Chapter 20
Appendix to Chapter 4

20.1 Factor-Intensity Reversals

We have shown in the text the crucial importance of the strong factor-intensity assumption (i.e., absence of factor-intensity reversals); here we examine formally the conditions under which reversals are present or absent. Let us begin by establishing the relationship between capital intensity and relative price of factors; for this purpose we employ the equilibrium conditions that state the equality between the value of marginal productivity of a factor and its price (this must be equal in both sectors). With the symbology introduced in Eq. (19.17), we have

$$\begin{aligned} g'_A \qquad &= pg'_B = p_K, \\ g_A - \varrho_A g'_A &= p\left(g_B - \varrho_B g'_B\right) = p_L, \end{aligned} \tag{20.1}$$

whence dividing the second equation by the first

$$\frac{p_L}{p_K} = \frac{g_i - \varrho_i g'_i}{g'_i} = \frac{g_i}{g'_i} - \varrho_i, \quad i = A, B. \tag{20.2}$$

Since g_i and g'_i are functions of ϱ_i, Eq. (20.2) expresses a relation between p_L/p_K and ϱ_i.This relation is increasing monotonically: in fact,

$$\frac{\mathrm{d}\left(p_L/p_K\right)}{\mathrm{d}\varrho_i} = \frac{\left(g'_i\right)^2 - g''_i g_i}{\left(g'_i\right)^2} - 1 = -\frac{g''_i g_i}{\left(g'_i\right)^2}, \tag{20.3}$$

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whence, by the inverse-function differentiation rule,[1]

$$\frac{\mathrm{d}\varrho_i}{\mathrm{d}\left(p_L/p_K\right)} = -\frac{\left(g_i'\right)^2}{g_i'' g_i}. \tag{20.4}$$

Since $g_i > 0$, $g_i'' < 0$ by the assumption of positive but decreasing marginal productivities, the derivatives (20.3) and (20.4) are positive. Equation (20.2) give rise to the curves drawn in Fig. 4.2.

We must now find the conditions under which these curves do or do not intersect (presence or absence of factor-intensity reversals, respectively). Interesting conditions are provided by the following theorem:

If the elasticity of substitution between factors is constant in each sector, no (one) factor-intensity reversal will occur when this elasticity is the same in (different between) the two sectors.

It should be remembered that the elasticity of substitution is defined as

$$\sigma_i = \frac{\mathrm{d}\varrho_i/\varrho_i}{\mathrm{d}MRTS/MRTS} = \frac{\mathrm{d}\varrho_i/\varrho_i}{\mathrm{d}\left(p_L/p_K\right)/\left(p_L/p_K\right)}, \tag{20.5}$$

where *MRTS* is the marginal rate of technical substitution along an isoquant, equal to the factor-price ratio in the situation of equilibrium.

From (20.5) we obtain

$$\frac{\mathrm{d}\varrho_i}{\varrho_i} = \sigma_i \frac{\mathrm{d}q}{q}, \tag{20.6}$$

where q denotes the factor-price ratio. Now, if we assume that σ_i is a constant, we can integrate throughout and obtain

$$\varrho_i = C_i q^{\sigma_i}, \tag{20.7}$$

where C_i depends on the arbitrary constants of integration. Then, if $\sigma_A = \sigma_B$, from Eq. (20.7) it follows that

$$\frac{\varrho_A}{\varrho_B} = C, \quad C \equiv C_A/C_B, \tag{20.8}$$

and so either ϱ_A will always be greater than ϱ_B (if $C > 1$) or vice versa (if $C < 1$): no factor-intensity reversal can occur. This is the case represented in Fig. 4.2a.

[1]For a complete proof that (20.2) is a one-to-one correspondence between ϱ_i and p_L/p_K see, for example, Gandolfo (1971, Appendix III, sect. 7, §7.5).

If, on the contrary, $\sigma_A \neq \sigma_B$ (for example we assume $\sigma_A > \sigma_B$), from (20.7) we get

$$\frac{\varrho_A}{\varrho_B} = C q^{\sigma_A - \sigma_B}, \quad C \equiv C_A / C_B. \tag{20.9}$$

Since the function $C q^{\sigma_A - \sigma_B}$ is increasing monotonically from zero to infinity, a unique value of q will exist, call it q^*, such that $\varrho_A / \varrho_B \lesseqgtr 1$ for $q \lesseqgtr q^*$. There will thus be one, and only one, factor intensity reversal, as is the case in Fig. 4.2b.

It is important to note that, when the elasticity of substitution is variable, the integration allowing the passage from (20.6) to (20.7) can no longer be performed, so that, in general, any number of reversals can occur.

As a typical example of production functions never giving rise to factor-intensity reversals, we recall the Cobb-Douglas function, $Y = HK^\alpha L^{1-\alpha}$ which has a constant elasticity of substitution equal to one, whilst the *CES* function, $Y = \left[\alpha K^{-\beta} + \gamma L^{-\beta}\right]^{-(1/\beta)}$, has a constant elasticity of substitution equal to $1/(1+\beta)$, and so can give rise to a reversal when the parameter β is different between the two sectors.

We must now demonstrate the one-to-one correspondence between the relative price of goods and the relative price of factors in the absence of factor-intensity reversals. This amounts to showing that there exists a monotonic relationship between the two variables if and only if no factor-intensity reversal occurs.

Let us consider the equilibrium conditions given in (19.59), namely

$$\begin{aligned} p_B &= a_{LB} p_L + a_{KB} p_K, \\ p_A &= a_{LA} p_L + a_{KA} p_K. \end{aligned} \tag{20.10}$$

If we divide the first equation by the second we get

$$\frac{p_B}{p_A} = \frac{q a_{LB} + a_{KB}}{q a_{LA} + a_{KA}}, \quad q \equiv p_L / p_K, \tag{20.11}$$

whence, by differentiation with respect to q (remember that the coefficients a_{ij} are functions of q through the optimization procedure), we obtain

$$\begin{aligned} &\frac{\mathrm{d}\left(p_B/p_A\right)}{\mathrm{d}q} \\ &= \frac{\left(a_{LB} + q a'_{LB} + a'_{KB}\right)\left(q a_{LA} + a_{KA}\right) - \left(a_{LA} + q a'_{LA} + a'_{KA}\right)\left(q a_{LB} + a_{KB}\right)}{\left(q a_{LA} + a_{KA}\right)^2}, \end{aligned} \tag{20.12}$$

where $a'_{ij} \equiv \mathrm{d}a_{ij}/\mathrm{d}q$. Now, from the optimum conditions,

$$\begin{aligned} p_L \mathrm{d}a_{LA} + p_K \mathrm{d}a_{KA} &= 0, \\ p_L \mathrm{d}a_{LB} + p_K \mathrm{d}a_{KB} &= 0, \end{aligned}$$

and so

$$\begin{aligned} qa'_{LA} + a'_{KA} &= 0, \\ qa'_{LB} + a'_{KB} &= 0. \end{aligned} \tag{20.13}$$

Thanks to (20.13), expression (20.12) simplifies to

$$\frac{\mathrm{d}\,(p_B/p_A)}{\mathrm{d}q} = \frac{a_{LB}a_{KA} - a_{LA}a_{KB}}{(qa_{LA} + a_{KA})^2} = a_{LA}a_{LB}\frac{\varrho_A - \varrho_B}{(qa_{LA} + a_{KA})^2}, \tag{20.14}$$

from which it can readily be seen that the derivative of the relative price of goods with respect to the relative price of factors is either always positive or always negative if and only if ϱ_A is either always greater or always smaller than ϱ_B, that is, if and only if no factor intensity reversal occurs. When, on the contrary, one or more reversals are present, the derivative (20.14) will change its sign (passing through zero) one or more times, and so the relation between p_B/p_A and p_L/p_K will be no longer monotonic. In Fig. 4.5a we have represented this relation when $\varrho_A > \varrho_B$ everywhere, whilst Fig. 4.5b represents the case of one factor-intensity reversal ($\varrho_B > \varrho_A$ initially, and then $\varrho_A > \varrho_B$).

20.2 Proof of the Fundamental Theorem

The basic Heckscher-Ohlin proposition to be proved is that a country abundant in a factor has a production bias in favour of the commodity which uses that factor more intensively. In what follows we are going to use the physical definition of factor abundance.

If we consider the full-employment relations (see Sect. 19.5)

$$\begin{aligned} a_{KA}A + a_{KB}B &= K, \\ a_{LA}A + a_{LB}B &= L, \end{aligned} \tag{20.15}$$

and divide through by L, we obtain

$$\begin{aligned} a_{KA}A/L + a_{KB}B/L &= K/L, \\ a_{LA}A/L + a_{LB}B/L &= 1. \end{aligned} \tag{20.16}$$

By solving this linear system we can express A/L and B/L in terms of the remaining quantities, namely

$$A/L = \frac{a_{LB}K/L - a_{KB}}{a_{KA}a_{LB} - a_{KB}a_{LA}}, \quad B/L = \frac{a_{KA} - a_{LA}K/L}{a_{KA}a_{LB} - a_{KB}a_{LA}}, \tag{20.17}$$

whence

$$\frac{A/L}{B/L} = \frac{A}{B} = \frac{a_{LB}K/L - a_{KB}}{a_{KA} - a_{LA}K/L}. \tag{20.18}$$

Equation (20.18) expresses the output ratio (A/B) in terms of the factor endowment ratio (K/L), given the technical coefficients a_{ij}. These coefficients depend on the factor-price ratio but, given this, are constant for any output level owing to the assumption of constant returns to scale. Therefore, for any factor-price ratio we can compute the derivative

$$\frac{\mathrm{d}(A/B)}{\mathrm{d}\,(K/L)} = \frac{a_{KA}a_{LB} - a_{LA}a_{KB}}{(a_{KA} - a_{LA}K/L)^2} = a_{LA}a_{LB}\frac{\varrho_A - \varrho_B}{(a_{KA} - a_{LA}K/L)^2}, \tag{20.19}$$

which will have an unambiguous sign thanks to the assumption of no factor intensity reversal; this assumption enables us to state that either ϱ_A is always greater than ϱ_B or ϱ_B is always greater than ϱ_A *independently of* the factor-price ratio. If we assume, as in the text, that commodity A is capital intensive, the derivative under consideration turns out to be positive, that is, the greater the factor endowment ratio (K/L) the higher the output of A relative to B, and vice versa. Since the production functions are assumed to be internationally identical, the above result holds for both countries; this proves the basic proposition, which can be used as a lemma in the proof of the fundamental theorem exactly as in the text.

We have also stated in the text—see Fig. 4.7—that in the pre-trade equilibrium situation, assuming that country 1 is capital abundant relative to country 2 ($\varrho_1 > \varrho_2$) and commodity A is capital-intensive relative to B ($\varrho_A > \varrho_B$), for any given A/B ratio the marginal rate of transformation, and so the relative price of goods (p_B/p_A), is higher in country 1 than in country 2. To show this it is sufficient to observe that, with no factor-intensity reversal, $\varrho_1 > \varrho_2$ implies $(p_L/p_K)_1 > (p_L/p_K)_2$—see Fig. 4.8—so that, by (20.1), $(p_B/p_A)_1 > (p_B/p_A)_2$ since by assumption $\varrho_A > \varrho_B$. It can also be seen that $(p_B/p_A)_1 = (p_B/p_A)_2$ when $\varrho_1 = \varrho_2$, so that no international trade can take place when the relative factor endowments of the countries coincide.

We conclude by observing that, as we have said in Sect. 1.2, the Heckscher-Ohlin model stresses the difference in factor endowments as the basis for trade, whilst the Ricardian theory emphasizes the differences in technology. however, Ford (1982) has argued that under certain conditions the two theories are in fact equivalent. This has generated considerable controversy (see, for example, Ford, 1985; Lloyd, 1985); for a balanced exposition of the issues involved we refer the reader to Neary (1985a), who argues that "what is at stake is not the *logical* but the *observational* equivalence of the two theories".

20.3 The Factor-Price-Equalization Theorem

Let us take up Eqs. (20.1) again (these postulate the absence of factor-intensity reversals) and rewrite them in the form

$$\begin{aligned} g'_A(\varrho_A) - p g'_B(\varrho_B) &= 0, \\ \left[g_A(\varrho_A) - \varrho_A g'_A(\varrho_A)\right] - p\left[g_B(\varrho_B) - \varrho_B g'_B(\varrho_B)\right] &= 0. \end{aligned} \tag{20.20}$$

We have here a set of two implicit functions in three variables: p, ϱ_A, ϱ_B. By using the implicit-function theorem, we can express ϱ_A and ϱ_B as single-valued and differentiable functions of p if the Jacobian of (20.20) with respect to ϱ_A, ϱ_B is different from zero. This Jacobian turns out to be

$$\begin{vmatrix} g''_A & -p g''_B \\ -\varrho_A g''_A & p \varrho_B g''_B \end{vmatrix} = p g''_A g''_B (\varrho_B - \varrho_A), \tag{20.21}$$

which is different from zero if and only if no factor-intensity reversal occurs. Thus, if we assume absence of reversals it follows that a unique value of ϱ_A and a unique value of ϱ_B will correspond to the relative price of commodities, p, determined as a consequence of international trade. By substituting these values in (20.1), the values of p_K and p_L can be uniquely determined. Now, given the assumption of internationally identical production functions, Eqs. (20.20) and, therefore, the single-valued relations between ϱ_A and p and between ϱ_B and p are identical in both countries; similarly identical are Eqs. (20.1). Therefore, as p is the same given the assumptions of free trade and no transports costs, the absolute prices of factors will be equalized between countries.

Alternatively, we could have used the one-to-one relation between the commodity price ratio and the factor price ratio demonstrated at the end of Sect. 20.1 and then the one-to-one relation between the relative price of factors and ϱ_i demonstrated at the beginning of the same section.

It should be noted, in conclusion, that the condition on the Jacobian ensures univalence only *locally*, that is, in the neighbourhood of the equilibrium point; for the conditions for global univalence see Gale and Nikaidô (1965) .

Further considerations on the FPE theorem are contained in Sect. 20.5.

20.4 A Brief Outline of the Generalizations of the Heckscher-Ohlin Model

The attempts at extending the Heckscher-Ohlin theorem and the factor price equalization theorem to the general multi-commodity, multi-factor, multi-country case, have given rise to an immense literature which it would be impossible to deal with here. Therefore we do no more than focus on what we feel are some of the most important points, referring the reader to the surveys by Chipman (1966) and

Ethier (1984) for the rest. In Sects. 20.5 and 20.6 we shall consider in more depth two important aspects of these generalizations.

Jones (1956) formulated the "chain proposition" in the many-commodity, two-factor, two-country model, whereby if the goods are ranked in order of factor intensities, then all of a country's exports must lie higher in this list than all its imports. Bhagwati (1972) showed this proposition to be incorrect, if factor-price equalization obtains. Deardoff (1979) gave a formal proof of the non-factor-price equalization case (in which the chain proposition is true) and provided an extension to the many-country case, showing that all of the exports of a country more abundant in a factor will be at least as intensive in that factor as each of the exports of all countries less abundant in that factor.

The reader will note that these extensions remain within the context of the *two-factor* assumption. In fact, except for special cases, the concept itself of factor intensity can no longer be clearly defined when there are many factors.

These difficulties have led to a search for an alternative formulation of the Heckscher-Ohlin theorem, which should be more or less equivalent to the original one in the $2 \times 2 \times 2$ case *and* be capable of easy generalization. Such a formulation (called the *factor-content* version of the Heckscher-Ohlin theorem) exists, and refers to the factors embodied in the goods traded internationally, instead of the goods themselves. In the simple $2 \times 2 \times 2$ case this formulation states that *each country is a net exporter of the (services of the) country's more abundant factor and a net importer of the (services of the) other factor*.

This is the path followed, for example, by Vanek (1968), who used the same basic assumptions as in the original theorem and assumed, in addition, factor-price equalization and productive diversification. We use "productive diversification" in Chipman's sense (1966, p. 21). The precise assumption of (Vanek, 1968, p. 750) is: "specialization (in the two-country world) in no more than $m - n$ products" where m is the number of products, n the number of factors, and $m \geq n$.

With these assumptions Vanek achieved interesting results in the context of a two-country model, but with any number of goods and factors. Let us denote by ${}^1V_i, {}^2V_i$ the endowments of the i-th factor ($i = 1, 2, \ldots, r$) in countries 1 and 2 respectively. Now, if the relation

$$\frac{{}^1V_1}{{}^2V_1} \geq \frac{{}^1V_2}{{}^2V_2} \geq \ldots \geq \frac{{}^1V_r}{{}^2V_r}, \tag{20.22}$$

holds with at least one strict inequality, then free international trade in commodities brings about the following consequences (amongst others):

(a) Country 1 is a net exporter of the services of factors $1, 2, \ldots, j$, with $j < r$, and a net importer of the services of factors $j + 1, \ldots, r$;
(b) j can be determined if we know the vector of factor prices;
(c) Knowing this vector, we can also compute exactly the net amounts of the services of the factors traded internationally.

These are interesting results (which can be extended to the case of more than two countries: see Horiba 1974), but are obtained at the cost of a serious limitation, that is, the assumption that factor price equalization obtains. What was an important result, demonstrated as another theorem in the original version of the theory, now becomes a basic assumption like, say, the international identity of production functions etc.

Other writers have tried to do without this very restrictive assumption, but only at the cost of introducing other and perhaps equally restrictive ones (see, for example, Harkness, 1978, 1983). Brecher and Choudhri (1982) have succeeded in proving the validity of the factor-content version of the Heckscher-Ohlin theorem without the assumption of factor price equalization or other restrictive assumptions, but only in the two-factor multi-commodity model of a two-country world.

Deardoff (1982), in the general case of the multi-factor multi-commodity multi-country model and without recourse to the assumption of factor price equalization or other restrictive assumptions, has proved that both the factor-content and the commodity version of the Heckscher-Ohlin theorem are valid in an *average* sense. More precisely, as regards the factor-content version, he has shown that the simple correlation between the vector containing the autarky factor prices (which inversely reflect the abundance of those factors: Deardoff is using the economic definition of abundance) of all countries and factors and the vector containing the net exports by each country of (the services of) each factor, arranged in the same order, is negative. The interpretation of this result is that countries will on average tend to be net exporters of their abundant factors and net importers of their scarce factors. As regards the commodity version, Deardoff shows that the "*comvariance*"[2] among the vector containing a measure of factor abundance (for each factor and country), the vector containing a measure of factor intensity, and the vector of net exports at world prices, is positive. The economic interpretation is that exported goods must *on average* use the relatively abundant factors relatively intensively, and imported goods must *on average* use the relatively scarce factors relatively intensively. This important result generalizes the Heckscher-Ohlin theorem as an explanation of the pattern of commodity trade in an "average" sense.

For results similar to Deardoff's, see Ethier (1982), Dixit and Woodland (1982), Helpman (1984a), and Svensson (1984).

An alternative approach to the general case is also possible, which consists in aggregating a higher dimensional model so as to obtain a model which exhibits all the properties of the two-by-two model (provided that suitable restrictions are imposed); for this line of research see, for example, Neary (1984, 1985b), and references therein.

Another important point is the generalization of the factor price equalization theorem. It is perhaps worth mentioning, in passing, that the debate on this generalization—beginning with an incorrect conjecture by Samuelson (1953)—has given origin to a new mathematical theorem, the Gale and Nikaidô (1965) on the global univalence of mappings.

[2]This is a term used by Deardoff (1982, p. 690) to denote a generalization (that he suggested) of the concept of covariance when one needs to correlate three variables symmetrically.

Three cases must be distinguished in examining factor price equalization in the general case.

1. The number of commodities is *equal* to the number of factors. In this case, if complete productive diversification obtains and the cost functions are *globally* invertible,[3] then—independently of the factor endowments of the various countries—the equalization of commodity prices will involve the equalization of factor prices.
2. The number of commodities is *smaller* than the number of factors. In this case the determination of factor prices depends not only on the (international) prices of commodities (assumed to be known), but also on the trading countries' factor endowments. Generally speaking, the difference in these endowments causes the non-equalization of factor prices. In other words, this equalization, though not impossible, is extremely unlikely.
3. The number of commodities n is *greater* than the number of factors r. In this case the determination of factor prices depends only on the prices of r commodities, but we do not know which the r commodities are. Thus to be sure that factor prices will be equalized, the global invertibility conditions must be verified for all square $r \times r$ submatrices drawn from the Jacobian of the system relating the vector of commodity prices to the vector of factor prices.

For further analysis of factor-price equalization see Deardoff (1994), Feenstra (2004).

Recent theoretical research on the generalizations of the Heckscher-Ohlin model has concentrated on the role of factor mobility, a topic dealt with in Sect. 6.8.1 and its appendix.

20.5 The Factor Price Equalization Set

Assume that there are N goods indexed by g and M factors indexed by j and that $N \geq M \geq 2$. Let $\mathbf{p}_v^*$ be the row vector of the integrated equilibrium factors price and let $a_{jg}\left(\mathbf{p}_v^*\right)$ be the unitary demand function for factor j in the production of good g. Let $\mathbf{\Delta}^*$ be the M by N technology matrix

$$\mathbf{\Delta}^* \equiv \begin{bmatrix} a_{11}\left(\mathbf{p}_v^*\right) & \ldots & a_{1N}\left(\mathbf{p}_v^*\right) \\ \ldots & \ldots & \ldots \\ a_{M1}\left(\mathbf{p}_v^*\right) & \ldots & a_{MN}\left(\mathbf{p}_v^*\right) \end{bmatrix}. \tag{20.23}$$

[3]The optimum conditions will give a differentiable mapping $\mathbf{p} = \mathbf{g}(\mathbf{w})$, where $\mathbf{p}$ is the vector of commodity prices and $\mathbf{w}$ is the vector of factor prices. Global invertibility (or univalence) ensures that the inverse mapping $\mathbf{w} = \mathbf{g}^{-1}(\mathbf{p})$ exists uniquely, namely a unique vector of factor prices corresponds to any vector of commodity prices exactly as a unique vector of commodity prices corresponds to any vector of factor prices; note that as we are considering *global* univalence, the conditions stated by the Gale-Nikaidô theorem must be satisfied.

Let $\mathbf{Z}^*$ be the integrated equilibrium output vector whose elements are Z_g^* and let $\mathbf{V}$ be the vector of world endowments whose elements are V_j. Lastly, let $\mathbf{Z}_i$ be the output vector and $\mathbf{V}_i$ the endowment vector of country i $(i = 1, 2)$ whose elements are V_{ji}. The factor price equalization set, denoted Φ, is defined as follows:

$$\Phi \equiv \{\mathbf{V}_i \mid \mathbf{\Delta}^*\mathbf{Z}_i = \mathbf{V}_i, \quad \mathbf{0} \leqslant \mathbf{Z}_i \leqslant \mathbf{Z}^*, \ \ i = 1, 2\}. \tag{20.24}$$

In words, the FPE set is the set of all possible endowment vectors $\mathbf{V}_i$ such that factors market clear at the integrated equilibrium factors price, and such that the output vector $\mathbf{Z}_i$ is between the null vector and the output vector of the integrated equilibrium. Expression (20.24) defines the set we are searching for. Now we turn to finding it starting from the integrated equilibrium. We begin by assigning arbitrarily to each country a non negative share of the integrated equilibrium output. These shares are denoted λ_{gi}. We then compute the factors endowment needed to produce the arbitrarily chosen quantity of output when the unitary factors input are those of integrated equilibrium; this is:

$$\mathbf{V}_i = \mathbf{\Delta}^* diag\left(\lambda_{gi}\right) \mathbf{Z}^*. \tag{20.25}$$

From linear algebra notation we recall that $diag\left(\lambda_{gi}\right)$ is a diagonal matrix whose elements are λ_{gi}. Let δ_g^* be the g-th column vector of $\mathbf{\Delta}^*$ and let $\mathbf{E}_g$ denote the sectorial employment vector of the integrated equilibrium, $\mathbf{E}_g = \delta_g^* Z_g$. It is now clear that the endowment vector obtained in expressions (20.25) satisfies the requirements of the FPE set defined in (20.24). Therefore

$$\Phi = \{\mathbf{V}_i \mid \exists \lambda_{gi} \geqslant 0, \ \sum_{i=1}^{2} \lambda_{gi} = 1 \ \forall g, \mathbf{V}_i = \sum_{g=1}^{N} \lambda_{gi} \mathbf{E}_g \ \forall \ i\} \tag{20.26}$$

In words, the FPE set is the set of endowment vectors obtained from all the convex combinations of the integrated equilibrium sectorial employment vectors. The FPE set in Sect. 4.3.2 has been constructed geometrically following expression (20.26), i.e., by finding the surface identified by all the convex combinations of the integrated equilibrium sectorial employment vectors.

20.6 The Heckscher-Ohlin-Vanek Theorem

Assume that there are N goods indexed by g and M factors indexed by j and that $N \geq M \geq 2$. Assume free trade and incomplete specialization. All variables are computed at the free trade equilibrium. Let $\mathbf{p}_v$ be the row vector of factor prices and let s_i be country i's share in world gross domestic product,

$$s_i \equiv \frac{\mathbf{p}_v \mathbf{V}_i}{\mathbf{p}_v \mathbf{V}}. \tag{20.27}$$

Let $\mathbf{C}_i$ and $\mathbf{C}$ be country i's and world's column vectors of consumption. With identical and homothetic preferences and trade balance equilibrium, $\mathbf{C}_i = s_i\mathbf{C}$. Let $\mathbf{T}_i$ be the vector of net exports (exports minus imports); net exports are equal to output minus consumption, thus, $\mathbf{T}_i = \mathbf{Z}_i - \mathbf{C}_i$. Let $\mathbf{F}_i$ denote the factor content of trade vector. The elements of $\mathbf{F}_i$ are the quantity of each factor services needed to produce the net exports of country i at the integrated equilibrium factor prices, $\mathbf{F}_i \equiv \mathbf{\Delta}^*\mathbf{T}_i$. Naturally, some elements of $\mathbf{F}_i$ are positive and some are negative. Free trade equilibrium in goods market requires $\mathbf{C} = \mathbf{Z}$ and equilibrium in factors market requires $\mathbf{\Delta} \boldsymbol{Z}_i = \mathbf{V}_i$. Therefore,

$$\mathbf{F}_i \equiv \mathbf{\Delta}^*\mathbf{T}_i = \mathbf{\Delta}^*\mathbf{Z}_i - \mathbf{\Delta}^*\mathbf{C}_i = \mathbf{V}_i - \mathbf{\Delta}^*\mathbf{C}_i = \mathbf{V}_i - s_i\mathbf{\Delta}^*\mathbf{Z} \tag{20.28}$$

Now, using $\mathbf{\Delta}^*\mathbf{Z} = \mathbf{V}$ we have

$$\mathbf{F}_i = \mathbf{V}_i - s_i\mathbf{V} \tag{20.29}$$

Expression (20.29) is known as the Heckscher-Ohlin-Vanek equation. It shows that the factor content of trade vector of a country is given by the difference between the endowment vector of the country and the world endowment vector, the latter multiplied by the country's share in world GDP. Let v_{ij} be country i's share in world endowment of factor j, i.e., $v_{ij} \equiv V_{ij}/V_j$.

Definition 20.1. A country is abundant in factor j iff

$$v_{ji} > s_i. \tag{20.30}$$

It is immediate from (20.29) and (20.30) that each country is a net exporter of the services of its abundant factors. Indeed, expression (20.29) may be written as

$$\mathbf{F}_i = diag\left(v_{ji} - s_i\right)\mathbf{V} \tag{20.31}$$

which proves the Heckscher-Ohlin-Vanek theorem.

If $M = N = 2$ expression (20.31) becomes

$$\mathbf{F}_i = \begin{bmatrix} (v_{Ki} - s_i)\,\bar{K} \\ (v_{Li} - s_i)\,\bar{L} \end{bmatrix} \tag{20.32}$$

and $s_i = \left(p_K v_{Ki}\bar{K} + p_L v_{Li}\bar{L}\right)/\left(p_K\bar{K} + p_L\bar{L}\right)$. It is clear that

$$\text{if } v_{Li} > v_{Ki}, \quad \text{then } \; v_{Li} > s_i > v_{Ki}, \tag{20.33}$$

$$\text{if } v_{Li} < v_{Ki}, \quad \text{then } \; v_{Li} < s_i < v_{Ki}, \tag{20.34}$$

Note that $v_{Li} > v_{Ki}$ ($v_{Li} < v_{Ki}$) implies that country i is *relatively* abundant in factor L (K). Thus, the two-by-two version of the Heckscher-Ohlin-Vanek theorem states that each country is the net exporter of the services of its *relatively* abundant factor, as we have already seen above.

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