Chapter 31
Appendix to Chapter 17

31.1 A Model of Economic Geography, International Trade, and Globalization

The following model is due to Krugman and Venables (1995).

The world consists of two countries (or regions: henceforth we shall use 'region' and 'country' interchangeably), North and South, assumed to be identical in technology, endowments, and preferences. Both countries produce two kinds of commodities: an agricultural good (produced under constant returns to scale with labour as the sole input) and a variety of manufacturing goods (produced under increasing returns to scale using labour and intermediate goods). Manufacturing goods can be used both as final goods by consumers and as intermediate goods in the manufacturing sector.

Given the identity of the two regions, we shall describe the equations concerning North, as analogous equations hold in South. Variables with a superscript asterisk refer to South.

31.1.1 The Demand Side

The representative consumer receives only labour income, and has preferences that can be represented by a Cobb-Douglas expenditure function $Q_A^{(1-\gamma)} Q_M^{\gamma} V$, where V is utility, Q_A is the price of the agricultural good, Q_M is the price index for manufacturing goods, and γ is the share of manufactures in the expenditure of consumers. Given L, the country's labour endowment, and assuming that the representative consumer receives only labour income at the wage rate w, the budget constraint is

$$wL = Q_A^{(1-\gamma)} Q_M^{\gamma} V. \tag{31.1}$$

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To determine Q_M, we begin by observing that, in equilibrium, all varieties of differentiated goods produced by the manufacturing sector are sold at the same price p (see Sect. 23.2.2). Because of increasing returns to scale coupled with the consumers' love for variety and the unlimited number of potential varieties, no firm will try to produce the same variety already produced by another firm (see Sect. 23.2.1). Hence the number of available varieties coincides with the number of firms in operation, that will usually be very large.

Now, as is typical of monopolistic competition models of international trade (see Sect. 23.2.3), there will be intra-industry trade in manufactures; we assume that manufacturing products of South sold in North incur iceberg transport costs at the rate $t \geq 1$. This means that a fraction $1/t$ of the good exported by South arrives to North, hence a consumer price in North of p^*t.

If we then aggregate all varieties of differentiated manufacturing products by a CES subutility function we obtain Q_M, which takes the form

$$Q_M = \left[np^{1-\sigma} + n^*(p^*t)^{1-\sigma}\right]^{1/(1-\sigma)}, \tag{31.2}$$

where n, n^* denote the number of varieties produced in North and South, respectively, and $\sigma > 1$ is the price elasticity of demand for a single variety. Let us note for future reference that Q_M is a decreasing function of the number of varieties.

31.1.2 The Supply Side

Agriculture is perfectly competitive and produces under constant returns to scale with labour as the sole input. The agricultural good can be costlessly traded, and is taken as the numéraire ($Q_A = 1$). Without loss of generality we can choose units so that one unit of labour produces one unit of output, which gives the equilibrium condition

$$w \geq 1, \tag{31.3}$$

where the equality sign must hold if the agricultural good is produced. Hence the wage rate equals unity if the country produces agriculture, and exceeds it only if there is no agricultural production.

The manufacturing sector uses labour and manufacturing intermediates to produce the composite final consumer good and intermediates. To keep the dimensionality of the model low, the major simplifying assumption is made that manufacturing output is an all-purpose composite commodity, that can be used both as composite final consumer good and as composite intermediate good. Thus the intermediate's price index is Q_M, as defined in Eq. (31.2).

Production is carried out by the representative firm combining labour and the intermediate with a Cobb-Douglas technology with shares $(1-\mu)$ and μ, respectively, with α units of the combined input used as fixed cost, and β units per unit output as variable cost. Since each firm produces output for both the domestic market (y) and exports (x), we can write each firm's total cost as

$$TC = w^{1-\mu} Q_M^{\mu} [\alpha + \beta(y + x)], \tag{31.4}$$

whence the marginal cost

$$MC = w^{1-\mu} Q_M^{\mu} \beta. \tag{31.5}$$

How do increasing returns to scale manifest themselves in these functions? Simply through the reduction in Q_M (and hence in total and marginal cost) due to the increase in n, the number of domestic products (= the number of domestic firms). Let us note, incidentally, that the reduction in cost due to the increase in the number of firms is typical of monopolistically competitive models with constant-elasticity demand functions, where all scale economies work through changes in the variety of goods produced.

31.1.3 Equilibrium

To characterize equilibrium we first note that, with free entry and exit of firms, a zero-profit situation obtains. We next define E, the total value of Northern expenditure on manufactured commodities, which is given by consumers' expenditure (a proportion γ of the wage bill) and intermediate demand, which is a proportion μ of costs (and hence of revenue, since there are no profits)

$$E = \gamma w L + \mu(x + y) p n. \tag{31.6}$$

Let us now consider price determination. It is well known that, under monopolistic competition, the equilibrium excess-price over marginal cost equals the reciprocal of the elasticity of demand, namely $(p - MC)/p = 1/\sigma$, whence the firm's price-setting rule

$$p = \frac{\sigma}{\sigma - 1} MC,$$

that is,

$$p = \frac{\sigma}{\sigma - 1} w^{1-\mu} Q_M^{\mu} \beta. \tag{31.7}$$

We now note that the demand for a single variety is

$$y = p^{-\sigma} Q_M^{\sigma-1} E, \qquad x = p^{-\sigma} t^{1-\sigma} \left(Q_M^*\right)^{\sigma-1} E^*, \tag{31.8}$$

in North and South, respectively.

The zero profit condition means $p(y + x) = TC$. Substituting TC from (31.4) and p from (31.7) we get

$$y + x = (\sigma - 1)\alpha/\beta. \tag{31.9}$$

Without loss of generality we can choose units of measurement such that $(\sigma - 1)\alpha/\beta = 1$, whence, at the zero-profit equilibrium,

$$y + x = 1. \tag{31.10}$$

Substitution of (31.8) in (31.10) gives

$$p^{-\sigma}\left[Q_M^{\sigma-1}E + t^{1-\sigma}\left(Q_M^*\right)^{\sigma-1}E^*\right] = 1, \tag{31.11}$$

whence

$$p^{\sigma} = Q_M^{\sigma-1}E + t^{1-\sigma}\left(Q_M^*\right)^{\sigma-1}E^*. \tag{31.12}$$

The equilibrium values of the endogenous variables Q_M, w, p, n, E, and of the analogous variables in the other country, are determined by Eqs. (31.2), (31.3), (31.6), (31.7), and (31.11), and analogous equations for the other region.

We now note that n, the number of firms (= varieties) in the manufacturing sector, influences firms' profitability in three ways.

(a) As shown by Eq. (31.2), an increase in n reduces Q_M. This shifts each firm's demand curve down—see Eq. (31.8)—and reduces firms' profitability, see Eq. (31.11). This is the standard channel. The two other are related to a positive μ, namely are operative only if manufacturing uses manufactures as input (see above).
(b) The reduction in Q_M due to the increase in n causes a decrease in total and marginal cost—see Eqs. (31.4) and (31.5)—and hence an increase in firms' profits. This is a *cost,* or *forward linkage* between firms.
(c) An increase in n increases total expenditure on manufactures—see Eq. (31.6)—that in turn raises demand and profits of each firm, as shown by Eqs. (31.8) and (31.11). This is a *demand*, or *backward linkage* between firms.

31.2 The Dynamics of the Model and the Emergence of a Core-Periphery Pattern

The dynamics of the model could be examined in terms of the number of firms in the manufacturing sector in the two countries, on the basis of the assumption that firms enter in the sector if profits are positive, exit in the opposite case. Hence the dynamic system

$$\begin{aligned} \dot{n} &= f(\pi), \quad \operatorname{sgn} f(\pi) = \operatorname{sgn} \pi, f(0) = 0, f'(0) > 0, \\ \dot{n}^* &= f^*(\pi^*), \operatorname{sgn} f^*(\pi) = \operatorname{sgn} \pi^*, f^*(0) = 0, f^{*\prime}(0) > 0, \end{aligned} \tag{31.13}$$

where the dot over a variable denotes its time derivative.

Since profits are total revenue minus total cost, using (31.4), (31.7), and (31.8) it is easy to see that profits (given transportation cost) ultimately depend on the number of firms, namely

$$\begin{aligned} \pi &= \pi(n, n^*; t), \\ \pi^* &= \pi^*(n, n^*; t), \end{aligned} \tag{31.14}$$

so that, substituting into (31.13), we can write our dynamic system as

$$\begin{aligned} \dot{n} &= \varphi(n, n^*; t), \\ \dot{n}^* &= \varphi^*(n, n^*; t). \end{aligned} \tag{31.15}$$

This is the typical form of a planar system involving a parameter (t), which can give rise to bifurcations (for a treatment of bifurcation theory see Gandolfo, 2009, chap. 24).

Loosely speaking, we have a bifurcation when, given a dynamic system involving a parameter, the passage of the parameter through a critical value causes a qualitative change in the nature of singular point of the system (for example, from stability to instability). The value(s) of the parameter at which such a change occurs are called bifurcation values.

A bifurcation point consistent with the formation of a core-periphery pattern due to the decline in transportation costs requires the system to be stable for $t > t_0$ (i.e., for cost of transport higher than a critical value t_0) and unstable for $t < t_0$.

However, the study of bifurcations in a 2×2 system is rather complicated, hence it is convenient to transform the model so as to reduce its dynamics to a single differential equation. This can be done by concentrating on manufacturing equilibrium in North.

31.2.1 The Manufacturing Sector

We first observe that in manufacturing, since proportion μ of costs is spent on intermediates (see above, Eq. (31.6)), the remaining proportion $(1 - \mu)$ is spent for the wage bill, which is

$$wL_M = (1 - \mu)np(y + x), \tag{31.16}$$

where L_M is manufacturing employment. Assuming an initially symmetric equilibrium, where both countries produce both commodities and have a wage equal to unity, the proportions L_M/L and L_A/L are equal to γ and $1-\gamma$, respectively. Given our choice of the units of measurement—see Eq. (31.10)—we have

$$wL_M = (1-\mu)np. \tag{31.17}$$

Let us now consider the ratios of Northern to Southern endogenous variables, defined as

$$\widetilde{Q}_M \equiv \frac{Q_M}{Q_M^*},\ \widetilde{p} \equiv \frac{p}{p^*},\ \widetilde{E} \equiv \frac{E}{E^*},\ \widetilde{w} \equiv \frac{w}{w^*}. \tag{31.18}$$

For future reference we also define

$$\tau \equiv t^{1-\sigma}, \tag{31.19}$$

and observe that, since $\sigma > 1$, $t \in (1,\infty)$ implies $\tau \in (0,1)$.

From Eq. (31.2) and its analogous for the other region we have

$$\widetilde{Q}_M = \frac{\left[np^{1-\sigma} + n^*(p^*t)^{1-\sigma}\right]^{1/(1-\sigma)}}{\left[n^*(p^*)^{1-\sigma} + n(pt)^{1-\sigma}\right]^{1/(1-\sigma)}},$$

whence

$$\widetilde{Q}_M^{1-\sigma} = \frac{np^{1-\sigma} + n^*(p^*t)^{1-\sigma}}{n(pt)^{1-\sigma} + n^*(p^*)^{1-\sigma}} = \frac{(np)p^{-\sigma} + n^*p^*\tau(p^*)^{-\sigma}}{np\tau p^{-\sigma} + n^*p^*(p^*)^{-\sigma}}.$$

Substituting $np = L_M w/(1-\mu), n^*p^* = L_M^* w^*/(1-\mu)$, that are derived from Eq. (31.17) and its South analogous, we obtain, after simple manipulations,

$$\widetilde{Q}_M^{1-\sigma} = \frac{L_M \widetilde{w}\widetilde{p}^{-\sigma} + \tau L_M^*}{\tau L_M \widetilde{w}\widetilde{p}^{-\sigma} + L_M^*}. \tag{31.20}$$

Consider now $\widetilde{E}$. From Eq. (31.6) and its analogous for the other region we have, using $np = L_M w/(1-\mu), n^*p^* = L_M^* w^*/(1-\mu)$ and taking (31.10) into account,

$$\widetilde{E} = \frac{\gamma wL + \frac{\mu}{1-\mu}L_M w}{\gamma w^* L^* + \frac{\mu}{1-\mu}L_M^* w^*} = \widetilde{w}\frac{\gamma(1-\mu)L + \mu L_M}{\gamma(1-\mu)L + \mu L_M^*}, \tag{31.21}$$

where we have used the fact that $L^* = L$ by our initial assumptions.

As regards $\widetilde{p}$, from Eq. (31.7) and its analogous we have

$$\widetilde{p} = \frac{\frac{\sigma}{\sigma-1}w^{1-\mu}Q_M^{\mu}\beta}{\frac{\sigma}{\sigma-1}w^{*1-\mu}Q_M^{*\mu}\beta} = \widetilde{w}^{1-\mu}\widetilde{Q}_M^{\mu}. \tag{31.22}$$

We finally have, using Eq. (31.12), its analogous for the other region, and the definition of τ,

$$\widetilde{p}^{\sigma} = \frac{Q_M^{\sigma-1}E + \tau\left(Q_M^*\right)^{\sigma-1}E^*}{\left(Q_M^*\right)^{\sigma-1}E^* + \tau\left(Q_M\right)^{\sigma-1}E} = \frac{\widetilde{Q}_M^{\sigma-1}\widetilde{E} + \tau}{\tau\widetilde{Q}_M^{\sigma-1}\widetilde{E} + 1}. \tag{31.23}$$

System (31.20)–(31.23) can be reduced to a two-equation system by eliminating $\widetilde{Q}_M$ and $\widetilde{E}$. From Eq. (31.22) we obtain

$$\widetilde{p}^{\frac{1-\sigma}{\mu}}\widetilde{w}^{(1-\sigma)(\mu-1)/\mu} = \widetilde{Q}_M^{1-\sigma}, \tag{31.24}$$

hence, substituting $\widetilde{Q}_M^{1-\sigma}$ from (31.20) and rearranging terms

$$\widetilde{p}^{\frac{1-\sigma}{\mu}}\widetilde{w}^{(1-\sigma)(\mu-1)/\mu} - \frac{L_M\widetilde{w}\widetilde{p}^{-\sigma} + \tau L_M^*}{\tau L_M\widetilde{w}\widetilde{p}^{-\sigma} + L_M^*}$$
$$\equiv \varphi_1\left(\widetilde{p}, \widetilde{w}, L_M, L_M^*\right) = 0. \tag{31.25}$$

Consider now Eq. (31.23), which yields $\widetilde{p}^{\sigma}\left[\tau\widetilde{Q}_M^{\sigma-1}\widetilde{E} + 1\right] = \widetilde{Q}_M^{\sigma-1}\widetilde{E} + \tau$, whence solving for $\widetilde{Q}_M^{\sigma-1}\widetilde{E}$ we get

$$\widetilde{Q}_M^{\sigma-1}\widetilde{E} = \frac{\tau - \widetilde{p}^{\sigma}}{\tau\widetilde{p}^{\sigma} - 1}. \tag{31.26}$$

Substitution from (31.24) and (31.21) into (31.26) yields

$$\widetilde{p}^{(\sigma-1)/\mu}\widetilde{w}^{(\sigma-1)(\mu-1)/\mu}\widetilde{w}\frac{\gamma(1-\mu)L + \mu L_M}{\gamma(1-\mu)L + \mu L_M^*} - \frac{\tau - \widetilde{p}^{\sigma}}{\tau\widetilde{p}^{\sigma} - 1}$$
$$\equiv \varphi_2\left(\widetilde{p}, \widetilde{w}, L_M, L_M^*\right) = 0. \tag{31.27}$$

Equations (31.25) and (31.27) are a set of two implicit functions in the four variables $\widetilde{p}, \widetilde{w}, L_M, L_M^*$. According to the implicit function theorem (see, for example, Gandolfo, 2009, chap. 20, sect. 20.2) we can express $\widetilde{p}, \widetilde{w}$ as differentiable functions of L_M, L_M^* provided that the Jacobian of the set is non-singular at the equilibrium point.

For this purpose we first observe that, by simple inspection, if $L_M = L_M^*$, Eqs. (31.25) and (31.27) are satisfied for $\widetilde{p} = \widetilde{w} = 1$. This we take as our (symmetric) equilibrium.

Let us now compute the Jacobian

$$\mathbf{J} \equiv \begin{bmatrix} \dfrac{\partial \varphi_1}{\partial \tilde{p}} & \dfrac{\partial \varphi_1}{\partial \tilde{w}} \\ \dfrac{\partial \varphi_2}{\partial \tilde{p}} & \dfrac{\partial \varphi_2}{\partial \tilde{w}} \end{bmatrix}, \tag{31.28}$$

where the partial derivatives are evaluated at the equilibrium point. Simple calculations yield

$$\mathbf{J} = \begin{bmatrix} \dfrac{1-\sigma}{\mu} + \dfrac{\sigma(1-\tau)}{1+\tau} & \dfrac{(\mu-1)(1-\sigma)}{\mu} + \dfrac{\tau-1}{1+\tau} \\ \dfrac{\sigma-1}{\mu} + \dfrac{\sigma(1+\tau)}{\tau-1} & \dfrac{\sigma(\mu-1)+1}{\mu} \end{bmatrix}, \tag{31.29}$$

from which we obtain, expanding the determinant of $\mathbf{J}$ and rearranging terms,

$$|\mathbf{J}| = \frac{2\tau}{\mu(1+\tau)(1-\tau)} \left\{2\sigma(\sigma-1)(1-\mu) + (\tau-1)\left[\sigma(\mu+1)-1\right]\right\}. \tag{31.30}$$

If we take the parameter restrictions into account (i.e., $\sigma > 1$, $0 < \mu < 1$, $\tau \in (0,1)$), we see that the fraction is positive, while the expression $\{2\sigma(\sigma-1)(1-\mu) + (\tau-1)[\sigma(\mu+1)-1]\}$ contains one positive and one negative term. To determine the sign of this expression, we first observe that it is a monotonically increasing function of τ. Hence if it is positive for $\tau = 0$ it will be positive for all positive τ. For $\tau = 0$ the expression becomes, after simple manipulations,

$$2\sigma(\sigma-1)(1-\mu) - [\sigma(\mu+1)-1] = (2\sigma-1)[\sigma(1-\mu)-1], \tag{31.31}$$

which will be positive when

$$\sigma(1-\mu) > 1 \quad \text{or,} \quad \frac{\sigma-1}{\sigma} > \mu, \tag{31.32}$$

a condition assumed by Krugman and Venables (1995, p. 878). This assumption turns out to be crucial—see below, Eqs. (31.43) and (31.44)—hence it is interesting to discuss its economic meaning. The condition requires either σ to be sufficiently high or μ to be sufficiently low.

Thus the first cause of the possible violation of condition (31.32) is that demand is insufficiently elastic (σ too low); the second cause (μ too high) is too high a share of intermediates in manufacturing costs, namely too strong backward and forward linkages. Now, when condition (31.32) is satisfied at the equilibrium point, a lower σ means a higher n (the equilibrium number of varieties produced), and so a lower σ means stronger economies of scale (economies of scale are positively related to n: see above, p. 619). Thus, in equilibrium, if σ is small economies of scale will be very high, and σ may become so small as to reverse inequality (31.32), which means that condition (31.32) will be violated when economies of scale are too strong (Krugman & Venables, 1995, p. 870).

Turning back to the mathematics, under (31.32) we have $|\mathbf{J}| \neq 0$, and there exist the differentiable functions

$$\begin{aligned} \widetilde{p} &= \widetilde{p}(L_M, L_M^*), \\ \widetilde{w} &= \widetilde{w}(L_M, L_M^*). \end{aligned} \tag{31.33}$$

We now consider a small change in manufacturing employment in North, $\mathrm{d}L_M$, with associated change $\mathrm{d}L_M^*$ in the opposite direction in the neighbourhood of the symmetric equilibrium. From the second function in (31.33) we have

$$\mathrm{d}\widetilde{w} = \frac{\partial \widetilde{w}}{\partial L_M}\mathrm{d}L_M + \frac{\partial \widetilde{w}}{\partial L_M^*}\mathrm{d}L_M^*.$$

Since we have assumed $\mathrm{d}L_M + \mathrm{d}L_M^* = 0$, we have

$$\mathrm{d}\widetilde{w} = \left(\frac{\partial \widetilde{w}}{\partial L_M} - \frac{\partial \widetilde{w}}{\partial L_M^*}\right)\mathrm{d}L_M,$$

whence

$$\frac{\mathrm{d}\widetilde{w}}{\mathrm{d}L_M} = \frac{\partial \widetilde{w}}{\partial L_M} - \frac{\partial \widetilde{w}}{\partial L_M^*}, \tag{31.34}$$

that gives the total effect on wages of the assumed change in manufacturing employment.

The comparative statics method (Gandolfo, 2009, chap. 20, sect. 20.2) gives us the way of rigorously computing $\partial\widetilde{w}/\partial L_M$, $\partial\widetilde{w}/\partial L_M^*$. Thus we have

$$\frac{\partial \widetilde{w}}{\partial L_M} = \frac{\begin{vmatrix} \dfrac{\partial \varphi_1}{\partial \widetilde{p}} & -\dfrac{\partial \varphi_1}{\partial L_M} \\ \dfrac{\partial \varphi_2}{\partial \widetilde{p}} & -\dfrac{\partial \varphi_2}{\partial L_M} \end{vmatrix}}{|\mathbf{J}|} \tag{31.35}$$

and

$$\frac{\partial \widetilde{w}}{\partial L_M^*} = \frac{\begin{vmatrix} \dfrac{\partial \varphi_1}{\partial \widetilde{p}} & -\dfrac{\partial \varphi_1}{\partial L_M^*} \\ \dfrac{\partial \varphi_2}{\partial \widetilde{p}} & -\dfrac{\partial \varphi_2}{\partial L_M^*} \end{vmatrix}}{|\mathbf{J}|}, \tag{31.36}$$

where $\partial\varphi_1/\partial L_M, \partial\varphi_2/\partial L_M, \partial\varphi_1/\partial L_M^*, \partial\varphi_2/\partial L_M^*$ are computed from Eqs. (31.25) and (31.27), and are evaluated at the symmetric equilibrium point. They turn out to be

$$\begin{aligned}\frac{\partial\varphi_1}{\partial L_M} &= \frac{\tau-1}{(1+\tau)L_M} = \frac{\tau-1}{(1+\tau)\gamma L},\\ \frac{\partial\varphi_2}{\partial L_M} &= \frac{\mu}{\gamma L},\\ \frac{\partial\varphi_1}{\partial L_M^*} &= \frac{1-\tau}{(1+\tau)L_M} = \frac{1-\tau}{(1+\tau)\gamma L},\\ \frac{\partial\varphi_2}{\partial L_M^*} &= -\frac{\mu}{\gamma L},\end{aligned} \tag{31.37}$$

where we have used the fact that $L_M = L_M^* = \gamma L$ at the symmetric equilibrium point. It is apparent from (31.37), given (31.35) and (31.36), that $\partial\widetilde{w}/\partial L_M = -\partial\widetilde{w}/\partial L_M^*$, hence (31.34) becomes

$$\frac{d\widetilde{w}}{dL_M} = 2\frac{\partial\widetilde{w}}{\partial L_M}. \tag{31.38}$$

Let us now calculate the numerator of the fraction in (31.35), call it N. Substituting the values of the partial derivatives found above and expanding the determinant we obtain, after rearrangement of terms,

$$N = \frac{1}{\gamma L\mu}\left\{(1-\mu)\left[\sigma(\mu-1)+1\right] + \tau(\mu+1)\left[\sigma(\mu+1)-1\right]\right\}. \tag{31.39}$$

Substitution of (31.39) and (31.30) in (31.35) and then in (31.38) yields, after rearrangement of terms,

$$\begin{aligned}\frac{d\widetilde{w}}{dL_M} &= \left(\frac{\tau-1}{\tau\gamma L}\right)\frac{(\mu-1)\left[\sigma(\mu-1)+1\right] - \tau(\mu+1)\left[\sigma(\mu+1)-1\right]}{2\sigma(\sigma-1)(1-\mu)+(\tau-1)\left[\sigma(\mu+1)-1\right]}\\ &= \left(\frac{\tau-1}{\tau\gamma L}\right)\frac{(1-\mu)\left[\sigma(1-\mu)-1\right] - \tau(\mu+1)\left[\sigma(\mu+1)-1\right]}{2\sigma(\sigma-1)(1-\mu)+(\tau-1)\left[\sigma(\mu+1)-1\right]}.\end{aligned} \tag{31.40}$$

We have already seen—see Eq. (31.30)—that the expression in the denominator of the second fraction on the r.h.s. is positive. Given that $(\tau-1)<0$, it follows that

$$\frac{d\widetilde{w}}{dL_M} \gtreqless 0 \text{ according as } (1-\mu)\left[\sigma(1-\mu)-1\right] - \tau(\mu+1)\left[\sigma(\mu+1)-1\right] \lesseqgtr 0, \tag{31.41}$$

i.e., recalling (31.32), according as

$$\tau^{-1} \lesseqgtr \frac{(\mu+1)\,[\sigma(\mu+1)-1]}{(1-\mu)\,[\sigma(1-\mu)-1]}, \tag{31.42}$$

which implies, given the definition of τ, that

$$\frac{\mathrm{d}\widetilde{w}}{\mathrm{d}L_M} \gtreqless 0 \text{ according as } t^{\sigma-1} \lesseqgtr \left(\frac{1+\mu}{1-\mu}\right)\left(\frac{\sigma(1+\mu)-1}{\sigma(1-\mu)-1}\right). \tag{31.43}$$

Let us denote by t_0 the value of t at which the above inequality is satisfied as an equality, and observe that, given (31.32) and $0 < \mu < 1$, both fractions in (31.43) are greater than unity, hence—since $\sigma > 1$—the critical value t_0 will certainly be greater than unity, which means a positive level of transportation cost.

It should now be pointed out the crucial role of assumption (31.32). Suppose that the contrary is true, namely $\sigma(1-\mu)-1 < 0$ (and suppose that this has no effect on $|\mathbf{J}|$, which remains non zero). Then division by a negative quantity would imply that in passing from (31.41) to (31.42) the order of the inequalities would have to be reversed, whence

$$\frac{\mathrm{d}\widetilde{w}}{\mathrm{d}L_M} \gtreqless 0 \text{ according as } t^{\sigma-1} \gtreqless \left(\frac{1+\mu}{1-\mu}\right)\left(\frac{\sigma(1+\mu)-1}{\sigma(1-\mu)-1}\right). \tag{31.44}$$

This means that, since the r.h.s. of the inequality is negative, there exists no positive critical value t_0, hence $\mathrm{d}\widetilde{w}/\mathrm{d}L_M > 0$ and the model would be always unstable (see the next section), i.e., a core-periphery pattern would always emerge no matter how high transport costs are. In other words, the forces driving to industrial agglomeration in North would always predominate: this region would become a kind of black hole for world industry. In economic terms, increasing returns to scale are so strong (let us recall that the crucial inequality is reversed when σ is too low, namely economies of scale are too strong) that an increase in manufacturing employment in North always causes an increase in the manufacturing wage rate in North relative to South.

31.2.2 The Dynamics: Bifurcation Analysis

We now come to the dynamics proper. We have seen that $\widetilde{w}$, the manufacturing wage rate in North relative to South, is a function of manufacturing employment L_M, given transport costs. Since in the symmetric equilibrium the wage rate is taken to be unity (see above), such a function will determine the corresponding equilibrium manufacturing employment, say L_M^e. There may be more than one equilibrium value of L_M.

Any such equilibrium will be stable if actual manufacturing employment tends to increase (decrease) when it falls short of (exceeds) the equilibrium value considered, unstable in the opposite case.

This is little more than a tautology, thus we must look for the forces that cause manufacturing employment to change. These are undoubtedly given by the wage rate: more precisely, actual manufacturing employment in North will tend to increase (decrease) if the wage rate there happens to be higher (lower) than in South, whatever the cause that has displaced the symmetric equilibrium. Given the functional relation between $\widetilde{w}$ and L_M, the variation in manufacturing employment will bring about a change in the wage rate, which will in turn feed back on manufacturing employment, and so forth.

The formal counterpart of this dynamic behaviour is the differential equation

$$\dot{L}_M = f\left(\widetilde{w}\right),\ f(1) = 0,\ f'(1) > 0,\ \text{sgn}\ f\left(\widetilde{w} - 1\right) = \text{sgn}\left(\widetilde{w} - 1\right), \quad (31.45)$$

where $\widetilde{w} = h(L_M; t)$, hence

$$\dot{L}_M = f\left(h(L_M; t)\right). \quad (31.46)$$

Equation (31.46) is a one-parameter differential equation, where a codimension-one bifurcation may occur. This is indeed the case. In fact, the characteristic root of its linear approximation at the equilibrium point is

$$\lambda = k\frac{\mathrm{d}\widetilde{w}}{\mathrm{d}L_M},$$

where $k \equiv f'(1)$, and $\mathrm{d}\widetilde{w}/\mathrm{d}L_M$ is evaluated at the equilibrium point. Equilibrium will be stable (unstable) when $\lambda \lessgtr 0$, respectively. This is equivalent to $\mathrm{d}\widetilde{w}/\mathrm{d}L_M \lessgtr 0$, given that $k > 0$. Using (31.43), we see that there is a bifurcation point at $t = t_0$, at which the equilibrium from stable ($t > t_0$ implies $\mathrm{d}\widetilde{w}/\mathrm{d}L_M < 0$) becomes unstable ($t < t_0$ implies $\mathrm{d}\widetilde{w}/\mathrm{d}L_M > 0$).

This proves that as transportation costs decline, there is a critical point at which the core-periphery pattern emerges.

For numerical simulations of this model see Krugman and Venables (1995).

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