commodity). Since followers undergo higher research costs, they will be crowded out by leaders.

Let us indicate by $\iota^S dt$ the probability of success of a Northern firm that undertakes research because its state-of-the-art product has been copied by a Southern imitator (the superscript S serves to indicate that the research effort is targeted at upgrading a Southern product). This Northern firm undergoes a research cost of $w^N a_L \iota^S dt$ and has an expected gain of $\upsilon^{NS}\iota^S dt$, where υ^{NS} denotes the value of a category (ii) firm, namely a Northern firm that has a Southern firm as its closest competitor. Maximization of net expected value $\upsilon^{NS}\iota^S dt - w^N a_L \iota^S dt$ gives

$$\upsilon^{NS} \leq w^N a_L, \quad \iota^S \geq 0, \tag{29.24}$$

with $\upsilon^{NS} = w^N a_L$ if $\iota^S > 0$.

We now consider the case in which a state-of-the-art product has escaped imitation by South (imitation activity is not always successful). Then not only the Northern leader but also Northern followers have an incentive to undertake research leading to the development of the next generation of products, as in such a situation they stand to gain more from a research success than do leaders.

In fact, the Northern leader would pass from its current positive profit (deriving from the production of the state-of-the-art commodity) to the higher profit deriving from the production of the next-generation commodity, hence it would make a one-step jump in profits. A Northern follower would instead pass from its current situation of zero profit (as it is currently producing nothing) to the profit deriving from the production of the next-generation commodity, hence it would make a two-step jump in profits.

On the other hand, the leader has a research advantage over the follower hence lower research costs as shown above. Thus both leaders and followers may undertake research. It is assumed (Grossman & Helpman, 1991a, p. 315) that only followers do. Since no leader undertakes research, this implies that there will be at most a single quality step between any leader and its closest competitor: if, in fact, research by a follower is successful, this follower will become a leader making a two-step jump, and the previous leader will become a follower with a one-step gap (the previous state-of-the-art-product is now second highest).

If we denote by υ^{NN} the value of a Northern firm (a follower that may become a leader thanks to successful innovation) that has another Northern firm (the previous leader) as its nearest competitor, this firm undergoes a research cost of $w^N a_F \iota^N dt$ and has an expected gain of $\upsilon^{NN}\iota^N dt$. Maximization of net expected value gives

$$\upsilon^{NN} \leq w^N a_F, \quad \text{with equality if } \iota^N > 0. \tag{29.25}$$

29.2.2.3 Imitation

Southern firms cannot develop next generation products but can copy any state-of-the-art product developed by Northern firms. Imitation, however, is a costly and risky venture that requires investment of resources in research with a related probability of success (i.e., of succeeding in developing a marketable copy of the

Northern state-of-the-art product), and is modelled like innovation. A Southern imitator can achieve a probability $m\mathrm{d}t$ of success in the time interval $\mathrm{d}t$ by devoting $a_m m$ units of labour to research during $\mathrm{d}t$. Maximization of net expected value $\upsilon^S m\mathrm{d}t - w^S a_m m\mathrm{d}t$ gives

$$\upsilon^S \leq w^S a_m, \quad \text{with equality if } m > 0. \tag{29.26}$$

29.2.3 *The No-Arbitrage Conditions*

Let us consider a category (i) firm, namely a Northern leader facing competition by a Northern follower. In a small interval of time $\mathrm{d}t$ the owners of the firm collect profits $\pi^{NN}\mathrm{d}t$, have a capital gain (or loss) given by the normal change in the firm's value $\dot{\upsilon}^{NN}\mathrm{d}t$, and have a probability $(\iota^N + m)\mathrm{d}t$ of suffering the total loss of the firm. In fact, the leading firm loses its market—and hence its value drops to zero—both when another Northern entrepreneur develops the next-generation product (an event that has the probability $\iota^N\mathrm{d}t$ of occurring, as shown above) and when a Southern imitator successfully copies its product (an event that has the probability $m\mathrm{d}t$ of occurring). The expected value of such a loss is $\upsilon^{NN}(\iota^N + m)\mathrm{d}t$.

Expressing these magnitudes as instantaneous rates, namely dividing them by $\upsilon^{NN}\mathrm{d}t$, the standard no-arbitrage condition (namely the condition that the firm's equities yield a normal rate of return) requires that the rate of profit plus the rate of normal capital gain or loss, minus the expected rate of total loss be equal to the current rate of interest in North, namely

$$\frac{\pi^{NN}}{\upsilon^{NN}} + \frac{\dot{\upsilon}^{NN}}{\upsilon^{NN}} - (\iota^N + m) = r^N. \tag{29.27}$$

Similarly a category (ii) firm, namely a Northern firm that has just upgraded a state-of-the-art commodity produced in South, gives rise to the condition

$$\frac{\pi^{NS}}{\upsilon^{NS}} + \frac{\dot{\upsilon}^{NS}}{\upsilon^{NS}} - (\iota^N + m) = r^N, \tag{29.28}$$

since the probability of total loss is the same as in the previous case.

We finally have a category (iii) firm, namely a Southern firm that has successfully copied a state-of-the-art product previously manufactured in North. As seen above, in the time interval $\mathrm{d}t$ this firm faces a probability $\iota^S\mathrm{d}t$ of displacement by a Northern firm that succeeds in upgrading the product. The usual no-arbitrage condition gives

$$\frac{\pi^S}{\upsilon^S} + \frac{\dot{\upsilon}^S}{\upsilon^S} - \iota^s = r^S, \tag{29.29}$$

where r^S is the rate of interest in South.

29.2.4 The Labour Markets

Labour markets are in equilibrium at every moment of time. Let us begin with North. Northern labour is employed in manufacturing and research.

In Northern manufacturing, the n^{NN} firms that have a Northern firm as their nearest competitor produce $1/\lambda w^N$ each. Since we have assumed that one unit of labour is required per unit of output, total labour demand by these firms is $n^{NN}/\lambda w^N$. Similarly the labour demand by the n^{NS} Northern firm that have a Southern firm as their closest competitor is $n^{NS}/\lambda w^S$.

To determine labour demand in the Northern R&D sector, we first recall that n^S is the number of successful Southern imitators and hence of imitated products. It follows that the measure of Northern firms engaged in research aimed at developing the next generation product to replace each imitated product will be n^S; each of these firms will demand $a_L\iota^S$ units of labour for research (see above), hence a total demand of $n^S a_L \iota^S$. Similarly total labour demand for research by Northern followers targeted at upgrading each of n^N state-of-the art goods produced by Northern leaders will be $n^N a_F \iota^N$.

In conclusion, clearing of the Northern labour market requires

$$\frac{n^{NN}}{\lambda w^N} + \frac{n^{NS}}{\lambda w^S} + n^S a_L \iota^S + n^N a_F \iota^N = L^N, \tag{29.30}$$

where L^N is the exogenously given labour supply in North.

Let us now consider South. The manufacturing sector produces $1/w^N$ units each of n^S imitated products, hence a total labour demand of $n^S\ /w^N$ units of labour.

As regards the Southern R&D sector, research for imitation is aimed at imitating n^N state-of-the art goods (currently produced by Northern leaders) and is carried out using $a_m m$ units of labour for each targeted good, hence a total labour demand of $n^N a_m m$ units.

Thus market clearing in South, given an exogenous labour supply L^S, requires

$$\frac{n^S}{w^N} + n^N a_m m = L^S. \tag{29.31}$$

29.2.5 Steady-State Equilibrium

Since the amount of resources (labour) is assumed to be constant, growth is due solely to R&D in innovation and imitation. In the steady state, resources are allocated in unchanging shares to the various activities (manufacturing and R&D). This in turn implies constant numbers of commodities in the various product categories in all regions as well as constant rates of innovation and imitation, and constant relative prices. Constancy of relative prices necessitates all nominal variables to grow at the same rate. This requires that the value of all firms grows at the same rate as nominal expenditure. Since we have assumed above (see Sect. 29.2.1) the normalization condition $E(t) = 1$ for all t, it follows that the values of firms remain constant. The normalization condition also implies that the interest rate equals the households' discount rate ρ (see Sect. 29.2.1), hence

$$r^N = r^S = \rho. \tag{29.32}$$

If we use these facts and combine each no-arbitrage condition with the corresponding value-maximization condition—namely Eq. (29.27) with Eq. (29.25), and so on—we get the steady-state relationships

$$\frac{\pi^{NN}}{a_F w^N} \leq \iota^N + m + \rho, \qquad \text{with equality if } \iota^N > 0, \tag{29.33}$$

$$\frac{\pi^{NS}}{a_L w^N} \leq \iota^N + m + \rho, \qquad \text{with equality if } \iota^S > 0, \tag{29.34}$$

$$\frac{\pi^S}{a_m w^S} \leq \rho + \iota^S, \qquad \text{with equality if } m > 0. \tag{29.35}$$

Let us now consider the number and composition of products, that have to remain constant as said above.

In North, category (i), namely Northern leaders that compete with Northern followers, expands whenever a follower succeeds in innovating in an industry where formerly a Southern firm was second to top. This takes place at the success rate $\iota^N n^{NS}$. This category, however, shrinks whenever a Southern firm succeeds in imitating one of the n^{NN} state-of-the-art commodities, an occurrence that takes place at the success rate mn^{NN}. Thus a constant number and composition of Northern industries requires equal inflows and outflows, that is,

$$\iota^N n^{NS} = mn^{NN}. \tag{29.36}$$

As regards South, the number of firms increases whenever an entrepreneur successfully imitates one of the

$$n^N \equiv n^{NS} + n^{NN} \tag{29.37}$$

Northern products. This occurs at the success rate mn^N. On the other hand, Southern firms drop out of the market at the rate $\iota^S n^S$, as next-generation products are developed in the labs of the n^S former Northern leaders that were displaced by Southern imitation. The number of Southern products (firms) remains constant when inflows match outflows, namely

$$mn^N = \iota^S n^S. \tag{29.38}$$

Equations (29.33)–(29.38) describe the steady-state of the model, which, however, is not uniquely determined. There are, in fact, several types of possible steady-state equilibria depending on the values of parameters, in particular of the labour input coefficients in research (parameters a_m, a_L, a_F). If these are too great, R&D is too costly, and no innovation or imitation will take place (a stationary equilibrium). Or a_m might be too large with respect to a_L, a_F, so that no imitation will take place and there will only be innovation. These cases are not really interesting, hence we shall assume that parameter values are such that both innovation and imitation are present.

Even so, however, there are two possible cases. In one, both leaders and followers undertake R&D because followers are also efficient at innovation, though less so than leaders (the "efficient followers" case). In the other, only leaders engage in research because of their relative superiority in the research lab (the "inefficient followers" case). Both cases are examined below.

29.2.5.1 Efficient Followers

In this case, conditions (29.33)–(29.35) hold as equalities. Using them together with the other equilibrium relationships it is possible to derive reduced-form equations for the variables we are interested in. These are:

- The aggregate rate of product improvement, $\iota \equiv \iota^N n^N + \iota^S n^S$;
- The aggregate rate of technology transfer to South, $\mu \equiv m n^N$;
- The relative wage rate of South, $\omega \equiv w^S / w^N$.

Using these definitions and the steady-state relationships (29.36)–(29.38), we immediately get

$$\begin{aligned} \iota^S n^S &= m n^N = \mu, \\ \iota^N n^N &= \iota - \iota^S n^S = \iota - m n^N = \iota - \mu. \end{aligned} \tag{29.39}$$

From Eqs. (29.36) and (29.37) we have

$$\begin{aligned} \iota^N n^{NS} &= m(n^N - n^{NS}) \\ &= m n^N - m n^{NS}, \end{aligned}$$

hence

$$(\iota^N - m) n^{NS} = m n^{NS}.$$

Multiplying through by n^N, using Eqs. (29.39), and solving for n^{NS} we get

$$n^{NS} = n^N \mu / \iota. \tag{29.40}$$

This last equation, together with Eq. (29.37), yields

$$\begin{aligned} n^{NN} &= n^N - n^{NS} \\ &= n^N - n^N \mu/\iota \\ &= n^N (1 - \mu/\iota). \end{aligned} \tag{29.41}$$

Using Eqs. (29.39)–(29.41) and the definition of ω we can rewrite the Northern labour-market equilibrium condition (29.30) as

$$\frac{n^N(1-\mu/\iota)}{\lambda w^N} + \frac{n^N \mu/\iota}{\lambda \omega w^N} + a_L\mu + a_F(\iota - \mu) = L^N. \tag{29.42}$$

As regards the Southern labour market, take Eq. (29.31), replace $n^N m$ with μ, and observe that $n^S \equiv 1 - n^N$ by (29.21) and (29.37). This gives

$$\frac{1 - n^N}{w^N} + a_m\mu = L^S. \tag{29.43}$$

The next step is to combine the profit expressions (29.21)–(29.23) with the no-arbitrage conditions (29.33)–(29.35) and the steady-state conditions (29.36) and (29.38).

We first rewrite the profit expressions as

$$\begin{aligned} \pi^S &= w^S(w^N/w^S - 1)/w^N, \\ \pi^{NS} &= (1 - w^N/\lambda w^S), \\ \pi^{NN} &= (1 - 1/\lambda). \end{aligned}$$

From the no-arbitrage conditions we have

$$\begin{aligned} \pi^S &= a_m w^S(\rho + \iota^S), \\ \pi^{NS} &= a_L w^N(\iota^N + m + \rho), \\ \pi^{NN} &= a_F w^N(\iota^N + m + \rho), \end{aligned}$$

hence

$$\begin{aligned} (w^N/w^S - 1)/w^N &= a_m(\rho + \iota^S), \\ (1 - w^N/\lambda w^S)/w^N &= a_L(\iota^N + m + \rho), \\ (1 - 1/\lambda)/w^N &= a_F(\iota^N + m + \rho). \end{aligned} \tag{29.44}$$

Given the definition of the relative wage rate ω we have

$$\begin{aligned} (1/\omega - 1)/w^N &= a_m(\rho + \iota^S), \\ (1 - 1/\lambda\omega)/w^N &= a_L(\iota^N + m + \rho), \\ (1 - 1/\lambda)/w^N &= a_F(\iota^N + m + \rho). \end{aligned} \tag{29.45}$$

Let us consider the first equation in (29.45) and multiply through by $(1 - n^N)$. We get

$$\frac{(1/\omega - 1)(1 - n^N)}{w^N} = a_m[(1 - n^N)\rho + (1 - n^N)\iota^S] \\ = a_m[(1 - n^N)\rho + \mu],$$

since $(1 - n^N)\iota^S = n^S\iota^S = \mu$ by the definitions.

We now consider the second and third equations in (29.45), multiply through by n^N and use the definitions, which give $\left(\iota^N + m + \rho\right) n^N = \iota + \rho n^N$. Putting these results together we can rewrite Eqs. (29.45) as

$$(1/\omega - 1)(1 - n^N)/w^N = a_m[(1 - n^N)\rho + \mu], \tag{29.46}$$

$$(1 - 1/\lambda\omega)n^N/w^N = a_L\left(\iota + \rho n^N\right), \tag{29.47}$$

$$(1 - 1/\lambda)n^N/w^N = a_F\left(\iota + \rho n^N\right). \tag{29.48}$$

If we divide Eq. (29.47) by Eq. (29.48), we get

$$\frac{1 - 1/\lambda\omega}{1 - 1/\lambda} = \frac{a_L}{a_F},$$

hence

$$\frac{\lambda}{\lambda - 1} - \frac{1}{(\lambda - 1)}\omega^{-1} = \frac{a_L}{a_F},$$

from which

$$\omega^{-1} = (1 - \lambda)\frac{a_L}{a_F} + \lambda. \tag{29.49}$$

Since λ, the quality index, is greater than 1, it follows that ω^{-1} is a decreasing linear function of a_L/a_F, which means that the relative wage of North is higher the lower is a_L relative to a_F, i.e., the more productive in R&D are leaders relative to followers.

29.2.5.2 Inefficient Followers

Since no research is carried out by followers, we have $\iota^N = 0$. Then conditions (29.36) and (29.38) imply $n^{NN} = 0$ and $\mu = \iota$, namely South learns to imitate precisely the same number of products that are improved in North per unit of time. The labour market clearing conditions (29.42) and (29.43) respectively simplify to

$$\frac{n^N}{\lambda\omega w^N} + a_L\iota = L^N, \tag{29.50}$$

$$\frac{1-n^N}{w^N} + a_m\iota = L^S. \tag{29.51}$$

As regards the no-arbitrage relations, (29.33) now holds as an inequality since $\iota^N = 0$. Conditions (29.34) and (29.35) hold as equalities and the same procedure followed above for Eqs. (29.46) and (29.47) gives, setting $\mu = \iota$,

$$(1/\omega - 1)(1-n^N)/w^N = a_m[(1-n^N)\rho + \iota], \tag{29.52}$$

$$(1 - 1/\lambda\omega)n^N/w^N = a_L\left(\iota + \rho n^N\right). \tag{29.53}$$

29.2.6 *Comparative Dynamics*

The interest of this model lies in the study of the effects of country size and research subsidies on the steady-state path. This is an exercise in comparative dynamics (see Gandolfo, 2009, chap. 20, sect. 20.6), and amounts to finding the partial derivatives of the steady-state values of the variables with respect to the size parameters and to parameters representing research subsidies.

As regards the former, country size is conveniently expressed by the labour force, hence our size parameters are L^N, L^S.

As regards research subsidies, we introduce parameters ϕ^N, ϕ^S representing the share of research costs subsidized by the government in North and South, respectively. Hence Northern and Southern entrepreneurs pay only the fraction $(1-\phi^N), (1-\phi^S)$ of R&D costs, respectively. Consequently, the no arbitrage conditions must be modified to take account of this lower cost, which can be done by multiplying the r.h.s. of (29.47) and (29.48) by $(1-\phi^N)$, and the r.h.s. of (29.46) by $(1-\phi^S)$.

Since the steady-state conditions are different according as followers are efficient or inefficient, these two cases must be treated separately.

29.2.6.1 Efficient Followers

We rewrite here the steady-state equations (29.42), (29.43), (29.47), and (29.46) as

$$
\begin{aligned}
&\varphi_1(\iota,\mu,n^N,w^N;L^N,L^S,\phi^N,\phi^S)\\
&\qquad\equiv \frac{n^N(1-\mu/\iota)}{\lambda w^N}+\frac{n^N\mu/\iota}{\lambda\omega w^N}+a_L\mu+a_F(\iota-\mu)-L^N=0,\\
&\varphi_2(\iota,\mu,n^N,w^N;L^N,L^S,\phi^N,\phi^S)\equiv\frac{1-n^N}{w^N}+a_m\mu-L^S=0,\\
&\varphi_3(\iota,\mu,n^N,w^N;L^N,L^S,\phi^N,\phi^S)\\
&\qquad\qquad\equiv a_L\left(\iota+\rho n^N\right)(1-\phi^N)-(1-1/\lambda\omega)n^N/w^N=0,\\
&\varphi_4(\iota,\mu,n^N,w^N;L^N,L^S,\phi^N,\phi^S)\\
&\quad\equiv a_m[(1-n^N)\rho+\mu](1-\phi^S)-(1/\omega-1)(1-n^N)/w^N=0.
\end{aligned}
\tag{29.54}
$$

The practical procedure usually followed is to differentiate these equations totally, and then compute the derivatives we are interested in as ratios of the relevant differentials. It is however more rigorous to proceed as follows. By the implicit function theorem we can express the variables ι,μ,n^N,w^N as differentiable functions of the parameters L^N,L^S,ϕ^N,ϕ^S in a neighbourhood of the equilibrium point provided that the Jacobian matrix of Eqs. (29.54) with respect to ι,μ,n^N,w^N is non-singular at the equilibrium point. We take as initial equilibrium point the steady-state where $\phi^N=\phi^S=0$, so as to determine the effects of the introduction of research subsidies with respect to the no-subsidy equilibrium.

The Jacobian matrix is

$$
\mathbf{J}=\begin{bmatrix}
\frac{\partial\varphi_1}{\partial\iota} & \frac{\partial\varphi_1}{\partial\mu} & \frac{\partial\varphi_1}{\partial n^N} & \frac{\partial\varphi_1}{\partial w^N}\\
\frac{\partial\varphi_2}{\partial\iota} & \frac{\partial\varphi_2}{\partial\mu} & \frac{\partial\varphi_2}{\partial n^N} & \frac{\partial\varphi_2}{\partial w^N}\\
\frac{\partial\varphi_3}{\partial\iota} & \frac{\partial\varphi_3}{\partial\mu} & \frac{\partial\varphi_3}{\partial n^N} & \frac{\partial\varphi_3}{\partial w^N}\\
\frac{\partial\varphi_4}{\partial\iota} & \frac{\partial\varphi_4}{\partial\mu} & \frac{\partial\varphi_4}{\partial n^N} & \frac{\partial\varphi_4}{\partial w^N}
\end{bmatrix}.
\tag{29.55}
$$

Simple calculations yield

$$
\begin{aligned}
&\frac{\partial\varphi_1}{\partial\iota}=b,\text{ where } b\equiv a_F-\frac{(1/\omega-1)n^N\mu}{\iota^2\lambda w^N}\text{ is assumed to be positive;}\\
&\frac{\partial\varphi_1}{\partial\mu}=\frac{-n^N/\iota}{\lambda w^N}+\frac{n^N/\iota}{\lambda\omega w^N}+(a_L-a_F)=\frac{n^N}{\lambda w^N}(1/\omega-1)/\iota+(a_L-a_F).
\end{aligned}
$$

Now, if we subtract (29.47) from (29.48) we have $(1-1/\lambda\omega)n^N/w^N-(1-1/\lambda)n^N/w^N=(a_L-a_F)\left(\iota+\rho n^N\right)$. Simple manipulations on the l.h.s. of this last expression yield $\frac{n^N}{\lambda w^N}(1-1/\omega)=(a_L-a_F)\left(\iota+\rho n^N\right)$. This can be substituted in the expression for $\partial\varphi_1/\partial\mu$ found above, thus obtaining

$$
\begin{aligned}
&\frac{\partial\varphi_1}{\partial\mu}=\frac{-(a_L-a_F)}{\iota}\left(\iota+\rho n^N\right)+(a_L-a_F)=\frac{(a_F-a_L)\rho n^N}{\iota};\\
&\frac{\partial\varphi_1}{\partial n^N}=\frac{(1-\mu/\iota)}{\lambda w^N}+\frac{\mu/\iota}{\lambda\omega w^N}=\frac{1+(\mu/\iota)(1/\omega-1)}{\lambda w^N}=\frac{\beta}{\lambda w^N},
\end{aligned}
$$

where $\beta \equiv 1 + (\mu/\iota)\,(1/\omega - 1) > 0$;

$\frac{\partial \varphi_1}{\partial w^N} = -\frac{n^N(1-\mu/\iota)}{\lambda(w^N)^2} - \frac{n^N\mu/\iota}{\lambda\omega(w^N)^2} = -n^N\frac{1+(\mu/\iota)\,(1/\omega-1)}{\lambda(w^N)^2} = \frac{-n^N\beta}{\lambda(w^N)^2}$;

$\frac{\partial \varphi_2}{\partial \iota} = 0$; $\frac{\partial \varphi_2}{\partial \mu} = a_m$; $\frac{\partial \varphi_2}{\partial n^N} = \frac{-1}{w^N}$;

$\frac{\partial \varphi_2}{\partial w^N} = -(w^N)^{-2}(1-n^N) = -(w^N)^{-2}(-n^S)$;

$\frac{\partial \varphi_3}{\partial \iota} = a_L(1-\phi^N) = a_L$ (since we are evaluating the Jacobian at $\phi^N = 0$);

$\frac{\partial \varphi_3}{\partial \mu} = 0$;

$\frac{\partial \varphi_3}{\partial n^N} = a_L\rho - (1-1/\lambda\omega)/w^N$. Now from Eq. (29.47) we easily obtain

$-(1-1/\lambda\omega)/w^N = -a_L\rho - \frac{a_L\iota}{n^N}$, hence $\frac{\partial \varphi_3}{\partial n^N} = -\frac{a_L\iota}{n^N}$;

$\frac{\partial \varphi_3}{\partial w^N} = (w^N)^{-2}(1-1/\lambda\omega)n^N$;

$\frac{\partial \varphi_4}{\partial \iota} = 0$; $\frac{\partial \varphi_4}{\partial \mu} = a_m$; $\frac{\partial \varphi_4}{\partial n^N} = -a_m\rho + \frac{1/\omega - 1}{w^N}$. From Eq. (29.46) we get

$\frac{1/\omega-1}{w^N} = a_m\rho + a_m\mu/(1-n^N) = a_m\rho + a_m\mu/n^S$ since $1 - n^N = n^S$.

Hence $\frac{\partial \varphi_4}{\partial n^N} = a_m\mu/n^S$;

$\frac{\partial \varphi_4}{\partial w^N} = (w^N)^{-2}(1/\omega - 1)n^S$.

These expressions can be substituted into (29.55) to obtain the explicit expression for the Jacobian. Using the property that multiplying one line (in our case the last column) of a determinant by the same constant is the same as multiplying the whole determinant by that constant, we can write the determinant of our Jacobian matrix as $|\mathbf{J}| = (w^N)^{-2}\Delta$ where

$$\Delta \equiv \begin{vmatrix} b & \frac{(a_F - a_L)\rho n^N}{\iota} & \frac{\beta}{\lambda w^N} & -\frac{n^N\beta}{\lambda} \\ 0 & a_m & -\frac{1}{w^N} & -n^S \\ a_L & 0 & -\frac{a_L\iota}{n^N} & \left(1-\frac{1}{\lambda\omega}\right)n^N \\ 0 & a_m & \frac{a_m\mu}{n^S} & (\frac{1}{\omega}-1)n^S \end{vmatrix}. \tag{29.56}$$

Expanding the determinant by the first column we obtain

$$\begin{aligned}\Delta = -ba_m\left[\frac{n^S a_L \iota}{\omega n^N} + (1-1/\lambda\omega)\frac{n^N a_m \mu}{n^S} + (1-1/\lambda\omega)\right] \\ -a_L\frac{(a_F - a_L)\rho n^N}{\iota}\left[\frac{n^S}{(1-1/\omega)w^N} + a_m\mu\right] \\ -a_L a_m\left[\frac{\beta n^S}{\lambda w^N(1/\omega - 1)} + \frac{a_m \mu n^N \beta}{n^S \lambda}\right] \\ -a_L a_m\left[\frac{\beta n^S}{\lambda w^N} + \frac{n^N \beta}{\lambda w^N}\right].\end{aligned} \tag{29.57}$$

If we remember that $\omega < 1$ (hence $(1/\omega - 1) > 0$), that $\lambda w^S > w^N$ (hence $\lambda\omega > 1$ and $(1 - 1/\lambda\omega) > 0$), and that $a_F - a_L > 0$, it is easy to check that all expressions in square brackets are positive and are multiplied by negative expressions, hence $\Delta < 0$.

We can now compute the partial derivatives of the variables with respect to the parameters (i.e., $\partial\iota/\partial L^N$, $\partial\iota/\partial L^S$, $\partial\iota/\partial\phi^N$, $\partial\iota/\partial\phi^S$, etc.) by applying the chain rule to Eqs. (29.54). We only calculate a subset of these 16 partial derivatives, beginning with the partial derivatives with respect to L^N. We have

$$\begin{aligned}\frac{\partial\varphi_1}{\partial\iota}\frac{\partial\iota}{\partial L^N} + \frac{\partial\varphi_1}{\partial\mu}\frac{\partial\mu}{\partial L^N} + \frac{\partial\varphi_1}{\partial n^N}\frac{\partial n^N}{\partial L^N} + \frac{\partial\varphi_1}{\partial w^N}\frac{\partial w^N}{\partial L^N} - \frac{\partial\varphi_1}{\partial L^N} = 0, \\ \frac{\partial\varphi_2}{\partial\iota}\frac{\partial\iota}{\partial L^N} + \frac{\partial\varphi_2}{\partial\mu}\frac{\partial\mu}{\partial L^N} + \frac{\partial\varphi_2}{\partial n^N}\frac{\partial n^N}{\partial L^N} + \frac{\partial\varphi_2}{\partial w^N}\frac{\partial w^N}{\partial L^N} - \frac{\partial\varphi_2}{\partial L^N} = 0, \\ \frac{\partial\varphi_3}{\partial\iota}\frac{\partial\iota}{\partial L^N} + \frac{\partial\varphi_3}{\partial\mu}\frac{\partial\mu}{\partial L^N} + \frac{\partial\varphi_3}{\partial n^N}\frac{\partial n^N}{\partial L^N} + \frac{\partial\varphi_3}{\partial w^N}\frac{\partial w^N}{\partial L^N} - \frac{\partial\varphi_3}{\partial L^N} = 0, \\ \frac{\partial\varphi_4}{\partial\iota}\frac{\partial\iota}{\partial L^N} + \frac{\partial\varphi_4}{\partial\mu}\frac{\partial\mu}{\partial L^N} + \frac{\partial\varphi_4}{\partial n^N}\frac{\partial n^N}{\partial L^N} + \frac{\partial\varphi_4}{\partial w^N}\frac{\partial w^N}{\partial L^N} - \frac{\partial\varphi_4}{\partial L^N} = 0,\end{aligned} \tag{29.58}$$

or

$$\mathbf{J}\mathbf{v}_{L^N} = \boldsymbol{\kappa}_{L^N}, \tag{29.59}$$

where $\mathbf{J}$ is the Jacobian matrix found above, $\mathbf{v}_{L^N} \equiv \left\{\frac{\partial\iota}{\partial L^N}\ \frac{\partial\mu}{\partial L^N}\ \frac{\partial n^N}{\partial L^N}\ \frac{\partial w^N}{\partial L^N}\right\}$ is the column vector of the partial derivatives of the endogenous variables with respect to the parameter L^N, and $\boldsymbol{\kappa}_{L^N} \equiv \left\{\frac{\partial\varphi_1}{\partial L^N}\ \frac{\partial\varphi_2}{\partial L^N}\ \frac{\partial\varphi_3}{\partial L^N}\ \frac{\partial\varphi_4}{\partial L^N}\right\} = \{1\ 0\ 0\ 0\}$ is the vector of known terms.

The solution to the linear system (29.59) exists, since $\mathbf{J}$ is non-singular, and is

$$\mathbf{v}_{L^N} = \mathbf{J}^{-1}\boldsymbol{\kappa}_{L^N}. \tag{29.60}$$

Similarly we have

$$\begin{aligned}\mathbf{v}_{L^S} &= \mathbf{J}^{-1}\boldsymbol{\kappa}_{L^S},\\ \mathbf{v}_{\phi^N} &= \mathbf{J}^{-1}\boldsymbol{\kappa}_{\phi^N},\\ \mathbf{v}_{\phi^S} &= \mathbf{J}^{-1}\boldsymbol{\kappa}_{\phi^S}.\end{aligned} \tag{29.61}$$

Using for example Cramer's rule we have

$$\frac{\partial \iota}{\partial L^N} = \frac{\begin{vmatrix} 1 & \frac{(a_F - a_L)\rho n^N}{\iota} & \frac{\beta}{\lambda w^N} & -\frac{n^N \beta}{\lambda} \\ 0 & a_m & -\frac{1}{w^N} & -n^S \\ 0 & 0 & -\frac{a_L \iota}{n^N} & \left(1 - \frac{1}{\lambda\omega}\right) n^N \\ 0 & a_m & \frac{a_m \mu}{n^S} & (\frac{1}{\omega} - 1) n^S \end{vmatrix}}{\Delta} \tag{29.62}$$
$$= -\frac{a_m}{\Delta}\left[\frac{a_L \iota}{\omega}\frac{n^S}{n^N} + \left(1 - \frac{1}{\lambda\omega}\right) n^N \left(\frac{1}{w^N} + \frac{a_m \mu}{n^S}\right)\right] > 0,$$

where the sign is obvious if we observe that the expression in square brackets is positive given that $\lambda\omega > 1$.

It would be tedious to repeat this step-by-step procedure for all the other partial derivatives, hence we only state the results (that can easily be obtained as shown in detail for $\partial\iota/\partial L^N$), with the explanation of some less obvious transformations. Thus we have

$$\frac{\partial \iota}{\partial L^S} = \frac{\rho n^N}{\Delta}\left\{(a_F - a_L)\left[a_L \frac{n^S}{n^N}(1/\omega - 1) + (1 - 1/\lambda\omega)\frac{a_m \mu}{\iota}\frac{n^N}{n^S}\right]\right.$$
$$\left. -\frac{a_m}{\rho}\left[(1 - 1/\lambda\omega)\beta/\lambda w^N - \beta a_L \iota/\lambda n^N\right]\right\},$$

hence

$$\frac{\partial \iota}{\partial L^S} = \frac{\rho n^N}{\Delta}\left\{(a_F - a_L)\left[a_L \frac{n^S}{n^N}(\frac{1}{\omega} - 1) + (1 - \frac{1}{\lambda\omega})\frac{a_m \mu}{\iota}\frac{n^N}{n^S}\right] - a_m \frac{a_L \beta}{\lambda}\right\} \gtrless 0, \tag{29.63}$$

where we have used the transformation

$$(1 - 1/\lambda\omega)/\lambda w^N - a_L \iota/\lambda n^N = a_L \rho/\lambda$$

that derives from (29.47).

Next we have

$$\frac{\partial \iota}{\partial \phi^N} = \frac{a_L(\iota+\rho n^N)}{\Delta}\left\{-a_m\left[\frac{\beta}{\lambda w^N}(\frac{1}{\omega}-1)n^S+\frac{n^N\beta}{\lambda}\frac{a_m\mu}{n^S}+\frac{\beta}{\lambda w^N}n^S\right.\right.$$
$$\left.\left.+\frac{1}{w^N}\frac{n^N\beta}{\lambda}\right]+\frac{(a_F-a_L)\rho n^N}{\iota}\left[-\frac{1}{w^N}(\frac{1}{\omega}-1)n^S+a_m\mu\right]\right\},$$

hence

$$\frac{\partial \iota}{\partial \phi^N} = \frac{a_m b_N}{\Delta}\left\{\frac{\beta}{\lambda}\left[\frac{n^S}{w^N}(1-\frac{1}{\omega})-a_m\mu\frac{n^N}{n^S}-\frac{n^N}{w^N}\right]+(a_L-a_F)n^N n^S\frac{\rho^2}{\iota}\right\}>0, \tag{29.64}$$

where $b_N \equiv a_L(\iota+\rho n^N)$, and where we have used the transformation

$$-\frac{1}{w^N}(\frac{1}{\omega}-1)(1-n^N)+a_m\mu = -a_m\rho(1-n^N),$$

that derives from (29.46).

The expression for $\partial\iota/\partial\phi^N$ is clearly positive because the expression in braces is negative given that $\omega<1$ and $a_L<a_F$. We now compute

$$\frac{\partial \iota}{\partial \phi^S} = -\frac{a_m[\rho(1-n^N)+\mu]}{\Delta}\left\{\frac{(a_F-a_L)\rho n^N}{\iota}\left[-\frac{1}{w^N}\left(1-\frac{1}{\lambda\omega}\right)n^N-\frac{a_L\iota}{n^N}n^S\right]\right.$$
$$\left.-a_m\left[\frac{\beta}{\lambda w^N}\left(1-\frac{1}{\lambda\omega}\right)n^N-\frac{a_L\iota\beta}{\lambda}\right]\right\}$$
$$=\frac{\rho b_S}{\Delta}\left\{(a_F-a_L)\left[\frac{(n^N)^2}{\iota w^N}\left(1-\frac{1}{\lambda\omega}\right)+a_L n^S\right]+\frac{\beta}{\lambda}a_m a_L n^N\right\}<0, \tag{29.65}$$

where we have used the transformation

$$\frac{1}{w^N}\left(1-\frac{1}{\lambda\omega}\right)n^N-a_L\iota = a_L\rho n^N$$

that derives from (29.47), and $b_S \equiv a_m[\rho(1-n^N)+\mu]$.

We next have

$$\frac{\partial\mu}{\partial L^N} = \frac{1}{\Delta}a_L\left[-(1/w^N)(1/\omega-1)n^S+a_m\mu\right].$$

Now, $(1/w^N)(1/\omega-1)=a_m\rho+a_m\iota/n^S$ by (29.46), hence

$$\frac{\partial\mu}{\partial L^N} = -\frac{1}{\Delta}a_L a_m\left[\rho n^S+(\iota-\mu)\right]>0, \tag{29.66}$$

given that $\iota-\mu>0$.

We then calculate

$$\begin{aligned}\frac{\partial\mu}{\partial L^S} = \frac{1}{\Delta}\{b\left[(a_L\iota/n^N)(1-1/\omega)n^S - (a_m\mu/n^S)(1-1/\lambda\omega)n^N\right] \\ -a_L\left[(\beta/\lambda w^N)(1/\omega-1)n^S + (a_m\mu/n^S)\beta n^N/\lambda\right]\} \\ = \left\{-\frac{1}{\Delta}\frac{a_m\mu n^N}{n^S}\left[\frac{a_L\beta}{\lambda}+b\left(1-\frac{1}{\lambda\omega}\right)\right]\right. \\ \left.+\left(\frac{1}{\omega}-1\right)n^S\left[\frac{a_L\beta}{\lambda w^N}+\frac{ba_L\iota}{n^N}\right]\right\} \qquad (29.67)\end{aligned}$$

which shows that $\partial\mu/\partial L^S > 0$.

The effect of ϕ^N on μ is given by

$$\begin{aligned}\frac{\partial\mu}{\partial\phi^N} &= \frac{a_L(\iota+\rho n^N)b\left[\frac{(1/\omega-1)n^S}{w^N}-a_m\mu\right]}{\Delta} \\ &= \frac{1}{\Delta}(b_N b a_m \rho n^S) < 0, \qquad (29.68)\end{aligned}$$

where we have used the transformation

$$\frac{(1/\omega-1)n^S}{w^N} - a_m\mu = a_m\rho n^S,$$

that derives from (29.46).

The effect of ϕ^S on μ is given by

$$\begin{aligned}\frac{\partial\mu}{\partial\phi^S} &= \frac{b_S}{\Delta}\{b\left[-(1-1/\lambda\omega)n^N/w^N - a_L\iota n^S/n^N\right] \\ &\quad +a_L\left[-\beta n^S/\lambda w^N - \beta n^N/\lambda w^N\right]\} \\ &= -\frac{b_S}{\Delta}\left\{b\left[\left(1-\frac{1}{\lambda\omega}\right)\frac{n^N}{w^N}+\frac{a_L\iota n^S}{n^N}\right]+\frac{a_L\beta}{\lambda w^N}\right\} > 0, \qquad (29.69)\end{aligned}$$

where we have used the fact that $n^S + n^N = 1$.

We finally compute

$$\frac{\partial n^N}{\partial L^S} = \frac{n^N}{\Delta}\left[ba_m\left(1-\frac{1}{\lambda\omega}\right)+a_L\left(\frac{1}{\omega}-1\right)n^S(a_F-a_L)\frac{\rho}{\iota}+a_La_m\frac{\beta}{\lambda}\right] < 0. \qquad (29.70)$$

The signs of the various partial derivatives give us the comparative dynamics results we are looking for. Strategic trade policy is a particularly interesting case.

Strategic Trade Policy Let us consider, for example, subsidies to innovation and imitation.

The introduction of a small research subsidy in North enhances R&D for both leaders and followers, hence the aggregate rate of innovation increases ($\partial\iota/\partial\phi^N > 0$). At the same time, the rate of imitation in South is adversely affected ($\partial\mu/\partial\phi^N < 0$). This is due to the following chain of effects. As North devotes more resources to R&D, less resources are devoted to the Northern manufacturing sector, which implies that the number of products manufactured in North declines while those manufactured in South increase. Thus the manufacturing sector expands in South and less resources are devoted there to research.

The effects of the introduction of a small subsidy to imitative research in South are exactly the opposite: the rate of imitation in South is favourably affected ($\partial\mu/\partial\phi^S > 0$) while the rate of innovation in North is adversely affected ($\partial\iota/\partial\phi^S < 0$).

These results point to a possible conflict between the two governments in their efforts to promote domestic growth via R&D, a result in contrast with a previous finding by the same authors in the context of another model (Grossman & Helpman, 1991a , chap. 11), where a subsidy to research by either government is favourable to technological progress in both regions.

We now examine the effects of size, as represented by L^N, L^S. An increase in the size of South increases both the rate of technological progress there ($\partial\mu/\partial L^S > 0$) and the intensity of imitation (i.e., $m = \mu/n^N$) increases, as can be seen from the fact that as μ increases, n^N decreases $\partial n^N/\partial L^S < 0$). The effect on the rate of technological progress in North is however uncertain ($\partial\iota/\partial L^S \gtrless 0$), while an increase in the size of North has a favourable effect on both the Northern and Southern rates of technological progress ($\partial\iota/\partial L^N > 0, \partial\mu/\partial L^N > 0$).

29.2.6.2 Inefficient Followers

In the case of inefficient followers, the steady-state equations are (29.50), (29.51), (29.53), and (29.52). In this case, since μ disappears from the equations, we take as endogenous variable the relative wage rate ω. Thus we have the system of implicit functions

$$\begin{aligned}
&\gamma_1(\iota,\omega,n^N,w^N;L^N,L^S,\phi^N,\phi^S) \equiv \frac{n^N}{\lambda\omega w^N} + a_L\iota - L^N &= 0,\\
&\gamma_2(\iota,\omega,n^N,w^N;L^N,L^S,\phi^N,\phi^S) \equiv \frac{1-n^N}{w^N} + a_m\iota - L^S &= 0,\\
&\gamma_3(\iota,\omega,n^N,w^N;L^N,L^S,\phi^N,\phi^S)\\
&\qquad \equiv -a_L\left(\iota+\rho n^N\right)(1-\phi^N) + (1-1/\lambda\omega)n^N/w^N &= 0,\\
&\gamma_4(\iota,\omega,n^N,w^N;L^N,L^S,\phi^N,\phi^S)\\
&\equiv -a_m[(1-n^N)\rho+\iota](1-\phi^S) + (1/\omega-1)(1-n^N)/w^N &= 0.
\end{aligned} \tag{29.71}$$

The partial derivatives making up the Jacobian evaluated at $\phi^N = \phi^S = 0$ are

$$\frac{\partial\gamma_1}{\partial\iota} = a_L; \frac{\partial\gamma_1}{\partial\omega} = -\frac{n^N}{\lambda w^N}\omega^{-2}; \frac{\partial\gamma_1}{\partial n^N} = \frac{1}{\lambda\omega w^N}; \frac{\partial\gamma_1}{\partial w^N} = -\frac{n^N}{\lambda\omega}\left(w^N\right)^{-2};$$
$$\frac{\partial\gamma_2}{\partial\iota} = a_m; \frac{\partial\gamma_2}{\partial\omega} = 0; \frac{\partial\gamma_2}{\partial n^N} = \frac{-1}{w^N}; \frac{\partial\gamma_2}{\partial w^N} = -n^S\left(w^N\right)^{-2};$$
$$\frac{\partial\gamma_3}{\partial\iota} = -a_L; \frac{\partial\gamma_3}{\partial\omega} = \frac{n^N}{\lambda w^N}\omega^{-2};$$
$$\frac{\partial\gamma_3}{\partial n^N} = -a_L\rho + (1 - 1/\lambda\omega)/w^N = \frac{a_L\iota}{n^N} \text{ using Eq. (29.53);}$$
$$\frac{\partial\gamma_3}{\partial w^N} = (1/\lambda\omega - 1)n^N\left(w^N\right)^{-2};$$
$$\frac{\partial\gamma_4}{\partial\iota} = -a_m; \frac{\partial\gamma_4}{\partial\omega} = -\frac{n^S}{w^N}\omega^{-2};$$
$$\frac{\partial\gamma_4}{\partial n^N} = \rho - (1/\omega - 1)/w^N = -\frac{a_m\iota}{n^S} \text{ using Eq. (29.52);}$$
$$\frac{\partial\gamma_4}{\partial w^N} = (1 - 1/\omega)n^S\left(w^N\right)^{-2}.$$

Using the property that multiplying one line (in our case the third column and then the fourth) of a determinant by the same constant is the same as multiplying the whole determinant by that constant. we can write the determinant of our Jacobian matrix as $|\mathbf{J}| = \omega^{-2}(w^N)^{-2}\tilde{\Delta}$, where

$$\tilde{\Delta} \equiv \begin{vmatrix} a_L & -\frac{n^N}{\lambda w^N} & \frac{1}{\lambda\omega w^N} & -\frac{n^N}{\lambda\omega} \\ a_m & 0 & -\frac{1}{w^N} & -n^S \\ -a_L & \frac{n^N}{\lambda w^N} & \frac{a_L\iota}{n^N} & \left(\frac{1}{\lambda\omega} - 1\right)n^N \\ -a_m & -\frac{n^S}{w^N} & -\frac{a_m\iota}{n^S} & \left(1 - \frac{1}{\omega}\right)n^S \end{vmatrix}. \quad (29.72)$$

If we expand $\tilde{\Delta}$ by the second row and then expand each of the resulting third-order determinants by the first row, we get, after simple manipulations (to obtain the expression in the third square brackets use the transformation $n^N(1/\lambda\omega - 1)/w^N + a_L\iota = -a_L\rho n^N$, that derives from Eq. (29.53))

$$\begin{aligned}\tilde{\Delta} =& \frac{a_m n^N}{\lambda w^N}\left[\frac{a_L\iota}{n^N}\left(1 - \frac{1}{\omega}\right)n^S + \left(\frac{1}{\lambda\omega} - 1\right)\frac{n^N a_m\iota}{n^S}\right] \\ &+ \frac{a_m n^N}{\lambda w^N}\left[\frac{1}{\lambda\omega w^N}(1 - \frac{1}{\omega})n^S - \frac{n^N a_m\iota}{n^S}\right]\end{aligned}$$

$$+a_m \frac{n^S}{\lambda\omega w^N}\left[-a_L \rho n^N\right]$$
$$+\frac{a_L}{w^N}\left[\frac{n^N n^S}{w^N}\left(\frac{1}{\lambda\omega}-1\right)\right]$$
$$+\frac{a_L}{w^N}\left[-\frac{n^N n^S}{\lambda w^N}\right]$$
$$+\frac{a_m}{w^N}\left[-\frac{n^N}{\lambda w^N}\right]$$
$$+n^S a_L\left[-\frac{n^S a_L \iota}{w^N n^N}-\frac{n^S}{w^N \lambda\omega w^N}\right]$$
$$+n^S a_m\left[-\frac{a_L \iota}{\lambda w^N}-\frac{n^N}{\lambda w^N \lambda\omega w^N}\right]. \quad (29.73)$$

If we recall that $\omega < 1, \lambda > 1, \lambda\omega > 1$, it is easy to see that all expressions in square brackets are negative, hence $\tilde{\Delta} < 0$.

We now proceed to calculate a subset of the partial derivatives of the endogenous variables with respect to the parameters by the same procedure whose details we have illustrated in the case of efficient followers. As before, all these derivatives are calculated at the point where $\phi^N = \phi^S = 0$.

Let us begin with the effects of size. We have

$$\frac{\partial n^N}{\partial L^N} = \frac{1}{\tilde{\Delta}}\left\{a_m\left[\frac{n^N n^S}{w^N}\left(\frac{1}{\lambda}-1\right)\right]-n^S\left[a_L\frac{n^S}{w^N}+a_m\frac{n^N}{\lambda w^N}\right]\right\} > 0 \quad (29.74)$$

and

$$\frac{\partial n^N}{\partial L^S} = \frac{1}{\tilde{\Delta}}\left[a_L\frac{n^S n^N}{w^N}\left(1+\frac{1}{\lambda}\right)+a_m\frac{\left(n^N\right)^2}{\lambda w^N}\right] < 0, \quad (29.75)$$

which (together with the identity $n^N + n^S = 1$) show that an increase in the size of a country has a favourable effect on the number of products manufactured in that country and, of course, the opposite effect on the other country.

Finally, nothing can be said on the effect of size on the relative wage rate, since the sign of the relevant partial derivatives is ambiguous.

Strategic Trade Policy We now turn to the effects of the introduction of research subsidies. We have

$$\frac{\partial n^N}{\partial \phi^N} = \frac{a_L\left(n^S\right)^2 b_N}{w^N \tilde{\Delta}} < 0, \quad (29.76)$$

$$\frac{\partial n^N}{\partial \phi^S} = -\frac{a_m \left(n^N\right)^2 b_S}{\lambda w^N \tilde{\Delta}} > 0, \tag{29.77}$$

$$\frac{\partial \omega}{\partial \phi^N} = \frac{\omega^2 a_m b_N}{\tilde{\Delta}} \left\{ \left[\frac{a_L}{a_m} (\frac{1}{\omega} - 1) \frac{n^S}{w^N} - a_L \iota \right] + \frac{n^S}{\lambda \omega^2 w^N} + \frac{n^N}{\lambda \omega} \frac{a_m \iota}{n^S} \right.$$
$$\left. + \frac{n^N}{\lambda \omega w^N} \right\} .$$

Using (29.52) and (29.51) we get

$$\frac{\partial \omega}{\partial \phi^N} = \frac{\omega^2 a_m b_N \left(\rho a_L n^S + n^S/\lambda \omega^2 w^N + L^S n^N/\lambda \omega n^S\right)}{\tilde{\Delta}} < 0. \tag{29.78}$$

$$\frac{\partial \omega}{\partial \phi^S} = \frac{a_L b_S}{\omega^{-2} \tilde{\Delta}} \left\{ \left[\frac{n^N}{w^N} + \frac{a_L \iota}{n^N} n^S \right] + \left[\frac{a_m}{a_L} \frac{1}{\lambda \omega} \left(1 - \frac{1}{\lambda \omega}\right) \frac{n^N}{w^N} - \frac{a_m \iota}{\lambda \omega} \right] \right.$$
$$\left. + \frac{n^S}{\lambda \omega w^N} \right\} ,$$

hence

$$\frac{\partial \omega}{\partial \phi^S} = -\frac{\omega^2 a_L b_S \left(\rho a_m n^N/\lambda \omega + n^N/w^N + L^N n^S/n^N\right)}{\tilde{\Delta}} > 0, \tag{29.79}$$

where we have used (29.53) and (29.50).

The signs of the partial derivatives can be interpreted in the following way. The introduction of a subsidy to R&D in North causes the industries there to devote more resources to research, hence less resources are employed in manufacturing. The number of products manufactured in North thus decreases ($\partial n^N/\partial \phi^N < 0$) while the relative wage of North increases ($\partial \omega/\partial \phi^N < 0$). A subsidy to imitative research in South has exactly the opposite result on product shares and the relative wage.

For an extension of this model see Lai (1995).

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