

# Chapter 19

## Appendix to Chapter 3

### 19.1 The Transformation Curve and the Box Diagram

#### 19.1.1 A Formal Derivation of the Transformation Curve and Its Properties

Let  $A = f_A(L_A, K_A)$  and  $B = f_B(L_B, K_B)$  be the aggregate production functions (twice continuously differentiable) of the two commodities, where  $L_A + L_B = L$  (the total amount of labour existing in the economy) and  $K_A + K_B = K$  (the total amount of capital existing in the economy). Consider then the following maximization problem

$$\text{Max } B = f_B(L_B, K_B) \text{ subject to } f_A(L - L_B, K - K_B) - A = 0, \quad (19.1)$$

where  $A$  is a *parametrically given* amount of commodity  $A$ .<sup>1</sup> The Lagrangian is

$$F = f_B(L_B, K_B) + \lambda [f_A(L - L_B, K - K_B) - A],$$

and the first-order conditions for a maximum are

$$\begin{aligned} \frac{\partial F}{\partial L_B} &= \frac{\partial f_B}{\partial L_B} - \lambda \frac{\partial f_A}{\partial L_A} = 0, \\ \frac{\partial F}{\partial K_B} &= \frac{\partial f_B}{\partial K_B} - \lambda \frac{\partial f_A}{\partial K_A} = 0, \\ \frac{\partial F}{\partial \lambda} &= f_A(L - L_B, K - K_A) - A = 0, \end{aligned} \quad (19.2)$$

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<sup>1</sup>Here, as everywhere in this book, we follow the commonly adopted practice of using the same symbol to denote both the commodity and its quantity.

whence

$$\frac{\partial f_B / \partial L_B}{\partial f_B / \partial K_B} = \frac{\partial f_A / \partial L_A}{\partial f_A / \partial K_A}, \quad (19.3)$$

which states that the MRTS have to be equal, namely that the isoquants in the Edgeworth-Bowley box must be tangent.

The second-order conditions require the following bordered Hessian determinant

$$\begin{vmatrix} \frac{\partial^2 f_B}{\partial L_B^2} + \lambda \frac{\partial^2 f_A}{\partial L_A^2} & \frac{\partial^2 f_B}{\partial L_B \partial K_B} + \lambda \frac{\partial^2 f_A}{\partial L_A \partial K_A} & -\frac{\partial f_A}{\partial L_A} \\ \frac{\partial^2 f_B}{\partial K_B \partial L_B} + \lambda \frac{\partial^2 f_A}{\partial K_A \partial L_A} & \frac{\partial^2 f_B}{\partial K_B^2} + \lambda \frac{\partial^2 f_A}{\partial K_A^2} & -\frac{\partial f_A}{\partial K_A} \\ -\frac{\partial f_A}{\partial L_A} & -\frac{\partial f_A}{\partial K_A} & 0 \end{vmatrix} \quad (19.4)$$

to be positive. We assume that this condition is satisfied.

The marginal rate of transformation is

$$-\frac{dB}{dA} = -\frac{(\partial f_B / \partial L_B) dL_B + (\partial f_B / \partial K_A) dK_B}{(\partial f_A / \partial L_A) dL_A + (\partial f_A / \partial K_A) dK_A}, \quad (19.5)$$

where of course the two total differentials must obey the optimum conditions (19.3) as well as the constraints  $dL_A + dL_B = 0$ ,  $dK_A + dK_B = 0$ . Therefore

$$\begin{aligned} -\frac{dB}{dA} &= -\frac{(\partial f_B / \partial K_B) \{[(\partial f_B / \partial L_B) / (\partial f_B / \partial K_B)] dL_B + dK_B\}}{(\partial f_A / \partial K_A) \{[(\partial f_A / \partial L_A) / (\partial f_A / \partial K_A)] dL_A + dK_A\}} \\ &= \frac{\partial f_B / \partial K_B}{\partial f_A / \partial K_A} = \frac{\partial f_B / \partial L_B}{\partial f_A / \partial L_A}. \end{aligned} \quad (19.6)$$

Consider now the marginal costs of  $A$  and  $B$ ,  $MC_A$  and  $MC_B$ . We know from microeconomics that

$$\begin{aligned} MC_A &= \frac{p_L}{\partial f_A / \partial L_A} = \frac{p_K}{\partial f_A / \partial K_A}, \\ MC_B &= \frac{p_L}{\partial f_B / \partial L_B} = \frac{p_K}{\partial f_B / \partial K_B}, \end{aligned} \quad (19.7)$$

where  $p_L$  and  $p_K$  are factor prices, and that—under perfect competition—

$$MC_A = p_A, \quad MC_B = p_B. \quad (19.8)$$

From (19.7) and (19.6) it follows that

$$-\frac{dB}{dA} = \frac{MC_A}{MC_B}, \quad (19.9)$$

and from (19.8) and (19.9) we have

$$-\frac{dB}{dA} = \frac{p_A}{p_B}. \quad (19.10)$$

These results have been commented on in Sect. 3.1. It goes without saying that the marginal rate of transformation could also be defined as  $-(dA/dB)$ , in which case the only modification is to take the reciprocal of the various fractions.

From (19.10) it follows that—provided the transformation curve is not linear and does not change its curvature—a unique point on the transformation curve corresponds to any price ratio  $p_B/p_A$ , so that the outputs of  $A$  and  $B$  can be considered as single-valued functions of this ratio.

### 19.1.2 Convexity or Concavity of the Transformation Curve

We now turn to examine the convexity or concavity of the transformation curve. From (19.2) and (19.6), we obtain

$$-\frac{dB}{dA} = \lambda, \quad (19.11)$$

in other words, the marginal rate of transformation is equal to the Lagrange multiplier. Therefore

$$\frac{d^2B}{dA^2} = -\frac{\partial\lambda}{\partial A}. \quad (19.12)$$

If we consider (19.2) as a set of implicit functions, we can express the variables  $L_B, K_B, \lambda$  as differentiable functions of the parameters  $A, L, K$ , provided that the Jacobian determinant of the set with respect to the variables is not zero at the equilibrium point (implicit function theorem: see, for example, [Gandolfo, 2009](#), Pt. III, chap. 20). As is always the case in maximization problems, this Jacobian coincides with the Hessian involved in the second order conditions, and so the condition required by the implicit function theorem is automatically satisfied.

Therefore we can differentiate totally the first-order conditions with respect to  $A$  and obtain

$$\begin{aligned} \left( \frac{\partial^2 f_B}{\partial L_B^2} + \lambda \frac{\partial^2 f_A}{\partial L_A^2} \right) \frac{\partial L_B}{\partial A} + \left( \frac{\partial^2 f_B}{\partial L_B \partial K_B} + \lambda \frac{\partial^2 f_A}{\partial L_A \partial K_A} \right) \frac{\partial K_B}{\partial A} - \frac{\partial f_A}{\partial L_A} \frac{\partial \lambda}{\partial A} &= 0, \\ \left( \frac{\partial^2 f_B}{\partial K_B L_B} + \lambda \frac{\partial^2 f_A}{\partial K_A L_A} \right) \frac{\partial L_B}{\partial A} + \left( \frac{\partial^2 f_B}{\partial K_B^2} + \lambda \frac{\partial^2 f_A}{\partial K_A^2} \right) \frac{\partial K_B}{\partial A} - \frac{\partial f_A}{\partial K_A} \frac{\partial \lambda}{\partial A} &= 0, \\ -\frac{\partial f_A}{\partial L_A} \frac{\partial L_B}{\partial A} - \frac{\partial f_A}{\partial K_A} \frac{\partial K_B}{\partial A} &= 1, \end{aligned} \quad (19.13)$$

whence, solving for  $\partial\lambda/\partial A$ ,

$$\frac{\partial\lambda}{\partial A} = \frac{\left(\frac{\partial^2 f_B}{\partial L_B^2} + \lambda \frac{\partial^2 f_A}{\partial L_A^2}\right) \left(\frac{\partial^2 f_B}{\partial K_B^2} + \lambda \frac{\partial^2 f_A}{\partial K_A^2}\right) - \left(\frac{\partial^2 f_B}{\partial L_B \partial K_B} + \lambda \frac{\partial^2 f_A}{\partial L_A \partial K_A}\right)^2}{|H|}, \quad (19.14)$$

where  $|H|$  is the Hessian determinant (19.4). Since  $|H|$  is positive, the sign of  $\partial\lambda/\partial A$  depends on the sign of the numerator of the fraction. In general, this sign is indeterminate, even if we assume that marginal productivities are positive and decreasing. In fact, since  $\lambda > 0$  by (19.2), the first two terms in the numerator of (19.14) are negative, so that their product is positive, and from this we must subtract the third term, which is also positive, if we rule out the exceptional case of its being zero.

### 19.1.3 Homogeneous Production Functions and Transformation Curve

This indeterminacy can be eliminated if we assume that the production functions are homogeneous of the first degree (constant returns to scale).

Since the properties of these functions are widely used in this and following chapters, we list them here for the reader's convenience, though they are well-known.

Given the twice-differentiable function

$$Y = F(X_1, X_2),$$

it is said to be (positively) homogeneous of degree  $n > 0$  if, for any  $\mu > 0$ ,

$$F(\mu X_1, \mu X_2) = \mu^n F(X_1, X_2).$$

It is easy to see that returns to scale are increasing, constant, decreasing, respectively for  $n \gtrless 1$ . Let us consider the case of a homogeneous function of the first degree. Such a function has the following properties:

$$(a) \textit{ Intensive form: } \begin{aligned} Y &= X_2 F(X_1/X_2, 1) = X_2 f(X_1/X_2), \\ Y &= X_1 F(1, X_2/X_1) = X_1 g(X_2/X_1), \end{aligned}$$

which allows us, for example, to express output per head as a function of capital per head only. If we let  $X_1$  denote capital,  $X_2$  labour,  $y$  output per head, and  $\rho$  capital per head, from (a) we have  $y = F(\rho, 1)$ .

$$(b) \textit{ Radiality: } \begin{aligned} \frac{\partial Y}{\partial X_1} &= f'(X_1/X_2) = g(X_2/X_1) - (X_2/X_1)g'(X_2/X_1), \\ \frac{\partial Y}{\partial X_2} &= f(X_1/X_2) - (X_1/X_2)f'(X_1/X_2) = g'(X_2/X_1), \end{aligned}$$

which means that marginal productivities are functions of the input ratio alone, so that the isoquants have an identical slope along any ray starting from the origin.

$$(c) \text{ Euler's theorem: } \frac{\partial Y}{\partial X_1} X_1 + \frac{\partial Y}{\partial X_2} X_2 = Y,$$

so that output is exhausted if—as is the case under perfect competition—each factor’s reward in real terms equals its marginal productivity.

(d) Relations between second-order pure and mixed derivatives:

$$\frac{\partial^2 Y}{\partial X_1^2} = -\frac{X_2}{X_1} \frac{\partial^2 Y}{\partial X_1 \partial X_2}, \quad \frac{\partial^2 Y}{\partial X_2^2} = -\frac{X_1}{X_2} \frac{\partial^2 Y}{\partial X_1 \partial X_2}.$$

This property means that, if we assume  $\partial^2 Y / \partial X_i^2 < 0$  (decreasing marginal productivities), then  $\partial^2 Y / \partial X_1 \partial X_2 = \partial^2 Y / \partial X_2 \partial X_1 > 0$ , namely an increase in a factor has a *positive* effect on the marginal productivity of the *other* factor.

Let us now define what is meant by a “well-behaved” production function. Mathematically, it is a first-degree homogeneous function with positive first-order partial derivatives and negative second-order pure partial derivatives, that—with reference to the intensive form—further satisfies the Inada-Uzawa conditions:

$$F(0, 1) = 0, \quad \lim_{\rho \rightarrow \infty} F(\rho, 1) = \infty, \\ F'(0, 1) = \infty, \quad \lim_{\rho \rightarrow 0} F'(\rho, 1) = \infty.$$

Going back to our problem, property (d) enables us to write

$$\frac{\partial^2 f_i}{\partial L_i^2} = -\frac{K_i}{L_i} \frac{\partial^2 f_i}{\partial L_i \partial K_i}, \quad \frac{\partial^2 f_i}{\partial K_i^2} = -\frac{L_i}{K_i} \frac{\partial^2 f_i}{\partial L_i \partial K_i}, \quad i = A, B. \quad (19.15)$$

By using these relations we can rewrite the numerator of (19.14), after some manipulations, in the form

$$\begin{aligned} & \lambda \frac{\partial^2 f_A}{\partial L_A \partial K_A} \frac{\partial^2 f_B}{\partial L_B \partial K_A} \left( \frac{K_B L_A}{K_A L_B} + \frac{K_A L_B}{K_B L_A} - 2 \right) \\ &= \lambda \frac{\partial^2 f_A}{\partial L_A \partial K_A} \frac{\partial^2 f_B}{\partial L_B \partial K_B} \frac{(K_B L_A - K_A L_B)^2}{K_A L_B K_B L_A} \\ &= \lambda \frac{\partial^2 f_A}{\partial L_A \partial K_A} \frac{\partial^2 f_B}{\partial L_B \partial K_B} \frac{(\varrho_B - \varrho_A)^2}{\varrho_A \varrho_B}, \end{aligned} \quad (19.16)$$

where  $\varrho_A \equiv K_A / L_A, \varrho_B \equiv K_B / L_B$  are the factor intensities in the two sectors. Now, since we have assumed decreasing marginal productivities, it follows from (19.15) that the second order mixed partial derivative must be positive in both sectors, and so (19.16) is a *positive* quantity, barring out the exceptional case of  $\varrho_A = \varrho_B$ . Therefore,  $\partial \lambda / \partial A \geq 0$  and consequently  $d^2 B / dA^2 \leq 0$ . This proves that,

with constant returns to scale, the transformation curve is concave to the origin, except for the case of equal factor intensities in the two sectors (in which case it is a straight line).

It can also be shown that, when the production functions are homogeneous of degree higher than the first (increasing returns to scale), the numerator of (19.14) is equal to (19.16) plus several other terms; these terms have different signs so that the numerator under consideration can be either negative or positive as well as change its sign. Therefore, with increasing returns to scale, the transformation curve can be either convex or concave to the origin as well as change its curvature; for further details see Sect. 3.5 and Herberg (1969).

## 19.2 A Simple Closed Economy

### 19.2.1 The Basic Model

The following model is derived from Kemp (1964, 1969b):

$$\begin{aligned}
 A &= L_A g_A(Q_A), \\
 B &= L_B g_B(Q_B), \\
 g'_A &= p g'_B, \\
 g_A - Q_A g'_A &= p (g_B - Q_B g'_B), \\
 L_A + L_B &= L, \\
 Q_A L_A + Q_B L_B &= K, \\
 I_A &= A + pB, \\
 A^D(I_A, p) &= A, \\
 B^D(I_A, p) &= B.
 \end{aligned} \tag{19.17}$$

The first two equations are the production functions which, thanks to the assumption of first-degree homogeneity, can be written in the intensive form  $L_i g_i(Q_i) \equiv L_i f_i(1, Q_i)$ ,  $i = A, B$ , where  $Q_i \equiv K_i/L_i$  are factor intensities in the two sectors.

Since  $g'_A \equiv \partial f_A / \partial K_A$ ,  $g'_B \equiv \partial f_B / \partial K_B$ , the third equation states that the value of the marginal product of capital, measured in terms of commodity  $A$  taken as *numéraire* ( $p \equiv p_B/p_A$ ), is equal in both sectors. In fact, under perfect competition the value of the marginal product of a factor must be equal to that factor's reward, which in turn must be equal in each sector. The fourth equation expresses the same condition for the marginal product of labour, since

$$g_A - Q_A g'_A \equiv \partial f_A / \partial L_A, \quad g_B - Q_B g'_B \equiv \partial f_B / \partial L_B.$$

The fifth and sixth equations state that both factors are fully employed;  $L$  and  $K$  are the given total amounts existing.

The seventh equation defines real aggregate income expressed in terms of the first commodity. The last two equations are the equilibrium conditions on the markets for

the commodities; the aggregate demand for each commodity is assumed to depend on aggregate income and on the relative price.

The model has nine equations and only eight unknowns ( $A, B, Q_A, Q_B, L_A, L_B, I_A, p$ ). However, one of the two demand = supply equations is not independent, for either one may be derived from the other if we take account of the budget restraint  $A + pB = A^D + pB^D$  (see Eq. (3.8), where the outputs of  $A$  and  $B$  have been called  $S_A$  and  $S_B$ ).

### 19.2.2 The Supply Side of the Model

Let us now consider the subset consisting of the first six equations of the model, which define the supply side of the economy. We see that it includes seven unknowns ( $A, B, Q_A, Q_B, L_A, L_B, p$ ) so that—assuming that its Jacobian determinant with respect to the first six variables is not zero—we can use the implicit function theorem and express the first six variables as continuously differentiable function of the seventh ( $p$ ); this proves rigorously that the supplies of  $A$  and  $B$  are functions of  $p$ , as shown heuristically in the text. It follows that  $I_A$  is ultimately a function of  $p$  only, and, consequently, that  $A^D$  and  $B^D$  can be expressed as general equilibrium functions of  $p$  only, as explained verbally in Sect. 3.2.2 (see also below, Sect. 19.2.3).

If we differentiate totally the first six equations with respect to  $p$  we obtain

$$\begin{aligned} \frac{dA}{dp} &= \frac{dL_A}{dp} g_A + L_A g'_A \frac{dQ_A}{dp}, \\ \frac{dB}{dp} &= \frac{dL_B}{dp} g_B + L_B g'_B \frac{dQ_B}{dp}, \\ g''_A \frac{dQ_A}{dp} &= g'_B + p g''_B \frac{dQ_B}{dp}, \\ -Q_A g''_A \frac{dQ_A}{dp} &= g_B - Q_B g'_B - p Q_B g''_B \frac{dQ_B}{dp}, \\ \frac{dL_A}{dp} + \frac{dL_B}{dp} &= 0, \\ \frac{dQ_A}{dp} L_A + Q_A \frac{dL_A}{dp} + \frac{dQ_B}{dp} L_B + Q_B \frac{dL_B}{dp} &= 0, \end{aligned} \tag{19.18}$$

from which we can compute the derivatives  $dA/dp$ ,  $dB/dp$ ,  $dQ_A/dp$ ,  $dQ_B/dp$ ,  $dL_A/dp$ ,  $dL_B/dp$ . We are interested in

$$\begin{aligned} \frac{dA}{dp} &= \frac{L_A g_B^2 p}{g''_A (Q_B - Q_A)^2} + \frac{g_A^2 L_B}{p^2 g''_B (Q_B - Q_A)^2}, \\ \frac{dB}{dp} &= -\frac{L_A g_B^2}{g''_A (Q_B - Q_A)^2} - \frac{g_A^2 L_B}{p^3 g''_B (Q_B - Q_A)^2}, \end{aligned} \tag{19.19}$$

where of course  $Q_B \neq Q_A$ . Since  $g_A''$  and  $g_B''$  are negative by the assumption of decreasing marginal products, it follows from (19.19) that  $dA/dp < 0$ ,  $dB/dp > 0$ , namely the supply of  $B$  increases, and the supply of  $A$  decreases, as  $p$  increases. It also follows from (19.19) that

$$\frac{dB}{dp} = -\frac{1}{p} \frac{dA}{dp}, \quad (19.20)$$

whence

$$-\frac{dA}{dB} = p, \quad (19.21)$$

as already shown in (19.10) and in Sect. 3.1. An alternative way of arriving at (19.20) is to start from the transformation curve,  $B = h(A)$ , whence

$$\frac{dB}{dp} = \frac{dB}{dA} \frac{dA}{dp},$$

and since  $dB/dA = -1/p$  from (19.10), we have

$$\frac{dB}{dp} = -\frac{1}{p} \frac{dA}{dp}.$$

### 19.2.3 The Demand Side of the Model

Let us now consider the demand side of the model. If we differentiate  $I_A$  with respect to  $p$ , we obtain

$$\frac{dI_A}{dp} = \frac{dA}{dp} + B + p \frac{dB}{dp} = B, \quad (19.22)$$

because  $dA/dp + p (dB/dp) = 0$  by (19.20). Therefore we can compute the total derivative of each demand function with respect to  $p$ :

$$\begin{aligned} \frac{dA^D}{dp} &= \frac{\partial A^D}{\partial I_A} \frac{dI_A}{dp} + \frac{\partial A^D}{\partial p} = \frac{\partial A^D}{\partial I_A} B + \frac{\partial A^D}{\partial p}, \\ \frac{dB^D}{dp} &= \frac{\partial B^D}{\partial I_A} \frac{dI_A}{dp} + \frac{\partial B^D}{\partial p} = \frac{\partial B^D}{\partial I_A} B + \frac{\partial B^D}{\partial p}. \end{aligned} \quad (19.23)$$

We assume that these demand functions are well behaved, namely  $\partial A^D/\partial I_A > 0$ ,  $\partial B^D/\partial I_A > 0$  (no inferior goods), and  $\partial A^D/\partial p > 0$ ,  $\partial B^D/\partial p < 0$  (normal price-effect: remember that  $p$  is  $p_B/p_A$ , so that  $\partial A^D/\partial p > 0$  means  $\partial A^D/\partial (1/p) < 0$ ). It follows that  $\partial A^D/\partial p > 0$ , as shown in Fig. 3.5b. The sign of  $dB^D/dp$  remains

indeterminate, for this derivative is the sum of a positive and a negative term; however, it can be shown that  $dB^D/dp$  must be negative at least in the neighbourhood of the equilibrium point. In fact, differentiation of the budget constraint yields

$$\frac{dA^D}{dp} + B^D + p \frac{dB^D}{dp} = B,$$

whence

$$\frac{dA^D}{dp} + p \frac{dB^D}{dp} = B - B^D. \quad (19.24)$$

Now,  $B - B^D = 0$  at the equilibrium point, so that  $dB^D/dp < 0$  since  $dA^D/dp > 0$ . Also note that  $dB^D/dp$  must a fortiori be negative below the equilibrium point, namely when  $B - B^D < 0$ .

## 19.3 International Trade and Offer Curves

### 19.3.1 *The Equilibrium Conditions: The Offer Curve and Its Slope*

Consider the excess demands for commodities  $A$  and  $B$  in country 1 and the budget constraint (Walras' law)

$$\begin{aligned} E_{1A}(p) &= A_1^D(I_{1A}, p) - A_1(p), \\ E_{1B}(p) &= B_1^D(I_{1A}, p) - B_1(p), \\ E_{1A}(p) + pE_{1B}(p) &= 0, \end{aligned} \quad (19.25)$$

which are written as functions of  $p$  only, because  $I_{1A}$  is a function of  $p$  as shown in Sect. 19.2.2. If the economy is closed, the equilibrium conditions (19.17) require that  $E_{1A} = E_{1B} = 0$ . If we introduce country 2, we have the relations

$$\begin{aligned} E_{2A}(p) &= A_2^D(I_{2A}, p) - A_2(p), \\ E_{2B}(p) &= B_2^D(I_{2A}, p) - B_2(p), \\ E_{2A}(p) + pE_{2B}(p) &= 0, \end{aligned} \quad (19.26)$$

where the terms of trade  $p$  must be the same in both countries as shown in the text.

International equilibrium requires that the world demands for the two commodities are equal to the respective world supplies, namely

$$\begin{aligned} E_{1A}(p) + E_{2A}(p) &= 0, \\ E_{1B}(p) + E_{2B}(p) &= 0. \end{aligned} \quad (19.27)$$

These conditions are not independent, for either one can be derived from the other given the two countries' budget constraints. By using these constraints, international equilibrium can also be expressed as

$$\begin{aligned} E_{1A}(p) &= pE_{2B}(p), \\ pE_{1B}(p) &= E_{2A}(p), \end{aligned} \quad (19.28)$$

where of course either one depends on the other. Hence  $p$  (the terms of trade) is determined, and by substituting back we determine all the other variables.

The graphic counterpart of (19.27) is Fig. 3.6; the graphic counterpart of (19.28) is Viner's terms-of-trade diagram (Viner, 1937, p. 362; see also Mosak, 1944, p. 77 and Kemp, 1964, pp. 62–63).

The derivation of the offer curve from the excess demand functions has already been shown in Sect. 3.4.1 and Fig. 3.7. Here we wish to demonstrate that *the offer curve is not necessarily well behaved* (namely monotonically increasing and concave to its import axis) *even if we assume underlying normal supply and demand schedules*.

Let us assume, as in the text, that country 1 wishes to import commodity  $A$  and to export commodity  $B$ , namely

$$E_{1A}(p) > 0, \quad E_{1B}(p) < 0, \quad (19.29)$$

so that we can write this country's offer curve as

$$-E_{1B} = G_1(E_{1A}), \quad (19.30)$$

where the minus sign serves to make  $-E_{1B}$  a positive quantity. Similarly,

$$E_{2A}(p) < 0, \quad E_{2B}(p) > 0, \quad (19.31)$$

and

$$-E_{2A} = G_2(E_{2B}). \quad (19.32)$$

For our purpose it suffices to consider one offer curve, say that of country 1. We have

$$G'_1 = \frac{d(-E_{1B})}{dE_{1A}} = -\frac{dE_{1B}/dp}{dE_{1A}/dp}. \quad (19.33)$$

By differentiating (19.25) and using (19.23) and (19.19) it can be seen that

$$\begin{aligned} \frac{dE_{1A}}{dp} &= \frac{\partial A_1^D}{\partial I_{1A}} B_1 + \frac{\partial A_1^D}{\partial p} - \frac{dA_1}{dp} > 0, \\ \frac{dE_{1B}}{dp} &= \frac{\partial B_1^D}{\partial I_{1A}} B_1 + \frac{\partial B_1^D}{\partial p} - \frac{dB_1}{dp} \geq 0. \end{aligned} \quad (19.34)$$

The ambiguity of the sign of  $dE_{1B}/dp$  derives from the fact that  $dB_1^D/dp$  has an ambiguous sign. Note that since  $dB_1^D/dp < 0$  at the equilibrium point, as shown above, then  $dE_{1B}/dp < 0$  at the equilibrium point, and so  $G_1' > 0$  at the origin. But, as we move away from the origin the sign becomes indeterminate, since we are considering  $E_{1B} < 0$  namely  $B_1 - B_1^D > 0$ , and Eq. (19.24) shows that  $dA_1^D/dp > 0$  and  $dB_1^D/dp > 0$  are perfectly compatible with the budget constraint.

We have thus proved that the offer curve, although increasing at the origin, need not be increasing everywhere, notwithstanding normal demand and supply functions for both commodities.

As regards its convexity or concavity, the sign of  $G_1'' = d^2(-E_{1B})/dE_{1A}^2$  is also indeterminate, for it involves the second derivatives of the demand and supply functions with respect to  $p$ , which are indeterminate.

The conclusion is that cases such as that depicted in Fig. 3.11 (multiple equilibria) as well as cases in which the offer curves are decreasing over some interval cannot be ruled out on the basis of normal underlying demand and supply functions.

### 19.3.2 Relationships Between the Various Elasticities

Let us finally examine the relations between the elasticity of the offer curve, the elasticity of import demand and the elasticity of export supply. We first examine country 1's elasticities.

The elasticity of the offer curve is defined—see Eq. (3.18)—as

$$e_1 = \frac{d(-E_{1B})}{dE_{1A}} \frac{E_{1A}}{(-E_{1B})} = \frac{dE_{1B}}{dE_{1A}} \frac{E_{1A}}{E_{1B}}. \tag{19.35}$$

The (total) price-elasticity of import demand is defined as

$$\xi_1 = \frac{dE_{1A}}{d(1/p)} \frac{1/p}{E_{1A}} = \frac{dE_{1A}}{d(1/p)} \frac{1}{pE_{1A}} = -\frac{dE_{1A}}{dp} \frac{p}{E_{1A}}, \tag{19.36}$$

and the (total) price-elasticity of export supply is defined as

$$\varepsilon_1 = \frac{d(-E_{1B})}{dp} \frac{p}{(-E_{1B})} = \frac{dE_{1B}}{dp} \frac{p}{E_{1B}}, \tag{19.37}$$

where the adjective total serves to remind us that when  $p$  changes  $I_A$  changes as well (as a function of  $p$ ), so that the quantity change includes both effects.

Since  $E_{1A} = -pE_{1B}$  from the budget constraint, we can write (19.36) as

$$\xi_1 = \frac{dE_{1A}}{d(-E_{1B}/E_{1A})} \frac{E_{1B}}{-E_{1A}^2} = \left( \frac{dE_{1B}}{dE_{1A}} \frac{E_{1A}}{E_{1B}} - 1 \right)^{-1}. \tag{19.38}$$

It follows from (19.38) and (19.35) that

$$\xi_1 = (e_1 - 1)^{-1}, \quad e_1 = \frac{1 + \xi_1}{\xi_1}. \quad (19.39)$$

Similarly we can write (19.37) as

$$\varepsilon_1 = \frac{dE_{1B}}{d(-E_{1A}/E_{1B})} \frac{-E_{1A}}{E_{1B}^2} = \left( \frac{dE_{1A}}{dE_{1B}} \frac{E_{1B}}{E_{1A}} - 1 \right)^{-1}. \quad (19.40)$$

It follows from (19.40) and (19.35) that

$$\varepsilon_1 = \left( \frac{1}{e_1} - 1 \right)^{-1} = \frac{e_1}{1 - e_1}. \quad (19.41)$$

Therefore

$$1 + \xi_1 + \varepsilon_1 = 0. \quad (19.42)$$

Similarly it can be shown that

$$\xi_2 = (e_2 - 1)^{-1}, \quad e_2 = \frac{1 + \xi_2}{\xi_2}, \quad \varepsilon_2 = \frac{e_2}{1 - e_2}, \quad 1 + \xi_2 + \varepsilon_2 = 0, \quad (19.43)$$

where

$$\begin{aligned} e_2 &= \frac{d(-E_{2A})}{dE_{2B}} \frac{E_{2B}}{(-E_{2A})}, & \xi_2 &= \frac{dE_{2B}}{dp} \frac{p}{E_{2B}}, \\ \varepsilon_2 &= \frac{d(-E_{2A})}{d(1/p)} \frac{p}{(-E_{2A})} = -\frac{dE_{2A}}{dp} \frac{p}{E_{2A}}. \end{aligned} \quad (19.44)$$

Consequently,

$$(1 + \xi_1 + \varepsilon_1) + (1 + \xi_2 + \varepsilon_2) = (1 + \xi_1 + \xi_2) + (1 + \varepsilon_1 + \varepsilon_2) = 0. \quad (19.45)$$

## 19.4 Stability

### 19.4.1 Terms-of-Trade Adjustment

In describing the determination of international equilibrium (Sect. 3.3) we assumed that  $p$  moves according to the pressure of world excess demands. The mathematical counterpart of this assumption is the following differential equation

$$\frac{dp}{dt} = \psi [E_{1B}(p) + E_{2B}(p)] = \psi \left[ E_{2B}(p) - \frac{1}{p} E_{1A}(p) \right], \quad (19.46)$$

where  $\psi$  is a sign-preserving function and  $\psi' [0] \equiv v > 0$ . To examine *local* stability, we expand the r.h.s. of (19.46) in Taylor's series at the equilibrium point and neglect all terms of order higher than the first, thus obtaining

$$\begin{aligned} \frac{d\bar{p}}{dt} &= v \left( \frac{dE_{2B}}{dp} + \frac{1}{p^2} E_{1A} - \frac{1}{p} \frac{dE_{1A}}{dp} \right) \bar{p} \\ &= v \frac{E_{2B}}{p} \left( \frac{dE_{2B}}{dp} \frac{p}{E_{2B}} + \frac{E_{1A}}{pE_{2B}} - \frac{p}{pE_{2B}} \frac{dE_{1A}}{dp} \right) \bar{p}, \end{aligned} \quad (19.47)$$

where  $\bar{p} \equiv p - p_E$  denotes the deviations from equilibrium, and all derivatives, etc., are evaluated at the equilibrium point. Since  $E_{1A} = pE_{2B}$  at the equilibrium point, by using the definitions of the price-elasticities of import demands—see (19.36) and (19.44)—we obtain

$$\frac{d\bar{p}}{dt} = v \frac{E_{2B}}{p} (1 + \xi_1 + \xi_2) \bar{p}. \quad (19.48)$$

Since  $E_{2B} > 0$  by assumption—see (19.31)—the necessary and sufficient stability condition is

$$1 + \xi_1 + \xi_2 < 0. \quad (19.49)$$

Condition (19.49) is sometimes referred to as the “*Marshall-Lerner condition*”. We completely agree with Kemp (1964, p. 70; 1969b, p. 84 fn. 5) when he writes that “Never were adjectives so incongruously applied. Marshall [...] developed a quite different stability condition; and Lerner was concerned neither with a barter nor with a dynamical economy”. We only add that this denomination is even more incongruous when it is applied to the stability condition concerning the foreign exchanges.

## 19.4.2 Quantity Adjustment

Let us now examine the stability of equilibrium when the variables which adjust themselves are the quantities of exports, as Marshall believed; the two behaviour assumptions have been described in Sect. 3.4.2. The mathematical counterpart of *behaviour assumption I* is the following system of differential equations

$$\begin{aligned} \frac{d(-E_{1B})}{dt} &= \varphi_1 [-E_{1B}(p) - (-E_{1B})], \\ \frac{d(-E_{1A})}{dt} &= \varphi_2 [-E_{2A}(p) - (-E_{2A})], \end{aligned} \quad (19.50)$$

where  $\varphi_1, \varphi_2$  are sign-preserving functions with  $\varphi_1'[0] \equiv v_1 > 0$ ,  $\varphi_2'[0] \equiv v_2 > 0$ ; the quantities  $-E_{1B}$  and  $-E_{2A}$  are the actual quantities of exports, whereas  $-E_{1B}(p)$  and  $-E_{2A}(p)$  are the *desired* quantities of exports at the *current* terms of trade  $p$ .

The linearization of system (19.50) at the equilibrium point is rather long-winded, and we refer the reader to Gandolfo (1971, p. 305) for the details. The result is

$$\begin{aligned} -\frac{dE_{1B}}{dt} &= (1 + \varepsilon_1) [E_{1B} - E_{1B}(p_E)] - p_E \varepsilon_1 [E_{2A} - E_{2A}(p_E)], \\ -\frac{dE_{2A}}{dt} &= -\frac{\varepsilon_2}{p_E} [E_{1B} - E_{1B}(p_E)] + (1 + \varepsilon_2) [E_{2A} - E_{2A}(p_E)], \end{aligned} \quad (19.51)$$

where  $\varepsilon_1, \varepsilon_2$  are defined in (19.37) and (19.44), and the units of quantity in both countries have been chosen so as to make  $v_1 = v_2 = 1$ . The characteristic equation of the differential equation system (19.51) is

$$\begin{vmatrix} 1 + \varepsilon_1 + \lambda & -p_E \varepsilon_1 \\ -\varepsilon_2/p_E & 1 + \varepsilon_2 + \lambda \end{vmatrix} = \lambda^2 + (2 + \varepsilon_1 + \varepsilon_2)\lambda + (1 + \varepsilon_1 + \varepsilon_2) = 0, \quad (19.52)$$

whose roots are  $-1, -(1 + \varepsilon_1 + \varepsilon_2)$ . Thus the movement will be monotonic, and will converge, if, and only if

$$1 + \varepsilon_1 + \varepsilon_2 > 0. \quad (19.53)$$

By using the relations between the  $\varepsilon$  and  $e$  elasticities—see (19.41) and (19.43)—the necessary and sufficient stability condition can be written as

$$\frac{1 - e_1 e_2}{(1 - e_1)(1 - e_2)} > 0. \quad (19.54)$$

Let us now consider *behaviour assumption II*, which gives rise to the following differential equation system

$$\begin{aligned} \frac{d(-E_{1B})}{dt} &= \varphi_1 [G_1(E_{1A}) - (-E_{1B})], \\ \frac{d(-E_{2A})}{dt} &= \varphi_2 [G_2(E_{2B}) - (-E_{2A})], \end{aligned} \quad (19.55)$$

where  $\varphi_1, \varphi_2$  are sign-preserving functions with  $\varphi_1'[0] \equiv s_1 > 0$ ,  $\varphi_2'[0] \equiv s_2 > 0$ ; the quantities  $-E_{1B}$  and  $-E_{2A}$  are the *actual* quantities of exports, whereas  $G_1(E_{1A})$  and  $G_2(E_{2B})$  are the *desired* quantities of exports corresponding to the *current* quantities of imports  $E_{1A}$  and  $E_{2B}$  respectively.

The linear approximation to system (19.54) is (for details of the procedure see Gandolfo, 1971, pp. 308–309):

$$\begin{aligned} -\frac{dE_{1B}}{dt} &= [E_{1B} - E_{1B}(p_E)] - p_E e_1 [E_{2A} - E_{2A}(p_E)], \\ -\frac{dE_{2A}}{dt} &= \frac{e_2}{p_E} [E_{1B} - E_{1B}(p_E)] + [E_{2A} - E_{2A}(p_E)], \end{aligned} \quad (19.56)$$

where  $e_1, e_2$  are defined in (19.35) and (19.44), and the units of quantity in both countries have been chosen so as to make  $s_1 = s_2 = 1$ . The characteristic equation of this linear differential equation system is

$$\begin{vmatrix} 1 + \lambda & p_E e_1 \\ e_2/p_E & 1 + \lambda \end{vmatrix} = \lambda^2 + 2\lambda + (1 - e_1 e_2) = 0, \quad (19.57)$$

whose roots are  $-1 \pm \sqrt{e_1 e_2}$ . The movement can be either monotonic or oscillatory according to whether  $e_1 e_2 \gtrless 0$ . The necessary and sufficient stability condition is

$$e_1 e_2 < 1. \quad (19.58)$$

Note that when  $e_1 e_2 < 0$  this condition is certainly satisfied, so that possible oscillatory movements are necessarily convergent.

## 19.5 Duality Approach

### 19.5.1 The Jones Model

We describe here the general equilibrium model due to Jones (1965), that will be put to use in subsequent chapters to prove some of the standard theorems in international trade theory.

Let  $a_{ij}, i = K, L, j = A, B$ , denote the quantity of factor  $i$  required to produce a unit of commodity  $j$ . Then we have

$$\begin{aligned} a_{LA}A + a_{LB}B &= L, \\ a_{KA}A + a_{KB}B &= K, \\ a_{LA}p_L + a_{KA}p_K &= p_A, \\ a_{LB}p_L + a_{KB}p_K &= p_B. \end{aligned} \quad (19.59)$$

These equations emphasize the dual relations between factor endowments and commodity outputs (first two equations, which derive from full factor employment), and between commodity prices and factor prices (last two equations, which derive from competitive equilibrium). Since, in general, the input coefficients  $a_{ij}$  are variable, Eq. (19.59) must be supplemented by four equations to determine these coefficients. Such equations derive from the firm's optimization procedure. In fact, with constant returns to scale, the input coefficients depend solely upon the factor-price ratio: as can be seen from property (a) in Sect. 19.1.3, the input

coefficients depend solely on the factor ratio which in turn—see property (b)—is uniquely determined, independently of the scale of production, by the factor-price ratio according to the cost minimization procedure. Therefore

$$a_{ij} = a_{ij} \left( \frac{p_L}{p_K} \right), \quad i = K, L; \quad j = A, B, \quad (19.60)$$

which are the four equations that we need. The eight Eqs. (19.59) and (19.60) describe the production side of the model, and make it possible to determine the eight unknowns  $a_{ij}$ ,  $A$ ,  $B$ ,  $p_L$ ,  $p_K$  given the four parameters  $L$ ,  $K$ ,  $p_A$ ,  $p_B$ . Let us now write the total differentials of Eq. (19.59):

$$\begin{aligned} Ada_{LA} + a_{LA}dA + Bda_{LB} + a_{LB}dB &= dL, \\ Ada_{KA} + a_{KA}dA + Bda_{KB} + a_{KB}dB &= dK, \\ p_L da_{LA} + a_{LA}dp_L + p_K da_{KA} + a_{KA}dp_K &= dp_A, \\ p_L da_{LB} + a_{LB}dp_L + p_K da_{KB} + a_{KB}dp_K &= dp_B. \end{aligned} \quad (19.61)$$

If we denote the relative changes by an asterisk (namely,  $a_{LA}^* \equiv da_{LA}/a_{LA}$ ,  $A^* \equiv dA/A$ , etc.), we can rewrite Eq. (19.61), after simple manipulations,<sup>2</sup> in the form

$$\begin{aligned} \lambda_{LA}A^* + \lambda_{LB}B^* &= L^* - (\lambda_{LA}a_{LA}^* + \lambda_{LB}a_{LB}^*), \\ \lambda_{KA}A^* + \lambda_{KB}B^* &= K^* - (\lambda_{KA}a_{KA}^* + \lambda_{KB}a_{KB}^*), \\ \theta_{LA}p_L^* + \theta_{KA}p_K^* &= p_A^* - (\theta_{LA}a_{LA}^* + \theta_{KA}a_{KA}^*), \\ \theta_{LB}p_L^* + \theta_{KB}p_K^* &= p_B^* - (\theta_{LB}a_{LB}^* + \theta_{KB}a_{KB}^*), \end{aligned} \quad (19.62)$$

where  $\lambda_{LA} \equiv a_{LA}A/L$ ,  $\lambda_{LB} \equiv a_{LB}B/L$  denote the fractions of the labour force used in sector  $A$  and in sector  $B$  respectively; by the first equation in (19.59) these fractions must add up to one,  $\lambda_{LA} + \lambda_{LB} = 1$ . Similarly, the sum of  $\lambda_{KA} \equiv a_{KA}A/K$  and  $\lambda_{KB} \equiv a_{KB}B/K$  must be equal to one. The  $\theta$ 's denote the factor shares in each sector:  $\theta_{LA} \equiv a_{LA}p_L/p_A$ ,  $\theta_{KA} \equiv a_{KA}p_K/p_A$  and so on; by the last two equations in (19.59) these shares must add up to one, namely  $\theta_{LA} + \theta_{KA} = 1$ ,  $\theta_{LB} + \theta_{KB} = 1$ .

If the input coefficients are fixed,  $a_{ij}^* \equiv 0$ , and so Eq. (19.62) are greatly simplified. But in the general case of variable coefficients we need four additional equations to determine the four  $a_{ij}^*$ . These are

<sup>2</sup>Consider for example the first equation and divide both sides by  $L$ , obtaining

$$\frac{A}{L} da_{LA} + a_{LA} \frac{1}{L} dA + \frac{B}{L} da_{LB} + a_{LB} \frac{1}{L} dB = \frac{dL}{L}.$$

Then multiply and divide the first term on the left by  $a_{LA}$  and so on; the result is

$$\frac{a_{LA}A}{L} \cdot \frac{da_{LA}}{a_{LA}} + \frac{a_{LA}A}{L} \cdot \frac{dA}{A} + \frac{a_{LB}B}{L} \cdot \frac{da_{LB}}{a_{LB}} + \frac{a_{LB}B}{L} \cdot \frac{dB}{B} = L,$$

which is the first equation in (19.62).

$$\begin{aligned}
\theta_{LA}a_{LA}^* + \theta_{KA}a_{KA}^* &= 0, \\
\theta_{LB}a_{LB}^* + \theta_{KB}a_{KB}^* &= 0, \\
\frac{a_{KA}^* - a_{LA}^*}{p_L^* - p_K^*} &= \sigma_A, \\
\frac{a_{KB}^* - a_{LB}^*}{p_L^* - p_K^*} &= \sigma_B.
\end{aligned} \tag{19.63}$$

The first two equations are easily derived by the usual cost minimization procedure. For any given output level the entrepreneur minimizes costs, treating factor prices as given. In other words, the entrepreneur chooses the input coefficients so as to minimize unit costs. The first order condition is, for commodity  $A$

$$d(a_{LA}p_L + a_{KA}p_K) = p_L da_{LA} + p_K da_{KA} = 0.$$

Dividing by  $p_A$  and expressing the changes in relative terms, we obtain the first equation in (19.63); we obtain the second equation in a similar way.

The third and fourth equations in (19.63) define the elasticity of substitution between factors in each sector by using the fact that, in equilibrium, the slope of the isoquant (the marginal rate of technical substitution) in each sector is equal to the ratio of factor prices, so that the proportional change in the marginal rate of substitution (which appears in the denominator of the formula defining the elasticity of substitution) can be expressed as  $p_L^* - p_K^*$ .

We can use Eq. (19.63) to express the proportional changes in the input coefficients in terms of the proportional changes in factor prices, namely

$$\begin{aligned}
a_{Lj}^* &= -\theta_{Kj}\sigma_j (p_L^* - p_K^*), \\
a_{Kj}^* &= \theta_{Lj}\sigma_j (p_L^* - p_K^*), \quad j = A, B
\end{aligned} \tag{19.64}$$

These expressions can be substituted in Eq. (19.62); the result is

$$\begin{aligned}
\lambda_{LA}A^* + \lambda_{LB}B^* &= L^* + \delta_L (p_L^* - p_K^*), \\
\lambda_{KA}A^* + \lambda_{KB}B^* &= K^* - \delta_K (p_L^* - p_K^*), \\
\theta_{LA}p_L^* + \theta_{KA}p_K^* &= p_A^*, \\
\theta_{LB}p_L^* + \theta_{KB}p_K^* &= p_B^*,
\end{aligned} \tag{19.65}$$

where  $\delta_L \equiv \lambda_{LA}\theta_{KA}\sigma_A + \lambda_{LB}\theta_{KB}\sigma_B$ ,  $\delta_K \equiv \lambda_{KA}\theta_{LA}\sigma_A + \lambda_{KB}\theta_{LB}\sigma_B$ ; note that  $\delta_L$  and  $\delta_K$  are zero in the case of fixed coefficients.

To close the model, the demand side must be introduced. To keep things as simple as possible, it is assumed that community taste patterns are homothetic and that no difference exists between the taste patterns of workers and capitalists. Therefore the ratio of the quantities demanded of  $A$  and  $B$  depends solely upon the commodity price-ratio:

$$\frac{A^D}{B^D} = h\left(\frac{p_A}{p_B}\right). \quad (19.66)$$

If we differentiate Eq. (19.66) and express the changes in relative form, we obtain

$$\frac{A^D}{B^D} (A^{D*} - B^{D*}) = h' \frac{p_A}{p_B} (p_A^* - p_B^*), \quad (19.67)$$

and if we use the definition of the elasticity of substitution between the two commodities on the demand side,

$$\sigma_D \equiv -\frac{\frac{d(A^D/B^D)}{A^D/B^D}}{\frac{d(p_A/p_B)}{p_A/p_B}} = h' \frac{B^D}{A^D} \frac{p_A}{p_B}, \quad (19.68)$$

we arrive at

$$A^{D*} - B^{D*} = -\sigma_D (p_A^* - p_B^*). \quad (19.69)$$

Equation (19.69) gives directly the change in the ratio of outputs consumed; to obtain the change in the ratio of outputs produced, we subtract the second equation in (19.65) from the first, which gives

$$(\lambda_{LA} - \lambda_{KA}) A^* + (\lambda_{LB} - \lambda_{KB}) B^* = (L^* - K^*) + (\delta_L + \delta_K) (p_L^* - p_K^*). \quad (19.70)$$

Now, from the fact that  $\lambda_{LA} + \lambda_{LB} = \lambda_{KA} + \lambda_{KB} = 1$  it follows that  $\lambda_{LA} - \lambda_{KA} = \lambda_{KB} - \lambda_{LB}$  and so

$$A^* - B^* = \frac{L^* - K^*}{\lambda_{LA} - \lambda_{KA}} + \frac{\delta_L + \delta_K}{\lambda_{LA} - \lambda_{KA}} (p_L^* - p_K^*). \quad (19.71)$$

Similarly, by subtracting the fourth equation in (19.65) from the third and noting that  $\theta_{LA} - \theta_{LB} = \theta_{KB} - \theta_{KA}$  (since  $\theta_{LA} + \theta_{KA} = \theta_{LB} + \theta_{KB} = 1$ ), we have

$$p_L^* - p_K^* = \frac{1}{\theta_{LA} - \theta_{LB}} (p_A^* - p_B^*). \quad (19.72)$$

Substitution of (19.72) into (19.71) gives

$$A^* - B^* = \frac{L^* - K^*}{\lambda_{LA} - \lambda_{KA}} + \sigma_S (p_A^* - p_B^*), \quad (19.73)$$

where  $\sigma_S \equiv \frac{\delta_L + \delta_K}{(\lambda_{LA} - \lambda_{KA})(\theta_{LA} - \theta_{LB})}$  represents the elasticity of substitution between the commodities on the supply side (along the transformation curve).

Finally observe that, in equilibrium, demand equals supply, so that  $A^{D^*} = A^*$ ,  $B^{D^*} = B^*$ . Therefore from (19.73) and (19.69) we obtain the change in the commodity price-ratio, which turns out to be

$$p_A^* - p_B^* = -\frac{1}{(\lambda_{LA} - \lambda_{KA})(\sigma_S + \sigma_D)}(L^* - K^*). \quad (19.74)$$

Consequently the change in the ratio of commodities produced is

$$A^* - B^* = \frac{1}{\lambda_{LA} - \lambda_{KA}} \frac{\sigma_D}{\sigma_S + \sigma_D} (L^* - K^*). \quad (19.75)$$

This completes the description of the equations of change of the model. As we said above, these will be put to use in the following chapters, in order to derive some important theorems in the theory of international trade; see Sects. 20.1–20.3, 21.1–21.3, 22.3, and 22.6; in some of these, further properties and extensions of the model are also examined.

## 19.5.2 Revenue Functions and Expenditure Functions

The duality approach is usually presented in terms of revenue functions (or GNP functions) and expenditure functions rather than in terms of the equations described in the previous section. These alternative presentations are, however, equivalent. The revenue and expenditure functions, in fact, respectively summarize the production side and the consumption side of the economy. Here we simply show their basic nature and relate it to the Jones model explained in the previous section. For an in-depth treatment see [Dixit and Norman \(1980\)](#); [Sgro \(1986\)](#); [Woodland \(1982\)](#), [Diewert \(1974, 1982\)](#) and [Cornes \(1992\)](#).

Let us begin with the production side. We have seen in the previous section that the production equations determine the eight unknowns  $a_{ij}$ ,  $A$ ,  $B$ ,  $p_L$ ,  $p_K$  given the four parameters  $L$ ,  $K$ ,  $p_A$ ,  $p_B$ . We can thus write the *value of GNP*, or *revenue function*, as

$$R = p_A A(p_A, p_B; L, K) + p_B B(p_A, p_B; L, K). \quad (19.76)$$

Since the production equations have been derived from an optimization process,  $R$  is clearly an optimal value.

If we now consider the (optimal) *unit cost functions*

$$\begin{aligned}c_A &= a_{LA}p_L + a_{KA}p_K, \\c_B &= a_{LB}p_L + a_{KB}p_K,\end{aligned}\tag{19.77}$$

we get the *total cost function*

$$\begin{aligned}C &= c_A A + c_B B \\&= R \\&= p_L(a_{LA}A + a_{LB}B) + p_K(a_{KA}A + a_{KB}B) \\&= p_L L + p_K K,\end{aligned}\tag{19.78}$$

since  $c_A = p_A$ ,  $c_B = p_B$ , and  $a_{LA}A + a_{LB}B = L$ ,  $a_{KA}A + a_{KB}B = K$  by the production Eq. (19.59). This expresses the fact that the (maximum) production revenue equals the (minimum) cost of production.

Equations (19.78) are implicit in Jones's model, and their properties of change have already been examined in the previous section.

As regards the demand side, the *expenditure function*  $E = f(u, \mathbf{p})$  is defined as the minimum required expenditure as a function of the utility target to be achieved ( $u$ ) and the vector of money prices ( $\mathbf{p}$ ) faced by the consumer. Given the dual nature of the consumer optimization problem (minimum expenditure to achieve a given utility level, or maximum achievable utility for a given money income), the value of  $E$  also expresses the money income  $Y$  just sufficient to achieve the utility level.

The expenditure function is related to the standard Hicksian compensated demand functions through *Shephard's lemma*, which shows that  $\partial E / \partial p_j$ , the partial derivative of the expenditure function with respect to any price  $p_j$ , coincides with the compensated demand function for commodity  $j$ ,  $q_j^d = q_j(u, \mathbf{p})$ . At the optimum point, the compensated and uncompensated demand functions coincide, hence we have

$$\partial E / \partial p_j = q_j(u, \mathbf{p}) = x_j(Y, \mathbf{p}),\tag{19.79}$$

where  $x_j^d = x_j(Y, \mathbf{p})$  is the standard uncompensated (or Marshallian) demand function. Since the expenditure function is homogeneous of the first degree in all prices, its partial derivatives are homogeneous of degree zero with respect to prices, namely the demand function is a function of relative rather than absolute prices, a well known result.

In the case of homothetic utility functions, the expenditure function takes on the particularly simple form

$$E = ue(\mathbf{p}),\tag{19.80}$$

i.e., similar to the cost function for firms with constant-returns-to scale technology; in fact, the expenditure function is homogeneous of degree one with respect to the utility level.

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