Chapter 6

Linear Algebra

6.1 Linear Functions and Dynamical Systems

In this chapter, we will be studying linear functions in n dimensions:

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

As we develop this subject, called linear algebra, we are always going to keep two applications in mind.

(1) discrete-time dynamical systems, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is the function giving the next state as a function of the previous state:

$$(X_1, X_2, \ldots, X_n)_{N+1} = f(X_1, X_2, \ldots, X_n)_N$$

(2) continuous-time differential equations, where f is the vector field giving the change vector as a function of the state vectors:

$$(X_1', X_2', \ldots, X_n') = f(X_1, X_2, \ldots, X_n)$$

Notation

When we want to refer to a point in $\mathbb{R}^n$, that is, a vector, we will denote it by a single **boldface** letter, such as $\mathbf{X}$ and $\mathbf{Y}$:

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \qquad \mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}$$

Note that we have started to write the vector $(X_1, X_2, \ldots, X_n)$ vertically, using round parentheses. The vertical expression means exactly the same thing as the horizontal expression; the horizontal one is common in dynamical systems theory, and the vertical one is common in linear algebra.

6.2 Linear Functions and Matrices

Points and Vectors

We know that the state space of a dynamical system is $\mathbb{R}^n$, the space of all n-tuples $(X_1, X_2, \ldots, X_n)$, with each X_i belonging to $\mathbb{R}$. This is the view of state space we developed in

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Chapter 1: state space is the space of all possible values of the state vector. This is true for both state space and tangent space, both of which are $\mathbb{R}^n$. For example, in the Romeo–Juliet models, the state space $\mathbb{R}^2$ consists of all possible pairs (R, J), where both R and J belong to $\mathbb{R}$, and the tangent space is also $\mathbb{R}^2$, the space of all possible pairs (R', J'), where both R' and J' belong to $\mathbb{R}$.

We also learned in Chapter 1 some elementary rules for manipulating vectors. We needed these rules, for example, in Euler's method, where we needed to multiply the change vector X' by the scalar Δt to get a small change vector, and then we needed to add the small change vector to the current state vector to get the next state vector. These rules for scalar multiplication and vector addition are the rules we will need for operating in $\mathbb{R}^n$.

The space of all n-vectors $\mathbb{R}^n$, together with the rules for scalar multiplication and vector addition, is called **n-dimensional vector space**. Note that the sum of n-vectors is also an n-vector, and the scalar multiple of an n-vector is also an n-vector. So the operations of scalar multiplication and vector addition keep us in the same space.

In this chapter, we will learn about the property of vector spaces and the linear functions that take $\mathbb{R}^n \to \mathbb{R}^k$, that is, take vectors in n-dimensional space (the domain) and assign to each of them a vector in k-dimensional space (the codomain). Most of the time, we will focus on the case $n = k$. To begin, let's recall the rules for operating with vectors from Chapter 1.

(1) If $\mathbf{X}$ and $\mathbf{Y}$ are two vectors in $\mathbb{R}^n$, then their **sum** is defined by

$$\mathbf{X} + \mathbf{Y} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} + \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} X_1 + Y_1 \\ X_2 + Y_2 \\ \vdots \\ X_n + Y_n \end{pmatrix}$$

(2) If $\mathbf{X}$ is a vector in $\mathbb{R}^n$ and a is a scalar in $\mathbb{R}$, we define the **multiplication of a vector by a scalar** as

$$a\mathbf{X} = a \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} aX_1 \\ aX_2 \\ \vdots \\ aX_n \end{pmatrix}$$

Exercise 6.2.1 Carry out the following operations, or say why they're impossible.

a) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}$ b) $-3\begin{pmatrix} 4 \\ 6 \\ -9 \end{pmatrix}$ c) $\begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$

d) $5(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 7 \\ 3 \end{pmatrix})$ e) $-4\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ f) $5\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 8\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Bases and Linear Combinations

In $\mathbb{R}^n$ there is a certain set of vectors that play a special role. It is the set

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \cdots \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

These n vectors are a *basis* for $\mathbb{R}^n$, by which we mean that every vector $\mathbf{X}$ can be written uniquely as

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + \cdots + X_n\mathbf{e}_n$$

To see why an arbitrary vector $\boldsymbol{X}$ can be represented uniquely in the $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ basis, recall that

$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} X_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ X_2 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ X_n \end{pmatrix}$$

by the rule of vector addition. This, in turn, means that

$$\boldsymbol{X} = X_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + X_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + X_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

by the rule of multiplication of a vector by a scalar.

There are many such sets of vectors, giving us many bases for $\mathbb{R}^n$. This particular basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is called the *standard basis*, but later in this chapter we will see other bases for $\mathbb{R}^n$.

For example, let's consider the 2D vector space $\mathbb{R}^2$ representing the juvenile (J) and adult (A) populations of some animal species. Then a point in (J, A) space represents a certain number of juveniles and a certain number of adults. So the point $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$ represents the state in which there are 5 juveniles and 10 adults. The standard basis for $\mathbb{R}^2$ is

$$\mathbf{e}_\mathrm{J} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{e}_\mathrm{A} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So we can write

$$\begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 10 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 10\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 5\mathbf{e}_\mathrm{J} + 10\mathbf{e}_\mathrm{A}$$

When we say that every vector $\mathbf{X}$ in $\mathbb{R}^n$ can be written uniquely as $X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + \cdots + X_n\mathbf{e}_n$, note that the only operations we have used are scalar multiplication and vector addition. When we use only scalar multiplication and vector addition to combine a set of vectors, the result is called a *linear combination* of those vectors.

Exercise 6.2.2 What are the standard basis vectors for $\mathbb{R}^4$?

Exercise 6.2.3 In $\mathbf{e}$ notation, what is the standard basis vector of $\mathbb{R}^6$ that has a 1 in position 5?

Exercise 6.2.4 Write the following vectors as the sum of scalar multiples of the standard basis vectors in $\mathbb{R}^2$.

a) $\begin{pmatrix} 45 \\ 12 \end{pmatrix}$ b) $\begin{pmatrix} 387 \\ 509 \end{pmatrix}$ c) $\begin{pmatrix} a \\ b \end{pmatrix}$

Exercise 6.2.5 Are the following expressions linear combinations? If so, of what variables?

a) $2a + 5b$ b) $e^X + 3Y$

c) $7Z + 6H - 3t^2$ d) $-6X + 4W + 5$

Exercise 6.2.6 Why does it make sense to describe a smoothie as a linear combination of ingredients?

Linear Functions: Definitions and Examples

In Chapter 2, we learned that a function f is called linear if and only if two conditions are met: 1) $f(X + Y) = f(X) + f(Y)$ and 2) $f(cX) = cf(X)$ for every scalar c. The same definition applies to functions that act on vectors.

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear if it has the properties

$$\begin{aligned} f(\mathbf{X} + \mathbf{Y}) &= f(\mathbf{X}) + f(\mathbf{Y}) \quad \text{for all } \mathbf{X}, \mathbf{Y} \text{ in } \mathbb{R}^n \\ f(c\mathbf{X}) &= cf(\mathbf{X}) \quad \text{for all } c \text{ in } \mathbb{R} \end{aligned}$$

Note that n and m don't have to be equal. In other words, the domain and codomain of f can have different dimensions, although in our applications, they usually won't.

Exercise 6.2.7 According to the definition of linearity, are the following functions linear?

a) $f(\begin{pmatrix} X \\ Y \end{pmatrix}) = \begin{pmatrix} X^2 \\ 2Y \end{pmatrix}$ b) $f(X) = \sqrt{X}$

c) $f(\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}) = \begin{pmatrix} 2X \\ XY \\ 3Z \end{pmatrix}$ d) $f(\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}) = \begin{pmatrix} 2X \\ 4Y \\ 3Z \end{pmatrix}$

What Do Linear Functions Look Like?

The definition of linearity tells us what it means for a function to be linear but doesn't give us an easy way to tell whether a particular function is linear without doing some work. We will now develop a way to do that. This will lead to a very useful notation for linear functions, one that we will use extensively for the next two chapters.

Linear functions $\mathbb{R}^1 \to \mathbb{R}^1$. We'll start with the simplest example, $f : \mathbb{R}^1 \to \mathbb{R}^1$. In this context, we think of numbers as one-dimensional vectors and write $\mathbb{R}^1$ instead of $\mathbb{R}$. Thinking of $\mathbb{R}^1$ as a one-dimensional vector space, we see that it has the standard basis $\{\mathbf{e}\} = \{(1)\}$.

If f is a linear function and $\mathbf{X}$ is any vector in $\mathbb{R}^1$, what is $f(\mathbf{X})$?

To start answering this question, we'll take the odd-seeming but useful step of writing $\mathbf{X}$ as $X \cdot \mathbf{e}$. Then, according to the definition of linearity, we have

$$f(\mathbf{X}) = f(X \cdot \mathbf{e}) = Xf(\mathbf{e})$$

Exercise 6.2.8 Which property of linear functions gives us this result?

But what is $f(\mathbf{e})$? We don't know what it is, but we do know that it belongs to $\mathbb{R}^1$. Let's just call it $\mathbf{k}$. Then

$$Xf(\mathbf{e}) = X\mathbf{k}$$

As before, we can rewrite $\mathbf{k}$ as $k\mathbf{e}$. Then, multiplying, we get

$$X\mathbf{k} = Xk\mathbf{e} = k\mathbf{X}$$

Putting it all together yields

$$f(\mathbf{X}) = Xf(\mathbf{e}) = X\mathbf{k} = Xk\mathbf{e} = k\mathbf{X}$$

Since $\mathbf{X}$ is in $\mathbb{R}^1$, it is the same as the scalar X, and we can drop the boldface notation and write $f(X) = kX$.

To summarize, if $f : \mathbb{R}^1 \to \mathbb{R}^1$ is linear, it must have the form $f(X) = kX$ for some scalar k in $\mathbb{R}$.

Linear functions $\mathbb{R}^2 \to \mathbb{R}^1$. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^1$ is a linear function. In $\mathbb{R}^2$, the standard basis is

$$\{\mathbf{e}_1, \mathbf{e}_2\} = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$$

A vector in $\mathbb{R}^2$ has the form $\begin{pmatrix}X\\Y\end{pmatrix}$ and can be written as

$$\begin{pmatrix}X\\Y\end{pmatrix} = X\begin{pmatrix}1\\0\end{pmatrix} + Y\begin{pmatrix}0\\1\end{pmatrix} = X\mathbf{e}_1 + Y\mathbf{e}_2$$

Then from the definition of linear function,

$$f(\begin{pmatrix}X\\Y\end{pmatrix}) = f(X\mathbf{e}_1 + Y\mathbf{e}_2) = f(X\mathbf{e}_1) + f(Y\mathbf{e}_2) = Xf(\mathbf{e}_1) + Yf(\mathbf{e}_2)$$

Exercise 6.2.9 Which property of linear functions gives us this result?

Now $f(\mathbf{e}_1)$ is some vector in $\mathbb{R}^1$; call it $\mathbf{a}$. Similarly, $f(\mathbf{e}_2)$ is some vector in $\mathbb{R}^1$; call it $\mathbf{b}$:

$$f(\begin{pmatrix}X\\Y\end{pmatrix}) = Xf(\mathbf{e}_1) + Yf(\mathbf{e}_2) = X\mathbf{a} + Y\mathbf{b} = Xa\mathbf{e} + Yb\mathbf{e} = aX + bY$$

To summarize, if $f : \mathbb{R}^2 \to \mathbb{R}^1$ is linear, it must have the form $f(\begin{pmatrix}X\\Y\end{pmatrix}) = aX + bY$ for two scalars a and b.

Exercise 6.2.10 Work through this procedure to find the form that a linear function $f : \mathbb{R}^3 \to \mathbb{R}^1$ must have.

Linear functions: $\mathbb{R}^n \to \mathbb{R}^1$. In general, if f is a linear function in $\mathbb{R}^n \to \mathbb{R}^1$, then

$$f(\mathbf{X}) = \mathbf{Y} \qquad \text{where } \mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}, \mathbf{Y} = \begin{pmatrix} Y \end{pmatrix}$$

In $\mathbb{R}^n$, the standard basis is $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. In $\mathbb{R}^1$, the standard basis is $\{\mathbf{e}\}$. Then there is a unique set of scalars $c_1, c_2, \ldots, c_n$ such that

$$\begin{aligned} f(\mathbf{X}) &= f(X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + \cdots + X_n\mathbf{e}_n) \\ &= X_1 f(\mathbf{e}_1) + X_2 f(\mathbf{e}_2) + \cdots + X_n f(\mathbf{e}_n) \\ &= X_1 c_1 \mathbf{e} + X_2 c_2 \mathbf{e} + \cdots + X_n c_n \mathbf{e} \\ &= c_1 X_1 \mathbf{e} + c_2 X_2 \mathbf{e} + \cdots + c_n X_n \mathbf{e} \\ &= c_1 X_1 + c_2 X_2 + \cdots + c_n X_n \qquad (\mathbf{e} \text{ is the same as the scalar } 1) \end{aligned}$$

Exercise 6.2.11 Explain what we are doing in each step in the series of equations above, paying special attention to places where we use vector operations and the properties of linear functions.

The representation of f as $f(\mathbf{X}) = f(X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + \cdots + X_n\mathbf{e}_n)$ is useful, because it explicitly shows the dependence on the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. If we change the basis to a nonstandard one $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then there will be a different unique set of scalars $a_1, a_2, \ldots, a_n$ and another unique set of scalars $b_1, b_2, \ldots, b_n$ such that

$$\begin{aligned} f(\mathbf{X}) &= f(a_1 X_1 \mathbf{v}_1 + a_2 X_2 \mathbf{v}_2 + \cdots + a_n X_n \mathbf{v}_n) \\ &= a_1 X_1 f(\mathbf{v}_1) + a_2 X_2 f(\mathbf{v}_2) + \cdots + a_n X_n f(\mathbf{v}_n) \\ &= a_1 X_1 b_1 \mathbf{e} + a_2 X_2 b_2 \mathbf{e} + \cdots + a_n X_n b_n \mathbf{e} \\ &= a_1 b_1 X_1 \mathbf{e} + a_2 b_2 X_2 \mathbf{e} + \cdots + a_n b_n X_n \mathbf{e} \\ &= a_1 b_1 X_1 + a_2 b_2 X_2 + \cdots + a_n b_n X_n \qquad (\mathbf{e} \text{ is the same as the scalar } 1) \end{aligned}$$

In summary, every linear function of $\mathbb{R}^n$ into $\mathbb{R}^1$ can be written as a linear combination of $X_1, X_2, \ldots, X_n$. The coefficients of the linear combination depend on the choice of basis, so we will absolutely have to keep track of the basis vectors that we are using.

The Matrix Representation of a Linear Function

Now that we understand linear functions from $\mathbb{R}^n$ to $\mathbb{R}^1$, we can extend this to a complete representation of all functions $\mathbb{R}^n$ to $\mathbb{R}^n$ (or even $\mathbb{R}^n$ to $\mathbb{R}^m$, although we will not often need that).

The case $f : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose f is a linear function $\mathbb{R}^2 \to \mathbb{R}^2$. In the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{R}^2$, we use the properties of linearity to get

$$f(\begin{pmatrix} X \\ Y \end{pmatrix}) = f(X\mathbf{e}_1 + Y\mathbf{e}_2) = Xf(\mathbf{e}_1) + Yf(\mathbf{e}_2)$$

Since both $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$ are vectors in $\mathbb{R}^2$, there are scalars a, b, c, and d such that

$$f(\mathbf{e}_1) = \begin{pmatrix} a \\ c \end{pmatrix} \qquad \text{and} \qquad f(\mathbf{e}_2) = \begin{pmatrix} b \\ d \end{pmatrix}$$

We can then say that

$$f(\begin{pmatrix} X \\ Y \end{pmatrix}) = X\begin{pmatrix} a \\ c \end{pmatrix} + Y\begin{pmatrix} b \\ d \end{pmatrix}$$

Applying scalar multiplication and vector addition, we get

$$f(\begin{pmatrix} X \\ Y \end{pmatrix}) = \begin{pmatrix} aX \\ cX \end{pmatrix} + \begin{pmatrix} bY \\ dY \end{pmatrix} = \begin{pmatrix} aX + bY \\ cX + dY \end{pmatrix}$$

Thus, the four numbers a, b, c, and d characterize f relative to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. Since X and Y are placeholders, in order to characterize the function f, we really need only the four numbers a, b, c, and d. We will write the four numbers as a 2×2 array in square brackets:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

When an array of numbers is used to characterize a linear function, the array is called a *matrix*. **We say that the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the matrix representation of f relative to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$.**

The operation of a linear function f on a vector is then calculated by applying the matrix representing f (relative to a given basis) to the representation of the vector. We can write

$$f(\begin{pmatrix} X \\ Y \end{pmatrix}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} aX + bY \\ cX + dY \end{pmatrix}$$

Exercise 6.2.12 Work through the reasoning of this section using numerical vectors of your choosing for $f(\begin{pmatrix} 1 \\ 0 \end{pmatrix})$ and $f(\begin{pmatrix} 0 \\ 1 \end{pmatrix})$.

When we want to talk about applying a matrix to a vector, we just write them next to each other, putting the matrix in square brackets on the left and the vector in round brackets on the right: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$. The action of f on a vector in the domain is found by applying the matrix representation of f to the vector, according to the rule shown in Figure 6.1.

Figure 6.1: Applying a matrix to a vector in $\mathbb{R}^2$.

Notice that the first column of the matrix is $f(\mathbf{e}_1)$, and the second column is $f(\mathbf{e}_2)$. This is a general principle of how matrices work.

Exercise 6.2.13 If $f(\mathbf{e}_1) = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ and $f(\mathbf{e}_2) = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, what is the matrix representation of f?

Exercise 6.2.14 If the matrix representing f is $\begin{bmatrix} 6 & 8 \\ 5 & 1 \end{bmatrix}$, what are $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$?

The case $f : \mathbb{R}^3 \to \mathbb{R}^3$. Suppose f is a linear function that takes vectors in $\mathbb{R}^3$ (the domain) to $\mathbb{R}^3$ (the codomain). And suppose $\mathbf{X}$ is a vector in $\mathbb{R}^3$. In the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, $\mathbf{X}$ can be written as

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = X_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + X_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + X_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3$$

To evaluate the action of f on $\mathbf{X}$, we know that

$$f(\mathbf{X}) = f(X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3)$$

By the rules of linearity, we can decompose $f(\mathbf{X})$ as

$$\begin{aligned} f(\mathbf{X}) &= f(X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3) \\ &= f(X_1\mathbf{e}_1) + f(X_2\mathbf{e}_2) + f(X_3\mathbf{e}_3) \\ &= X_1 f(\mathbf{e}_1) + X_2 f(\mathbf{e}_2) + X_3 f(\mathbf{e}_3) \end{aligned}$$

We can say that $f(\mathbf{e}_1)$ is some vector in $\mathbb{R}^3$. Therefore, there are scalars a_{11}, a_{21}, and a_{31} such that

$$f(\mathbf{e}_1) = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$$

The vector $f(\mathbf{e}_2)$ is also some vector in $\mathbb{R}^3$. So there are scalars a_{12}, a_{22}, and a_{32} such that

$$f(\mathbf{e}_2) = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}$$

Similarly, for $f(\mathbf{e}_3)$, there are scalars a_{13}, a_{23}, and a_{33} such that

$$f(\mathbf{e}_3) = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

Consequently, plugging the expressions for $f(\mathbf{e}_1)$, $f(\mathbf{e}_2)$, and $f(\mathbf{e}_3)$ into $f(\mathbf{X})$, we get

$$\begin{aligned} f(\mathbf{X}) &= X_1 f(\mathbf{e}_1) + X_2 f(\mathbf{e}_2) + X_3 f(\mathbf{e}_3) \\ &= X_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + X_2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + X_3 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} a_{11}X_1 \\ a_{21}X_1 \\ a_{31}X_1 \end{pmatrix} + \begin{pmatrix} a_{12}X_2 \\ a_{22}X_2 \\ a_{32}X_2 \end{pmatrix} + \begin{pmatrix} a_{13}X_3 \\ a_{23}X_3 \\ a_{33}X_3 \end{pmatrix}$$
$$= \begin{pmatrix} a_{11}X_1 + a_{12}X_2 + a_{13}X_3 \\ a_{21}X_1 + a_{22}X_2 + a_{23}X_3 \\ a_{31}X_1 + a_{32}X_2 + a_{32}X_2 \end{pmatrix}$$
$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

Therefore, the 3×3 matrix $[a_{ij}]$ is the matrix[1] representation of $f : \mathbb{R}^3 \to \mathbb{R}^3$ relative to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Exercise 6.2.15 For a function $f : \mathbb{R}^3 \to \mathbb{R}^2$, choose vectors for $f(\mathbf{e}_1), f(\mathbf{e}_2), f(\mathbf{e}_3)$ and work through the reasoning above to find the matrix representation of f. What are the dimensions of this matrix?

Exercise 6.2.16 Similarly, for another function $g : \mathbb{R}^3 \to \mathbb{R}^2$, choose vectors for $g(\mathbf{e}_1), g(\mathbf{e}_2)$ and work through the reasoning above to find the matrix representation of g. What are the dimensions of this matrix?

Generalizing to $f : \mathbb{R}^n \to \mathbb{R}^n$. We can generalize these ideas to $f : \mathbb{R}^n \to \mathbb{R}^n$. Suppose f is a linear function $\mathbb{R}^n \to \mathbb{R}^n$. If $\mathbf{X}$ is any vector in $\mathbb{R}^n$, then it can be written in the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ as

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + \cdots + X_n\mathbf{e}_n$$

To find $f(\mathbf{X})$, we use the fact that we know that there are always scalars $a_{ij} (i, j = 1, 2, \ldots, n)$ such that

$$f(\mathbf{e}_1) = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} \quad f(\mathbf{e}_2) = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} \quad \cdots \quad f(\mathbf{e}_n) = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}$$

Then

$$f\left(\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}\right) = f(X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + \cdots + X_n\mathbf{e}_n) \qquad \text{linear combination}$$
$$= X_1 f(\mathbf{e}_1) + X_2 f(\mathbf{e}_2) + \cdots + X_n f(\mathbf{e}_n) \qquad \text{properties of linearity}$$

[1] We will often write the matrix whose components are a_{ij} as the matrix $[a_{ij}]$.

$$= X_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + X_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + X_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}$$

representation of $f(\mathbf{e}_1), f(\mathbf{e}_2), \ldots, f(\mathbf{e}_n)$

$$= \begin{pmatrix} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n \\ \cdots \\ a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nn}X_n \end{pmatrix}$$

scalar multiplication vector addition

$$= \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

We say that the $n \times n$ matrix $[a_{ij}]$ is the matrix representation of $f : \mathbb{R}^n \to \mathbb{R}^n$ relative to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

Similar to the $\mathbb{R}^2 \to \mathbb{R}^2$ and the $\mathbb{R}^3 \to \mathbb{R}^3$ cases, the action of $f : \mathbb{R}^n \to \mathbb{R}^n$ on a vector in $\mathbb{R}^n$ is found by applying the matrix representation of f to the vector, according to the rule shown in Figure 6.2.

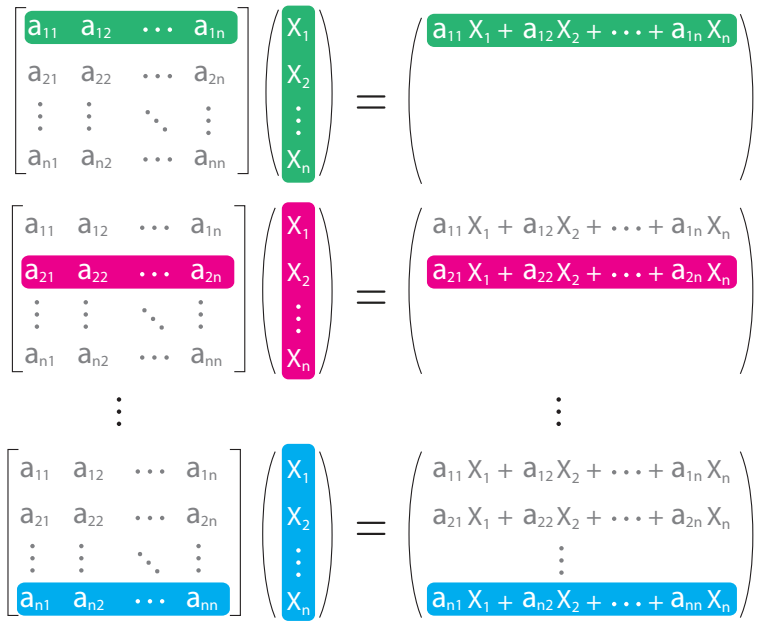


Figure 6.2: Applying a matrix to a vector in $\mathbb{R}^n$.

If f is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^n$, the columns of the matrix representing f are $f(\mathbf{e}_1)$, $f(\mathbf{e}_2)$, ..., $f(\mathbf{e}_n)$.

What all this abstract work buys us is the ability to say what a function does to *any* vector by knowing what it does to the standard basis vectors. For example, in the $f : \mathbb{R}^2 \to \mathbb{R}^2$ case, it means that we can say what the function does to an infinity of possible vectors by knowing what it does to just two vectors, $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$. This is powerful, and it will enable us to understand techniques for working with matrices instead of just memorizing them.

We will now develop an example of the use of matrices in biology that we will refer to throughout this chapter.

A Matrix Population Model: Black Bears

As an example of a linear function $\mathbb{R}^2 \to \mathbb{R}^2$, we will consider a state space (J, A), where J is the number of juveniles, and A is the number of adults of a species of black bear.

Black bears are a common and highly adaptable species found throughout North America, from the Appalachian Mountains to suburban Los Angeles. Females become sexually mature at three or four years of age and live 15 to 20 years in the wild. Approximately every two years, a female will give birth, most commonly to two cubs. We are interested in developing a simple mathematical model of a black bear population.

To model a black bear population, we divide it up into juveniles J (cubs and subadults who are not yet sexually mature) and adults A. Then the state of the system is given by a point in juvenile–adult (J, A) space, that is, as a vector $\begin{pmatrix}J\\A\end{pmatrix}$.

Suppose that on average, a female black bear gives birth to two cubs every two years. This averages out to one cub per year. However, it would simplify our work to focus only on females, as many population models do. Therefore, we will say that a female bear gives birth to 0.5 female cubs each year, on average. Each year, about 10% of juveniles die and 25% mature into adults, leaving 65% as juveniles.

Representing the juvenile population in the Nth year as J_N and that of adults as A_N, we have the juvenile population in the $(N+1)$st year as

$$J_{N+1} = 0.65J_N + 0.5A_N$$

If an adult bear's life expectancy is around 14 years and bears become adults at age 4, they average 10 years as adults. This makes the per capita death rate $1/10 = 0.1$ adults per year, so each year, $1 - 0.1 = 90\%$ of adults remain adults. In addition, as we mentioned before, 25% of juveniles mature into adults each year. This gives the adult population in the $(N+1)$st year as

$$A_{N+1} = 0.25J_N + 0.9A_N$$

Therefore, the black bear population model is given by a linear function f:

$$\begin{pmatrix}J_{N+1}\\A_{N+1}\end{pmatrix} = f\left(\begin{pmatrix}J_N\\A_N\end{pmatrix}\right) = \begin{pmatrix}0.65J_N + 0.5A_N\\0.25J_N + 0.9A_N\end{pmatrix}$$

which can be written in matrix form

$$\begin{pmatrix}J_{N+1}\\A_{N+1}\end{pmatrix} = \begin{bmatrix}0.65 & 0.5\\0.25 & 0.9\end{bmatrix}\begin{pmatrix}J_N\\A_N\end{pmatrix} = \boldsymbol{M}\begin{pmatrix}J_N\\A_N\end{pmatrix}$$

Exercise 6.2.17 What are the matrices representing the following systems of equations?

a) $X_{N+1} = 2X_N + 6Y_N$ and $Y_{N+1} = 3X_N + 8Y_N$

b) $X_{N+1} = -1.5X_N$ and $Y_{N+1} = 6X_N + Y_N$

c) $Z_{N+1} = 18Z_N + 5W_N$ and $W_{N+1} = -7Z_N + 2.2W_N$

d) $a_{N+1} = -3a_N$ and $b_{N+1} = b_N$

e) $a_{N+1} = -2b_N$ and $b_{N+1} = 4a_N$

Exercise 6.2.18 What systems of equations are represented by the following matrices? (You can use X and Y as your variables.)

a) $\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$ b) $\begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 4 \\ -5 & 0 \end{bmatrix}$

d) $\begin{bmatrix} -1 & 0 \\ 0 & 2.5 \end{bmatrix}$ e) $\begin{bmatrix} 0 & 4 & 0 \\ -7 & 0 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

Applying Matrices to Vectors

Suppose during one year, we have a population of 100 juvenile bears and 50 adult bears and want to know what the population will be next year. The current state of the population can be written in the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ as

$$\begin{pmatrix} J_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} 100 \\ 50 \end{pmatrix} = 100\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 50\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 100\mathbf{e}_1 + 50\mathbf{e}_2$$

We now need to apply the function f to this vector to find the next year's population.

From the matrix representation of this function, we can immediately say that $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$ are the first and second columns of $\boldsymbol{M}$, respectively.

$$\boldsymbol{M} = \begin{pmatrix} 0.65 & 0.5 \\ 0.25 & 0.9 \end{pmatrix} \qquad f(\mathbf{e}_1) = \begin{pmatrix} 0.65 \\ 0.25 \end{pmatrix} \qquad f(\mathbf{e}_2) = \begin{pmatrix} 0.5 \\ 0.9 \end{pmatrix}$$

Then the next year's population is

$$\begin{aligned} f\left(\begin{pmatrix} J_0 \\ A_0 \end{pmatrix}\right) &= f(100\mathbf{e}_1 + 50\mathbf{e}_2) \\ &= 100f(\mathbf{e}_1) + 50f(\mathbf{e}_2) \\ &= 100\begin{pmatrix} 0.65 \\ 0.25 \end{pmatrix} + 50\begin{pmatrix} 0.5 \\ 0.9 \end{pmatrix} \\ &= \begin{pmatrix} 100 \times 0.65 + 50 \times 0.5 \\ 100 \times 0.25 + 50 \times 0.9 \end{pmatrix} \\ &= \begin{pmatrix} 90 \\ 70 \end{pmatrix} \end{aligned}$$

Therefore, next year's population will be 90 juveniles and 70 adults.

Exercise 6.2.19 Use the method we used here to find the next year's population if this year's population consists of 15 juveniles and 8 adults.

We can also use the rule for applying a matrix to a vector (Figure 6.1) to calculate the populations of the two age groups in the following year:

$$\begin{pmatrix} J_1 \\ A_1 \end{pmatrix} = \begin{bmatrix} 0.65 & 0.5 \\ 0.25 & 0.9 \end{bmatrix} \begin{pmatrix} 100 \\ 50 \end{pmatrix} = \begin{pmatrix} 0.65 \times 100 + 0.5 \times 50 \\ 0.25 \times 100 + 0.9 \times 50 \end{pmatrix} = \begin{pmatrix} 90 \\ 70 \end{pmatrix}$$

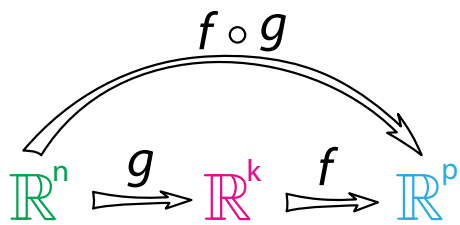
Exercise 6.2.20 Evaluate:

a) $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{pmatrix} 10 \\ 10 \end{pmatrix}$ b) $\begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ c) $\begin{bmatrix} 4 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 4 & 2 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$

Composition of Linear Functions, Multiplication of Matrices

It is a crucial property of functions that we can "chain" them; that is, we can apply functions repeatedly. In Chapters 1 and 2, we saw that if f and g are functions $\mathbb{R} \to \mathbb{R}$, then we can define $f(g(X))$, the result of applying f to $g(X)$, which is written as "$f \circ g$" and called "f composed with g."

In higher dimensions, the idea of chaining functions and applying them successively also makes perfect sense. If f takes $\mathbb{R}^n$ to $\mathbb{R}^k$ and g takes $\mathbb{R}^k$ to $\mathbb{R}^p$, we can define $f \circ g(\mathbf{X}) = f\,(g(\mathbf{X}))$.



This is the general case, but in this text, we are mostly interested in the case $\mathbb{R}^n \to \mathbb{R}^n \to \mathbb{R}^n$.

If f and g are *linear* functions, represented (in the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$) by matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, then their composition $f \circ g$ is also a linear function, which is therefore represented by a matrix we will call $\boldsymbol{C}$. As always, the columns of this matrix show what the function does to the standard basis vectors. The first column is $(f \circ g)(\mathbf{e}_1)$, the second is $(f \circ g)(\mathbf{e}_2)$, and the nth column is $(f \circ g)(\mathbf{e}_n)$.

How do we find the matrix of $f \circ g$? We already know $g(\mathbf{e}_1)$; it's just the first column of $\boldsymbol{B}$. Now all we need to do is apply f to this vector, which we can do using the shortcut of applying the matrix $\boldsymbol{A}$ to $g(\mathbf{e}_1)$. Similarly, to find the second column of the matrix of $f \circ g$, we apply the matrix $\boldsymbol{A}$ to $g(\mathbf{e}_2)$, which is the second column of $\boldsymbol{B}$. Repeating this process, we generate the n columns of the matrix that represents $f \circ g$.

We can also develop this idea algebraically to calculate the matrix $\boldsymbol{C} = [c_{ij}]$ from $\boldsymbol{A}$ and $\boldsymbol{B}$. Suppose $\boldsymbol{A} = [a_{ij}]$ and $\boldsymbol{B} = [b_{ij}]$. If we take an arbitrary vector $\mathbf{X}$ in $\mathbb{R}^n$, apply $\boldsymbol{B}$ to it, and then apply $\boldsymbol{A}$ to the result, we get

$$
\begin{aligned}
\boldsymbol{ABX} &= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{pmatrix} b_{11}X_1 + b_{12}X_2 + \cdots + b_{1n}X_n \\ b_{21}X_1 + b_{22}X_2 + \cdots + b_{2n}X_n \\ \vdots \\ b_{n1}X_1 + b_{n2}X_2 + \cdots + b_{nn}X_n \end{pmatrix} \qquad \text{apply } \boldsymbol{B} \text{ to } \mathbf{X} \\
&= \begin{pmatrix} c_{11}X_1 + c_{12}X_2 + \cdots + c_{1n}X_n \\ c_{21}X_1 + c_{22}X_2 + \cdots + c_{2n}X_n \\ \vdots \\ c_{n1}X_1 + c_{n2}X_2 + \cdots + c_{nn}X_n \end{pmatrix} \qquad \text{apply } \boldsymbol{A} \text{ to } \boldsymbol{BX} \\
&= \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \boldsymbol{CX}
\end{aligned}
$$

where $c_{ij} = a_{i1}b_{1j} + \cdots + a_{ii}b_{ij} + \cdots + a_{ij}b_{jj} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{k=n} a_{ik}b_{kj}$
We can think of this matrix multiplication graphically (Figure 6.3). To find c_{ij}, take row i of matrix $\boldsymbol{A}$ and column j of matrix $\boldsymbol{B}$, line the two up, and then multiply them componentwise, adding up the results.

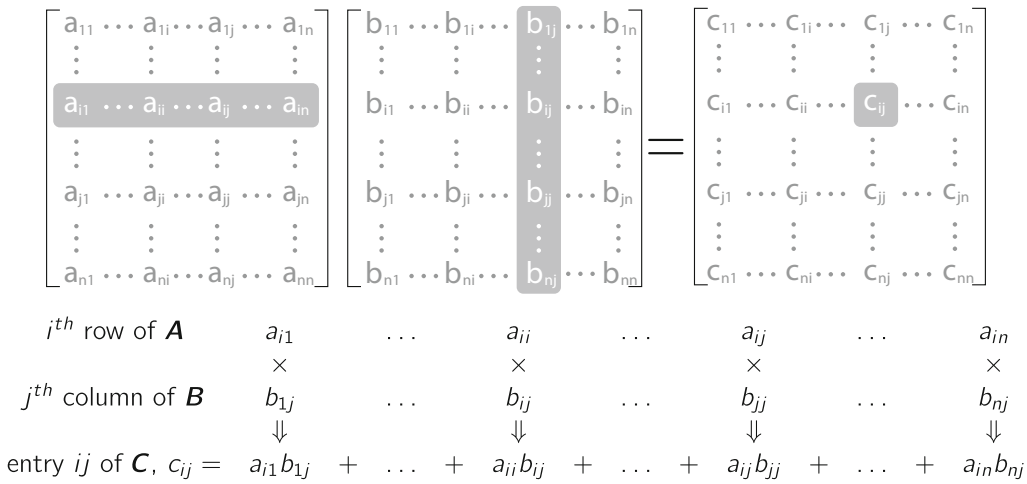


Figure 6.3: Multiplication of two $n \times n$ matrices.

Matrix Multiplication

If a linear function f is represented by the matrix $\boldsymbol{A}$ and another linear function g is represented by the matrix $\boldsymbol{B}$, then the composition $f \circ g(\mathbf{X})$ is represented by the matrix product $\boldsymbol{ABX}$.

Exercise 6.2.21 For the following functions, can $f(g(x))$ exist?

a) $f : \mathbb{R}^2 \to \mathbb{R}^5$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$

b) $f : \mathbb{R}^4 \to \mathbb{R}^3$ and $g : \mathbb{R}^2 \to \mathbb{R}^3$

c) $f : \mathbb{R}^7 \to \mathbb{R}^{138}$ and $g : \mathbb{R}^{26} \to \mathbb{R}^7$

Exercise 6.2.22 If the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ have the following dimensions, does $\boldsymbol{AB}$ exist?

a) $\boldsymbol{A}$ is a 5×2 matrix and $\boldsymbol{B}$ is a 2×3 matrix.

b) $\boldsymbol{A}$ is a 3×4 matrix and $\boldsymbol{B}$ is a 3×2 matrix.

c) $\boldsymbol{A}$ is a 138×7 matrix and $\boldsymbol{B}$ is a 7×26 matrix.

Exercise 6.2.23 Multiply:

a) $\begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 4 & 5 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -2 & 5 \\ 1 & -3 \end{bmatrix}$

An Application of Matrix Multiplication

We can illustrate the principle of multiplication of matrices by considering an alternative scenario for the black bear, in a bad year. We will model "bad year" by lowering the birth rate from 0.5 to 0.4 and increasing the death rate for juveniles to 40%, with 50% of them remaining juvenile and only 10% maturing to adults. The juvenile population model is

$$J_{N+1} = 0.5J_N + 0.4A_N$$

We also increase the adult death rate to 20%, so the survival rate will be $100\% - 20\% = 80\%$. The adult population model is therefore

$$A_{N+1} = 0.1J_N + 0.8A_N$$

Putting these together, we get

$$\begin{pmatrix} J_{N+1} \\ A_{N+1} \end{pmatrix} = \begin{pmatrix} 0.5J_N + 0.4A_N \\ 0.1J_N + 0.8A_N \end{pmatrix}$$

The matrix that describes the "bad year" dynamics is therefore

$$\boldsymbol{M_{bad}} = \begin{bmatrix} 0.5 & 0.4 \\ 0.1 & 0.8 \end{bmatrix}$$

We can then calculate the populations after a good year followed by a bad year. The two-year forecast for an initial population of 100 juveniles and 50 adults is

$$\boldsymbol{M_{bad}}\ \boldsymbol{M}\begin{pmatrix} J_0 \\ A_0 \end{pmatrix} = \begin{bmatrix} 0.5 & 0.4 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 0.65 & 0.5 \\ 0.25 & 0.9 \end{bmatrix} \begin{pmatrix} 100 \\ 50 \end{pmatrix} = \begin{bmatrix} 0.425 & 0.61 \\ 0.265 & 0.77 \end{bmatrix} \begin{pmatrix} 100 \\ 50 \end{pmatrix} = \begin{pmatrix} 73 \\ 65 \end{pmatrix}$$

Exercise 6.2.24 Verify that this calculation is correct by applying the good-year matrix $\boldsymbol{M}$ to the initial condition, and then applying the bad-year matrix $\boldsymbol{M_{bad}}$ to the result. How does your result compare to the above calculation?

Exercise 6.2.25 What does the matrix $\boldsymbol{M}\ \boldsymbol{M_{bad}}$ represent?

Exercise 6.2.26 What matrix product represents a sequence of two good years, followed by two bad years, followed by a good year? Be careful about the order of multiplication.

Notation

matrix symbol	matrix	vector symbol	vector	matrix operating on vector
$\boldsymbol{M}$	$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$	$\mathbf{X}$	$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$	$\boldsymbol{M}\mathbf{X} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

Once we have the matrix representation of a function, we can then talk about what would happen if we applied the function repeatedly to get the long-term behavior of the system. This is our next topic.

Further Exercises 6.2

1. If f is linear, what is $f\left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right)$?
2. Give two everyday or scientific examples of linear combinations not mentioned in the text and briefly explain why each is a linear combination.
3. You are making smoothies. (Be sure to justify your answers to the questions that follow.)
   a) A smoothie recipe can be seen as a linear combination of ingredients. Explain why this is true.
   b) Is the cost to make a smoothie a linear function of the costs of the ingredients?
   c) Is the caloric content of the smoothie a linear function of the caloric content of the ingredients?
   d) Iron is absorbed better in the presence of vitamin C. Is the amount of available iron in your smoothie a linear function of the amount of available iron in the ingredients?
   e) You get your friends to taste your creations. Is the number of friends who like a smoothie likely to be a linear function of the number who like each ingredient?

f) Your smoothies are a hit and you decide to go into business. If you want to keep prices simple, so that all smoothies of a given size cost the same, will your prices be a linear function of the prices of the ingredients?

4. While going to a teaching assistant's office hours, you get lost in the bowels of the School of Engineering. You are walking through the Materials Science Department when you find a strip of a material that looks like nothing you have ever seen before. You pocket it for later examination. Back in your room, you decide to study how the material responds to stretching and compression. Design an experiment to see whether its response to these forces is linear.

5. You are studying how temperature affects the growth of your state flower in order to predict the species's response to climate change. You have a greenhouse and can grow the plants at any temperature you want.

   a) Suppose you call the average temperature at which the plants grow 0, so below-average temperatures are negative and above-average ones are positive. Similarly, below-average growth rates are negative and above-average ones are positive. Design an experiment to test whether the response of change in growth rate to change in temperature is linear.

   b) What result do you expect this experiment to produce? Justify your answer.

6. The function $g : \mathbb{R}^2 \to \mathbb{R}^2$ is linear.

$$g(\begin{pmatrix}-1\\4\end{pmatrix}) = \begin{pmatrix}5\\-2\\3\end{pmatrix} \text{ and } g(\begin{pmatrix}3\\2\end{pmatrix}) = \begin{pmatrix}3\\-3\\0\end{pmatrix}$$

   Since $\begin{pmatrix}-2\\22\end{pmatrix} = 5\begin{pmatrix}-1\\4\end{pmatrix} + \begin{pmatrix}3\\2\end{pmatrix}$, what is $g(\begin{pmatrix}-2\\22\end{pmatrix})$?

7. Assume that f is a linear function. Without using matrices, do the following:

   a) If $f(\begin{pmatrix}1\\0\end{pmatrix}) = \begin{pmatrix}2\\3\end{pmatrix}$ and $f(\begin{pmatrix}0\\1\end{pmatrix}) = \begin{pmatrix}-4\\7\end{pmatrix}$, find $f(\begin{pmatrix}5\\6\end{pmatrix})$.

   b) If $f(\begin{pmatrix}1\\0\end{pmatrix}) = \begin{pmatrix}7\\5\\9\end{pmatrix}$ and $f(\begin{pmatrix}0\\1\end{pmatrix}) = \begin{pmatrix}2\\4\\6\end{pmatrix}$, find $f(\begin{pmatrix}3\\4\end{pmatrix})$.

   c) If $f(\begin{pmatrix}1\\0\\0\end{pmatrix}) = \begin{pmatrix}1\\2\end{pmatrix}$, $f(\begin{pmatrix}0\\1\\0\end{pmatrix}) = \begin{pmatrix}3\\5\end{pmatrix}$, and $f(\begin{pmatrix}0\\0\\1\end{pmatrix}) = \begin{pmatrix}-9\\-2\end{pmatrix}$, find $f(\begin{pmatrix}8\\-5\\7\end{pmatrix})$.

8. Could the functions described below be linear? Justify your answers.

   a) $f(\begin{pmatrix}12\\3\end{pmatrix}) = \begin{pmatrix}6\\-5\end{pmatrix}$ and $f(\begin{pmatrix}-4\\-1\end{pmatrix}) = \begin{pmatrix}-2\\3\end{pmatrix}$

   b) $f(\begin{pmatrix}2\\-5\\3\end{pmatrix}) = \begin{pmatrix}-1\\2\end{pmatrix}$, $f(\begin{pmatrix}4\\1\\3\end{pmatrix}) = \begin{pmatrix}5\\2\end{pmatrix}$ and $f(\begin{pmatrix}6\\-4\\0\end{pmatrix}) = \begin{pmatrix}3\\4\end{pmatrix}$

9. Multiply:

   a) $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

   b) $\begin{bmatrix} 5 & 8 \\ 0 & 4 \end{bmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

   c) $\begin{bmatrix} 6 & -2 & 7 \\ 1 & 0 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$

   d) $\begin{bmatrix} 0 & 1 & 3 \\ -4 & 2 & 1 \\ 3 & 6 & -2 \end{bmatrix} \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$

10. Carry out the following matrix multiplications. For each problem, say what the function represented by each matrix does to the standard basis vectors and what the product of the two matrices would do to these vectors.

    a) $\begin{bmatrix} 7 & 9 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 4 & 6 \end{bmatrix}$

    b) $\begin{bmatrix} 5 & -4 \\ 2 & 0.5 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & -1 \end{bmatrix}$

    c) $\begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

11. Multiply:

    a) $\begin{bmatrix} 7 & 8 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -2 & -3 \end{bmatrix}$

    b) $\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 5 & 2 & -1 \\ 4 & 2 & 1 \end{bmatrix}$

    c) $\begin{bmatrix} -2 & 1 \\ 0 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} -6 & 3 & 7 \\ 9 & -4 & -5 \end{bmatrix}$

    d) $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 0 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 4 & 6 & -7 \\ -2 & 0 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

12. What is the difference between multiplying a matrix times a vector and multiplying two matrices?

13. We have two linear functions, $f : \mathbb{R}^2 \to \mathbb{R}^4$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$. The matrix representing f is

$$\begin{bmatrix} -2 & 3 \\ 5 & 4 \\ 2 & 1 \\ 0 & 3 \end{bmatrix}$$

    a) Suppose

$$g(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) = \begin{pmatrix} 5 \\ 7 \end{pmatrix}, g(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } g(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

    Find the matrix of g.

    b) Find the matrix of $f \circ g$ or explain in terms of functions why it does not exist.

    c) Find the matrix of $g \circ f$ or explain in terms of functions why it does not exist.

14. The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear.

    a) If $f(\begin{pmatrix} 2 \\ 0 \end{pmatrix}) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $f(\begin{pmatrix} 0 \\ 5 \end{pmatrix}) = \begin{pmatrix} -15 \\ 5 \end{pmatrix}$, find the matrix representing f.

b) What is $f(\begin{pmatrix}3\\4\end{pmatrix})$?

c) $g : \mathbb{R}^2 \to \mathbb{R}^2$ is also a linear function. If $g(\begin{pmatrix}1\\0\end{pmatrix}) = \begin{pmatrix}2\\3\end{pmatrix}$ and $g(\begin{pmatrix}0\\1\end{pmatrix}) = \begin{pmatrix}7\\-1\end{pmatrix}$, what is the matrix of $g \circ f$?

6.3 Long-Term Behaviors of Matrix Models

With an understanding of vectors and matrices, we can now use them to model biological processes and explore the long-term dynamics of these systems.

The long-term behavior of a matrix model is revealed by applying the matrix many times over. This is called an *iterated matrix* or *iterated function*. If we begin with an initial condition $\mathbf{X}$, then the long-term behavior is

$$\underbrace{M \cdots M}_{N}\mathbf{X} = M^N\mathbf{X}$$

for large values of N.

Matrix models can exhibit three basic types of long-term dynamics: stable and unstable equilibrium behavior, neutral equilibria, and neutral oscillations. We will study examples of each of these in turn.

Stable and Unstable Equilibria

The black bear population model developed in the previous section is an example of a *Leslie matrix*. A Leslie matrix model of a population gives the rates at which individuals go from one life stage to another. In this case, we have two life stages, juvenile and adult. The diagonal entries give the fraction of the population that stays within the same life stage, while the off-diagonal entry in the top row gives the birth rate of juveniles. The off-diagonal entry in the bottom row is the transition rate from the juvenile stage to the adult stage. Therefore, in the model

$$M = \begin{bmatrix}0.65 & 0.5\\0.25 & 0.9\end{bmatrix}$$

65% of juveniles remain juveniles and 90% of adults remain adults in any given year. Furthermore, 25% of juveniles in a given year mature into adults, and the average adult has 0.5 (female) offspring.

Exercise 6.3.1 Come up with a Leslie matrix model for a fictional species with two life stages and describe the meaning of its entries, as above.

Let's look at the long-term behavior of this model. If we iterate M from an initial condition of 10 juveniles and 50 adults for 15 times, we see that both juvenile and adult populations grow with time (Figure 6.4, left). Notice that the trajectory consists of isolated points. This is because a Leslie matrix is a discrete-time model. If we plot these points in J-A state space, we see that after the first few values, all the points fall on a straight line passing through the origin, implying that the ratio of juveniles to adults remains constant as the population grows (Figure 6.4, right).

Moreover, the distance between successive state points increases with time, meaning that the population growth rate increases with population size.

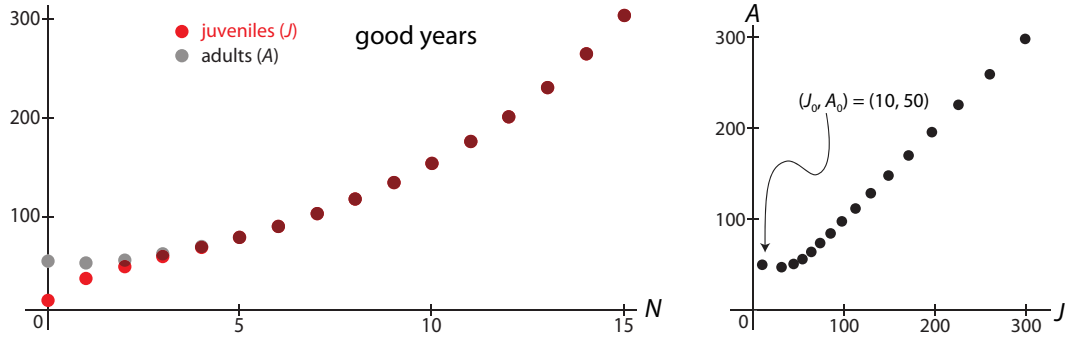


Figure 6.4: Time series (left) and corresponding trajectory (right) produced by iterating the matrix $\boldsymbol{M}$, modeling the black bear population in a good year. Notice that both consist of discrete points.

Now let's consider a bad year, which, as we saw, is modeled by the matrix

$$\boldsymbol{M}_{bad} = \begin{bmatrix} 0.5 & 0.4 \\ 0.1 & 0.8 \end{bmatrix}$$

Iterating this matrix, we see that both juvenile and adult populations go to zero with time (Figure 6.5, left). However, this decline doesn't initially affect both age groups in the same way. The juvenile population grows for a time, while the adult population just shrinks. Of course, this can't go on forever, so after a few years, both populations enter long-term decline. (The system's behavior before it enters this long-term pattern is called a *transient*.)

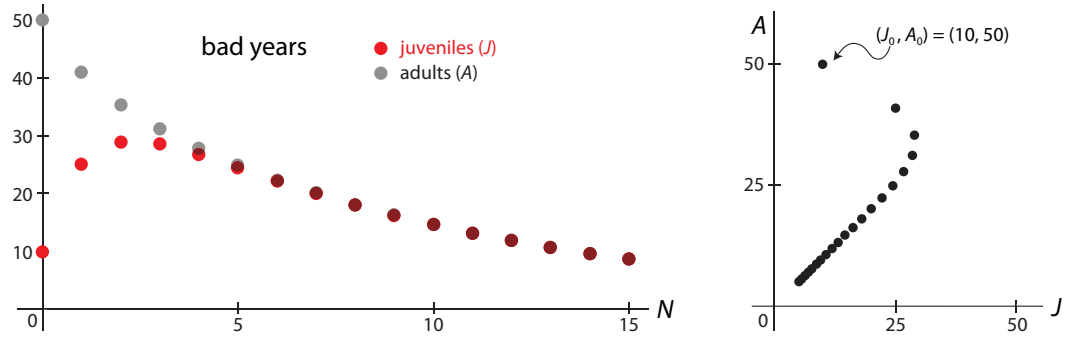


Figure 6.5: Time series (left) and corresponding trajectory (right) produced by iterating the matrix $\boldsymbol{M}_{\text{bad}}$, modeling the black bear population in a bad year.

Let's consider another Leslie matrix for a two-stage population. Here we will consider a situation in which 10% of juveniles remain juvenile, 40% become adults, and the rest die. The birth rate is 1.4 offspring per adult, and only 20% of adults survive each year. This gives us a

matrix

$$M_{osc} = \begin{bmatrix} 0.1 & 1.4 \\ 0.4 & 0.2 \end{bmatrix}$$

If we iterate M_{osc}, we see that both juvenile and adult populations approach the stable equilibrium at $(0, 0)$ in an oscillatory manner (Figure 6.6).

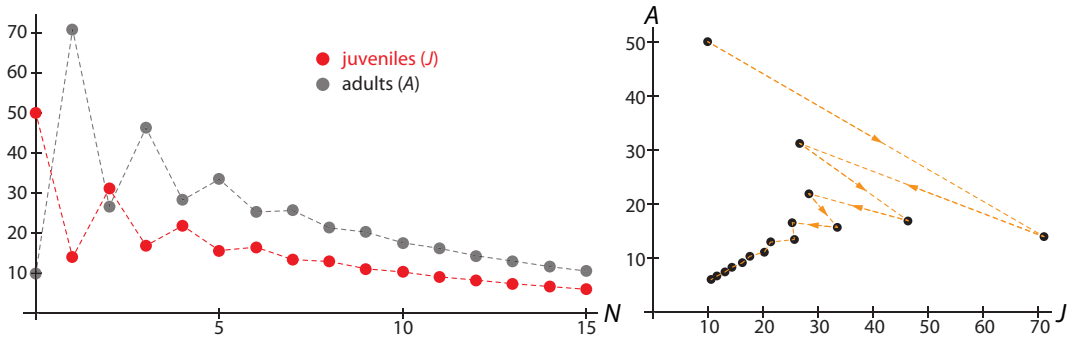


Figure 6.6: Time series (left) and corresponding trajectory (right) produced by iterating the matrix M_{osc}.

Neutral Equilibria

We will now consider an important class of models whose equilibria are not the isolated equilibrium points we have been seeing all along. In these models, called *Markov processes*, the final equilibrium value depends on the initial condition, so there is an infinity of equilibrium points.

All of the models we have seen so far can be thought of as *compartmental models*. In a compartmental model, a large number of objects are transferred from one compartment to another, according to rules. In the discrete-time version of compartmental modeling, these transfers take place at discrete time points, $1, 2, 3, \ldots, N$.

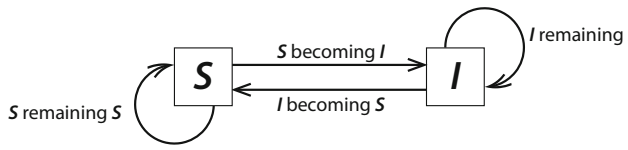
In epidemiology, the study of infectious diseases, many models use compartments called *susceptibles* (those who can become infected), and *infecteds*. We will represent these two populations by S and I.

In epidemiology, linear models of disease transmission are used to predict whether a disease will initially spread. Epidemiologists will make an estimate of the rate of "new cases per old case," the quantity called R_0 (read "R-zero" or "R-nought") and then model the epidemic as

$$I_{N+1} = R_0 \, I_N$$

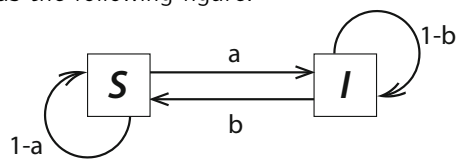
where I_N is the number of infected people at the Nth time point. If $R_0 > 1$, the epidemic spreads, while if $R_0 < 1$, the epidemic will tend to die out.

In the more general case, we can write a simple compartmental model representing the transfers from the susceptibles compartment S to the infecteds compartment I and vice versa.



We will make the extremely strong assumption that at each time point, a constant fraction a of the susceptibles become infected and a constant fraction b of the infecteds recover to become

susceptibles again. If a is the fraction of S that become I, then the fraction of S that remain S must be $1 - a$. If b is the fraction of I that become S, then the fraction of I that remain I must be $1 - b$. This gives us the following figure.



The discrete-time dynamics for this S-I compartmental model are

$$S_{N+1} = (1 - a)S_N + bI_N$$
$$I_{N+1} = aS_N + (1 - b)I_N$$

This can be written in matrix form:

$$\begin{pmatrix} S_{N+1} \\ I_{N+1} \end{pmatrix} = \begin{bmatrix} 1 - a & b \\ a & 1 - b \end{bmatrix} \begin{pmatrix} S_N \\ I_N \end{pmatrix}$$

Let's choose $a = 0.1$ and $b = 0.2$, which means that at each time point, 10% of susceptible individuals become infected, and 90% remain susceptible. Similarly, 20% of infected individuals recover, with 80% remaining infected. Notice that the disease is nonlethal, because there are no death terms in this model. And there is no immunity, since infecteds return to the susceptible compartment.

This gives us the matrix

$$\boldsymbol{M_{SI}} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \tag{6.1}$$

If we iterate $\boldsymbol{M_{SI}}$, we see a new kind of behavior. If we begin with an initial condition of 10 susceptibles and 50 infecteds, the system stabilizes at an equilibrium point. And if we begin with a different initial condition, at 30 susceptibles and 80 infecteds, the system also stabilizes at an equilibrium point, *but a different one*.

Exercise 6.3.2 Explain why the entries in each column of a transition matrix such as equation (6.1) must add up to one. *(Hint: Label the rows and columns, writing "from" and "to" where appropriate.)*

Exercise 6.3.3 Starting with 20 susceptible and 40 infected individuals, iterate $\boldsymbol{M_{SI}}$ 15 times in SageMath. What steady state does the system reach? Do the same for 50 susceptible and 60 infected individuals. How do your results compare to the simulations in Figure 6.7?

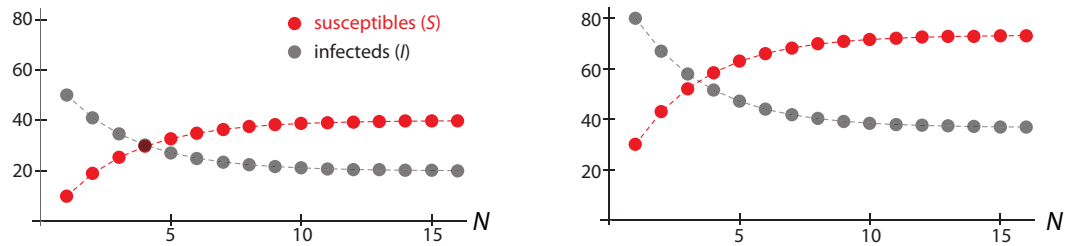


Figure 6.7: Time series from two simulations of the susceptible-infected model. Starting from different initial conditions, the system converges to different equilibrium points.

Exercise 6.3.4 What is the behavior of the total population $(S + I)$ over time?

Why does this susceptible–infected system behave so differently from the black bear Leslie matrices we studied at the beginning of this section? One key difference is that Leslie matrices involve births and deaths. A population modeled by a Leslie matrix model *must* grow or decline unless the birth and death rates exactly balance. In this particular disease model, on the other hand, individuals are just shuffled from one compartment to another, without any overall increase or decrease in population size.

Neutral Oscillations

Our final example of a matrix model is one that gives neutral oscillations (Bodine et al. 2014). By "neutral," we mean that here, as in the previous example of neutral equilibria, the final outcome depends on the initial condition, only here the final outcome is an oscillation. These "neutral oscillations" are therefore a discrete-time analogue to the neutral oscillations we saw in the frictionless spring and the shark–tuna models.

Locusts, which are important agricultural pests, have three stages in their life cycle: eggs (E), hoppers (juveniles) (H), and adults (A). In a certain locust species, the egg and hopper stages each last one year, with 2% of eggs surviving to become hoppers and 5% of hoppers surviving to become adults. Adults lay 1000 eggs (as before, we are modeling only females) and then die.

From these principles, we can write a 3-variable linear equation

$$\begin{aligned} E_{N+1} &= 0 \cdot E_N + 0 \cdot H_N + 1000 A_N \\ H_{N+1} &= 0.02 E_N + 0 \cdot H_N + 0 \cdot A_N \\ A_{N+1} &= 0 \cdot E_N + 0.05 H_N + 0 \cdot A_N \end{aligned}$$

which gives rise to a 3×3 Leslie matrix:

$$\boldsymbol{L} = \begin{bmatrix} 0 & 0 & 1000 \\ 0.02 & 0 & 0 \\ 0 & 0.05 & 0 \end{bmatrix}$$

Simulating the model, iterating $\boldsymbol{L}$ with an initial population of 50 eggs, 100 hoppers, and 50 adults results in oscillatory dynamics of the populations over time. Consider, for example, the adult population (Figure 6.8, black dots). As you can see, the adult population oscillates with no overall growth or decline.

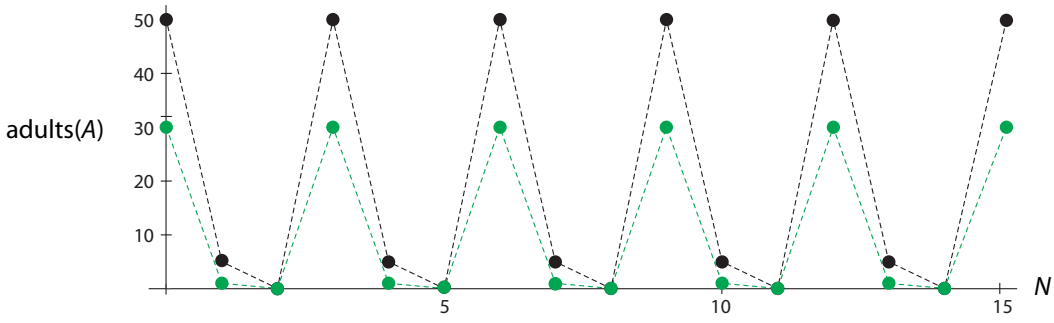


Figure 6.8: Time series of adult populations from two simulations (black and green) of the locust population model from two different initial conditions.

If we try a different initial condition, say 50 eggs, 20 hoppers, and 30 adults, we get a different oscillation, also with no overall growth or decay, but with different values (Figure 6.8, green dots).

Exercise 6.3.5 Simulate the discrete-time dynamical system described by the matrix $\boldsymbol{L}$, and plot all three populations.

Exercise 6.3.6 Calculate the total population $E + H + A$ at each time point. How does it change?

We have now seen the repertoire of long-term behaviors that linear models can exhibit: stable and unstable equilibria, neutral equilibria, and neutral oscillations.

Matrix Models in Ecology and Conservation Biology

One interesting example of the use of matrix models in real scientific research involves the extinction of moas, giant birds that inhabited New Zealand until shortly after it was colonized by humans in the late 1200s AD. Archaeological data suggested that moas went extinct less than 200 years after human colonization. But could a small population really hunt moas to extinction so rapidly?

Researchers used data from present-day moa relatives and analysis of fossil remains to build a Leslie matrix model of moa population dynamics (Holdaway and Jacomb 2000). The goal of the model was to study the relative importance of two different factors in the extinction of the moa, namely, human hunting and habitat loss. This is a type of question that is ideally suited to modeling: we can try different combinations of the two factors and see what happens.

The study used model parameters that changed over time to represent the effects of a growing human population on moa survivorship. The results indicated that even low hunting pressure by a population of a few hundred people was enough to drive moas to extinction in 160 years or less (Figure 6.9).

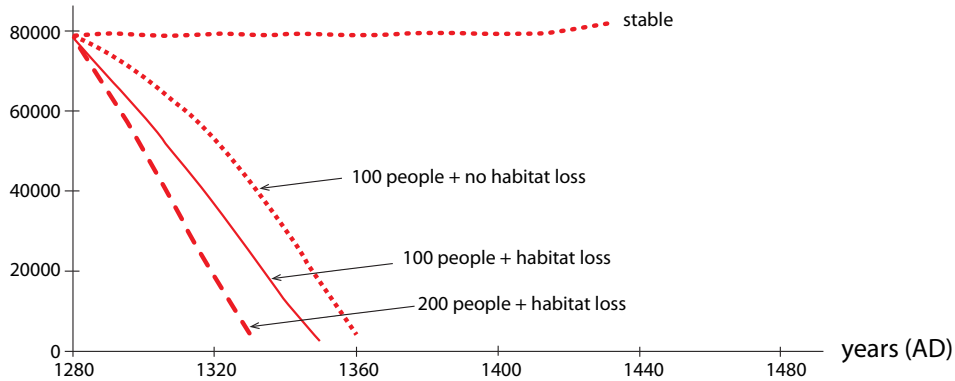


Figure 6.9: Simulated effects of different human colonization scenarios on moa populations. Redrawn from "Rapid extinction of the moas (Aves: Dinornithiformes): model, test, and implications," by R.N. Holdaway and C. Jacomb, 2000, *Science* 287(5461):2250–2254. Reprinted with permission from AAAS.

Note from their simulations that even without habitat loss, the hunting pressure of even 100 humans, growing at 2.2% per year, with no habitat loss, was enough to drive the moa to extinction, albeit in a slightly longer time. Habitat loss made it worse, and if they considered an initial population of 200 humans and included habitat loss, the decline was even more catastrophic. The authors conclude that "Long-lived birds are very vulnerable to human predation of adults."

Exercise 6.3.7 If a species is going extinct, what equilibrium is the population size approaching? Is this equilibrium stable or unstable?

Matrix models are also helping to prevent sea turtles from going the way of the moa. Loggerhead sea turtles are an endangered species. Adult females build nests on beaches, lay eggs, and leave. Hatchlings then go out to sea, where they grow into juveniles and then adults.

In the 1980s, sea turtle conservation efforts focused on protecting nests and hatchlings. Then a group of ecologists decided to test whether such efforts, even if extremely successful, could actually save the species from extinction (Crouse et al. 1987). They used field data to build a matrix model consisting of seven life stages (eggs and hatchlings, small juveniles, large juveniles, subadults, novice breeders, first-year remigrants, and mature breeders), and for each stage in turn, they reduced mortality to zero. This is obviously impossible, but it's the most basic test a conservation strategy must pass. If eliminating all mortality in a life stage can't save the species, neither can merely reducing the mortality.

Simulations showed that if nothing was done, the population would decline. However, eliminating all mortality in the eggs and hatchlings stage didn't reverse the decline. To do so, it was necessary to protect large juveniles and subadults. Since most preventable mortality at this stage came from turtles getting caught in fishing and shrimping nets, mandating the installation of turtle excluder devices that allow sea turtles to escape from nets is a much better strategy for protecting the species. The United States currently requires the use of these devices, but some countries in loggerhead habitat do not.

Further Exercises 6.3

1. Giant pandas are a vulnerable species famous for their consumption of large amounts of bamboo. Write a discrete-time matrix model of a giant panda population using the following assumptions. We are modeling only the female population.
   - Pandas have three life stages: cubs, subadults, and reproductively mature adults.
   - Cubs remain cubs for only one year. They have a mortality rate of 17%.
   - Pandas remain subadults for three years. Thus, about 33% of subadults mature into adults each year.
   - 28% of subadults die each year.
   - On average, adults give birth to 0.5 female cubs each year.
   - 97.7% of adults survive from one year to the next.

2. Nitrogen is a key element in all organisms. Use the following assumptions to set up a matrix model of nitrogen flow in an ecosystem consisting of producers (P), consumers (C) and decomposers (D).
   - 25% of the nitrogen in plants goes to consumers and 50% goes to decomposers.
   - 75% of the nitrogen in consumers goes to decomposers.
   - 5% of the nitrogen in decomposers goes to consumers, and 15% is lost from the ecosystem. The rest goes to plants.

3. In epidemiology, a common way to model the spread of an infectious disease is to track the number of *susceptible* individuals (S), the number of currently *infected* individuals (I), and the number of individuals who have *recovered* from the disease with immunity (R). Assume the following:
   - Each day, 2% of susceptible individuals get infected.
   - On average, a person remains infected for five days, so each day roughly 20% of infected individuals recover. Most (say 18%) will have developed immunity to the disease, but a few (2%) will not be immune, and thus will immediately be susceptible again.
   - A person's immunity does not last forever. Each day 1% of recovered individuals become susceptible again.

     a) Draw a compartment diagram for this model and label each of the arrows appropriately.

     b) What is the matrix of this model?

4. Black-lip oysters (*Pinctada margaritifera*) are born male, but may become female later in life (a phenomenon known as *protandrous hermaphroditism*). We can therefore divide their population into three life stages: juveniles (which are all male), adult males, and adult females. Assume the following:
   - Each year, about 9% of juveniles remain juveniles, 0.9% grow to become adult males, and 0.1% grow into adult females. The rest die.
   - Each year, about 4% of adult males become female, and about 10% of them die.
   - About 10% of adult females die each year. Females never change back into males.
   - Each female lays enough eggs to yield about 200 juveniles per year.

   Write a discrete-time matrix model based on these assumptions.

6.4 Eigenvalues and Eigenvectors

We have now seen a variety of matrix models, with a variety of long-term behaviors, such as equilibrium point behaviors and oscillatory behaviors. We simulated these long-term behaviors by simply iterating the matrix over and over again from an initial condition. Our goal now is to understand these long-term behaviors and to be able to predict them, by studying the structure of the model itself. In order to do this, we need to develop one more critical piece of linear algebra: the concepts of *eigenvalues and eigenvectors*.

Linear Functions in One Dimension

Recall from Chapter 2 that the linear functions in one dimension are exactly the functions $f(X) = rX$, where r is in $\mathbb{R}$. Those are the only functions that can pass the stringent test for linearity:

$$\begin{aligned} f(X+Y) &= f(X) + f(Y) \quad && \text{for all } X, Y \\ f(kX) &= kf(X) && \text{for all } k \text{ in } \mathbb{R} \end{aligned}$$

Linear Functions in Two Dimensions

Let's consider an arbitrary linear function $f : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\begin{pmatrix} U \\ V \end{pmatrix} = f\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right) = \begin{pmatrix} aX + bY \\ cX + dY \end{pmatrix}$$

As we saw, this function can also be represented in matrix form:

$$\begin{pmatrix} U \\ V \end{pmatrix} = \boldsymbol{M}\begin{pmatrix} X \\ Y \end{pmatrix}$$

where

$$\boldsymbol{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

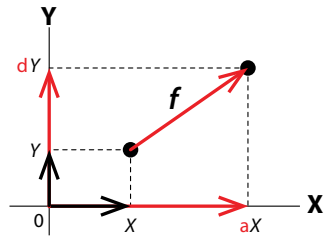
The easiest way to make a 2D function is to take two 1D functions and join them together. So if $U = aX$ and $V = dY$, then we can make the function

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} aX \\ dY \end{pmatrix}$$

This represents a very special case in which U depends only on X, and V depends only on Y. In this special case, the function is represented by a *diagonal matrix*, which is a matrix whose entries are all 0 except those on the descending diagonal:

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} aX \\ dY \end{pmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

In this case, it is easy to determine the action of function f: it acts like multiplication by a along the X axis and like multiplication by d along the Y axis.



For example, consider a linear discrete-time dynamical system consisting of two species that don't interact with each other, such as sharks and rabbits. Let S_N be the number of sharks in the Nth year, and let R_N be the number of rabbits in the Nth year. Because there is no interaction, S_{N+1} is purely a function of S_N, and R_{N+1} is purely a function of R_N. If the shark population grows at a rate a and the rabbit population grows at a rate d, then $S_{N+1} = aS_N$ and $R_{N+1} = dR_N$.

The matrix representation of this system of two noninteracting species is then

$$\begin{pmatrix} S_{N+1} \\ R_{N+1} \end{pmatrix} = \begin{pmatrix} aS_N \\ dR_N \end{pmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{pmatrix} S_N \\ R_N \end{pmatrix}$$

A diagonal matrix represents a function that can be decomposed into two 1-dimensional functions along the axes **X** and **Y**. Diagonal matrices represent systems in which the variables are noninteracting.

Exercise 6.4.1 Consider the matrix $\boldsymbol{M} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

a) Compute $\boldsymbol{M}\mathbf{e}_1$, $\boldsymbol{M}\mathbf{e}_2$, and $\boldsymbol{M}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b) Draw $\boldsymbol{e}_1$, $\boldsymbol{e}_1$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and the vectors you obtained in the first part of this problem.

c) Describe what $\boldsymbol{M}$ does to $\mathbf{e}_1$, $\mathbf{e}_2$, and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

d) What will $\boldsymbol{M}$ do to other vectors that lie along the X axis? The Y axis?

e) What will $\boldsymbol{M}$ do to vectors that do not lie along the axes?

Exercise 6.4.2 Repeat the previous exercise for $\boldsymbol{M} = \begin{bmatrix} 0.5 & 0 \\ 0 & -2 \end{bmatrix}$.

Eigenvalues

Understanding the action of a diagonal matrix is easy. But what about the general case? The typical matrix is not a diagonal matrix, so it is hard to guess what the action of the matrix looks like. Since U is a function of both X and Y, and so is V, we cannot simply decompose f into two 1D systems acting *along the X and Y axes*. We can't just look at the X and Y axes and stretch or compress the standard basis vectors.

But what if we could find two new axes? Specifically, what if we could find two vectors **U** and **V** such that f *is* decomposable into two 1D systems acting along the **U** and **V** axes?

If two such axes did exist, then by definition, they would have to have the property that

$$\boldsymbol{M}\mathbf{U} = \lambda_1 \mathbf{U} \quad \text{and} \quad \boldsymbol{M}\mathbf{V} = \lambda_2 \mathbf{V}$$

for some real numbers λ_1 and λ_2, which means that $\boldsymbol{M}$ would be acting along the vector **U** as multiplication by λ_1, and acting along the vector **V** as multiplication by λ_2.

When this can be done, we call **U** and **V** the *eigenvectors* of $\boldsymbol{M}$, and λ_1 and λ_2 are the corresponding *eigenvalues*.

Exercise 6.4.3 One of the eigenvalues of the matrix $\boldsymbol{M}$ is 3, and a corresponding eigenvector is $\mathbf{V} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Find $\boldsymbol{M}\mathbf{V}$.

In other words, we are looking for solutions to the linear equation

$$\boldsymbol{M}\mathbf{E} = \lambda\mathbf{E} \tag{6.2}$$

where $\mathbf{E}$ is the axis we are looking for (Figure 6.10). We will solve this equation for λ and $\mathbf{E}$. Let

$$\boldsymbol{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \mathbf{E} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

We can write

$$\boldsymbol{M}\mathbf{E} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} aX + bY \\ cX + dY \end{pmatrix}$$

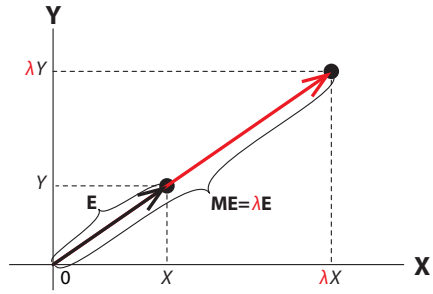


Figure 6.10: The effect of applying the matrix $\boldsymbol{M}$ to the vector $\mathbf{E}$ (black arrow) is a new vector that is $\mathbf{E}$ multiplied by a scalar λ.

and

$$\lambda\mathbf{E} = \lambda \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \lambda X \\ \lambda Y \end{pmatrix}$$

Since $\boldsymbol{M}\mathbf{E} = \lambda\mathbf{E}$,

$$\begin{pmatrix} aX + bY \\ cX + dY \end{pmatrix} = \begin{pmatrix} \lambda X \\ \lambda Y \end{pmatrix}$$

From this vector equation, we get the following two equations:

$$aX + bY = \lambda X$$
$$cX + dY = \lambda Y$$

We want to manipulate these equations to give us an expression in terms of λ. The first expression is

$$\begin{aligned} & aX + bY = \lambda X \\ \Longrightarrow\ & \lambda X - aX = bY \\ \Longrightarrow\ & (\lambda - a)X = bY \\ \Longrightarrow\ & X = \frac{bY}{\lambda - a} \end{aligned}$$

which gives us X in terms of Y. We will now use that to substitute for X in the second expression,

$$cX + dY = \lambda Y$$

which gives us

$$\begin{aligned} & c\frac{bY}{\lambda - a} + dY = \lambda Y \\ \implies & \frac{cb\cancel{Y}}{\lambda - a} = (\lambda - d)\cancel{Y} \\ \implies & \frac{cb}{\lambda - a} = (\lambda - d) \\ \implies & cb = (\lambda - a)(\lambda - d) \\ \implies & cb = \lambda^2 - a\lambda - d\lambda + ad \end{aligned}$$

which finally gives us

$$\lambda^2 - (a + d)\lambda + (ad - cb) = 0$$

This is a quadratic equation in λ, called the *characteristic equation*, which must be satisfied if λ is a solution to equation (6.2).

We know how to solve quadratic equations. Using the quadratic formula, we get

$$\lambda = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(1)(ad - cb)}}{2(1)}$$

which can be simplified to

$$\lambda = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - cb)}}{2} = (\lambda_1, \lambda_2) \tag{6.3}$$

We have found a very fundamental relationship. For every matrix $\boldsymbol{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ there is a set of axes[2] $\mathbf{U}$, $\mathbf{V}$ such that $\boldsymbol{M}\mathbf{U} = \lambda_1\mathbf{U}$ and $\boldsymbol{M}\mathbf{V} = \lambda_2\mathbf{V}$, and we have found λ_1 and λ_2 in terms of the coefficients a, b, c, and d. The quadratic formula gives us two values of λ (note the $\pm$ sign in the expression). These two values, which we call λ_1 and λ_2, are called the two *eigenvalues* of the matrix $\boldsymbol{M}$.

For the matrix $\boldsymbol{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the *characteristic equation* (or *characteristic polynomial*) for an eigenvalue λ in 2D is

$$\lambda^2 - (a + d)\lambda + (ad - cb) = 0$$

Eigenvalues are solutions to this equation.

Let's try an example. Consider the matrix

$$\boldsymbol{M} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

This is obviously an undecomposable function of X_1 and X_2. Can we find two new axes along which it is decomposable? Plugging the coefficient values into equation (6.3), we get

$$\lambda = \frac{4 \pm \sqrt{4^2 - 4(3 - 8)}}{2} = \frac{4 \pm 6}{2} = (\lambda_1, \lambda_2) = (5, -1)$$

[2] We will later see that these may not be axes in the usual sense, since they could involve complex numbers, but we can still write them down symbolically.

We have now found that there are two axes **U**, **V** such that the matrix acts like multiplication by $\lambda_1 = 5$ along **U**, and acts like multiplication by $\lambda_2 = -1$ along **V**.

But we do not know what **U** and **V** are yet.

Exercise 6.4.4 Compute the eigenvalues of the following matrices: $\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ and $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$.

Eigenvectors

We now need to find **U** and **V**. Let's say $\mathbf{U} = \begin{pmatrix} X \\ Y \end{pmatrix}$. Since we said that M acts like multiplication by 5 along **U**, this means that

$$M\mathbf{U} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X + 2Y \\ 4X + 3Y \end{pmatrix} = \lambda_1 \mathbf{U} = 5 \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 5X \\ 5Y \end{pmatrix}$$

So

$$\begin{aligned} X + 2Y = 5X \quad &\Longrightarrow \quad Y = 2X \\ 4X + 3Y = 5Y \quad &\Longrightarrow \quad Y = 2X \end{aligned}$$

Now $Y = 2X$ is the equation for the line in (X, Y) space that has slope 2 and passes through the origin. This line is the axis **U**. We can choose any nonzero vector on the **U** axis to represent it, for example, the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. **This vector is then called an** *eigenvector* **of the matrix** M **corresponding to the eigenvalue** $\lambda_1 = 5$.

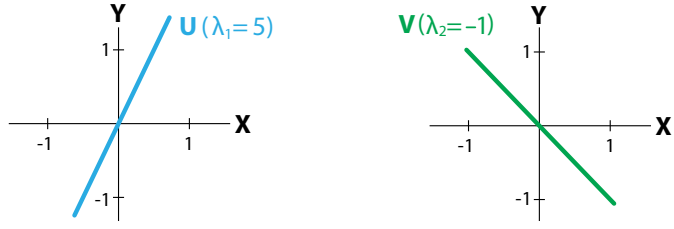
An eigenvector corresponding to the second eigenvalue $\lambda_2 = -1$ can be found in a similar manner. Let's assume $\mathbf{V} = \begin{pmatrix} X \\ Y \end{pmatrix}$. Then

$$M\mathbf{V} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X + 2Y \\ 4X + 3Y \end{pmatrix} = \lambda_2 \mathbf{V} = -1 \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -X \\ -Y \end{pmatrix}$$

So

$$\begin{aligned} X + 2Y = -X \quad &\Longrightarrow \quad Y = -X \\ 4X + 3Y = -Y \quad &\Longrightarrow \quad Y = -X \end{aligned}$$

$Y = -X$ is the equation for the line in (X, Y) space that has slope -1 and passes through the origin. This line is the axis **V**. As before, we can choose any nonzero vector on the **U** axis to represent it, for example, the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which is then called an *eigenvector* of the matrix M corresponding the eigenvalue $\lambda_2 = -1$.



We have now accomplished a basic task: given an indecomposable nondiagonal matrix, we have found two new axes, $\mathbf{U}$ and $\mathbf{V}$, along which the matrix is diagonal. Let's call this diagonal matrix $\boldsymbol{D}$. This new set of axes can be seen as a new coordinate system for $\mathbb{R}^2$; call it $\{\mathbf{U}, \mathbf{V}\}$. In the $\{\mathbf{U}, \mathbf{V}\}$ coordinate system, the matrix $\boldsymbol{D}$ is diagonal:

$$\boldsymbol{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

matrix in $\{\mathbf{X}, \mathbf{Y}\}$	eigenvalues	eigenvectors	diagonalized matrix in $\{\mathbf{U}, \mathbf{V}\}$
$\boldsymbol{M} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$	$\lambda_1 = 5$	$\mathbf{U} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$	$\boldsymbol{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$
	$\lambda_2 = -1$	$\mathbf{V} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$	

If a matrix $\boldsymbol{M}$ has two real eigenvalues λ_1 and λ_2, this implies that $\boldsymbol{M}$ can be decomposed using two new axes, $\mathbf{U}$ and $\mathbf{V}$, such that $\boldsymbol{M}$ acts like multiplication by λ_1 along $\mathbf{U}$ and like multiplication by λ_2 along $\mathbf{V}$.

New coordinate systems. We can navigate in $\mathbb{R}^2$ using these two new axes. The standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the most familiar coordinate system for $\mathbb{R}^2$: to get to any point, go a certain distance horizontally (parallel to $\mathbf{e}_1$) and a certain distance vertically (parallel to $\mathbf{e}_2$). The eigenvectors $\mathbf{U}$ and $\mathbf{V}$ also form a coordinate system, and we can get to any point in $\mathbb{R}^2$ by moving a certain distance in the $\mathbf{U}$-direction and a certain distance in the $\mathbf{V}$-direction.

We will now illustrate the process of navigating in $\mathbb{R}^2$ using two different coordinate systems. As our $\{\mathbf{X}, \mathbf{Y}\}$ coordinate system, we will use the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. For the $\{\mathbf{U}, \mathbf{V}\}$ coordinate system, we will use the eigenvectors we just calculated:

$$\{\mathbf{X}, \mathbf{Y}\} = \{\mathbf{e}_1, \mathbf{e}_2\} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\} \qquad \{\mathbf{U}, \mathbf{V}\} = \{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$$

Consider the point $\mathbf{p}$ represented in the standard $\{\mathbf{X}, \mathbf{Y}\}$ coordinate system as $\mathbf{p}_{\{X,Y\}} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$. To navigate from the origin to $\mathbf{p}$, go three units in the $\mathbf{X}$ direction, and zero units in the $\mathbf{Y}$ direction (Figure 6.11, left).

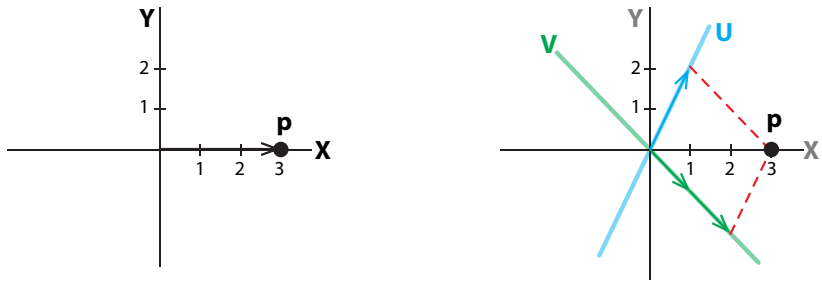


Figure 6.11: Finding the coordinates of the point $\mathbf{p}$ in a new coordinate system $\{\mathbf{U}, \mathbf{V}\}$.

In order to navigate to p in the $\{\mathbf{U}, \mathbf{V}\}$ coordinate system, suppose that the coordinates of $\mathbf{p}$ are c_1 and c_2. We have

$$\mathbf{p}_{\{U,V\}} = c_1\mathbf{U} + c_2\mathbf{V} = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}c_1 \times 1 + c_2 \times 1\\ c_1 \times 2 + c_2 \times (-1)\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}$$

Solving this algebraically, we get

$$\left.\begin{array}{l} c_1 \times 1 + c_2 \times 1 = 3 \\ c_1 \times 2 + c_2 \times (-1) = 0 \end{array}\right\} \implies c_1 = 1, c_2 = 2$$

Therefore, to navigate from the origin to $\mathbf{p}$ in the $\{\mathbf{U}, \mathbf{V}\}$ coordinate system, go one unit in the $\mathbf{U}$ direction and two units in the $\mathbf{V}$ direction (Figure 6.11, right).

Exercise 6.4.5 Find the eigenvectors of the matrices whose eigenvalues you found in Exercise 6.4.4 on page 303.

We will now use the ability to change coordinate systems to map the action of $\boldsymbol{M}$.

Using eigenvalues and eigenvectors to calculate the action of a matrix

We will now show how to use the eigenvectors and corresponding eigenvalues of a matrix to calculate the action of the matrix on a test point.

The following discussion is somewhat technical; the details can be skimmed over, and the reader can skip to "Are All Matrices Diagonalizable?" on page 312. However, the high-level summary of what we will do here is important. What we are going to do, for an arbitrary matrix $\boldsymbol{M}$ and a test point $\mathbf{p}$, is find $\mathbf{q} = \boldsymbol{M}\mathbf{p}$. We will do this by the following procedure:

(1) Pick a test point $\mathbf{p}$. Let $\{\mathbf{X}, \mathbf{Y}\}$ be an arbitrary coordinate system (it could be the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ or any other). Suppose we have the coordinates of $\mathbf{p}$ in the $\{\mathbf{X}, \mathbf{Y}\}$ coordinate system as $\mathbf{p}_{\{X,Y\}} = \begin{pmatrix}p_X\\p_Y\end{pmatrix}$.

(2) Calculate the eigenvectors $\mathbf{U}, \mathbf{V}$ of the matrix $\boldsymbol{M}$ and their corresponding eigenvalues λ_1 and λ_2.

(3) Find the representation of the test point $\mathbf{p}$ in the $\{\mathbf{U}, \mathbf{V}\}$ coordinate system to obtain $\mathbf{p}_{\{U,V\}} = \begin{pmatrix}p_U\\p_V\end{pmatrix}$.

(4) Evaluate the action of $\boldsymbol{M}$ by multiplying the $\mathbf{U}$-component p_U by λ_1, and the $\mathbf{V}$-component p_V by λ_2. This gives us the location of the point $\mathbf{q}$ in the $\{\mathbf{U}, \mathbf{V}\}$ coordinate system, $\mathbf{q}_{\{U,V\}} = \begin{pmatrix}\lambda_1 p_U\\ \lambda_2 p_V\end{pmatrix}$.

(5) Transform the $\{\mathbf{U}, \mathbf{V}\}$ coordinate representation of $\mathbf{q}$, $\mathbf{q}_{\{U,V\}}$ back into the $\{\mathbf{X}, \mathbf{Y}\}$ coordinate system to obtain $\mathbf{q}_{\{X,Y\}}$.

An example. Let's compute what the matrix $\boldsymbol{M} = \begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$ does to the test point $\mathbf{p}$. For the $\{\mathbf{X}, \mathbf{Y}\}$ coordinate system, we will use $\{\mathbf{e}_1, \mathbf{e}_2\}$. In this standard coordinate system, we pick the test point $\mathbf{p}_{\{X,Y\}} = \begin{pmatrix}p_X\\p_Y\end{pmatrix} = \begin{pmatrix}1\\0.5\end{pmatrix}$.

In order to calculate the action of $\boldsymbol{M}$, we need to locate this point on the **U** and **V** axes (Figure 6.12). To do this, we need a way of transforming from the $\{\mathbf{X}, \mathbf{Y}\}$ coordinate system to the new $\{\mathbf{U}, \mathbf{V}\}$ coordinate system to get $\mathbf{p}_{\{U,V\}} = \begin{pmatrix} p_U \\ p_V \end{pmatrix}$.

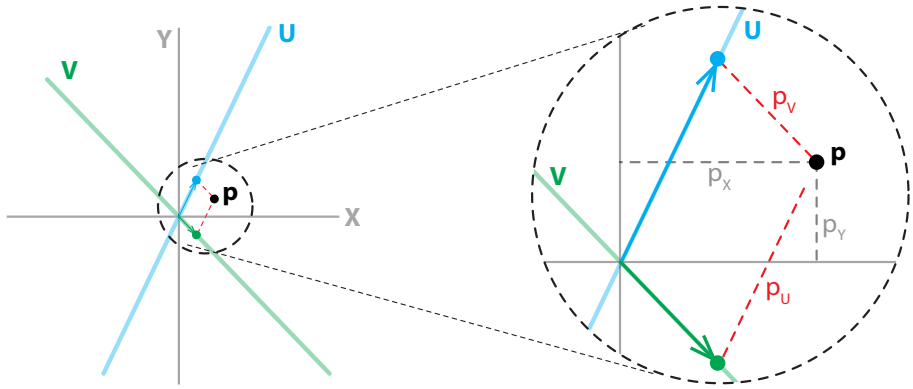


Figure 6.12: The coordinates of the point **p** in the $\{\mathbf{X}, \mathbf{Y}\}$ and $\{\mathbf{U}, \mathbf{V}\}$ coordinate systems.

Once we have the test point **p** represented in the $\{\mathbf{U}, \mathbf{V}\}$ coordinate system, we then just multiply its components by the corresponding eigenvalues λ_1 and λ_2 (Figure 6.13). Here, the **U**-component is multiplied by $\lambda_1 = 5$, and the **V**-component is multiplied by $\lambda_2 = -1$. Thus, the image under $\boldsymbol{M}$ of the test point $\mathbf{p}_{\{U,V\}} = \begin{pmatrix} p_U \\ p_V \end{pmatrix}$ is the point $\mathbf{q}_{\{U,V\}} = \begin{pmatrix} q_U \\ q_V \end{pmatrix} = \begin{pmatrix} \lambda_1 \cdot p_U \\ \lambda_2 \cdot p_V \end{pmatrix} = \begin{pmatrix} 5p_U \\ -p_V \end{pmatrix}$.

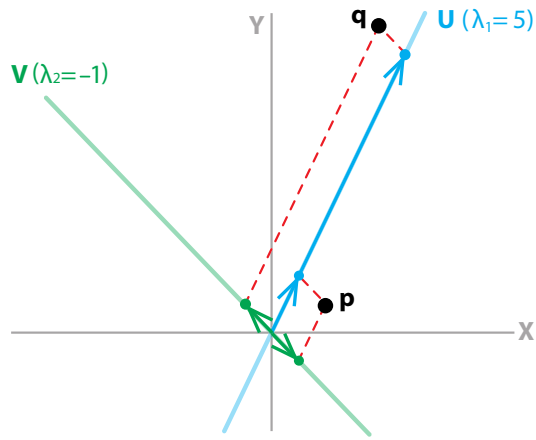


Figure 6.13: Using eigenvalues and corresponding eigenvectors to find the action of $\boldsymbol{M}$ on the point **p** in the $\{\mathbf{U}, \mathbf{V}\}$ coordinate system.

We now have the point $\mathbf{q}$ represented in the $\{\mathbf{U}, \mathbf{V}\}$ coordinate system, that is, $\mathbf{q}_{\{U,V\}}$. The final step is to transform the point $\mathbf{q}$ back into the original $\{\mathbf{X}, \mathbf{Y}\}$ coordinate system to get $\mathbf{q}_{\{X,Y\}} = \begin{pmatrix} q_X \\ q_Y \end{pmatrix}$ (Figure 6.14).

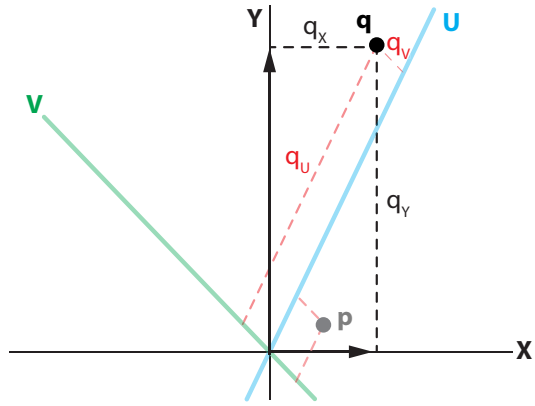


Figure 6.14: Transforming the point $\mathbf{q}$ back into the original $\{\mathbf{X}, \mathbf{Y}\}$ coordinate system.

These figures graphically illustrate the process of finding the new point using the $\{\mathbf{U}, \mathbf{V}\}$ coordinate system. Now, in order to actually calculate that point, we have to do it algebraically, using the linear algebra of coordinate transforms.

Changing bases: coordinate transforms. In $\mathbb{R}^2$, we have been using as our basis vectors the standard basis

$$\{\mathbf{X}, \mathbf{Y}\} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$$

The key to calling this set of vectors a basis is that every vector $\mathbf{p}$ can be written in the $\{\mathbf{X}, \mathbf{Y}\}$ coordinate system as

$$\mathbf{p}_{\{X,Y\}} = \begin{pmatrix} p_X \\ p_Y \end{pmatrix} = p_X \begin{pmatrix} 1 \\ 0 \end{pmatrix} + p_Y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = p_X \mathbf{X} + p_Y \mathbf{Y}$$

But the standard basis isn't the only possible one. In fact, *any* two vectors that aren't multiples of each other can serve as a basis for $\mathbb{R}^2$.

If we pick $\mathbf{U}$ and $\mathbf{V}$ as two such vectors, then every vector $\mathbf{p}$ that had coordinates $\begin{pmatrix} p_X \\ p_Y \end{pmatrix}$ in the $\{\mathbf{X}, \mathbf{Y}\}$ basis now has a new set of coordinates $\begin{pmatrix} p_U \\ p_V \end{pmatrix}$ in the $\{\mathbf{U}, \mathbf{V}\}$ basis. We want to find those new coordinates.

In general, there is always a matrix transform that will take the representation of a point expressed in any basis in $\mathbb{R}^n$ to any other basis. Here we will illustrate this for the case in $\mathbb{R}^2$ in which the two coordinate systems are $\{\mathbf{Z}, \mathbf{W}\}$ and $\{\mathbf{U}, \mathbf{V}\}$.

Suppose we have a vector $\mathbf{p}$ and we know its coordinates in $\{\mathbf{Z}, \mathbf{W}\}$ space as $\mathbf{p}_{\{Z,W\}}$. We would like to know the vector $\mathbf{p}$ expressed in the $\{\mathbf{U}, \mathbf{V}\}$ coordinate system, that is, $\mathbf{p}_{\{U,V\}}$. In other words, we want to find the transformation matrix $\boldsymbol{T}$ such that $\mathbf{p}_{\{U,V\}} = \boldsymbol{T}\mathbf{p}_{\{Z,W\}}$.

In order to find the transformation matrix $\boldsymbol{T}$, the key is to express the "old" coordinates $\{\mathbf{Z}, \mathbf{W}\}$ in terms of the "new" $\{\mathbf{U}, \mathbf{V}\}$ coordinates. Assuming that there are a, b, c, d such that

$$\begin{aligned} \mathbf{Z} &= a\mathbf{U} + b\mathbf{V} \\ \mathbf{W} &= c\mathbf{U} + d\mathbf{V} \end{aligned}$$

we can substitute for $\mathbf{Z}$ and $\mathbf{W}$ the corresponding expressions in $\mathbf{U}$ and $\mathbf{V}$ to get an expression for $\mathbf{p}$ in the $\{\mathbf{U}, \mathbf{V}\}$ coordinates as

$$\begin{aligned} \mathbf{p}_{\{\mathrm{Z,W}\}} &= p_Z\mathbf{Z} + p_W\mathbf{W} \\ &= p_Z\big(a\mathbf{U} + b\mathbf{V}\big) + p_W\big(c\mathbf{U} + d\mathbf{V}\big) \\ &= \big(a \cdot p_Z + c \cdot p_W\big)\mathbf{U} + \big(b \cdot p_Z + d \cdot p_W\big)\mathbf{V} \\ &= p_U\mathbf{U} + p_V\mathbf{V} \end{aligned}$$

So

$$\mathbf{p}_{\{\mathrm{U,V}\}} = \begin{pmatrix} p_U \\ p_V \end{pmatrix} = \begin{pmatrix} a \cdot p_Z + c \cdot p_W \\ b \cdot p_Z + d \cdot p_W \end{pmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{pmatrix} p_Z \\ p_W \end{pmatrix}$$

Therefore, the transformation matrix $\boldsymbol{T}$ that gives us $\mathbf{p}_{\{\mathrm{U,V}\}}$ in terms of $\mathbf{p}_{\{\mathrm{Z,W}\}}$ is

$$\boldsymbol{T} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

and the required transformation is

$$\mathbf{p}_{\{\mathrm{U,V}\}} = \boldsymbol{T}\mathbf{p}_{\{\mathrm{Z,W}\}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \mathbf{p}_{\{\mathrm{Z,W}\}}$$

Now we need to find a, c, b, and d. First, let's recall the definition of each of the coordinates in terms of their components:

$$\mathbf{Z} = \begin{pmatrix} Z_X \\ Z_Y \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} W_X \\ W_Y \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} U_X \\ U_Y \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} V_X \\ V_Y \end{pmatrix}$$

Notice that in each case, we are expressing the coordinate vector in terms of its representation in the standard $\{\mathbf{X}, \mathbf{Y}\}$ basis. So, while we are transforming from one arbitrary $\{\mathbf{Z}, \mathbf{W}\}$ basis to another arbitrary $\{\mathbf{U}, \mathbf{V}\}$ basis, we are keeping track of both of them in terms of their representation in the standard $\{\mathbf{X}, \mathbf{Y}\}$ basis.

Substituting the component definitions of each coordinate into the definition of a, b, c, and d, we get

$$\begin{aligned} \mathbf{Z} &= a\mathbf{U} + b\mathbf{V} \\ \mathbf{W} &= c\mathbf{U} + d\mathbf{V} \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \begin{pmatrix} Z_X \\ Z_Y \end{pmatrix} &= a\begin{pmatrix} U_X \\ U_Y \end{pmatrix} + b\begin{pmatrix} V_X \\ V_Y \end{pmatrix} \\ \begin{pmatrix} W_X \\ W_Y \end{pmatrix} &= c\begin{pmatrix} U_X \\ U_Y \end{pmatrix} + b\begin{pmatrix} V_X \\ V_Y \end{pmatrix} \end{aligned}$$

If we multiply this out, we get

$$\begin{aligned} Z_X &= aU_X + bV_X \\ Z_Y &= aU_Y + bV_Y \\ W_X &= cU_X + dV_X \\ W_Y &= cU_Y + dV_Y \end{aligned}$$

These are four linear equations in four unknowns. We can solve this problem by hand, or we can use the computer algebra function of SageMath to do all the messy work. The result of this

algebra is that we now have a, b, c, and d in terms of the components of $\mathbf{U}, \mathbf{V}, \mathbf{Z}$ and $\mathbf{W}$:

$$a = \frac{-V_X Z_Y + V_Y Z_X}{U_X V_Y - U_Y V_X},\ b = \frac{V_Y W_X - V_X W_Y}{U_X V_Y - U_Y V_X},\ c = \frac{V_Y W_X - V_X W_Y}{U_X V_Y - U_Y V_X},\ d = \frac{U_X W_Y - U_Y W_X}{U_X V_Y - U_Y V_X}$$

If we assemble these into the transformation matrix $\boldsymbol{T}$, we get

$$\boldsymbol{T} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \frac{1}{U_X V_Y - U_Y V_X} \begin{bmatrix} -V_X Z_Y + V_Y Z_X & V_Y W_X - V_X W_Y \\ U_X Z_Y - U_Y Z_X & U_X W_Y - U_Y W_X \end{bmatrix}$$

This is a complete expression for the transformation matrix. It cannot fail to give us the transformation matrix, unless, of course, the expression in the denominator $U_X V_Y - U_Y V_X$ equals 0.

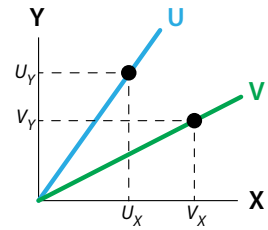
What does it mean for $U_X V_Y - U_Y V_X$ to be equal to zero?

$$U_X V_Y - U_Y V_X = 0 \quad \Longleftrightarrow \quad U_X V_Y = U_Y V_X$$

If we assume that neither $\mathbf{U}$ nor $\mathbf{V}$ is the $\mathbf{Y}$ axis, which would otherwise make $U_X = 0$ or $V_X = 0$, then we can divide by each of them and get

$$\frac{U_Y}{U_X} = \frac{V_Y}{V_X}$$

Notice that $\frac{U_Y}{U_X}$ is the slope of the $\mathbf{U}$ vector, and $\frac{V_Y}{V_X}$ is the slope of the $\mathbf{V}$ vector.



If the slope of $\mathbf{U}$ is equal to the slope of $\mathbf{V}$, then $\mathbf{U}$ and $\mathbf{V}$ are multiples of each other, and therefore they are not a basis for $\mathbb{R}^2$.

Exercise 6.4.6 Show that under the condition $U_X V_Y - U_Y V_X = 0$, if $\mathbf{U}$ is the $\mathbf{Y}$ axis ($U_X = 0$), then $\mathbf{V}$ has to be the $\mathbf{Y}$ axis as well ($V_X = 0$), and vice versa, which contradicts our assumption that $\mathbf{U}$ and $\mathbf{V}$ serves as a basis in $\mathbb{R}^2$.

The action of M. We can now return to our problem of evaluating the action of $\boldsymbol{M}$ on the test point $\mathbf{p} = (1, 0.5)$ in the $\{\mathbf{X}, \mathbf{Y}\}$ coordinate system, that is,

$$\mathbf{p}_{\{X,Y\}} = \begin{pmatrix} p_X \\ p_Y \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$$

using the eigenvalues and eigenvectors of $\boldsymbol{M}$. Our first task is to find the test point $\mathbf{p}$ expressed in the coordinate system of the eigenvectors $\mathbf{U}$ and $\mathbf{V}$ of the matrix $\boldsymbol{M}$. This is a straightforward application of the transformation matrix $\boldsymbol{T}$ we just developed.

Here the "old" coordinate system $\{\mathbf{Z}, \mathbf{W}\}$ is

$$\{\mathbf{Z}, \mathbf{W}\} = \{\mathbf{X}, \mathbf{Y}\} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$$

and the "new" coordinate system is the system of eigenvectors $\mathbf{U}$ and $\mathbf{V}$ of the matrix $\boldsymbol{M}$:

$$\{\mathbf{U}, \mathbf{V}\} = \{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\}$$

The coordinate components we need to calculate $\boldsymbol{T}$ are

$$\begin{aligned}\mathbf{Z} &= \begin{pmatrix}Z_X\\Z_Y\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}\\ \mathbf{W} &= \begin{pmatrix}W_X\\W_Y\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}\\ \mathbf{U} &= \begin{pmatrix}U_X\\U_Y\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}\\ \mathbf{V} &= \begin{pmatrix}V_X\\V_Y\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}\end{aligned} \quad \Longleftrightarrow \quad \begin{cases} Z_X = 1\\ Z_Y = 0\\ W_X = 0\\ W_Y = 1\\ U_X = 1\\ U_Y = 2\\ V_X = 1\\ V_Y = -1\end{cases}$$

So the transformation matrix $\boldsymbol{T}$ from the "old" $\{\mathbf{X}, \mathbf{Y}\}$ coordinate system to the "new" $\{\mathbf{U}, \mathbf{V}\}$ coordinate system is

$$\boldsymbol{T} = \begin{bmatrix}a & c\\ b & d\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & -\frac{1}{3}\end{bmatrix}$$

Then we can use this transformation matrix $\boldsymbol{T}$ to give us the test point $\mathbf{p}$ expressed in the $\{\mathbf{U}, \mathbf{V}\}$ coordinate system, $\mathbf{p}_{\{U,V\}}$, in terms of $\mathbf{p}_{\{X,Y\}}$:

$$\mathbf{p}_{\{U,V\}} = \boldsymbol{T}\ \mathbf{p}_{\{X,Y\}} = \begin{bmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & -\frac{1}{3}\end{bmatrix}\begin{pmatrix}1\\0.5\end{pmatrix} = \begin{pmatrix}0.5\\0.5\end{pmatrix}$$

Therefore, our test point is

$$\mathbf{p}_{\{U,V\}} = \begin{pmatrix}p_U\\p_V\end{pmatrix} = \begin{pmatrix}0.5\\0.5\end{pmatrix} = p_U\mathbf{U} + p_V\mathbf{V}$$

Now that we have the point expressed in the eigenvector $\{\mathbf{U}, \mathbf{V}\}$ coordinate system, we can use the eigenvalues to calculate the action of the matrix. We said that the action of that matrix $\boldsymbol{M}$ is that it acts like multiplication by λ_1 along its corresponding $\mathbf{U}$ eigenvector, and multiplication by λ_2 along its corresponding $\mathbf{V}$ eigenvector.

Therefore, in order to find the point, which we will call $\mathbf{q}$, that results from the action of the matrix $\boldsymbol{M}$ on the test point $\mathbf{p}$, we simply multiply the $\mathbf{U}$-component of $\mathbf{p}$ by λ_1 and the $\mathbf{V}$-component of $\mathbf{p}$ by λ_2 to find $\mathbf{q}_{\{U,V\}}$:

$$\mathbf{q}_{\{U,V\}} = \begin{pmatrix}q_U\\q_V\end{pmatrix} = \boldsymbol{D}\ \mathbf{p}_{\{\mathrm{U,V}\}} = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{pmatrix}q_U\\q_V\end{pmatrix} = \begin{pmatrix}\lambda_1 \cdot p_U\\ \lambda_2 \cdot p_V\end{pmatrix} = \begin{pmatrix}5 \times 0.5\\ -1 \times 0.5\end{pmatrix} = \begin{pmatrix}2.5\\-0.5\end{pmatrix}$$

To confirm this and check our work, let's calculate the action of $\boldsymbol{M}$ in the $\{\mathbf{X}, \mathbf{Y}\}$ coordinate system and then transform the result into the $\{\mathbf{U}, \mathbf{V}\}$ coordinate system and see whether the two calculations agree.

First, we find $\mathbf{q}_{\{X,Y\}}$ by applying $\boldsymbol{M}$ to the test point $\mathbf{p}_{\{X,Y\}}$:

$$\mathbf{q}_{\{X,Y\}} = \boldsymbol{M}\,\mathbf{p}_{\{X,Y\}} = \begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}\begin{pmatrix}1\\ 0.5\end{pmatrix} = \begin{pmatrix}2\\ 5.5\end{pmatrix}$$

Then we use the transformation matrix $\boldsymbol{T}$ to transform $\mathbf{q}_{\{X,Y\}}$ into $\mathbf{q}_{\{U,V\}}$,

$$\mathbf{q}_{\{U,V\}} = \boldsymbol{T}\,\mathbf{q}_{\{X,Y\}} = \begin{bmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & -\frac{1}{3}\end{bmatrix}\begin{pmatrix}2\\ 5.5\end{pmatrix} = \begin{pmatrix}2.5\\ -0.5\end{pmatrix}$$

which agrees exactly with our calculation of $\mathbf{q}_{\{U,V\}}$ using the eigenvalues. The two methods of calculating $\mathbf{q}_{\{U,V\}}$ are equivalent:

$$\begin{array}{ccc}\mathbf{q}_{\{X,Y\}} & \xrightarrow{\quad\boldsymbol{T}\quad} & \mathbf{q}_{\{U,V\}}\\ \boldsymbol{M}\uparrow & & \uparrow\boldsymbol{D} = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\\ \mathbf{p}_{\{X,Y\}} & \xrightarrow{\quad\boldsymbol{T}\quad} & \mathbf{p}_{\{U,V\}}\end{array}$$

However, $\mathbf{q}_{\{U,V\}}$ is not what we originally wanted; we wanted $\mathbf{q}_{\{X,Y\}}$. We need to take one step further and somehow get back to the $\{\mathbf{X},\mathbf{Y}\}$ coordinate system from $\mathbf{q}_{\{U,V\}}$. To do this, we need the *inverse of the matrix* $\boldsymbol{T}$, that is, the matrix that "undoes" the action of $\boldsymbol{T}$. To find this matrix, called $\boldsymbol{T}^{-1}$, realize that

$$\boldsymbol{T}^{-1}\boldsymbol{T} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

If we let

$$\boldsymbol{T}^{-1} = \begin{bmatrix}c_1 & c_2\\ c_3 & c_4\end{bmatrix}$$

then we have

$$\begin{bmatrix}c_1 & c_2\\ c_3 & c_4\end{bmatrix}\begin{bmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & -\frac{1}{3}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

which implies

$$\begin{bmatrix}c_1\frac{1}{3}+c_2\frac{2}{3} & c_1\frac{1}{3}-c_2\frac{1}{3}\\ c_3\frac{1}{3}+c_4\frac{2}{3} & c_3\frac{1}{3}-c_4\frac{1}{3}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \Longrightarrow \begin{cases}c_1\frac{1}{3}+c_2\frac{2}{3} = 1\\ c_1\frac{1}{3}-c_2\frac{2}{3} = 0\\ c_3\frac{1}{3}+c_4\frac{2}{3} = 0\\ c_3\frac{1}{3}-c_4\frac{1}{3} = 1\end{cases} \Longrightarrow \begin{cases}c_1 = 1\\ c_2 = 1\\ c_3 = 2\\ c_4 = -1\end{cases}$$

So

$$\boldsymbol{T}^{-1} = \begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}$$

Consequently, we can go from $\mathbf{p}_{\{X,Y\}}$ to $\mathbf{q}_{\{X,Y\}}$ by transforming into the $\{\mathbf{U},\mathbf{V}\}$ system by $\boldsymbol{T}$, applying $\boldsymbol{D}$, and then transforming back into the $\{\mathbf{X},\mathbf{Y}\}$ coordinate system using $\boldsymbol{T}^{-1}$:

$$\mathbf{q}_{\{X,Y\}} = \boldsymbol{M}\mathbf{p}_{\{X,Y\}} = \boldsymbol{T}^{-1}\boldsymbol{D}\boldsymbol{T}\mathbf{p}_{\{X,Y\}}$$

In summary, we can evaluate the action of the matrix $\boldsymbol{M}$ on a point by applying the diagonal matrix $\boldsymbol{D}$:

$$\begin{array}{ccc} & \boldsymbol{T}^{-1} & \\ \mathbf{q}_{\{X,Y\}} & \longleftarrow & \mathbf{q}_{\{U,V\}} \\ \boldsymbol{M}\uparrow & & \uparrow \boldsymbol{D} = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} \\ & \boldsymbol{T} & \\ \mathbf{p}_{\{X,Y\}} & \longrightarrow & \mathbf{p}_{\{U,V\}} \end{array}$$

This may seem as though we are not saving much effort, because we also have to figure out $\boldsymbol{T}$ and $\boldsymbol{T}^{-1}$. However, if $\boldsymbol{M}$ is a matrix representing a dynamical system, then we need to iterate $\boldsymbol{M}$ many times to simulate the dynamics. In this case, the advantage is clear: we need to calculate and apply $\boldsymbol{T}$ and $\boldsymbol{T}^{-1}$ only once, and the rest of the iteration process is simply applying the diagonal matrix $\boldsymbol{D}$ many times, which is easy:

$$\underbrace{\boldsymbol{M}\cdots\boldsymbol{M}}_{N}\mathbf{p}_{\{X,Y\}} = \boldsymbol{T}^{-1}\underbrace{\boldsymbol{D}\cdots\boldsymbol{D}}_{N}\boldsymbol{T}\mathbf{p}_{\{X,Y\}}$$

Are all matrices diagonalizable?

We have successfully diagonalized the matrix $\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$, and it makes sense to ask, are all matrices diagonalizable in this way?

The answer is no. Consider the matrix

$$\boldsymbol{M} = \begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}$$

Let's calculate its eigenvalues. Plugging the matrix coefficients into the characteristic equation (equation (6.3) on page 302),

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-cb)}}{2}$$

we get

$$\lambda = \frac{(-1) \pm \sqrt{(-1)^2 - 4(0-(-1))}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$$

and here we have a problem. Notice the "$\sqrt{-3}$" term. As you know, there is no such real number. There is a concept of *imaginary numbers*, like $\boldsymbol{i} = \sqrt{-1}$, and in that notation, we can write our eigenvalue as

$$\lambda = \frac{-1 \pm \sqrt{3}\sqrt{-1}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\boldsymbol{i}$$

But what can this mean? It certainly does not look good for our goal of decomposing the matrix into two 1D multiplications.

In fact, the appearance of imaginary numbers is an infallible sign that we are dealing with a type of motion that is indecomposable, namely, *rotation*.

The reason why complex numbers are associated with rotations can be made intuitive. Think of a function f that has an eigenvalue $\lambda = -1$ along the eigenvector X. The action of f is to flip the direction of any vector along this axis, for example, it would flip $(1, 0)$ to $(-1, 0)$; see Figure 6.15.

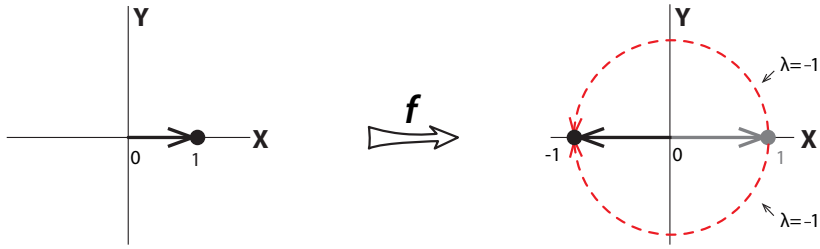


Figure 6.15: The function f, whose eigenvalue is -1 along its eigenvector (which is the **X** axis) flips a positive vector (left) to a negative one (right).

Now think about this function not as a flip, but as a rotation through 180°, *say counterclockwise.* And now let's consider a rotation of 90°, say counterclockwise. What would be the eigenvalue of this 90° rotation? It has the property that applying it twice has the effect of a flip, that is, $\lambda = -1$. But as we saw earlier, if $f(X) = \lambda X$, then the effect of applying f twice is

$$f\Big(f(X)\Big) = \lambda\Big(f(X)\Big) = \lambda\Big(\lambda X\Big) = \lambda^2 X$$

The 90-degree rotation applied twice is the 180° rotation. *So if* λ_{90° *were the eigenvalue of the* 90° *rotation, it would have to have the property that*

$$(\lambda_{90^\circ})^2 = -1$$

That, of course, implies that λ_{90° is imaginary. The equation has two solutions,

$$\lambda_{90^\circ} = \pm \boldsymbol{i}$$

The two solutions $+\boldsymbol{i}$ and $-\boldsymbol{i}$ correspond to the counterclockwise and clockwise rotations (Figure 6.16).

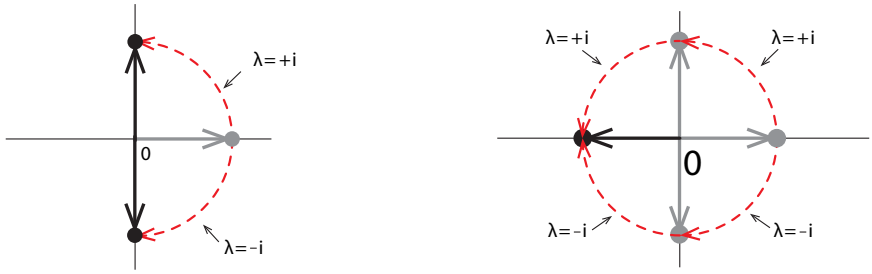


Figure 6.16: Left: the imaginary eigenvalues $\lambda = \pm\boldsymbol{i}$ represent a 90 degree rotation, either clockwise ($\lambda = -\boldsymbol{i}$) or counterclockwise ($\lambda = +\boldsymbol{i}$). Right: applying either rotation twice has the effect of flipping the horizontal vector, that is, multiplying it by -1.

It makes sense that rotation could not have real eigenvalues, because two real eigenvalues would mean that the function could be split into two 1D expansions and contractions. But rotation is an action that is essentially two-dimensional, and therefore indecomposable.

Think about the rotation matrices that we discussed earlier. For example, the matrix

$$\boldsymbol{M} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

represents counterclockwise rotation through the angle θ (Figure 6.17).

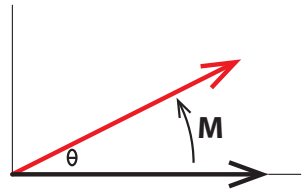


Figure 6.17: The effect of the rotation matrix $\boldsymbol{M}$ is to rotate the black vector counterclockwise by θ, producing the red vector.

What are its eigenvalues? Plugging the matrix coefficients into the characteristic equation (equation (6.3) on page 302),

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-cb)}}{2}$$

we get

$$\lambda = \frac{(2\cos\theta) \pm \sqrt{\Big(2\cos\theta\Big)^2 - 4\Big((\cos\theta)^2 - (-(\sin\theta)^2)\Big)}}{2}$$

But recall that

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

so

$$\lambda = \frac{(\cancel{2}^{1}\cos\theta) \pm \sqrt{\cancel{4}^{1}\Big(\cos\theta\Big)^2 - \cancel{4}^{1}}}{\cancel{2}^{1}} = (\cos\theta) \pm \sqrt{(\cos\theta)^2 - 1}$$
$$= \cos\theta \pm \sqrt{-(\sin\theta)^2}$$
$$= \cos\theta \pm \sin\theta\sqrt{-1}$$

Therefore, the eigenvalues for this rotation matrix consist of a pair of complex conjugate values:

$$\lambda = \cos\theta \pm \sin\theta\, \boldsymbol{i}$$

And when the rotation angle is $\theta = 90°$, the eigenvalues are

$$\lambda_{90°} = \cos 90° \pm \sin 90°\, \boldsymbol{i} = \pm\boldsymbol{i}$$

This confirms our earlier remark that the λ for a 90° rotation would have to be $\lambda = \pm\boldsymbol{i}$.

We can now return to our original example:

$$M = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

We calculated its eigenvalues as

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

which implies that the action of M must be a rotation. We can confirm this by applying M to some random test points.

Note that successive applications of the matrix M bring the point back to its original position after three iterations (Figure 6.18).

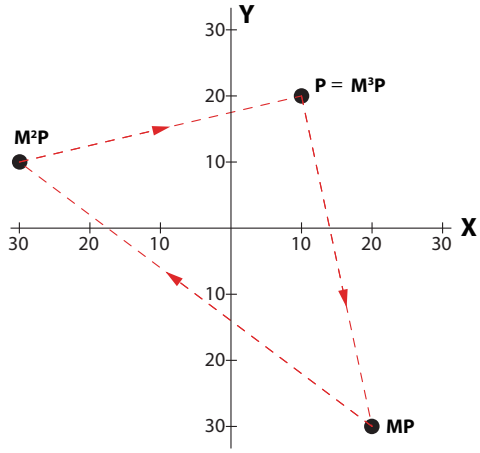


Figure 6.18: Applying the matrix M to the point $\mathbf{p}$ three times brings it back to $\mathbf{p}$.

Exercise 6.4.7 Show that $M^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Exercise 6.4.8 Using the point $\mathbf{p} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ as the test point, apply M three times to calculate $M\mathbf{p}$, $M^2\mathbf{p}$, and $M^3\mathbf{p}$.

Thus, we confirm that complex eigenvalues imply the existence of rotation. To put it another way, what is an eigenvector? It's a vector whose direction is unchanged by the action of M, which merely stretches, contracts, and/or flips it. But obviously, in the action of a rotation, no direction stays the same! So a rotation cannot have real eigenvalues or real eigenvectors.

So we can now give a definite answer to our question, are all matrices diagonalizable? The answer is no. Instead there is a weaker condition that is true: every 2D matrix is either

(1) diagonalizable, which means that it has two real eigenvalues, or
(2) a rotation (possibly together with expansion and/or contraction), which means that it has a pair of complex conjugate eigenvalues.

Eigenvalues in n Dimensions

We have focused so far on 2D linear functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ and used the variables $\mathbf{X}$ and $\mathbf{Y}$ to describe the domain and $\mathbf{U}$ and $\mathbf{V}$ to describe the codomain.

Now we want to study the n-dimensional case, and we will need a new terminology for the variables. We want to consider an n-dimensional linear function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

We will call the domain variables $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and the codomain variables $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$, so

$$f(\mathbf{X}) = \mathbf{Y}$$
$$f(X_1, X_2, \ldots, X_n) = (Y_1, Y_2, \ldots, Y_n)$$

From the definition of linear function, we know that there are constants

$$a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, a_{22}, \ldots, a_{2n}, a_{n1}, a_{n2}, \ldots, a_{nn}$$

such that

$$\begin{aligned} Y_1 &= a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n \\ Y_2 &= a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n \\ &\vdots \quad\quad \vdots \\ Y_n &= a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nn}X_n \end{aligned}$$

so that f is represented by the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

The application of f to the vector $\mathbf{X}$ is then represented by

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}$$

Do n-dimensional linear functions have eigenvalues and eigenvectors? The answer is that the n-dimensional case is remarkably like the 2-dimensional case. We will need some theorems and principles from a linear algebra course or text. We will state them here as we need them; the interested reader is encouraged to look them up for fuller treatment.

The first question is, can we find eigenvalues? Recall that in 2D, we wrote down the equation

$$\boldsymbol{M}\mathbf{E} = \lambda\mathbf{E}$$

where $\boldsymbol{M}$ is the matrix in question and λ and $\mathbf{E}$ are the desired eigenvalue and corresponding eigenvector. In 2D, we wrote this matrix as

$$\boldsymbol{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We then brute-force solved the linear equations and got the characteristic polynomial

$$\lambda^2 + (a+d)\lambda + (ad - cb) = 0$$

In order to generalize this process to n dimensions, we have to go back and restate our argument in more general language. We were looking for eigenvectors and eigenvalues by trying to solve

$$\boldsymbol{M}\mathbf{E} = \lambda\mathbf{E}$$

This is equivalent to saying

$$\boldsymbol{M}\mathbf{E} = (\lambda\mathbb{I})\mathbf{E}$$

where $\mathbb{I}$ is the identity matrix

$$\mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

but that implies

$$\begin{aligned} \boldsymbol{M}\mathbf{E} - (\lambda\mathbb{I})\mathbf{E} &= 0 \\ \implies \quad (\boldsymbol{M} - \lambda\mathbb{I})\mathbf{E} &= 0 \end{aligned}$$

For every matrix, linear algebra defines a quantity, called the *determinant*. The determinant of a matrix is a number that provides certain information about the matrix. Linear algebra defines this number, called $det(\boldsymbol{M})$ or $|\boldsymbol{M}|$, for an arbitrary n-dimensional matrix $\boldsymbol{M}$.

The details of the definition need not concern us here. What is important is two facts about the determinant:

(1) The equation $(\boldsymbol{M} - \lambda\mathbf{I})\mathbf{E} = 0$ has a nontrivial solution if and only if

$$det(\boldsymbol{M} - \lambda\mathbb{I}) = 0$$

(2) the determinant of a 2D matrix $\boldsymbol{M} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is

$$det(\boldsymbol{M}) = a_{11}a_{22} - a_{21}a_{12}$$

We can now redescribe our brute-force derivation of the characteristic polynomial in 2D by realizing that we are looking for solutions to

$$(\boldsymbol{M} - \lambda\mathbb{I})\mathbf{E} = 0$$

Since $\boldsymbol{M}$ is the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the requirement

$$det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

implies

$$\begin{aligned} (a-\lambda)(d-\lambda) - cb &= 0 \\ \implies \quad \lambda^2 + (a+d)\lambda + (ad - cb) &= 0 \end{aligned}$$

which is exactly the characteristic polynomial!

The format $det(\boldsymbol{M}-\lambda\mathbb{I}) = 0$ generalizes to n dimensions: the eigenvalues of the n-dimensional matrix

$$\boldsymbol{M} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

are exactly the solutions to this equation.

The actual calculation of the determinant in higher dimensions is messy and is best left to computer algebra programs, such as SageMath. This is especially true because just as the 2D characteristic polynomial contains a λ^2 term, the n-dimensional characteristic polynomial contains a λ^n term. Solving higher-order polynomial equations is extremely tedious and difficult by hand.

We do know one very important fact, so important that it is sometimes called the fundamental theorem of algebra: An nth-order polynomial equation

$$a_1X^n + a_2X^{n-1} + \cdots + a_n = 0 \qquad \text{(where the } a_1, a_2, \dots, a_n \text{ are real numbers)}$$

has exactly n solutions. Moreover, these solutions are either real or pairs of complex conjugates.

These n solutions are exactly the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the $n \times n$ matrix $\boldsymbol{M}$.

Therefore, an n-dimensional matrix has exactly n eigenvalues, and each of them is either a real number or half of a pair of complex conjugate numbers.

Further Exercises 6.4

1. If $\boldsymbol{M}$ is a 3×3 matrix and $\begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$ is an eigenvector of $\boldsymbol{M}$ with corresponding eigenvalue 5, what is $\boldsymbol{M}\begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$?

2. If $f : \mathbb{R}^4 \to \mathbb{R}^4$ is a linear function and -2 is an eigenvalue of f with corresponding eigenvector $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ -3 \\ -7 \end{pmatrix}$, what is $f(\mathbf{v})$?

3. The matrix $\boldsymbol{A} = \begin{bmatrix} -7 & 3 \\ -18 & 8 \end{bmatrix}$ has an eigenvector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. What is its corresponding eigenvalue?

4. The matrix $\boldsymbol{A} = \begin{bmatrix} 2 & -5 & -4 \\ 0 & 3 & 2 \\ 0 & -4 & -3 \end{bmatrix}$ has an eigenvector $\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$. What is its corresponding eigenvalue?

5. Which of the following are eigenvectors of $\begin{bmatrix} 7 & -5 \\ 10 & -8 \end{bmatrix}$? What are their corresponding eigenvalues?

   a) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ b) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ c) $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ d) $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$

6. Compute the eigenvalues and, if they exist, eigenvectors of the following matrices:

   a) $\begin{bmatrix} 7 & 9 \\ 3 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 2 \\ 4 & 6 \end{bmatrix}$ c) $\begin{bmatrix} 5 & -4 \\ 2 & 0.5 \end{bmatrix}$ d) $\begin{bmatrix} 3 & 4 \\ 2 & -1 \end{bmatrix}$

   e) $\begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix}$ f) $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

7. Compute the eigenvalues of the linear function

$$f\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right) = \begin{pmatrix} 4X - 5Y \\ 2X - 2Y \end{pmatrix}$$

8. One of the eigenvalues of the matrix $\begin{bmatrix} -9 & -8 \\ 12 & 11 \end{bmatrix}$ is 3. What is a corresponding eigenvector for it?

9. One of the eigenvalues of the matrix $\begin{bmatrix} 4 & 5 & -3 \\ 4 & 6 & -4 \\ 8 & 11 & -7 \end{bmatrix}$ is 2. What is a corresponding eigenvector for it?

10. a) Solve for a and b in the equation

$$a\begin{pmatrix} 2 \\ 5 \end{pmatrix} + b\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 14 \end{pmatrix}$$

    b) Use your answer to part (a) to give the *coordinates* of $\begin{pmatrix} 9 \\ 14 \end{pmatrix}$ with respect to the basis $\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

11. Give the coordinates of $\begin{pmatrix} -7 \\ 5 \end{pmatrix}$ with respect to the basis $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, (-1, 2)$.

12. The point of this problem is to demonstrate that if you know all the eigenvalues and eigenvectors of a linear function f (or a matrix $\boldsymbol{M}$), you can compute $f(\mathbf{W})$ (which is $\boldsymbol{M}\mathbf{W}$) for every vector $\mathbf{W}$. In short, knowing all the eigenvalues and eigenvectors is equivalent to knowing the function.

    a) Solve for u and v in the equation

$$u\begin{pmatrix} 2 \\ 5 \end{pmatrix} + v\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 14 \end{pmatrix}$$

(Hint: You will probably want to rewrite this as a system of equations and "solve simultaneously.")

b) Explain what your answer to part (a) means about the *coordinates* of $\begin{pmatrix} 9 \\ 14 \end{pmatrix}$ in some nonstandard coordinate system. *(Hint: Which one?)*

c) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear function and its eigenvectors are as follows:

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix} \text{ with eigenvalue } 2, \text{ and } \begin{pmatrix} -3 \\ 1 \end{pmatrix} \text{ with eigenvalue } -3$$

What is $f(\begin{pmatrix} 2 \\ 5 \end{pmatrix})$? What is $f(\begin{pmatrix} -3 \\ 1 \end{pmatrix})$?

d) Continuing from part (c), what is $f(\begin{pmatrix} 9 \\ 14 \end{pmatrix})$? *(Hint: Use your answers to parts (a) and (c) and the two defining properties of a linear function.)*

13. Diagonalize the following matrices:

a) $\begin{bmatrix} 8 & -3 \\ 10 & -3 \end{bmatrix}$ b) $\begin{bmatrix} 2 & -2 \\ 0 & -1 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

6.5 Linear Discrete-Time Dynamics

We will now develop an application of linear algebra to linear discrete-time dynamical systems. Here $f : \mathbb{R}^n \to \mathbb{R}^n$ is the function that tells us that

$$\mathbf{X}_{N+1} = f(\mathbf{X}_N)$$

In 1D, we saw that the only functions that can pass the stringent test for linearity are the functions $f(X) = kX$, where k is some constant in $\mathbb{R}$. If $k \neq 0$, these functions can equal 0 only once, and that is when $X = 0$. The definition of an equilibrium point for a discrete-time dynamical system is

$$X_{N+1} = X_N$$

But if $X_{N+1} = kX_N$, then this would imply $kX_N = X_N$. If $X_N \neq 0$, then k must equal 1. And $k = 1$ is a very special value that is atypical and to be avoided; note that if $k = 1$, every point is an equilibrium point. As we saw in our discussion of discrete-time dynamical systems ("Discrete-Time Dynamical Systems" in Chapter 5 on page 225), the fact that $f(X) = kX$ can be zero only when $X = 0$ means that the discrete-time system $X_{N+1} = kX_N$ has exactly one equilibrium point, at $X = 0$. As we saw, this equilibrium point is stable if $|k| < 1$, and unstable if $|k| > 1$.

Linear Uncoupled Two-Dimensional Systems

Let's consider the two-dimensional case. To create our first example, we will take two 1D discrete-time systems and join them together into an uncoupled (or decoupled) 2D system. "Uncoupled" means that the growth of X depends only on X, and the growth of Y depends only

on Y:

$$\begin{aligned} X_{N+1} &= \alpha X_N \\ Y_{N+1} &= \beta Y_N \end{aligned} \quad \Longrightarrow \quad \begin{pmatrix} X_{N+1} \\ Y_{N+1} \end{pmatrix} = \begin{pmatrix} \alpha X_N \\ \beta Y_N \end{pmatrix}$$

This can also be written in matrix form:

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}$$

Notice that all the nonzero entries of this matrix are located on the diagonal going from the top left corner to the bottom right (the main diagonal). It's a *diagonal* matrix. If the matrix representing a system of equations is diagonal, the variables in the equations are uncoupled.

So for example, if there are two noninteracting populations, one of which is growing at 40% a year and the other at 20% a year, the system is described by the 2×2 matrix

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} 1.4 & 0 \\ 0 & 1.2 \end{bmatrix}$$

If we begin with an initial condition

$$\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} 50 \\ 50 \end{pmatrix}$$

then the population of the two species in the following year is

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{bmatrix} 1.4 & 0 \\ 0 & 1.2 \end{bmatrix} \begin{pmatrix} 50 \\ 50 \end{pmatrix} = \begin{pmatrix} 1.4 \times 50 + 0 \times 50 \\ 0 \times 50 + 1.2 \times 50 \end{pmatrix} = \begin{pmatrix} 70 \\ 60 \end{pmatrix}$$

If we iterate this matrix repeatedly, we see that if we start at an initial condition of $(X_0, Y_0) = (50, 50)$, the trajectory quickly flattens out, and the growth becomes mostly in the X direction (Figure 6.19). The lesson here is that if a diagonal matrix has unequal growth rates, then the dynamics will be eventually dominated by the larger growth rate. Here the growth rate along the X axis is 40% and the growth rate along the Y axis is 20%, so the dynamics will eventually be dominated by the growth in X.

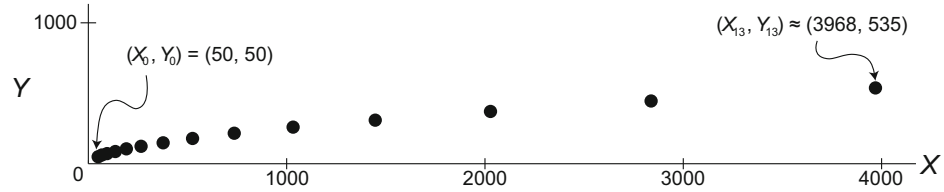


Figure 6.19: Repeated applications of a matrix will result in a trajectory that lies along the direction of the dominant eigenvector. Here both populations are growing.

We can also have declining populations. If one population is growing at 40% a year and the other is declining at 20% a year, the matrix describing the system is

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} 1.4 & 0 \\ 0 & 0.8 \end{bmatrix}$$

If we iterate this matrix repeatedly, we see that there is growth in the X direction and shrinking in the Y direction, and once again, the growth dynamics are eventually dominated by the dimension with the larger growth (Figure 6.20).

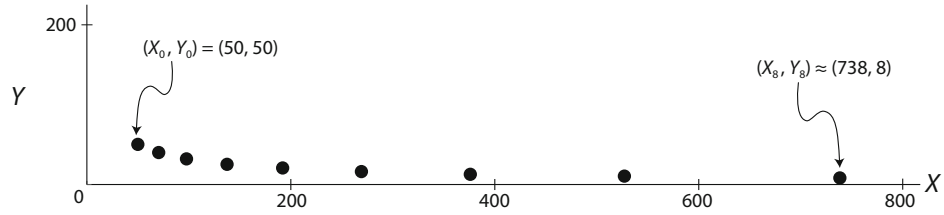


Figure 6.20: When one population is declining, the long-term trajectory still lies along the direction of the dominant eigenvector.

Uncoupled systems are therefore easy to analyze, because the behavior of each variable can be studied separately and the system then reassembled. Each variable is growing or shrinking exponentially, and the overall system behavior is just a combination of the behaviors of the variables making it up. (For simplicity, we use the word "grow" from now on to mean either positive or negative growth.)

Exercise 6.5.1 If there are two noninteracting populations, one of which is growing at 20% a year and the other at 25% a year, derive the matrix that describes the dynamics of the system and simulate a trajectory of this system.

Exercise 6.5.2 If one population is growing at 20% a year and the other is declining at 10% a year. What would be the matrix that describes this system? Draw a trajectory of this system.

Exercise 6.5.3 For the exercise above, plot time series graphs for each population separately to show that it is undergoing exponential growth or decline.

To understand this long-term behavior better, we can examine geometrically how a system's state vector is transformed by a matrix. Let's use the matrix

$$\begin{bmatrix} 1.4 & 0 \\ 0 & 0.8 \end{bmatrix}$$

and apply it to three test vectors $\begin{pmatrix} 50 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 50 \\ 50 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 50 \end{pmatrix}$. We get

$$\begin{bmatrix} 1.4 & 0 \\ 0 & 0.8 \end{bmatrix}\begin{pmatrix} 50 \\ 0 \end{pmatrix} = \begin{pmatrix} 70 \\ 0 \end{pmatrix} \qquad \begin{bmatrix} 1.4 & 0 \\ 0 & 0.8 \end{bmatrix}\begin{pmatrix} 50 \\ 50 \end{pmatrix} = \begin{pmatrix} 70 \\ 40 \end{pmatrix} \qquad \begin{bmatrix} 1.4 & 0 \\ 0 & 0.8 \end{bmatrix}\begin{pmatrix} 0 \\ 50 \end{pmatrix} = \begin{pmatrix} 0 \\ 40 \end{pmatrix}$$

If we plot these (Figure 6.21), we see that if a vector is along the X or Y axis, it just grows or shrinks when multiplied by the matrix. However, a vector in general position is rotated in addition to growing.

There is one more case we have to deal with. So far, all the entries in our matrices have been positive real numbers. We have been thinking of examples in population dynamics, and the only multipliers that make sense in population dynamics are positive real numbers. Suppose, for example, that one of the matrix entries was negative. Then when we applied the matrix to a vector of populations, one of the populations would become negative, which makes no sense in the real world. But in general, state variables can take on any values, positive or negative, and in these cases, negative multipliers make sense.

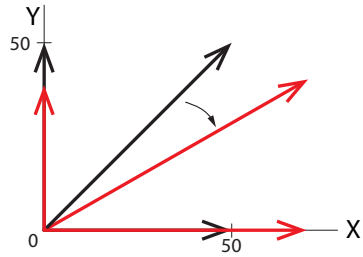


Figure 6.21: Black colors denote the three test vectors. Red colors denote the vectors that result after applying the matrix to these test vectors.

Consider, for example, the matrix

$$\begin{bmatrix} -1.4 & 0 \\ 0 & 0.8 \end{bmatrix}$$

If we begin with an initial condition

$$\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} 50 \\ 50 \end{pmatrix}$$

then the next value is

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{bmatrix} -1.4 & 0 \\ 0 & 0.8 \end{bmatrix} \begin{pmatrix} 50 \\ 50 \end{pmatrix} = \begin{pmatrix} -1.4 \times 50 + 0 \times 50 \\ 0 \times 50 + 0.8 \times 50 \end{pmatrix} = \begin{pmatrix} -70 \\ 40 \end{pmatrix}$$

If we apply the matrix repeatedly, we get a trajectory that flips back and forth between positive and negative X values, since multiplying twice by a negative number gives a positive number. This results in an oscillation. This particular oscillation has a growing amplitude, since $|-1.4| > 1$. At the same time, the dynamics along the Y axis are shrinking, since $0.8 < 1$ (Figure 6.22).

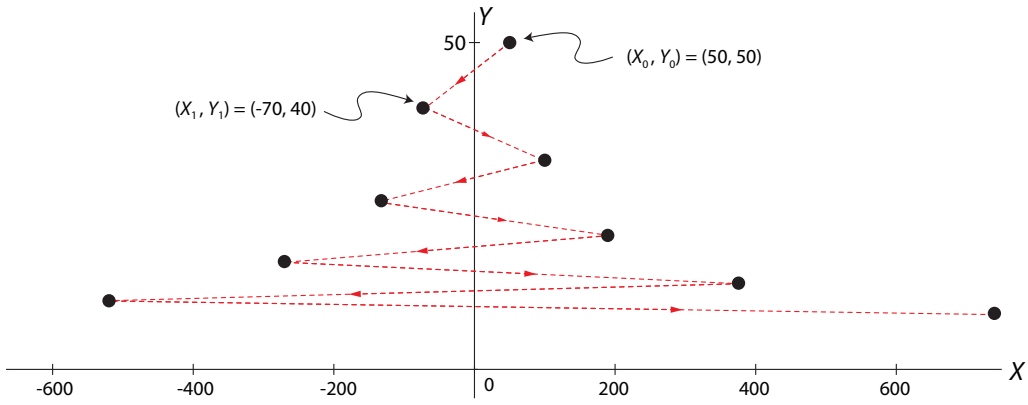


Figure 6.22: When the dominant eigenvalue is negative, repeated applications of the matrix still result in a trajectory that lies along the dominant eigenvector (here the **X** axis), while flipping back and forth between positive and negative X values.

Note that as the number of iterations grows, the trajectory grows flatter and flatter, and it clings more and more to the X axis. Thus the long-term behavior of this system will be dominated by the changes in X, because $|-1.4| > |0.8|$ and -1.4 is the eigenvalue in the X direction.

For every matrix, let's define its **principal eigenvector** as the eigenvector whose eigenvalue has the largest absolute value. (Since these matrices are diagonal, their eigenvalues are simply the matrix entries on the main diagonal, and the corresponding eigenvectors are the X and Y axes.)

We can now make a general statement, which is illustrated by all three examples: *the long-term behavior of an iterated matrix dynamical system is dominated by the principal eigenvalue, and the state point will evolve until its motion lies along the principal eigenvector.*

We can now summarize the behavior of 2D decoupled linear discrete-time systems. These are the systems represented by the matrix

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

They have a unique equilibrium point at $(0, 0)$, and the stability of that equilibrium point is determined by the absolute value of α and β:

- If $|\alpha| > 1$ and $|\beta| > 1$, then the equilibrium point is purely unstable.
- If $|\alpha| < 1$ and $|\beta| < 1$, then the equilibrium point is purely stable.
- If $|\alpha| < 1$ and $|\beta| > 1$ (or the reverse, $|\alpha| > 1$ and $|\beta| < 1$), then the equilibrium point is an unstable saddle point.

Moreover, the signs of α and β determine whether the state point oscillates on its way toward or away from the equilibrium point.

- If $\alpha < 0$, there is oscillation along the X axis.
- If $\beta < 0$, there is oscillation along the Y axis.
- If $\alpha > 0$, there is no oscillation along the X axis.
- If $\beta > 0$, there is no oscillation along the Y axis.

Exercise 6.5.4 By determining the absolute value and the signs of α and β, predict the long-term behavior of the four discrete dynamical systems described by the following matrices:

a) $\begin{bmatrix} -2 & 0 \\ 0 & 0.5 \end{bmatrix}$ b) $\begin{bmatrix} 1.3 & 0 \\ 0 & 0.6 \end{bmatrix}$ c) $\begin{bmatrix} -0.2 & 0 \\ 0 & 0.8 \end{bmatrix}$ d) $\begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix}$

and then verify this prediction by iterating the matrix to simulate the dynamical systems.

Linear Coupled Two-Dimensional Systems

In the more general case, of course, X and Y are coupled: the next X value depends on both the previous X value and the previous Y value, and so does the next Y value. This gives us a matrix

$$\begin{aligned} X_{N+1} &= aX_N + bY_N \\ Y_{N+1} &= cX_N + dY_N \end{aligned} \quad \Longrightarrow \quad \begin{pmatrix} X_{N+1} \\ Y_{N+1} \end{pmatrix} = \begin{pmatrix} aX_N + bY_N \\ cX_N + dY_N \end{pmatrix}$$

which can then be written in the matrix form

$$\begin{pmatrix} X_{N+1} \\ Y_{N+1} \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} X_N \\ Y_N \end{pmatrix}$$

where the off-diagonal entries are not zero.

We have already seen that such a matrix has eigenvalues λ_1 and λ_2. Generically, these completely determine the action of the matrix.

- If λ_1 and λ_2 are real numbers, then there exist eigenvectors $\mathbf{U}$ and $\mathbf{V}$ corresponding to those eigenvalues.
- If an eigenvalue has absolute value less than 1, the matrix shrinks vectors lying along that eigenvector.
- If an eigenvalue has absolute value greater than 1, the matrix expands vectors lying along that eigenvector.
- If an eigenvalue is negative, the action of the matrix is to flip back and forth between negative and positive values along that eigenvector.
- The other case was that λ_1 and λ_2 are a pair of complex conjugate eigenvalues, and then the action of the matrix was a rotation.

We will now use exactly these insights to draw conclusions about matrices as discret-time dynamical systems: to determine the stability of the equilibrium point at $(0, 0)$*, find the eigenvalues of the matrix and infer stability.*

Let's look at some examples.

A Saddle Point: The Black Bear Model

We previously saw a model of black bear populations in which the juvenile and adult populations in the $(N + 1)$st year were given as a linear function of the populations in the Nth year:

$$\begin{aligned} J_{N+1} &= 0.65J_N + 0.5A_N \\ A_{N+1} &= 0.25J_N + 0.9A_N \end{aligned} \quad \Longrightarrow \quad \begin{pmatrix} J_{N+1} \\ A_{N+1} \end{pmatrix} = \begin{pmatrix} 0.65J_N + 0.5A_N \\ 0.25J_N + 0.9A_N \end{pmatrix}$$

where 0.65 is the fraction of juveniles who remain alive as juveniles in the next year, and 0.25 is the fraction of juveniles who mature into adults that year. Furthermore, 0.5 is the birth rate with which adults give birth to juveniles, and 0.9 is the fraction of adults who survive into the next year.

The matrix form is

$$\begin{pmatrix} J_{N+1} \\ A_{N+1} \end{pmatrix} = \begin{bmatrix} 0.65 & 0.5 \\ 0.25 & 0.9 \end{bmatrix} \begin{pmatrix} J_N \\ A_N \end{pmatrix}$$

We saw that if we iterated $\boldsymbol{M}$ repeatedly, the juvenile and adult populations went to infinity (Figure 6.4 on page 292). We will now explain why that is the case by looking at the eigenvalues and corresponding eigenvectors of $\boldsymbol{M}$.

First, let's find the eigenvalues for the matrix

$$\boldsymbol{M} = \begin{bmatrix} 0.65 & 0.5 \\ 0.25 & 0.9 \end{bmatrix}$$

by plugging the matrix coefficients into the characteristic equation (equation (6.3) on page 302):

$$\lambda = \frac{(0.65 + 0.9) \pm \sqrt{(0.65 + 0.9)^2 - 4(0.65 \times 0.9 - 0.25 \times 0.5)}}{2}$$

$$= \frac{1.6 \pm \sqrt{(0.75)^2}}{2} = \frac{1.55 \pm 0.75}{2}$$
$$= (1.15, 0.4)$$

Therefore, the two eigenvalues are

$$\lambda_1 = 1.15 \text{ and } \lambda_2 = 0.4$$

Note that these are real numbers and that $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$. *Therefore, the behavior must have one stable direction and one unstable direction. In other words, it must be a saddle point.*

To find the axes of the saddle point, we will calculate the eigenvectors $\mathbf{U}$ and $\mathbf{V}$ corresponding to each eigenvalue. Let's say that $\mathbf{U} = \begin{pmatrix} J \\ A \end{pmatrix}$. The matrix $\boldsymbol{M}$ acts like multiplication by λ_1 along $\mathbf{U}$, which means that

$$\boldsymbol{M}\mathbf{U} = \lambda_1 \mathbf{U}$$

So we can say

$$\boldsymbol{M}\mathbf{U} = \begin{bmatrix} 0.65 & 0.5 \\ 0.25 & 0.9 \end{bmatrix} \begin{pmatrix} J \\ A \end{pmatrix} = \begin{pmatrix} 0.65J + 0.5A \\ 0.25J + 0.9A \end{pmatrix} = \lambda_1 \mathbf{U} = 1.15 \begin{pmatrix} J \\ A \end{pmatrix} = \begin{pmatrix} 1.15J \\ 1.15A \end{pmatrix}$$

So

$$0.65J + 0.5A = 1.15J \quad \Longrightarrow \quad A = J$$
$$0.25J + 0.9A = 1.15A \quad \Longrightarrow \quad A = J$$

Now $A = J$ is the equation for a line in (J, A) space that has slope $= +1$. This line is the axis $\mathbf{U}$. We can choose any vector on the $\mathbf{U}$ axis to represent it, for example the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is then an eigenvector of the matrix $\boldsymbol{M}$ corresponding to the eigenvalue $\lambda_1 = 1.15$.

The eigenvector corresponding to the second eigenvalue $\lambda_2 = 0.4$ can be found in a similar manner. It satisfies

$$\boldsymbol{M}\mathbf{V} = \lambda_2 \mathbf{V}$$

Let's assume $\mathbf{V} = \begin{pmatrix} J \\ A \end{pmatrix}$. Then

$$\boldsymbol{M}\mathbf{V} = \begin{bmatrix} 0.65 & 0.5 \\ 0.25 & 0.9 \end{bmatrix} \begin{pmatrix} J \\ A \end{pmatrix} = \begin{pmatrix} 0.65J + 0.5A \\ 0.25J + 0.9A \end{pmatrix} = \lambda_2 \mathbf{V} = 0.4 \begin{pmatrix} J \\ A \end{pmatrix} = \begin{pmatrix} 0.4J \\ 0.4A \end{pmatrix}$$

So

$$0.65J + 0.5A = 0.4J \quad \Longrightarrow \quad A = -0.5J$$
$$0.25J + 0.9A = 0.4A \quad \Longrightarrow \quad A = -0.5J$$

The equation $A = -0.5J$ is the equation for a line in (J, A) space that has slope $= -0.5$. This line is the axis $\mathbf{V}$. We can choose any vector on the $\mathbf{V}$ axis to represent it, for example the vector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, which is then an eigenvector of the matrix $\boldsymbol{M}$ corresponding to the eigenvalue $\lambda_2 = 0.4$.

If we plot these eigenvectors and choose a point, let's say

$$\begin{pmatrix} J_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} 10 \\ 50 \end{pmatrix}$$

as our initial condition and apply the matrix on this vector once, we get

$$\begin{pmatrix} J_1 \\ A_1 \end{pmatrix} = \begin{bmatrix} 0.65 & 0.5 \\ 0.25 & 0.9 \end{bmatrix} \begin{pmatrix} 10 \\ 50 \end{pmatrix} = \begin{pmatrix} 0.65 \times 10 + 0.5 \times 50 \\ 0.25 \times 10 + 0.9 \times 50 \end{pmatrix} = \begin{pmatrix} 31.5 \\ 47.5 \end{pmatrix}$$

We see that the action of this matrix is to push the state point closer to the **U** axis while moving away from the **V** axis. Thus, for this initial condition, the action of the matrix is to increase the number of juveniles and decrease the number of adults in the first year (Figure 6.23).

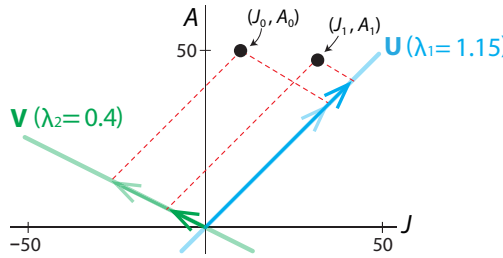


Figure 6.23: One application of the matrix $\boldsymbol{M}$ to the point (J_0, A_0) takes it to (J_1, A_1) which is closer to the dominant eigenvector **U** axis and further from the **V** axis.

If we iterate the matrix many times from two different initial conditions, we see that successive points march toward the **U** axis and out along it. Since the **U** axis is the line $A = J$, we can say that the populations of the two age groups approach a 1 : 1 ratio, while the whole population grows larger and larger (Figure 6.24).

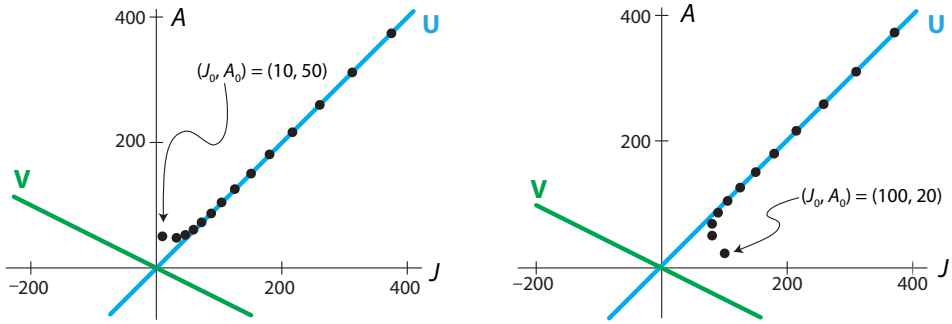


Figure 6.24: After repeated iterations of the matrix $\boldsymbol{M}$, the long-term trajectory lies along the direction of the dominant eigenvector **U** axis, regardless of the initial conditions. Eventually, the ratio of adults to juveniles approaches a constant value.

Finally, our theoretical prediction of "saddle point" can be confirmed by applying the matrix repeatedly to a set of initial conditions lying on a circle. In this way, we can construct a graphical picture of the action of $\boldsymbol{M}$. We see that the action is to squeeze along the **V** axis and expand along the **U** axis (Figure 6.25).

Notice an interesting feature of Figure 6.25. We started with a circle of initial conditions, but by the fifth iteration, the original circle had flattened nearly into a line, and **that line was lying along the principal eigenvector.**

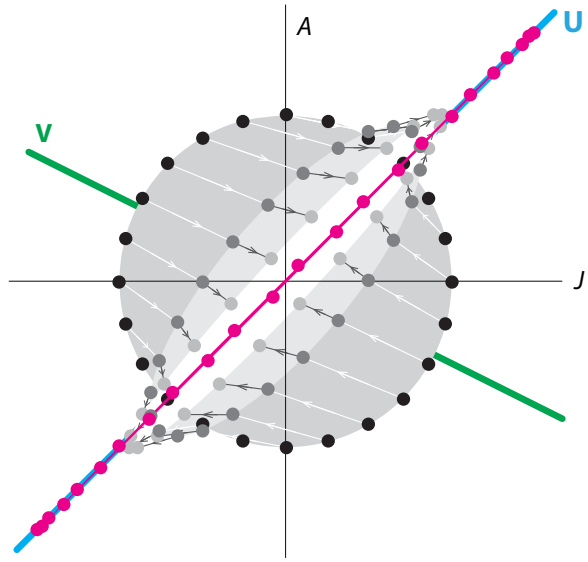


Figure 6.25: One application of the *J*-*A* matrix to a circle of initial conditions (black dots) transforms them into an oval (dark gray dots). Applying the matrix for the second time, it flattens the oval even further and expands it along the **U** axis (light gray dots). By the fifth iteration (red dots), the initial circle has been transformed into a line lying along the principal eigenvector and expanding in that direction.

A Stable Equilibrium Point: Black Bear in a Bad Year

Let's consider the alternative scenario for the black bear, in a bad year.

We modeled "bad year" by lowering the birth rate from 0.5 to 0.4, and increasing the death rate for juveniles to 40%, with 50% of them remaining juvenile and only 10% maturing to adults. The juvenile population dynamics are

$$J_{N+1} = 0.5J_N + 0.4A_N$$

We also increased the adult death rate to 20%, and therefore, the survival rate will be $1 - 20\% = 80\%$. The adult population dynamics are therefore

$$A_{N+1} = 0.1J_N + 0.8A_N$$

Putting these together, we get

$$\begin{pmatrix} J_{N+1} \\ A_{N+1} \end{pmatrix} = \begin{pmatrix} 0.5J_N + 0.4A_N \\ 0.1J_N + 0.8A_N \end{pmatrix}$$

The matrix that describes the "bad year" dynamics is

$$\boldsymbol{M_{bad}} = \begin{bmatrix} 0.5 & 0.4 \\ 0.1 & 0.8 \end{bmatrix}$$

Recall that when we iterated $\boldsymbol{M_{bad}}$ repeatedly, both juvenile and adult populations appeared to go to extinction (Figure 6.5 on page 292). We can explain this long-term behavior by studying the eigenvalues and corresponding eigenvectors of $\boldsymbol{M_{bad}}$.

What are the dynamics of this system? First, let's find the eigenvalues for the matrix by plugging the matrix coefficients into the characteristic equation

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-cb)}}{2}$$

We get

$$\begin{aligned}\lambda &= \frac{(0.5+0.8) \pm \sqrt{(0.5+0.8)^2 - 4(0.5\times 0.8 - 0.1\times 0.4)}}{2} \\ &= \frac{1.3 \pm \sqrt{(0.25)}}{2} = \frac{1.3 \pm 0.5}{2} \\ &= (0.9, 0.4)\end{aligned}$$

Therefore, the two eigenvalues are

$$\lambda_1 = 0.9 \text{ and } \lambda_2 = 0.4$$

Note that these are real numbers and that both $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$. *Therefore, the behavior must have two stable directions. In other words, it must be a purely stable node.*

To find the axes of the node, we will calculate the eigenvectors $\mathbf{U}$ and $\mathbf{V}$ corresponding to each eigenvalue. Let's say $\mathbf{U} = \begin{pmatrix} J \\ A \end{pmatrix}$. The matrix $\boldsymbol{M_{bad}}$ acts like multiplication by λ_1 along $\mathbf{U}$, which means that

$$\boldsymbol{M_{bad}}\ \mathbf{U} = \lambda_1 \mathbf{U}$$

So we can say

$$\boldsymbol{M_{bad}}\mathbf{U} = \begin{bmatrix} 0.5 & 0.4 \\ 0.1 & 0.8 \end{bmatrix} \begin{pmatrix} J \\ A \end{pmatrix} = \begin{pmatrix} 0.5J + 0.4A \\ 0.1J + 0.8A \end{pmatrix} = \lambda_1 \mathbf{U} = 0.9 \begin{bmatrix} J \\ A \end{bmatrix} = \begin{pmatrix} 0.9J \\ 0.9A \end{pmatrix}$$

So

$$\begin{aligned} 0.5J + 0.4A = 0.9J \quad &\Longrightarrow \quad A = J \\ 0.1J + 0.8A = 0.9A \quad &\Longrightarrow \quad A = J \end{aligned}$$

Now "$A = J$" is the equation for the line in (J, A) space that has slope $= +1$. This line is the axis $\mathbf{U}$. We can choose any vector on the $\mathbf{U}$ axis to represent it, for example the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is then an eigenvector of the matrix $\boldsymbol{M_{bad}}$ corresponding to the eigenvalue $\lambda_1 = 0.9$.

The eigenvector corresponding to the second eigenvalue $\lambda_2 = 0.4$ can be found in a similar manner. It satisfies

$$\boldsymbol{M_{bad}}\ \mathbf{V} = \lambda_2 \mathbf{V}$$

Let's assume $\mathbf{V} = \begin{pmatrix} J \\ A \end{pmatrix}$. Then

$$\boldsymbol{M_{bad}}\mathbf{V} = \begin{bmatrix} 0.5 & 0.4 \\ 0.1 & 0.8 \end{bmatrix} \begin{pmatrix} J \\ A \end{pmatrix} = \begin{pmatrix} 0.5J + 0.4A \\ 0.1J + 0.8A \end{pmatrix} = \lambda_2 \mathbf{V} = 0.4 \begin{pmatrix} J \\ A \end{pmatrix} = \begin{pmatrix} 0.4J \\ 0.4A \end{pmatrix}$$

So

$$\begin{aligned} 0.5J + 0.4A = 0.4J \quad &\Longrightarrow \quad A = -0.25J \\ 0.1J + 0.8A = 0.4A \quad &\Longrightarrow \quad A = -0.25J \end{aligned}$$

The equation $A = -0.25J$ is the equation for the line in (J, A) space that has slope $= -0.25$. This line is the axis $\mathbf{V}$. We can choose any vector on the $\mathbf{V}$ axis to represent it, for example the

vector $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$, which is then an eigenvector of the matrix $\boldsymbol{M_{bad}}$ corresponding to the eigenvalue $\lambda_2 = 0.4$.

If we plot these eigenvectors and choose a point

$$\begin{pmatrix} J_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} 10 \\ 50 \end{pmatrix}$$

as our initial condition and apply the matrix to this vector once, we get the population of the two age groups in the next year:

$$\begin{pmatrix} J_1 \\ A_1 \end{pmatrix} = \begin{bmatrix} 0.5 & 0.4 \\ 0.1 & 0.8 \end{bmatrix} \begin{pmatrix} 10 \\ 50 \end{pmatrix} = \begin{pmatrix} 0.5 \times 10 + 0.4 \times 50 \\ 0.1 \times 10 + 0.8 \times 50 \end{pmatrix} = \begin{pmatrix} 25 \\ 41 \end{pmatrix}$$

We see that the action of this matrix is to push the state point closer to the **U** axis while moving away from the **V** axis. So the action of $\boldsymbol{M_{bad}}$ is to move the state point toward the **U** axis, but in contrast to the good year case, $\boldsymbol{M_{bad}}$ moves the state point to a *lower* **V**-value (Figure 6.26).

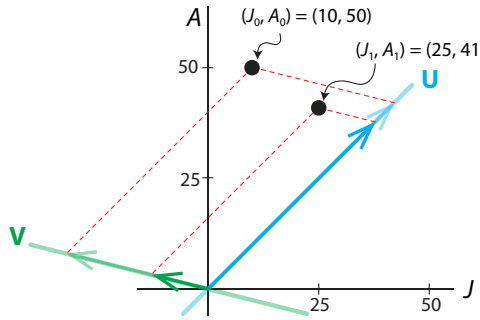


Figure 6.26: One application of the matrix $\boldsymbol{M_{bad}}$ to the point (J_0, A_0) takes it to (J_1, A_1) which is closer to both the **U** and **V** axes.

If we iterate $\boldsymbol{M_{bad}}$ repeatedly, the state point always walks toward the **U** axis while approaching $(0, 0)$, which means extinction (Figure 6.27).

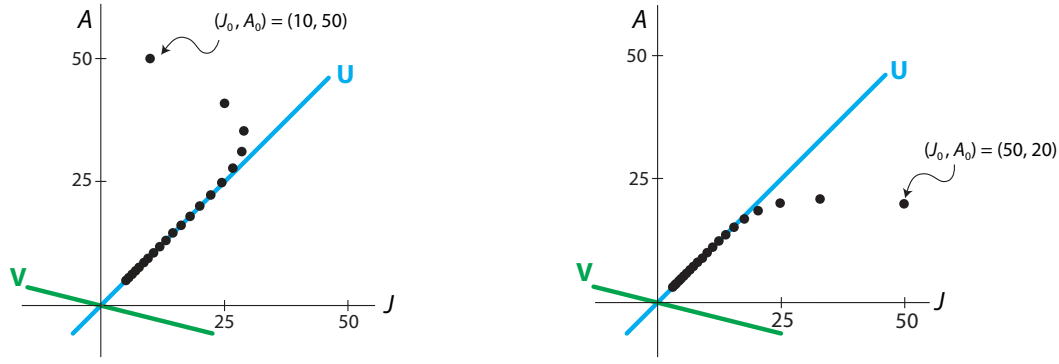


Figure 6.27: After repeated iterations of the matrix $\boldsymbol{M_{bad}}$, the ratio of adults to juveniles is approaching a constant value, regardless of the initial conditions. Notice that the both populations are decreasing.

Finally, we confirm our theoretical prediction of "stable node" by applying the $\boldsymbol{M_{bad}}$ matrix repeatedly to a set of initial conditions that lie on a circle. The effect is to collapse the circle onto the **U** axis along the direction of the **V** axis while shrinking along the **U** axis. The overall effect is to shrink the circle to the point $(0, 0)$ (Figure 6.28).

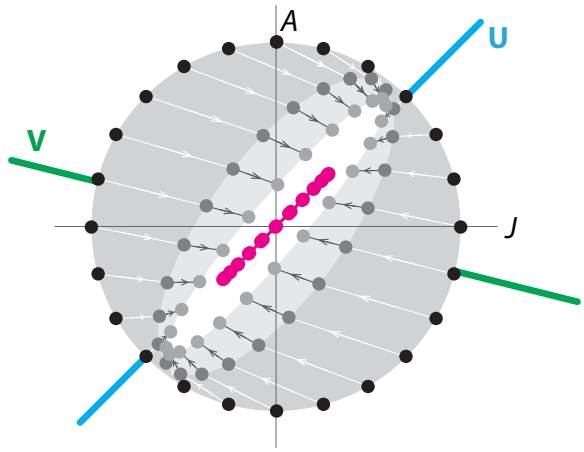


Figure 6.28: One application of the $\boldsymbol{M_{bad}}$ matrix to a circle of initial conditions (black dots) transforms them into an oval (dark gray dots). Applying the matrix for the second time, it flattens the oval even further and shrinks it along the **U** axis (light gray dots). By the fifth iteration (red dots), the initial circle has been transformed into a line lying along the principal eigenvector and continually shrinking along that direction.

Stable Equilibrium Point with Oscillatory Approach

We also simulated another Leslie matrix for a two-stage population. In this case, 10% of juveniles remain juvenile, 40% become adults, and the rest die. The birth rate is 1.4 offspring per adult, and only 20% of adults survive each year. This gives us the matrix

$$\boldsymbol{M_{osc}} = \begin{bmatrix} 0.1 & 1.4 \\ 0.4 & 0.2 \end{bmatrix}$$

Repeated iteration of $\boldsymbol{M_{osc}}$ resulted in an oscillatory approach to a stable equilibrium point at $(0, 0)$ (Figure 6.6 on page 293). We can understand this behavior by considering the eigenvalues and corresponding eigenvectors of $\boldsymbol{M_{osc}}$.

First, let's find the eigenvalues for the matrix by plugging the matrix coefficients into the characteristic equation

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-cb)}}{2}$$

We get

$$\begin{aligned} \lambda &= \frac{(0.1+0.2) \pm \sqrt{(0.1+0.2)^2 - 4(0.1 \times 0.2 - 0.4 \times 1.4)}}{2} \\ &= \frac{0.3 \pm \sqrt{(2.25)}}{2} = \frac{0.3 \pm 1.5}{2} \\ &= (0.9, -0.6) \end{aligned}$$

Therefore, the two eigenvalues are

$$\lambda_1 = 0.9 \text{ and } \lambda_2 = -0.6$$

These two eigenvalues are both real, and both have absolute value less than 1*, so we know that the equilibrium point is stable.* To find the axes of the equilibrium point, we need to find the corresponding eigenvectors.

First

$$\boldsymbol{M_{osc}}\mathbf{U} = \lambda_1 \mathbf{U}$$

We can say that

$$\boldsymbol{M_{osc}}\mathbf{U} = \begin{bmatrix} 0.1 & 1.4 \\ 0.4 & 0.2 \end{bmatrix} \begin{pmatrix} J \\ A \end{pmatrix} = \begin{pmatrix} 0.1J + 1.4A \\ 0.4J + 0.2A \end{pmatrix} = \lambda_1 \mathbf{U} = 0.9 \begin{pmatrix} J \\ A \end{pmatrix} = \begin{pmatrix} 0.9J \\ 0.9A \end{pmatrix}$$

This gives us

$$\begin{aligned} 0.1J + 1.4A = 0.9J \quad &\Longrightarrow \quad A = 1.75J \\ 0.4J + 0.2A = 0.9A \quad &\Longrightarrow \quad A = 1.75J \end{aligned}$$

which implies that the eigenvector $\mathbf{U}$ lies on the line $A = 1.75J$, which has slope 1.75. The vector $(J, A) = (4, 7)$ will serve nicely as an eigenvector on this line.

For the second eigenvector, we solve

$$\boldsymbol{M_{osc}}\mathbf{V} = \lambda_2 \mathbf{V}$$

We can say that

$$\boldsymbol{M_{osc}}\boldsymbol{V} = \begin{bmatrix} 0.1 & 1.4 \\ 0.4 & 0.2 \end{bmatrix} \begin{pmatrix} J \\ A \end{pmatrix} = \begin{pmatrix} 0.1J + 1.4A \\ 0.4J + 0.2A \end{pmatrix} = \lambda_2 \mathbf{U} = -0.6 \begin{pmatrix} J \\ A \end{pmatrix} = \begin{pmatrix} -0.6J \\ -0.6A \end{pmatrix}$$

yielding

$$\begin{aligned} 0.1J + 1.4A = -0.6J \quad &\Longrightarrow \quad A = -0.5J \\ 0.4J + 0.2A = -0.6A \quad &\Longrightarrow \quad A = -0.5J \end{aligned}$$

The second eigenvector is therefore any vector on the line $A = -0.5J$, which is the line of slope -0.5. For example, we can take $(J, A) = (2, -1)$ as our eigenvector $\mathbf{V}$.

Having calculated the eigenvalues and the eigenvectors, we can now make the theoretical prediction that this matrix will shrink slowly *along* $\mathbf{U}$ and collapse more quickly *toward* the $\mathbf{U}$ axis in an oscillating manner. The presence of a negative eigenvalue means that the matrix will flip the state point back and forth on either side of the $\mathbf{U}$ axis. This flipping will occur with ever-decreasing amplitudes, since $|\lambda_2| < 1$.

Let's verify these predictions by applying the matrix to a test point (Figure 6.29). We see, exactly as predicted, that the state point oscillates around the $\mathbf{U}$ axis with diminishing amplitude as it approaches the origin.

Finally, if we apply the matrix repeatedly to a circle of initial conditions, we see that the first iteration has flattened the circle into an oval, which is pointing below the $\mathbf{U}$ axis. The second iteration flattens and shrinks the oval further and tilts it upward, so that it is pointing above

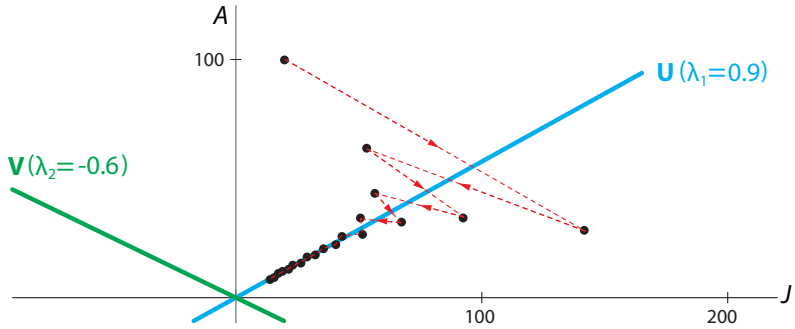


Figure 6.29: Iteration of the matrix $\boldsymbol{M_{osc}}$ causes the state point to diminish continually along the **U** axis, while also diminishing along the **V** axis, but in an oscillatory manner.

the **U** axis, while the third iteration further shrinks and flattens the oval and tilts it back to point below the **U** axis. The oscillatory tilt above and below the **U** axis is caused by the negative eigenvalue along the **V** direction (Figure 6.30).

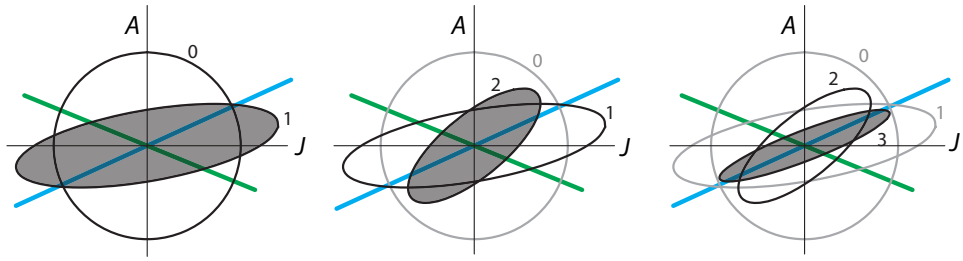


Figure 6.30: Starting with a circle of initial conditions (0), repeated action of the matrix $\boldsymbol{M_{osc}}$ flattens the circle into an ellipse (1), and flips the ever-flattening ellipse back and forth across the **U** axis, in a diminishing manner (2, 3).

Thus the overall behavior is an oscillatory approach to the stable equilibrium point at $(0, 0)$, so both populations shrink to zero.

Unstable Equilibrium Point with Oscillatory Departure

In the previous example of $\boldsymbol{M_{osc}}$, the black bear population collapse is due partly to the low birth rate of 1.4 offspring per adult. If we raise this birth rate to 2 offspring per adult, we get the matrix

$$\boldsymbol{M_{osc2}} = \begin{bmatrix} 0.1 & 2 \\ 0.4 & 0.2 \end{bmatrix}$$

and this new system has a distinctly different behavior. Now we have an unstable oscillatory equilibrium point (Figure 6.31).

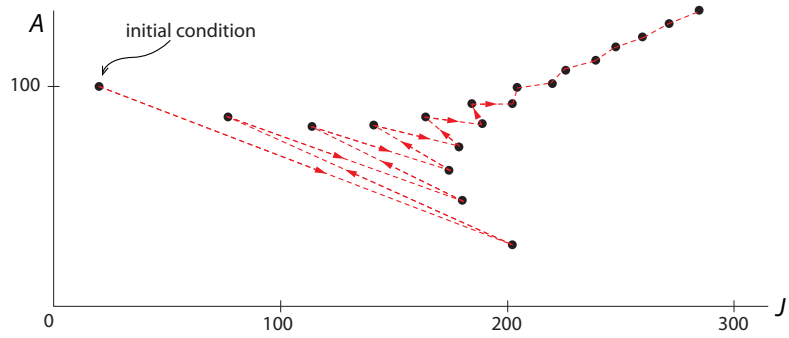


Figure 6.31: Iteration of the matrix $\boldsymbol{M_{osc2}}$ results in a trajectory that is oscillatory/stable in one direction, and expanding (unstable) in the other.

Exercise 6.5.5 Calculate the eigenvalues and eigenvectors of this matrix with increased birth rate, and use them to explain the behavior in Figure 6.31.

Neutral Equilibria: Markov Processes, and an Example in Epidemiology

We modeled the susceptible and infected populations in an epidemic, using a Markov process model ("Neutral Equilibria" on page 293). We saw that when we iterated the matrix $\boldsymbol{M_{SI}}$ repeatedly, we observed that the system would go to a stable equilibrium, but the equilibrium depended on the initial condition (Figure 6.7 on page 294). We can explain why this occurs by studying the eigenvalues and corresponding eigenvectors of $\boldsymbol{M_{SI}}$. We will see that in Markov process models, there is always an eigenvalue $\lambda = 1$ that gives us *a line of equilibrium points* along its corresponding eigenvector.

As before, the discrete-time dynamics for this S-I compartmental model is written in matrix form as

$$\begin{pmatrix} S_{N+1} \\ I_{N+1} \end{pmatrix} = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix} \begin{pmatrix} S_N \\ I_N \end{pmatrix}$$

We made the assumption that at each time point (such as day, week, or month), a constant fraction a of the susceptibles become infected and a constant fraction b of the infecteds recover to become susceptible again. If a is the fraction of S that become I, then the fraction of S that remain S must be $1 - a$. If b is the fraction of I that become S, then the fraction of I that remain I must be $1 - b$.

We chose $a = 0.1$ and $b = 0.2$, giving us the matrix

$$\boldsymbol{M_{SI}} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$

What are the dynamics of this system? Let's find the eigenvalues for this matrix by plugging the matrix coefficients into the characteristic equation (equation (6.3) on page 302); we get

$$\begin{aligned} \lambda &= \frac{(0.9+0.8) \pm \sqrt{(0.9+0.8)^2 - 4(0.9 \times 0.8 - 0.1 \times 0.2)}}{2} \\ &= \frac{1.7 \pm \sqrt{0.09}}{2} = \frac{1.7 \pm 0.3}{2} \\ &= (1, 0.7) \end{aligned}$$

Therefore, the two eigenvalues are

$$\lambda_1 = 1 \text{ and } \lambda_2 = 0.7$$

To find their corresponding eigenvectors $\mathbf{U}$ and $\mathbf{V}$, let's say $\mathbf{U} = \begin{pmatrix} S \\ I \end{pmatrix}$. The matrix $\boldsymbol{M_{SI}}$ acts like multiplication by λ_1 along $\mathbf{U}$. This means that

$$\boldsymbol{M_{SI}}\mathbf{U} = \lambda_1 \mathbf{U}$$

$$\boldsymbol{M_{SI}}\mathbf{U} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} 0.9S + 0.2I \\ 0.1S + 0.8I \end{pmatrix} = \lambda_1 \mathbf{U} = 1 \begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} S \\ I \end{pmatrix}$$

So

$$\begin{aligned} 0.9S + 0.2I = S \quad &\Longrightarrow \quad I = 0.5S \\ 0.1S + 0.8I = I \quad &\Longrightarrow \quad I = 0.5S \end{aligned}$$

Now $I = 0.5S$ is the equation of a line in (S, I) space that has slope 0.5. This line is the axis $\mathbf{U}$. We can choose any vector on the $\mathbf{U}$ axis to represent it, for example the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, which is then called an **eigenvector** of the matrix $\boldsymbol{M_{SI}}$ corresponding the eigenvalue $\lambda_1 = 1$.

The eigenvector corresponding to the second eigenvalue $\lambda_2 = 0.7$ can be found in a similar manner. It satisfies

$$\boldsymbol{M_{SI}}\mathbf{V} = \lambda_2 \mathbf{V}$$

Let's assume $\mathbf{V} = \begin{pmatrix} S \\ I \end{pmatrix}$. Then

$$\boldsymbol{M_{SI}}\mathbf{V} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} 0.9S + 0.2I \\ 0.1S + 0.8I \end{pmatrix} = \lambda_2 \mathbf{V} = 0.7 \begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} 0.7S \\ 0.7I \end{pmatrix}$$

So

$$\begin{aligned} 0.9S + 0.2I = 0.7S \quad &\Longrightarrow \quad I = -S \\ 0.1S + 0.8I = 0.7I \quad &\Longrightarrow \quad I = -S \end{aligned}$$

Since $I = -S$ is the equation of a line in (S, I) space that has slope $= -1$, this line is the axis $\mathbf{V}$. We can choose any vector on the $\mathbf{U}$ axis to represent it, for example the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which is then called an **eigenvector** of the matrix $\boldsymbol{M_{SI}}$ corresponding the eigenvalue $\lambda_2 = 0.7$.

The action of $\boldsymbol{M_{SI}}$. The matrix acts as multiplication by $\lambda_1 = 1$ along $\mathbf{U}$, and it acts as multiplication by $\lambda_2 = 0.7$ along $\mathbf{V}$. The problem comes when we try to say whether the point $(0, 0)$ is stable or unstable. Along the $\mathbf{V}$ eigenvector, it has $|\lambda_2| = 0.7 < 1$, so it is clearly stable in the $\mathbf{V}$ direction. But in the $\mathbf{U}$ direction, it is neither expanding nor shrinking! The eigenvalue $\lambda_1 = 1$ along the $\mathbf{U}$ direction means that *every point on* $\mathbf{U}$ *is an equilibrium point*.

According to this analysis, the action of the matrix $\boldsymbol{M_{SI}}$ on a point should be to compress it along $\mathbf{V}$ axis and leave it unchanged (that is, multiplied by $\lambda_1 = 1$) along the $\mathbf{U}$ axis.

This prediction is confirmed by some experiments with the matrix $\boldsymbol{M_{SI}}$.

If we start with an initial condition

$$\begin{pmatrix} S_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} 50 \\ 50 \end{pmatrix}$$

and apply the matrix to this vector once, we get

$$\begin{pmatrix} S_1 \\ I_1 \end{pmatrix} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{pmatrix} 50 \\ 50 \end{pmatrix} = \begin{pmatrix} 0.9 \times 50 + 0.2 \times 50 \\ 0.2 \times 50 + 0.8 \times 50 \end{pmatrix} = \begin{pmatrix} 55 \\ 45 \end{pmatrix}$$

If we decompose this initial condition along the directions of the two eigenvectors, we get the **U**-component and the **V**-component. The action of the matrix has no effect on the **U**-component, and it shrinks the **V**-component to 70% of its previous value (Figure 6.32, left). If we now apply $\boldsymbol{M_{SI}}$ repeatedly, we see that the overall effect is to walk the point down along the **V** direction toward the **U** axis (Figure 6.32, right).

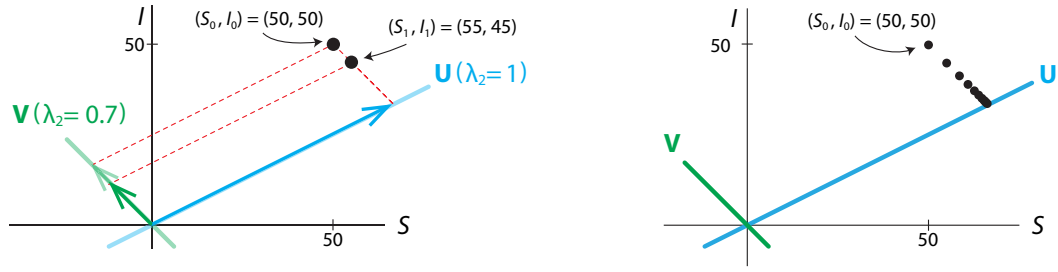


Figure 6.32: Left: Application of the S-I matrix $\boldsymbol{M_{SI}}$ to the initial condition (S_0, I_0) results in the state point (S_1, I_1), closer to the **U** axis but at a constant distance from the **V** axis. Right: Repeated applications of $\boldsymbol{M_{SI}}$ approach the **U** axis while remaining a constant distance from **V**.

Indeed, if we start with any initial condition on the line parallel to the **V** axis passing through $(50, 50)$, the dynamical system will converge to the same equilibrium point. For example, if we take an initial condition on the other side of the **U** axis, say $(90, 10)$, we see that the action of the matrix is to walk the point *up* along the **V** direction toward the **U** axis (Figure 6.33).

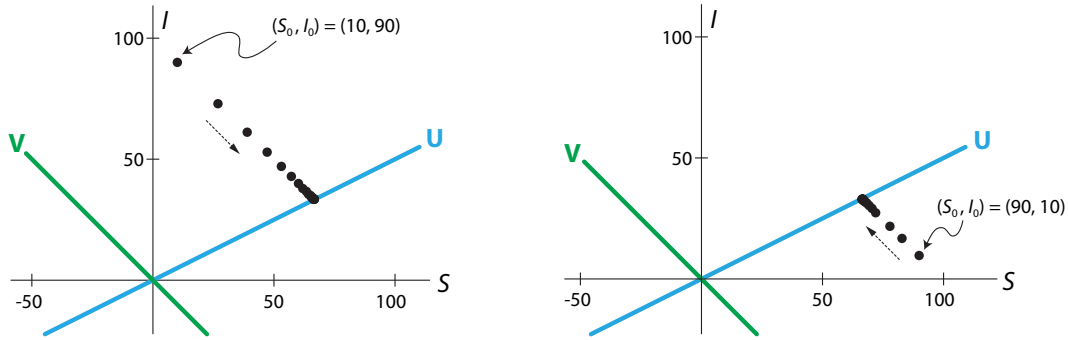


Figure 6.33: The **U** axis is a line of stable equilibrium points for the matrix $\boldsymbol{M_{SI}}$. Any initial condition on a given line parallel to the **V** axis will approach the same equilibrium point on the **U** axis.

Thus it is clear from both theoretical prediction and experiments that it is only the **U** component of the initial condition that determines the final equilibrium point.

Therefore, if we start from an initial condition along a different line, say (10, 60), we see that the action of $\boldsymbol{M_{SI}}$ is to walk the state point toward a different equilibrium point on the **U** axis (Figure 6.34).

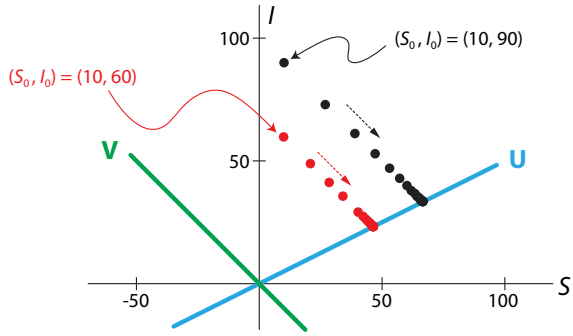


Figure 6.34: Two trajectories (red and black) starting from different initial conditions that do not lie on the same line parallel to the **V** axis, will both approach the **U** axis but toward different equilibrium points.

An effective way to visualize the action of any matrix $\boldsymbol{M}$ is to take a large number of initial conditions in a circle and look at what repeated iterations of $\boldsymbol{M}$ do to the circle.

When we make this plot for the *S*-*I* matrix, we see that the action of $\boldsymbol{M_{SI}}$ is to flatten the circle into an oval. If we apply $\boldsymbol{M_{SI}}$ repeatedly, the oval gets thinner and thinner and shifts its axis slightly until it begins to resemble a thick flat line lying exactly along the **U** axis (Figure 6.35).

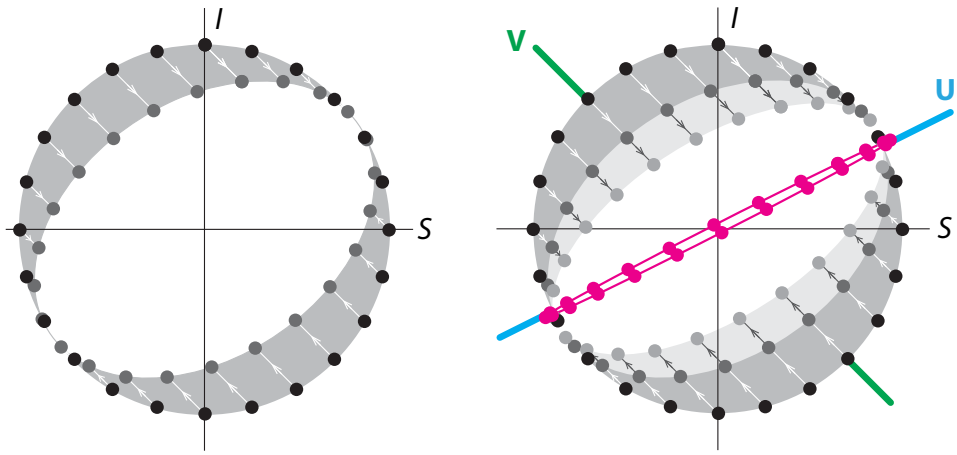


Figure 6.35: Left: one application of the *S*-*I* matrix to a circle of initial conditions (dark gray) transforms them into a oval. Right: repeated applications flatten and rotate the oval. By the tenth iteration (red dots), the initial circle has been transformed into a line lying along the principal eigenvector.

Thus, we see here again the fact that repeated iteration of a matrix from any set of initial conditions results in a thin oval whose principal axis moves closer and closer to the principal eigenvector. Finally, for a large number of iterations, the resulting structure resembles a line, a thin finger pointing along the principal eigenvector. And so once again, *when you iterate a matrix many times, you are looking at its principal eigenvector.*

Markov processes Note that in this case, there are no births or deaths; the number of people remains constant. Therefore, the sum of the entries in each column of the matrix must be equal to 1, because each person in the compartment must go somewhere.

$$\begin{bmatrix} \underbrace{\begin{matrix}\text{fraction of}\\ S \text{ who remain } S\\ +\\ \text{fraction of}\\ S \text{ who become } I\end{matrix}}_{\Sigma=1} & \underbrace{\begin{matrix}\text{fraction of}\\ I \text{ who become } S\\ +\\ \text{fraction of}\\ I \text{ who remain } I\end{matrix}}_{\Sigma=1} \end{bmatrix} \qquad \begin{bmatrix} \underbrace{\begin{matrix}1\text{-}a\\ +\\ a\end{matrix}}_{\Sigma=1} & \underbrace{\begin{matrix}b\\ +\\ 1\text{-}b\end{matrix}}_{\Sigma=1} \end{bmatrix}$$

A matrix whose columns all add up to 1 is called a *stochastic matrix*. It's called "stochastic" (which means involving chance or probability) because we can interpret the matrix entries as transition probabilities from one compartment to another.

We can imagine a large number of particles, in this case people, hopping from one compartment to another, with hopping probabilities given by the elements of the matrix. Every matrix of transition probabilities like this one will have the property that the columns all add to 1, because probabilities must add to 1. When we interpret the matrix as a matrix of transition probabilities, the process is called a Markov process.

In all such processes, $\lambda = 1$ will always be an eigenvalue, and hence all equilibria are neutral equilibria. In a neutral equilibrium system, the behavior will always be to go to a stable final state, but the stable final state depends on the initial condition.

Neutral Oscillations from the Locust Model

We saw that the three-variable locust model consists of three stages: eggs (E), hoppers (juveniles) (H), and adults (A) (Bodine et al. 2014). The egg and hopper stages each last one year, with 2% of eggs surviving to become hoppers and 5% of hoppers surviving to become adults. Adults lay 1000 eggs (as before, we are modeling only females) and then die. The model was

$$\begin{aligned} E_{N+1} &= 0 \cdot E_N + 0 \cdot H_N + 1000 A_N \\ H_{N+1} &= 0.02 E_N + 0 \cdot H_N + 0 \cdot A_N \\ A_{N+1} &= 0 \cdot E_N + 0.05 H_N + 0 \cdot A_N \end{aligned} \quad \Longrightarrow \quad \boldsymbol{L} = \begin{bmatrix} 0 & 0 & 1000 \\ 0.02 & 0 & 0 \\ 0 & 0.05 & 0 \end{bmatrix}$$

We saw that the model gave us neutral oscillations, which depended on the initial conditions (Figure 6.8 on page 295). We can confirm this by plotting the trajectory of repeated applications of $\boldsymbol{L}$ to two different initial conditions in 3-dimensional (E, H, A) state space (Figure 6.36).

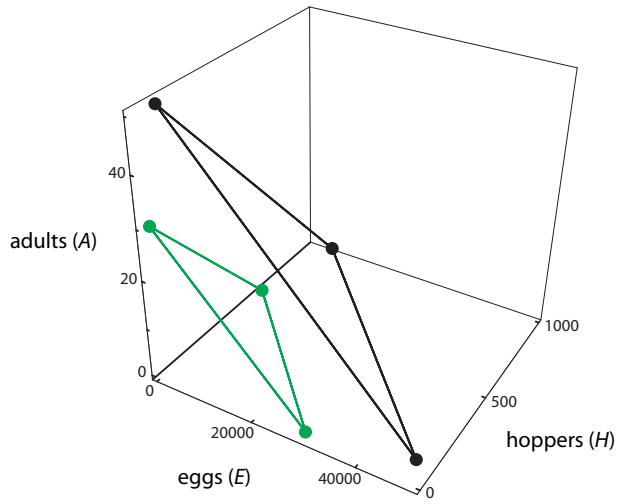


Figure 6.36: Two trajectories resulting from simulations of the locust population model with two different initial conditions.

To explain this neutral oscillatory behavior, we need to study the eigenvalues of the matrix $\boldsymbol{L}$; see Exercise 6.5.6 below.

Exercise 6.5.6 Use SageMath to calculate the eigenvalues of $\boldsymbol{L}$. Verify that they are

$$\lambda_1 = 1, \quad \lambda_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\boldsymbol{i}, \quad \lambda_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\boldsymbol{i}$$

What do the eigenvalues tell you about the behavior you have just seen? Relate each of the phenomena you saw above to specific properties of the eigenvalues.

Lessons

We have seen that the equilibrium point behavior of a linear discrete-time dynamical system is entirely determined by the eigenvalue and eigenvector decomposition of its matrix representation.

There is also an important lesson about the long-term behavior of linear (or matrix) discrete-time systems that we remarked on in each of our examples: if you take a blob of points and apply a matrix $\boldsymbol{M}$ to them many times, you will be looking at the principal eigenvector of $\boldsymbol{M}$. Put another way, the long-term behavior of a linear discrete-time system is dominated by its largest eigenvalue and the corresponding eigenvector.

There is a nice algebraic way to see why this is true. Suppose our n-dimensional dynamical system is

$$\mathbf{X}_{N+1} = f(\mathbf{X}_N) = \boldsymbol{M}\mathbf{X}_N$$

If we start with an initial condition $\mathbf{X}_0$, then

$$\mathbf{X}_\mathrm{N} = \boldsymbol{M}^{\boldsymbol{N}}\mathbf{X}_0$$

Now suppose that the eigenvalues of $\boldsymbol{M}$, in descending order of magnitude (absolute value), are $\lambda_1, \lambda_2, \ldots, \lambda_n$, and the corresponding eigenvectors are $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_\mathrm{n}$. In the basis

$\{\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n\}$ formed by the n eigenvectors, there are constants $c_1, c_2, \ldots, c_n$ such that we can decompose the initial condition $\mathbf{X}_0$ into

$$\mathbf{X}_0 = c_1\mathbf{E}_1 + c_2\mathbf{E}_2 + \cdots + c_n\mathbf{E}_n$$

Then applying M to $\mathbf{X}_0$ once, we get

$$\begin{aligned}\mathbf{X}_1 = M(\mathbf{X}_0) &= M\Big(c_1\mathbf{E}_1 + c_2\mathbf{E}_2 + \cdots + c_n\mathbf{E}_n\Big)\\ &= M\Big(c_1\mathbf{E}_1\Big) + M\Big(c_2\mathbf{E}_2\Big) + \cdots + M\Big(c_n\mathbf{E}_n\Big)\\ &= c_1M\mathbf{E}_1 + c_2M\mathbf{E}_2 + \cdots + c_nM\mathbf{E}_n\\ &= c_1\lambda_1\mathbf{E}_1 + c_2\lambda_2\mathbf{E}_2 + \cdots + c_n\lambda_n\mathbf{E}_n\end{aligned}$$

And similarly,

$$\begin{aligned}\mathbf{X}_2 = M(\mathbf{X}_1) &= M\Big(c_1\lambda_1\mathbf{E}_1 + c_2\lambda_2\mathbf{E}_2 + \cdots + c_n\lambda_n\mathbf{E}_n\Big)\\ &= M\Big(c_1\lambda_1\mathbf{E}_1\Big) + M\Big(c_2\lambda_2\mathbf{E}_2\Big) + \cdots + M\Big(c_n\lambda_n\mathbf{E}_n\Big)\\ &= c_1\lambda_1^2\mathbf{E}_1 + c_2\lambda_2^2\mathbf{E}_2 + \cdots + c_n\lambda_n^2\mathbf{E}_n\end{aligned}$$

If we iterate M 100 times, we get

$$\begin{aligned}\mathbf{X}_{100} = M(\mathbf{X}_{99}) &= M\Big(c_1\lambda_1^{99}\mathbf{E}_1 + c_2\lambda_2^{99}\mathbf{E}_2 + \cdots + c_n^{99}\lambda_n\mathbf{E}_n\Big)\\ &= M\Big(c_1\lambda_1^{99}\mathbf{E}_1\Big) + M\Big(c_2\lambda_2^{99}\mathbf{E}_2\Big) + \cdots + M\Big(c_n\lambda_n^{99}\mathbf{E}_n\Big)\\ &= c_1\lambda_1^{100}\mathbf{E}_1 + c_2\lambda_2^{100}\mathbf{E}_2 + \cdots + c_n\lambda_n^{100}\mathbf{E}_n\end{aligned}$$

If λ_1 is even slightly larger than λ_2, then λ_1^{100} will be *much* larger then λ_2^{100}. Therefore, the dynamics along the principal eigenvector will dominate the long-term behavior of the matrix. This principle is beautifully illustrated in the following example.

Further Exercises 6.5

1. A swan population can be subdivided into young swans (Y) and mature swans (M). We can then set up a discrete-time model of these populations as follows:

$$\begin{pmatrix} Y_{N+1} \\ M_{N+1} \end{pmatrix} = \begin{bmatrix} 0.57 & 1.5 \\ 0.25 & 0.88 \end{bmatrix} \begin{pmatrix} Y_N \\ M_N \end{pmatrix}$$

   a) Explain the biological meaning of each of the four numbers in the matrix of this model.

   b) It turns out that the eigenvectors of this matrix are approximately as follows (you can check this using SageMath if you wish): $\begin{pmatrix} 1.9 \\ -0.6 \end{pmatrix}$ with eigenvalue 0.09 and $\begin{pmatrix} 1.9 \\ 1.0 \end{pmatrix}$ with eigenvalue 1.36. What will happen to the swan population in the long run?

   c) Many years in the future, if there are 2000 mature swans, approximately how many young swans would you expect there to be?

2. A blobfish population consists of juveniles and adults. Each year, 50% of juveniles become adults and 10% die. Adults have a 75% chance of surviving from one year to the next and have, on average, four offspring a year.
   a) Write a discrete-time matrix model describing this population.
   b) If the population this year consists of 50 juveniles and 35 adults, what will next year's population be?
   c) What will happen to the population in the long run?

6.6 Google PageRank

Shortly after the invention of the World Wide Web, programs began to appear that would enable you to search over the web to find websites, or "pages," that mentioned a specified key word or phrase.

The early versions of these "web browsers" or "search engines" were not very good. If you typed in "Paris, France" you were as likely to be directed to the personal web page of a couple from Seattle who had recently visited Paris and posted photos as to, say, the French government website or the official site of the city of Paris.

Something had to be done to enable the search engine to rank websites according to how "important" they are. But what does "important" mean? One answer to this was provided by two graduate students in computer science, Sergey Brin and Larry Page, who published an article in the journal *Computer Networks* in 1998, called "The Anatomy of a Large-Scale Hypertextual Web Search Engine" (Brin and Page 1998). They began their paper thus: "In this paper, we present Google, a prototype of a large-scale search engine which makes heavy use of the structure present in hypertext. Google is designed to crawl and index the Web efficiently and produce much more satisfying search results than existing systems."

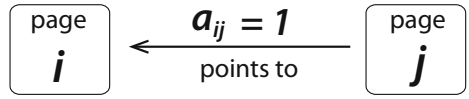
Their key idea is that we want not just websites, but websites that are themselves pointed to or voted for by other important websites, that is, by websites that are themselves pointed to or "voted for" by other important websites, which then "pass on" their importance to the sites that they point to. This regress suggests a dynamical system or iterated matrix system, iterating the "points to" function over and over.

As we saw in the discussion of discrete-time dynamical systems, the result of iterating a matrix M over and over is the principal eigenvector of M. Indeed, Page and Brin describe their new concept, called PageRank, which assigns an importance PR(A) to every page A, as follows: "PageRank or PR(A) can be calculated using a simple iterative algorithm, and corresponds to the principal eigenvector of the normalized link matrix of the web."

The key idea is that we can represent networks with matrices. So let's consider a net that is composed of pages $p_1, p_2, \ldots, p_n$. First, we will create the "points to" matrix, which is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

where $a_{ij} = 1$ if page p_j points to page p_i, and $a_{ij} = 0$ if not.



Note that the sum of the elements in each row i is the total number of pages that point to page i, and the sum of the elements in each column j is the total number of pages that page j points to:

$$\begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

$$\underbrace{a_{i1} + \cdots + a_{ij} + \cdots + a_{in}}_{\text{sum of the } i\text{th row}} = \begin{array}{c}\text{total number of}\\ \text{pages that}\\ \text{point to page } i\end{array}$$

$$\begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

$$\underbrace{a_{1j} + \cdots + a_{ij} + \cdots + a_{nj}}_{\text{sum of the } j\text{th column}} = \begin{array}{c}\text{total number of}\\ \text{pages that}\\ \text{page } j \text{ points to}\end{array}$$

Then we have to account for the fact that a webpage might point to many other pages. A "vote" from a selective page counts more than a "vote" from a page that points to lots of other pages, so if one page points to many others, the importance score that it passes on to the other pages must be diluted by the total number of outbound links. For example, if page j points to page i, then $a_{ij} = 1$. But this will need to be diluted by the total number of pages that page j points to, which is $a_{1j} + a_{2j} + \cdots + a_{nj}$.

So we define $L_{i,j}$ as the normalized weight of page j's vote on page i:

$$L_{i,j} = \frac{\text{page } j\text{'s vote on page } i \text{ (0 or 1)}}{\text{total number of pages that page } j \text{ pointed to}} = \frac{a_{ij}}{a_{1j} + a_{2j} + \cdots + a_{nj}}$$

We now define the *"links to" matrix*

$$\boldsymbol{L} = [L_{ij}] = \begin{bmatrix} L_{11} & L_{12} & \dots & L_{1n} \\ L_{21} & L_{22} & \dots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \dots & L_{nn} \end{bmatrix}$$

Then let's define the *PageRank vector* as the vector made up of the "importance" of each page $p_1, p_2, \dots, p_n$. This is the vector that the search engine needs to calculate to assign an importance to each page in the network. Its components $PR_1, PR_2, \dots, PR_n$ are the importance scores of each page. The higher the score, the more important the page. The more important the page, the higher it appears in the search engine results:

$$\mathbf{PR} = \begin{pmatrix} \text{importance of } p_1 \\ \text{importance of } p_2 \\ \vdots \\ \text{importance of } p_n \end{pmatrix} = \begin{pmatrix} PR_1 \\ PR_2 \\ \vdots \\ PR_n \end{pmatrix}$$

To start with, we will assume an initial condition, which we will call "old **PR**," in which all n pages have equal importance. We will normalize the total importance to 1, so

$$\text{old } \mathbf{PR} = \begin{pmatrix} \text{old } PR_1 \\ \text{old } PR_2 \\ \vdots \\ \text{old } PR_n \end{pmatrix} = \begin{pmatrix} \frac{1}{n} \\ \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{pmatrix}$$

Then we update the old **PR** vector. The new value of PR_i is the sum of the normalized incoming links to page i. In this way, each page that points to page i "passes on" a fraction of its own importance to page i.

That is, we update the page rank PR_i by assigning the new value

$$\text{new } PR_i = L_{i1} \cdot \Big(\text{old } PR_1\Big) + L_{i2} \cdot \Big(\text{old } PR_2\Big) + \cdots + L_{in} \cdot \Big(\text{old } PR_n\Big)$$

which is the sum of the normalized weight of each page j's vote on page i $\times$ page j's page rank.

If we do this update for each of the old page ranks, we get a "new" page rank vector

$$\text{new } \mathbf{PR} = \begin{pmatrix} \text{new } PR_1 \\ \text{new } PR_2 \\ \vdots \\ \text{new } PR_n \end{pmatrix} = \begin{pmatrix} L_{11} \cdot (\text{old } PR_1) + L_{12} \cdot (\text{old } PR_2) + \cdots + L_{1n} \cdot (\text{old } PR_n) \\ L_{21} \cdot (\text{old } PR_1) + L_{22} \cdot (\text{old } PR_2) + \cdots + L_{2n} \cdot (\text{old } PR_n) \\ \vdots \\ L_{n1} \cdot (\text{old } PR_1) + L_{n2} \cdot (\text{old } PR_2) + \cdots + L_{nn} \cdot (\text{old } PR_n) \end{pmatrix}$$

This can be rewritten as

$$\text{new } \mathbf{PR} = \begin{pmatrix} \text{new } PR_1 \\ \text{new } PR_2 \\ \vdots \\ \text{new } PR_n \end{pmatrix} = \begin{bmatrix} L_{11} & L_{12} & \dots & L_{1n} \\ L_{21} & L_{22} & \dots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \dots & L_{nn} \end{bmatrix} \begin{pmatrix} \text{old } PR_1 \\ \text{old } PR_2 \\ \vdots \\ \text{old } PR_n \end{pmatrix}$$

or in vector form

$$\text{new } \mathbf{PR} = \boldsymbol{L} \Big(\text{old } \mathbf{PR}\Big)$$

But as Page and Brin saw, this is only a first estimate. The next question is, how important are the sites that pointed to the sites that pointed to site i? To take that factor into account, we replace the "new" page rank vector by a "new new" page rank vector

$$\begin{aligned} \text{new new PR} = \begin{pmatrix} \text{new new } PR_1 \\ \text{new new } PR_2 \\ \vdots \\ \text{new new } PR_n \end{pmatrix} &= \begin{bmatrix} L_{11} & L_{12} & \dots & L_{1n} \\ L_{21} & L_{22} & \dots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \dots & L_{nn} \end{bmatrix} \begin{pmatrix} \text{new } PR_1 \\ \text{new } PR_2 \\ \vdots \\ \text{new } PR_n \end{pmatrix} \\ &= \begin{bmatrix} L_{11} & L_{12} & \dots & L_{1n} \\ L_{21} & L_{22} & \dots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \dots & L_{nn} \end{bmatrix} \begin{bmatrix} L_{11} & L_{12} & \dots & L_{1n} \\ L_{21} & L_{22} & \dots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \dots & L_{nn} \end{bmatrix} \begin{pmatrix} \text{old } PR_1 \\ \text{old } PR_2 \\ \vdots \\ \text{old } PR_n \end{pmatrix} \end{aligned}$$

or in vector form

$$\text{new new } \mathbf{PR} = \boldsymbol{L}\Big(\text{new } \mathbf{PR}\Big) = \boldsymbol{L}^2 \Big(\text{old } \mathbf{PR}\Big)$$

In other words, the infinite regress that is contained in the idea of "sites that are linked to by sites that are linked to by . . . " is actually a model for a discrete-time dynamical system that is an iteration of the "link to" matrix.

What happens when we iterate this link matrix $\boldsymbol{L}$ many times? Suppose the eigenvectors of $\boldsymbol{L}$ are $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n$, in descending order of their corresponding eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$. So λ_1 is the largest eigenvalue.

The action of applying $\boldsymbol{L}$ to the initial condition "old **PR**" many times is then dominated by the principal eigenvector of $\boldsymbol{L}$, which is $\mathbf{E}_1$. Indeed, there are constants $c_1, c_2, \ldots, c_n$ that enable us to express the initial condition

$$\text{old } \mathbf{PR} = c_1\mathbf{E}_1 + c_2\mathbf{E}_2 + \cdots + c_n\mathbf{E}_n$$

in the eigenvector basis $\{\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n\}$. After iterating the matrix many times, say 100, we get

$$\begin{aligned}\boldsymbol{L}^{100}\Big(\text{old } \mathbf{PR}\Big) &= \boldsymbol{L}^{100}\Big(c_1\mathbf{E}_1\Big) + \boldsymbol{L}^{100}\Big(c_2\mathbf{E}_2\Big) + \cdots + \boldsymbol{L}^{100}\Big(c_n\mathbf{E}_n\Big) \\ &= c_1\lambda_1^{100}\mathbf{E}_1 + c_2\lambda_2^{100}\mathbf{E}_2 + \cdots + c_n\lambda_n^{100}\mathbf{E}_n\end{aligned}$$

So the long-term behavior of repeatedly iterating $\boldsymbol{L}$ is dominated by the principal eigenvector $\mathbf{E}_1$.

We call the principal eigenvector of the matrix $\boldsymbol{L}$ the **page rank vector.** Thus the vector $\mathbf{E}_1$,

$$\mathbf{E}_1 = \begin{pmatrix} PR_1 \\ PR_2 \\ \vdots \\ PR_n \end{pmatrix}$$

and its components $PR_1, PR_2, \ldots, PR_n$ are the page ranks, the final importance scores assigned to each page. When you search a term, Google presents pages to you in the order of their page rank eigenvector.

Surfer Model

In our discussion of Markov processes, we saw that a Markov process can be represented by a matrix ($\boldsymbol{M}$) each element m_{ij} of which is the probability of a person "hopping" from compartment j to compartment i in the next time interval.

The long-term behavior of the system is given by the iteration of the matrix, which will tend to some outcome. As we saw, the results of that iteration are determined by the eigenvector and eigenvalue decomposition of the matrix.

Brin and Page realized that their "links to" matrix could also be seen as a model of a Markov process, in which a random web surfer "hops" from one page j to another page i with a probability equal to the normalized weight of page j's vote on page i, which is L_{ij}.

Notice that the "links to" matrix satisfies the key condition that defines a "stochastic" matrix: each column adds up to 1. For example, the elements of the jth column of the "links to" matrix are $L_{1j}, \ldots L_{ij}, \ldots L_{nj}$. Recall that the definition of L_{ij} is

$$L_{i,j} = \frac{\text{page } j\text{'s vote on page } i \text{ (0 or 1)}}{\text{total number of pages that page } j \text{ pointed to}} = \frac{a_{ij}}{a_{1j} + \cdots + a_{nj}}$$

So the sum of the jth column of the "links to" matrix is

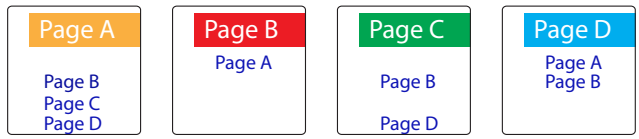
$$\begin{aligned} L_{1j} + \cdots + L_{ij} + \cdots + L_{nj} &= \frac{a_{1j}}{a_{1j} + \cdots + a_{nj}} + \cdots + \frac{a_{ij}}{a_{1j} + \cdots + a_{nj}} + \cdots + \frac{a_{nj}}{a_{1j} + \cdots + a_{nj}} \\ &= \frac{a_{1j} + \cdots + a_{nj}}{a_{1j} + \cdots + a_{nj}} \\ &= 1 \end{aligned}$$

$$\begin{bmatrix} L_{11} & \cdots & L_{1j} & \cdots & L_{1n} \\ \vdots & & \vdots & & \vdots \\ L_{i1} & \cdots & L_{ij} & \cdots & L_{in} \\ \vdots & & \vdots & & \vdots \\ L_{n1} & \cdots & L_{nj} & \cdots & L_{nn} \end{bmatrix} \qquad \begin{array}{c} \text{sum of the } j\text{th column} \\ L_{1j} + \cdots + L_{ij} + \cdots + L_{nj} \end{array} = \mathbf{1}$$

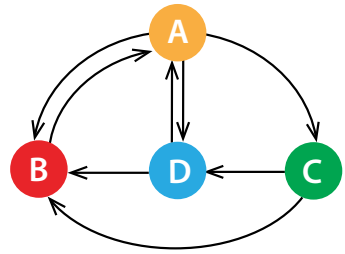
So the page rank vector can be interpreted in this surfer model as the probability that the surfer, clicking randomly on each page, will end up on a given page.

An Example of the PageRank Algorithm

Suppose we have a network of four web pages, A, B, C, and D, with links to the other pages as shown below.



In this network, the "points to" relationship is summarized as



where the arrow means "points to." We can then derive the "points to" matrix, more commonly called a *directed adjacency matrix* because it shows which pages are linked and the direction of the link. For example, from the diagram, we know that page A points to page C; therefore, in the "points to" matrix, we have $a_{C \leftarrow A} = 1$:

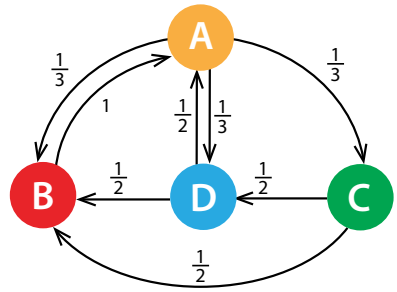
$$\text{"points to" matrix} = \begin{bmatrix} a_{A \leftarrow A} & a_{A \leftarrow B} & a_{A \leftarrow C} & a_{A \leftarrow D} \\ a_{B \leftarrow A} & a_{B \leftarrow B} & a_{B \leftarrow C} & a_{B \leftarrow D} \\ a_{C \leftarrow A} & a_{C \leftarrow B} & a_{C \leftarrow C} & a_{C \leftarrow D} \\ a_{D \leftarrow A} & a_{D \leftarrow B} & a_{D \leftarrow C} & a_{D \leftarrow D} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

From the "points to" matrix, we can derive the "links to" matrix $\boldsymbol{L}$ by normalizing each "vote" from page j to page i by the total number of "votes" cast by page j. So for example, the sum

of the first column of the "points to" matrix is the total number of pages that page A points to, which is 3. So each vote that A casts has to be divided by 3.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$
$$\Sigma=3 \quad \Sigma=1 \quad \Sigma=2 \quad \Sigma=2$$

In this manner, we derive the normalized weights as



which gives rise to the "links to" matrix

$$\boldsymbol{L} = \begin{bmatrix} L_{A\leftarrow A} & L_{A\leftarrow B} & L_{A\leftarrow C} & L_{A\leftarrow D} \\ L_{B\leftarrow A} & L_{B\leftarrow B} & L_{B\leftarrow C} & L_{B\leftarrow D} \\ L_{C\leftarrow A} & a_{C\leftarrow B} & L_{C\leftarrow C} & L_{C\leftarrow D} \\ L_{D\leftarrow A} & L_{D\leftarrow B} & L_{D\leftarrow C} & L_{D\leftarrow D} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

If we begin with an initial condition that is the vector of equal weights to each page (0.25), then the results of repeatedly iterating the matrix $\boldsymbol{L}$ are

$$\mathbf{PR} = \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{pmatrix} \quad \boldsymbol{L}\,\mathbf{PR} = \begin{pmatrix} 0.38 \\ 0.33 \\ 0.08 \\ 0.21 \end{pmatrix} \quad \boldsymbol{L}^2\,\mathbf{PR} = \begin{pmatrix} 0.44 \\ 0.27 \\ 0.13 \\ 0.17 \end{pmatrix} \quad \boldsymbol{L}^3\,\mathbf{PR} = \begin{pmatrix} 0.35 \\ 0.29 \\ 0.15 \\ 0.21 \end{pmatrix}$$

$$\boldsymbol{L}^4\,\mathbf{PR} = \begin{pmatrix} 0.40 \\ 0.30 \\ 0.12 \\ 0.19 \end{pmatrix} \quad \boldsymbol{L}^5\,\mathbf{PR} = \begin{pmatrix} 0.39 \\ 0.29 \\ 0.13 \\ 0.19 \end{pmatrix} \quad \boldsymbol{L}^6\,\mathbf{PR} = \begin{pmatrix} 0.38 \\ 0.29 \\ 0.13 \\ 0.20 \end{pmatrix} \quad \boldsymbol{L}^7\,\mathbf{PR} = \begin{pmatrix} 0.39 \\ 0.29 \\ 0.13 \\ 0.19 \end{pmatrix}$$

$$\boldsymbol{L}^8\,\mathbf{PR} = \begin{pmatrix} 0.39 \\ 0.29 \\ 0.13 \\ 0.19 \end{pmatrix} \quad \boldsymbol{L}^9\,\mathbf{PR} = \begin{pmatrix} 0.39 \\ 0.29 \\ 0.13 \\ 0.19 \end{pmatrix} \quad \boldsymbol{L}^{10}\,\mathbf{PR} = \begin{pmatrix} 0.39 \\ 0.29 \\ 0.13 \\ 0.19 \end{pmatrix}$$

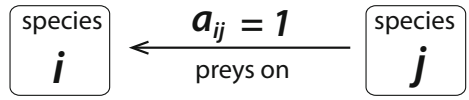
Note that the iteration process stabilizes after only a few iterations and reaches a "stationary distribution" that is the principal eigenvector, which gives us the final page ranks.

Food Webs

Another example of a Google-style eigenvector-based ranking system can be found in the analysis of *food webs* in ecology.

In a food web, nutrients move from one species to another. In an application of the Google eigenvector concept, Allesina and Pascual wanted to find out whether a given species was "important for co-extinctions" (Allesina and Pascual 2009). That is, they wanted to know which species had the biggest impact on the food web and whose loss would therefore be the most catastrophic.

If the food web has species $1, 2, \ldots, k$ that interact with each other, we will let $[a_{ij}]$ be the $k \times k$ matrix that represents the "preys on" hierarchy, in other words, the ith row and jth column entry of the "preys on" matrix is given by $a_{ij} = 1$ if species j preys on species i.



Just as Google wants the web pages that are pointed to by web pages that are pointed to ..., so in food webs we are interested in species that are preyed on by species that are preyed on ....

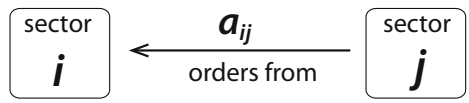
We find these "important" species by the same method: start with the "preys on" matrix of 0's and 1's, normalize it to a stochastic matrix (all columns add to 1), and then find the principal eigenvector. Each species' importance in this food web is then its corresponding component in this principal eigenvector.

The ranking that is produced by the principal eigenvector is then interpretable as "the sequence of the losses that results in the fastest collapse of the network." Allesina and Pascual argue that this dominant eigenvector analysis is superior to other approaches to food webs, for example, those that focus on "hub" or "keystone" species, which are defined as those species that have the largest number of links to other species.

Input/Output Matrices and Complex Networks

Economics

The history of matrix analysis of networks begins in economics. The economist Wassily Leontieff produced an input/output matrix analysis of the Unites States economy in 1941. In a matrix representation of an economy, we have a list of "sectors" $s_1, s_2, \ldots, s_k$, such as steel, water, rubber, oil. Then we form the $k \times k$ matrix $[a_{ij}]$ in which each entry a_{ij} represents the quantity of resources that sector j orders from sector i.



The first practical application came two years later, during World War II. The US government asked Leontieff to create an input/output matrix representing the Nazi war economy in order to identify which sectors were the most critical. This was done, and the eigenvector calculation of this large-dimensional matrix was one of the very early uses of automated computing.

Leontieff used "the first commercial electro-mechanical computer, the IBM Automatic Sequence Controlled Calculator (called the Mark I), originally designed under the direction of

Harvard mathematician Howard Aiken in 1939, built and operated by IBM engineers in Endicott, New York for the US Navy" (Miller and Blair 2009).

The results of his eigenvector analysis would not have been immediately obvious: the critical sectors were oil and ball bearings. Ball bearings were critical components of machinery and vehicles, and no substitutes for them existed. In accord with this analysis, the US Army Air Forces designated ball bearing factories and oil refineries as the major targets for their bombing campaign in Europe.

Ecological Networks

In the 1970s, ecologists studying the flow of energy and nutrients (substances like carbon, nitrogen, and phosphorus) in ecosystems discovered Leontief's work and began using it to study ecosystems as input/output systems (Hannon 1973), creating the field of *ecological network analysis*. (See Fath and Patten (1999) for a readable introduction.) The first step in doing so is to decide what substance to study (this substance is called the *currency* of the model), and if we are studying a whole ecosystem, decide how to partition it into compartments. *Compartments* can be species, collections of species, or nonliving ecosystem components such as dissolved nitrate in water.

We then measure or estimate how much of our currency flows between each pair of compartments. This gives what is called the *flow matrix* $\boldsymbol{F}$. Entry f_{ij} of this matrix tells us how much currency flows from compartment j to compartment i. For example, the ecological interactions that make up an oyster–mussel community in a reef have been modeled as consisting of six compartments. The currency in this case is energy, and the flows from one compartment to another are shown in Figure 6.37.

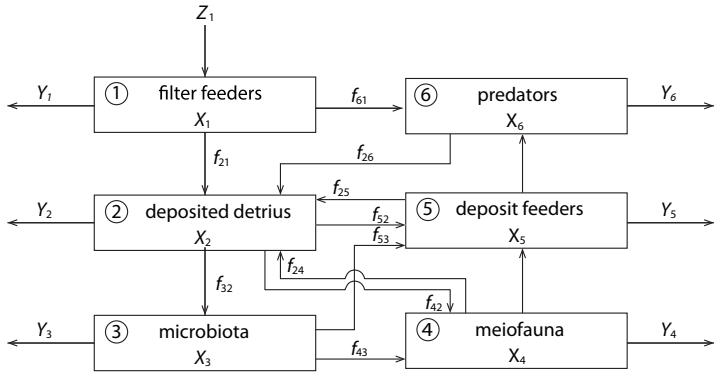


Figure 6.37: Six compartment model of reef community (redrawn from Patten (1985)).

Based on the graph of the network, we can make an input–output matrix for the compartments in the system. We can then iterate this matrix to find the long-term behavior predicted by the model.

Suppose we iterate the matrix many times and the system stabilizes at some equilibrium point. When the system is at equilibrium, the sum of all the outflows (or inflows) from a compartment is called the compartment's *throughflow*.

We can make a vector, $\mathbf{T}$, of these throughflows. Dividing each entry in the $\boldsymbol{F}$ matrix by the throughflow of the donor compartment gives a matrix called the $\boldsymbol{G}$ matrix, where $G_{ij} = \frac{f_{ij}}{T_j}$. This matrix gives us the probability that a unit of currency leaving compartment j enters compartment i, or the fraction of the currency that does so.

The $\boldsymbol{G}$ matrix tells us about the currency going from compartment j to compartment i in one direct step. However, ecologists are interested in more than just the question of how much flows from j to i. They also want to know about second-order flows, in which currency transfer happens in two steps: $j \to k \to i$; the currency first has to get from j to k and then from k to i. The probability of going from j to k is G_{kj}, and the probability of going from k to i is G_{ik}. And the probability of going from $j \to i$ through k is the product of G_{kj} and G_{ik}. Adding up these products for all the compartments that could play the role of k gives the fraction of currency leaving j that gets to i in *two* steps. We can do this for every compartment in the model simply by multiplying the $\boldsymbol{G}$ matrix by itself. The resulting matrix is written as $\boldsymbol{G}^2$. More generally, the amount of currency going from j to i in n steps is entry i, j of the matrix $\boldsymbol{G}^n$.

Why is this interesting? Well, all powers of $\boldsymbol{G}$ tell us about indirect flows between j and i. We may sum all these matrices to obtain the sum of all indirect flows as $\boldsymbol{G}^2 + \boldsymbol{G}^3 + \cdots$. Because real ecosystems leak energy and nutrients, the entries in $\boldsymbol{G}^{n+1}$ are generally smaller than those in $\boldsymbol{G}^n$, and the sum $\boldsymbol{G}^2 + \boldsymbol{G}^3 + \cdots$ converges to some limiting matrix. Comparing the entries of this matrix to those of $\boldsymbol{G}$ itself lets us compare the relative importance of direct and indirect flows. It turns out that in many ecosystem models, indirect flows are significant and can even carry more energy or nutrients than direct flows!

Why does this happen, despite the fact that currency is lost at every step? It's true that a longer path will typically carry less currency than a shorter one. But how many long paths are there? We can find out by taking powers of the adjacency matrix $\boldsymbol{A}$. The i, jth entry of $\boldsymbol{A}^n$ tells us the number of paths of length n between j and i. For most ecosystem and food web models, these numbers rapidly become astronomical. For example, in the 29-species food web in Figure 6.38, there are at least 28 million paths between seals and the fish hake (Yodzis 1998).

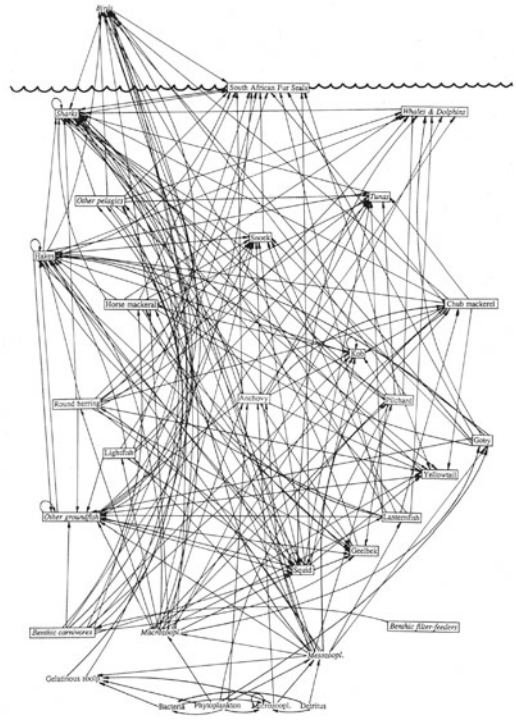


Figure 6.38: A food web for an ecosystem off the coast of southern Africa. Reprinted from "Local trophodynamics and the interaction of marine mammals and fisheries in the Benguela ecosystem," by P. Yodzis, 1998, *Journal of Animal Ecology* 67(4):635–658. Copyright 1998 John Wiley & Sons. Reprinted with permission from John Wiley & Sons.

This proliferation of paths allows indirect paths taken together to carry a large amount of energy or nutrients, even though no individual path may be very significant. This is one of the reasons why predicting how an ecosystem or other complex system will respond to an intervention is difficult.

6.7 Linear Differential Equations

Our second major application of linear algebra is the subject of linear differential equations. Here, the function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is the vector field that assigns the n-dimensional change vector

$$\mathbf{X}' = (X_1', X_2', \ldots, X_n')$$

to the n-dimensional state vector

$$\mathbf{X} = (X_1, X_2, \ldots, X_n)$$

Since both $\mathbf{X}'$ and $\mathbf{X}$ are vectors in $\mathbb{R}^n$, the vector field f truly is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$.

We can decompose the function $f : \mathbb{R}^n \to \mathbb{R}^n$ into n component functions $f_1, f_2, \ldots, f_n$, each of which is a function $\mathbb{R}^n \to \mathbb{R}$. This amounts to writing the vector differential equation

$$\mathbf{X}' = f(\mathbf{X})$$

as the n-component differential equations

$$\begin{aligned} X_1' &= f_1(X_1, X_2, \ldots, X_n) \\ X_2' &= f_2(X_1, X_2, \ldots, X_n) \\ &\vdots \quad\quad \vdots \\ X_n' &= f_n(X_1, X_2, \ldots, X_n) \end{aligned}$$

Linear dynamical systems have particularly simple behaviors and can be completely classified.

Equilibrium Points

First of all, let's discuss equilibrium points. If we think about one-dimensional linear vector fields, then we are talking about either

$$X' = rX \quad \text{or} \quad X' = -rX \qquad (\text{assuming r} > 0)$$

It is clear that the only equilibrium points these systems can have are $X = 0$.

But what about two-dimensional or even n-dimensional cases? In the n-dimensional case, if we are looking for equilibrium points, we are looking for solutions to

$$\mathbf{X}' = 0 = f(\mathbf{X})$$

which implies

$$\begin{aligned} X_1' &= 0 = f_1(X_1, X_2, \ldots, X_n) \\ X_2' &= 0 = f_2(X_1, X_2, \ldots, X_n) \\ &\vdots \quad\quad \vdots \quad\quad \vdots \\ X_n' &= 0 = f_n(X_1, X_2, \ldots, X_n) \end{aligned}$$

where $f_1, f_2, \ldots, f_n$ are all linear functions $\mathbb{R}^n \to \mathbb{R}$.

How many solutions can this set of equations have? Here, a theorem from elementary algebra comes to the rescue:[3] *setting n linear functions of n unknowns equal to zero can have only one solution*, and that is

$$X_1 = X_2 = \cdots = X_n = 0$$

(We can find this by using the first equation to eliminate X_1 in terms of the other variables, then using the second equation to eliminate X_2, and finally we get an equation of the form $aX_n = 0$, which can have only the solution $X_1 = X_2 = \cdots = X_n = 0$.)

A linear system of differential equations has a unique equilibrium point, at

$$X_1 = X_2 = \cdots = X_n = 0$$

Stability

Having found the equilibrium point, we now need to determine its stability.

In the one-dimensional case, we have already seen that $X' = rX$ has a stable equilibrium point at $X = 0$ if and only if $r < 0$.

If we now pass to the decoupled 2D case,

$$X' = aX$$
$$Y' = dY$$

we can say that since the system decouples into two 1D subsystems along the X and Y axes, the behavior of the equilibrium point is given by the behaviors along the two axes. The two 1D subsystems are $X' = aX$ and $Y' = dY$. And if we join them, we get

$$\left.\begin{aligned} X' &= aX \\ Y' &= dY \end{aligned}\right\} \quad \Longrightarrow \quad \begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

As we saw in Chapter 3, these equilibrium points can be purely stable nodes ($a < 0$ and $d < 0$), purely unstable nodes ($a > 0$ and $d > 0$), and saddle points ($a < 0$ and $d > 0$ or $a > 0$ and $d < 0$).

Exercise 6.7.1 Why does it makes sense that these signs of a and d give rise to the equilibrium types listed above? *(Hint: Draw some phase portraits.)*

Exercise 6.7.2 Classify the equilibria of the following systems:

a) $\begin{cases} X' = 2X \\ Y' = -3Y \end{cases}$ b) $\begin{cases} X' = 0.5X \\ Y' = 1.8Y \end{cases}$ c) $\begin{cases} X' = -1.2X \\ Y' = -0.3Y \end{cases}$

[3] Almost all the time. The exceptions are cases in which two equations are multiples of each other, such as $0 = X + Y$ and $0 = 2X + 2Y$. Try solving these for X and Y; you don't get a definite answer.

The Flow Associated with a Linear Differential Equation

1D Recall a very important fact about the differential equation

$$X' = rX$$

As we saw in Chapter 2, *this differential equation has an explicit solution.* In other words, it's possible to actually write out a function $X(t)$ such that

$$X'(t) = rX(t)$$

In this case, the explicit solution to the differential equations is the function

$$X(t) = X(0)e^{rt}$$

where $X(0)$ is the initial condition.

We call $X(0)e^{rt}$ the *flow* corresponding to the differential equation $X' = rX$.

Exercise 6.7.3 Find the flow of the differential equation $X' = 0.25X$.

2D Let's go on to discuss the two-dimensional case. The simplest case is two uncoupled systems

$$X' = aX$$
$$Y' = dY$$

This can be represented as the matrix differential equation

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

The flow corresponding to the diagonal matrix differential equation is then just the combination of the flows in the two components:

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} X(0)e^{at} \\ Y(0)e^{dt} \end{pmatrix}$$

where $X(0), Y(0)$ are the initial conditions.

Exercise 6.7.4 Find the flow of the differential equation $\begin{cases} X' = 0.3X \\ Y' = -0.5Y \end{cases}$

Exercise 6.7.5 What differential equation has the flow $\begin{cases} X(t) = X(0)e^{2t} \\ Y(t) = Y(0)e^{-0.7t} \end{cases}$

Eigenbehavior

We can look at the equation

$$X' = rX$$

represented by the linear function

$$f(X) = rX$$

and ask something that may seem redundant and pointless. We will ask whether this 1D linear function has an eigenvalue and an eigenvector. The answer is that of course it does. An eigenvector is a subspace along which f acts like multiplication by λ, and X obviously satisfies this, with $\lambda = r$.

Therefore, for the differential equation $X' = rX$, we can rewrite the equation for the flow as

$$X(t) = X(0)e^{\lambda t} \qquad (\text{where } \lambda = r)$$

Similarly, in the 2D uncoupled case, for the matrix differential equation

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

we can ask whether the matrix has eigenvalues and eigenvectors. And again, the answer is that of course it does: the vectors

$$\{\mathbf{X}, \mathbf{Y}\} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$$

are eigenvectors, and the corresponding eigenvalues are

$$\lambda_X = a \quad \lambda_Y = d$$

Then we can rewrite the equation for the flow for this uncoupled 2D system as

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} X(0)e^{\lambda_X t} \\ Y(0)e^{\lambda_Y t} \end{pmatrix}$$

Exercise 6.7.6 Construct the flow for the matrix differential equation

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

This form is the key to understanding the general 2D case. By mixing and matching various values of λ_X and λ_Y, we get a gallery of equilibrium points in diagonal linear systems (Figure 6.39).

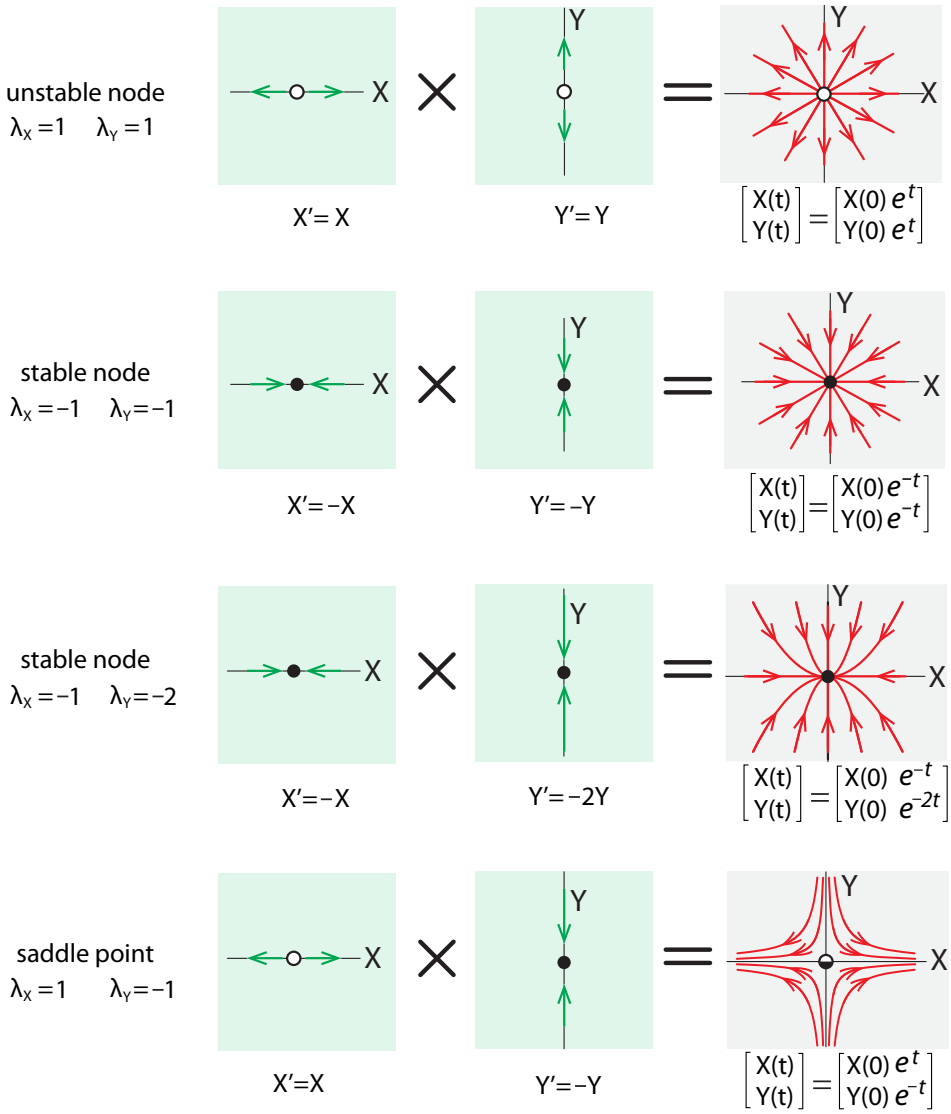


Figure 6.39: Equilibrium points and flows in 2D uncoupled systems.

We can now go on to the general case:

$$\left.\begin{aligned} X' &= aX + bY \\ Y' &= cX + dY \end{aligned}\right\} \quad \Longrightarrow \quad \begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

The key to understanding behavior in this general case is to decompose the system into its eigenvalues and eigenvectors, and then infer the flow from the "eigenbehavior" just as we have been doing. So, for example, if λ_1 and λ_2 are both real numbers, we find their corresponding

eigenvectors **U** and **V**, and conclude that the flow is $U(0)e^{\lambda_1 t}$ on the **U** axis and $V(0)e^{\lambda_2 t}$ on the **V** axis. This completely determines the behavior in the 2D state space.

An example in two dimensions. Consider the linear differential equation

$$X' = \frac{9}{7}X - \frac{4}{7}Y$$
$$Y' = \frac{8}{7}X - \frac{9}{7}Y$$

represented by the matrix differential equation

$$\left.\begin{aligned} X' &= \frac{9}{7}X - \frac{4}{7}Y \\ Y' &= \frac{8}{7}X - \frac{9}{7}Y \end{aligned}\right\} \implies \begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{bmatrix} \frac{9}{7} & -\frac{4}{7} \\ \frac{8}{7} & -\frac{9}{7} \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

How will this system behave? We need to study the eigenvalues and corresponding eigenvectors of the matrix

$$\boldsymbol{M} = \begin{bmatrix} \frac{9}{7} & -\frac{4}{7} \\ \frac{8}{7} & -\frac{9}{7} \end{bmatrix}$$

The eigenvalues of this matrix are obtained by plugging the matrix entries into the characteristic equation (equation (6.2) on page 299). We get

$$\lambda_1 = 1 \text{ and } \lambda_2 = -1$$

Exercise 6.7.7 Confirm this.

Next, we calculate the eigenvectors. The eigenvector $\boldsymbol{U}$ corresponding to λ_1 satisfies

$$\boldsymbol{M}\mathbf{U} = \lambda_1 \mathbf{U}$$

We can say that

$$\boldsymbol{M}\mathbf{U} = \begin{bmatrix} \frac{9}{7} & -\frac{4}{7} \\ \frac{8}{7} & -\frac{9}{7} \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{9}{7}X - \frac{4}{7}Y \\ \frac{8}{7}X - \frac{9}{7}Y \end{pmatrix} = \lambda_1 \mathbf{U} = 1 \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

This gives us

$$\frac{9}{7}X - \frac{4}{7}Y = X \quad \implies \quad Y = 0.5X$$
$$\frac{8}{7}X - \frac{9}{7}Y = Y \quad \implies \quad Y = 0.5X$$

which implies that the eigenvector **U** lies on the line $Y = 0.5X$, which has slope 0.5. The vector $(X, Y) = (2, 1)$ will serve nicely as an eigenvector on this line.

The eigenvector $\boldsymbol{V}$ corresponding to λ_2 must satisfy

$$\boldsymbol{M}\,\mathbf{V} = \lambda_2 \mathbf{V}$$

We can say that

$$\boldsymbol{M}\mathbf{V} = \begin{bmatrix} \frac{9}{7} & -\frac{4}{7} \\ \frac{8}{7} & -\frac{9}{7} \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{9}{7}X - \frac{4}{7}Y \\ \frac{8}{7}X - \frac{9}{7}Y \end{pmatrix} = \lambda_2 \mathbf{V} = -1 \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

This gives us

$$\frac{9}{7}X - \frac{4}{7}Y = -X \quad \Longrightarrow \quad Y = 4X$$

$$\frac{8}{7}X - \frac{9}{7}Y = -Y \quad \Longrightarrow \quad Y = 4X$$

which implies that the eigenvector $\mathbf{V}$ lies on the line $Y = 4X$, which has slope 4. The vector $(X, Y) = (1, 4)$ will serve nicely as an eigenvector on this line.

The resulting equilibrium point structure therefore has a stable direction along the $\mathbf{V}$ axis ($\lambda_V = \lambda_2 = -1$) and an unstable direction along the $\mathbf{U}$ axis ($\lambda_U = \lambda_1 = 1$). *Therefore, the equilibrium point is a saddle point whose axes are* $\mathbf{U}$ *and* $\mathbf{V}$.

The flow corresponding to this saddle point is then exactly as in the uncoupled 2D system

$$\begin{pmatrix} \mathbf{U}(t) \\ \mathbf{V}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{U}(0)e^{\lambda_U t} \\ \mathbf{V}(0)e^{\lambda_V t} \end{pmatrix}$$

where $U(0)$ and $V(0)$ are initial conditions expressed in the $\{\mathbf{U}, \mathbf{V}\}$ coordinate system (Figure 6.40).

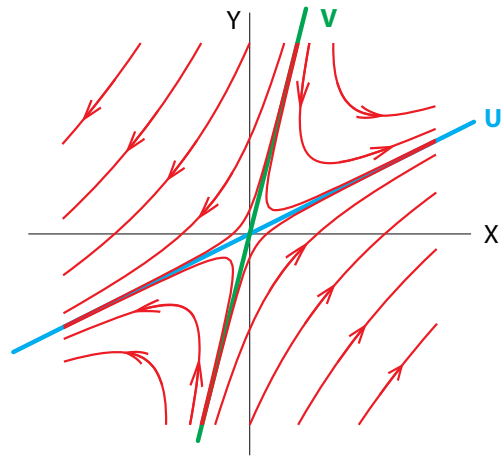


Figure 6.40: The flow around a saddle point. $\mathbf{U}$ and $\mathbf{V}$ are the unstable and stable eigenvectors.

Suppose we are given a matrix differential equation

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} X \\ Y \end{pmatrix}$$

and we want to know the behavior from an initial condition $(X(0), Y(0))$. In order to find it:

(1) Use the coordinate transformation matrix $\boldsymbol{T}$ (see Changing bases: coordinate transforms in section 6.4) to transform the initial conditions from the $\{\mathbf{X}, \mathbf{Y}\}$ coordinate system to

the $\{\mathbf{U}, \mathbf{V}\}$ coordinate system:

$$\begin{pmatrix}\mathbf{X}(0)\\ \mathbf{Y}(0)\end{pmatrix} \xrightarrow{\quad \boldsymbol{T} \quad} \begin{pmatrix}\mathbf{U}(0)\\ \mathbf{V}(0)\end{pmatrix}$$

(2) Evolve the differential equation along the $\mathbf{U}$, $\mathbf{V}$ axes by the exponential flows

$$\begin{pmatrix}\mathbf{U}(t)\\ \mathbf{V}(t)\end{pmatrix} = \begin{pmatrix}\mathbf{U}(0)e^{\lambda_U t}\\ \mathbf{V}(0)e^{\lambda_V t}\end{pmatrix}$$

(3) Use the inverse coordinate transformation matrix $\boldsymbol{T}^{-1}$ to transform the result from the $\{\mathbf{U}, \mathbf{V}\}$ coordinate system back into the $\{\mathbf{X}, \mathbf{Y}\}$ coordinate system:

$$\begin{pmatrix}\mathbf{X}(t)\\ \mathbf{Y}(t)\end{pmatrix} \xleftarrow{\quad \boldsymbol{T}^{-1} \quad} \begin{pmatrix}\mathbf{U}(0)e^{\lambda_U t}\\ \mathbf{V}(0)e^{\lambda_V t}\end{pmatrix}$$

Exercise 6.7.8 Classify the equilibria of the following linear differential equations:

a) $\begin{cases} X' = Y \\ Y' = -2X - 3Y \end{cases}$ b) $\begin{cases} X' &= 4X + 3Y \\ Y' &= X - 2Y \end{cases}$

Complex eigenvalues Finally, let's consider the nondiagonalizable cases. Consider, for example, the spring with friction:

$$\left.\begin{matrix} X' = V \\ V' = -X - V \end{matrix}\right\} \Longrightarrow \begin{pmatrix}X'\\ V'\end{pmatrix} = \begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}\begin{pmatrix}X\\ V\end{pmatrix}$$

$$\boldsymbol{M} = \begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}$$

The eigenvalues of $\boldsymbol{M}$ are

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\,\boldsymbol{i} \approx -0.5 \pm 0.866\,\boldsymbol{i}$$

So the eigenvalues are a pair of complex conjugate numbers with negative real parts.

How are we to understand the flow in the case of complex conjugate eigenvalues? The key is that it is really the same as in the case of real eigenvalues. There, we saw that the flow has the general form

$$e^{\lambda t}$$

along the corresponding eigenvectors. The same is true for imaginary eigenvalues: if $\lambda = a + b\,\boldsymbol{i}$, then the flow is

$$e^{\lambda t} = e^{(a+b\boldsymbol{i})t} = e^{at}e^{b\boldsymbol{i}\,t}$$

The key to the dynamics is in the expression $e^{at}e^{bit}$. Notice that it is the product of two terms.

The first term e^{at} is an exponential in time, and its exponent is the real part of the eigenvalue. Therefore, if the real part of the eigenvalue is positive, the solution has a term that is exponentially growing with time, whereas if the real part of the eigenvalue is negative, the

term becomes a negative exponential, decaying in time. So the sign of a, the real part of the eigenvalue, determines whether the dynamics are growing or shrinking.

The second term, $e^{b\,\boldsymbol{i}\,t}$, which contains the imaginary part of the eigenvalue, $b\,\boldsymbol{i}$, contributes rotation to the flow. We can see this by recalling Euler's formula $e^{\boldsymbol{i}x} = \cos(x) + \boldsymbol{i}\sin(x)$. So

$$e^{b\,\boldsymbol{i}\,t} = \cos(bt) + \boldsymbol{i}\sin(bt)$$

The presence of cosine and sine functions of time guarantees that the solution is a periodic function of time, which gives the solution its oscillatory component.

So, to return to our example of the spring with friction, we can say that the equilibrium point at $(0, 0)$ is

(1) oscillatory, because the eigenvalues are complex conjugates;
(2) shrinking, because the real part of the eigenvalues is less than 0.

Therefore, the equilibrium point is a stable spiral, which we confirm with simulation (Figure 6.41).

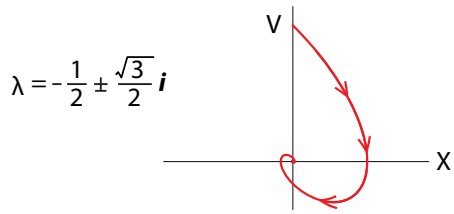


Figure 6.41: Simulation of the spring with friction verifies the prediction of a stable spiral equilibrium point.

As another example, in the spring with negative friction,

$$\left.\begin{aligned} X' &= V \\ V' &= -X + V \end{aligned}\right\} \quad \Longrightarrow \quad \begin{pmatrix} X' \\ V' \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} X \\ V \end{pmatrix}$$

the dynamics are given by the eigenvalues of the matrix

$$\boldsymbol{M} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

which are

$$\lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\,\boldsymbol{i} \approx 0.5 \pm 0.866\,\boldsymbol{i}$$

We conclude that the equilibrium point at $(0, 0)$ is

(1) oscillatory, because the eigenvalues are complex conjugates;
(2) expanding, because the real part of the eigenvalues is greater than 0.

Therefore, the equilibrium point is an unstable spiral (Figure 6.42).

$$\lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\boldsymbol{i}$$

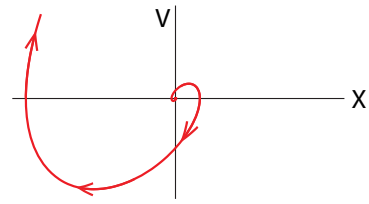


Figure 6.42: Simulation of the spring with negative friction verifies the prediction of an unstable spiral equilibrium point.

Finally, for the frictionless spring,

$$\left.\begin{aligned} X' &= V \\ V' &= -X \end{aligned}\right\} \Longrightarrow \begin{pmatrix} X' \\ V' \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} X \\ V \end{pmatrix}$$

the dynamics are given by the eigenvalues of the matrix

$$\boldsymbol{M} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

which are

$$\lambda = \pm\boldsymbol{i}$$

We conclude that the equilibrium point at $(0, 0)$ is
(1) oscillatory, because the eigenvalues are complex conjugates;
(2) neither expanding nor shrinking, because the real part of the eigenvalues is equal to 0.
Therefore, the equilibrium point is a center (Figure 6.43).

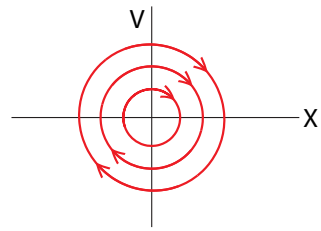


Figure 6.43: Simulation of the frictionless spring verifies the prediction of a neutral equilibrium point.

Exercise 6.7.9 Classify the equilibria of the linear differential equations whose eigenvalues are given below:

a) $2 \pm -3\boldsymbol{i}$ b) $0.5 \pm 2.6\boldsymbol{i}$ c) $-3 \pm -0.75\boldsymbol{i}$ d) $-0.25 \pm -0.1\boldsymbol{i}$

A Compartmental Model in Pharmacokinetics

A simple test for liver function is to inject a dye into the bloodstream and see how fast the liver clears it from the blood and excretes it into the bile. If it clears the dye quickly, liver function is normal. In the case of the liver, this test is possible because there is a dye (bromsulphthalein, BSP) that is absorbed only by the liver (Watt and Young 1962).

In order to understand the dynamics of this process, we make a simple linear model. The model is compartmental, with a blood compartment X and a liver compartment Y. (We don't need a bile compartment, since nothing depends on it; we can view it as excretion.)

We've seen compartmental models before, in the discrete-time setting. In the epidemiology model, for example, we had an S (susceptible) compartment and an I (infected) compartment, and we imagined particles (that is, people) "hopping" from one compartment to another at different rates. Here we imagine not particles but a continuous fluid, "flowing" from one compartment to another at different rates.

The compartmental model is shown in Figure 6.44, where a is the transfer rate of the dye from the blood (X) to the liver (Y), b is the transfer rate from the liver (Y) to the blood (X), and h is the clearance rate from the liver into the bile. To measure liver function, h is the quantity we really want to know.

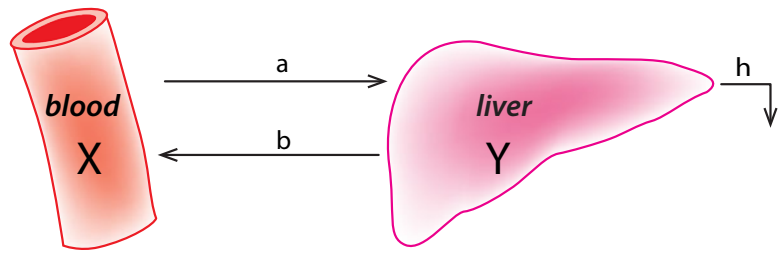


Figure 6.44: Compartmental model of the movement of a tracer dye between the liver and the bloodstream.

The problem is that we can't observe h. All we can observe is $X(t)$, the concentration of the dye in the blood. We can estimate $X(t)$ by making a number of blood draws over time, measuring the dye level at each time point and then using curve-fitting software to estimate the smooth curve that best fits the data points.

In order to get from an observation of $X(t)$ to an estimation of h, we need to solve this model. The differential equations are

$$\begin{aligned} X' &= -\underbrace{aX}_{\text{blood}\to\text{liver}} + \underbrace{bY}_{\text{liver}\to\text{blood}} \\ Y' &= \underbrace{aX}_{\text{blood}\to\text{liver}} - \underbrace{bY}_{\text{liver}\to\text{blood}} - \underbrace{hY}_{\text{liver}\to\text{bile}} \end{aligned}$$

which we can write as a matrix differential equation

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{bmatrix} -a & b \\ a & -(b+h) \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

To model a single injection of the dye (BSP), we set the initial condition of the dye concentration in the blood compartment to a nonzero value $X(0) = c$, and the initial condition of the dye concentration in the liver compartment $Y(0) = 0$.

We will solve for the long-term dynamics by finding the eigenvalues of the matrix

$$\boldsymbol{M} = \begin{bmatrix} -a & b \\ a & -(b+h) \end{bmatrix} \qquad (a > 0, b > 0, h > 0)$$

Plugging the four entries of $\boldsymbol{M}$ into the characteristic polynomial (equation (6.3) on page 302), we get the two eigenvalues as

$$\lambda_1, \lambda_2 = \frac{1}{2}\Big(-(a+b+h) \pm \sqrt{(a+b+h)^2 - 4ah} \Big)$$

First of all, let's note that both eigenvalues are real. In order for this to be true, the expression under the $\sqrt{}$ sign has to be nonnegative. This is easily checked:

$$\begin{aligned} (a+b+h)^2 - 4ah &= a^2 + b^2 + h^2 + 2ab + 2ah + 2bh - 4ah \\ &= a^2 + b^2 + h^2 + 2ab - 2ah + 2bh \\ &= a^2 - 2ah + h^2 + b^2 + 2ab + 2bh \\ &= (a-h)^2 + 2ab + 2bh + b^2 \\ &> 0 \end{aligned}$$

The next question is whether the eigenvalues are negative or positive. That depends upon whether $\sqrt{(a+b+h)^2 - 4ah}$ is less than $(a+b+h)$. It is certainly true that

$$(a+b+h)^2 - 4ah < (a+b+h)^2$$

since $4ah$ is a positive number. This implies

$$\sqrt{(a+b+h)^2 - 4ah} < a+b+h$$

which implies

$$-(a+b+h) \pm \sqrt{(a+b+h)^2 - 4ah} < 0$$

So both eigenvalues λ_1, λ_2 are negative real numbers, which means that $(0,0)$, the state in which all dye is cleared, is a stable equilibrium point. Therefore, the behavior in approach to the stable equilibrium point is the sum of two exponentially decaying terms. The question is how fast the state point goes to the stable equilibrium point, for which we need the explicit solution.

Suppose that the eigenvectors corresponding to λ_1 and λ_2 are $\mathbf{U}$ and $\mathbf{V}$. Then we can write the explicit solution to the differential equation as

$$\begin{pmatrix} \mathbf{U}(t) \\ \mathbf{V}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{U}(0)e^{\lambda_1 t} \\ \mathbf{V}(0)e^{\lambda_2 t} \end{pmatrix}$$

But what we need, to compare it to the experimentally measured data, is $\mathbf{X}(t)$. So we need $\mathbf{X}(t)$ and $\mathbf{Y}(t)$, not $\mathbf{U}(t)$ and $\mathbf{V}(t)$.

We go from one coordinate system to the other just as we did before by means of the coordinate transformation matrix $\boldsymbol{T}$ that takes the $\{\mathbf{X}, \mathbf{Y}\}$ basis into the $\{\mathbf{U}, \mathbf{V}\}$ basis:

$$\begin{array}{ccc} \begin{pmatrix} \mathbf{X}(0) \\ \mathbf{Y}(0) \end{pmatrix} & \xrightarrow{\quad \boldsymbol{T} \quad} & \begin{pmatrix} \mathbf{U}(0) \\ \mathbf{V}(0) \end{pmatrix} \\ & & \Big\downarrow \lambda_1, \lambda_2 \\ \begin{pmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{pmatrix} & \xleftarrow{\quad \boldsymbol{T}^{-1} \quad} & \begin{pmatrix} \mathbf{U}(t) \\ \mathbf{V}(t) \end{pmatrix} \end{array}$$

When we carry this out, we get explicit solutions

$$X(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

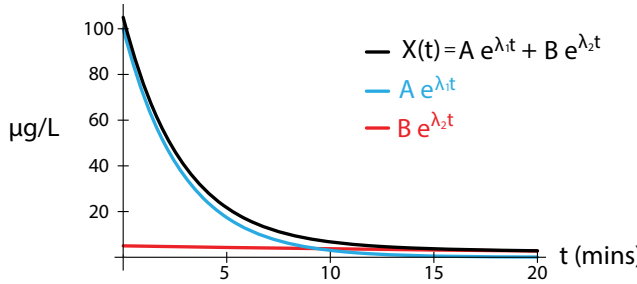
$$Y(t) = \frac{1}{b}\left(A(a-\lambda_1)e^{\lambda_1 t} + B(a-\lambda_2)e^{\lambda_2 t}\right)$$

where $$A = \frac{(a-\lambda_2)X(0) - bY(0)}{\lambda_1 - \lambda_2} \qquad B = \frac{(a-\lambda_1)X(0) - bY(0)}{\lambda_2 - \lambda_1}$$

In order to compare $X(t)$ to the experimental data, we face a problem. There are four unknown parameters in the $X(t)$ equation, and it is very difficult to infer four unknown parameters from a single curve.

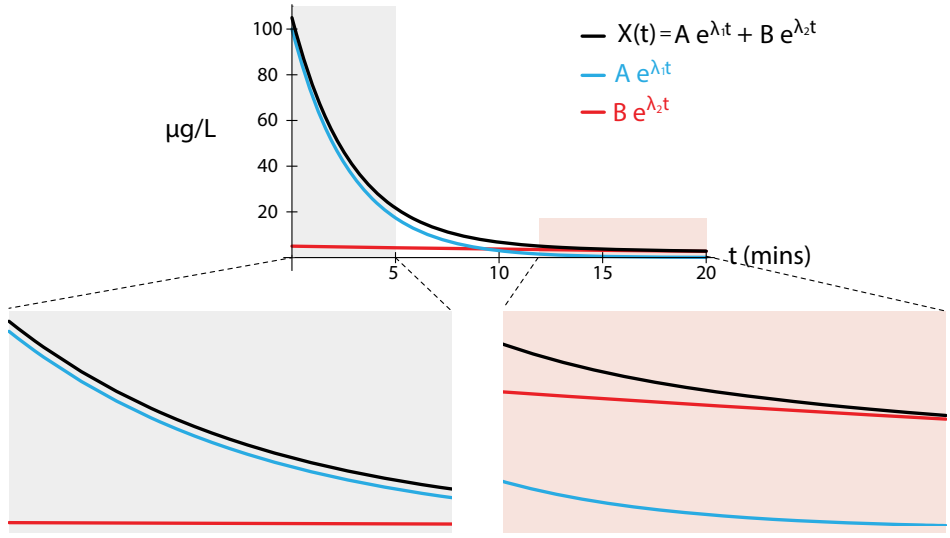
The key step in doing this is to think about the graph of a process that is represented by the sum of two negative exponentials.

Choosing typical numbers for the parameters, and assuming that $|\lambda_1|$ is significantly greater than $|\lambda_2|$, so that λ_1 is a rapidly decaying process and λ_2 is a slowly decaying process (which is the case in the liver), we obtain the following graph:



The trick is to notice that in the early part of the curve, say the first five minutes, the curve $X(t)$ is very close to the fast negative exponential, while for $t > 10$ minutes, the curve $X(t)$ is very close to the slowly decaying process.

We then use the first segment of the $X(t)$ curve to estimate $Ae^{\lambda_1 t}$, and the second segment of the $X(t)$ curve to estimate $Be^{\lambda_2 t}$. A simple calculation then gives us h, which is the liver's clearance rate.



Linear Differential Equations in n Dimensions

The extension to n-dimensional linear differential equations is straightforward: the situation in n dimensions is very similar to that in two dimensions, and no really new phenomena occur.

We already saw that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then the matrix $\boldsymbol{M}$ that represents f has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. We saw that each eigenvalue is either a real number or one of a pair of complex conjugate eigenvalues.

We can then say that the equilibrium point at $(0, 0, \ldots, 0)$ can be decomposed into

(1) stable 1D directions (eigenvectors whose eigenvalues $\lambda < 0$);
(2) unstable 1D directions (eigenvectors whose eigenvalues $\lambda > 0$);
(3) 2D spiraling behaviors corresponding to pairs of complex conjugate eigenvalues, which are stable (spiraling in) if the real part of the eigenvalues is negative, and unstable (spiraling out) if the real part of the eigenvalues is positive.

In this way, we can completely classify every equilibrium point of a linear differential equation.

Further Exercises 6.7

1. Suppose Romeo and Juliet's love obeys the differential equation

$$\begin{pmatrix} R' \\ J' \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} R \\ J \end{pmatrix}$$

where $\boldsymbol{A}$ is a 2×2 matrix with the following eigenvectors:

$$\begin{pmatrix} -2 \\ 3 \end{pmatrix} \text{ with eigenvalue } -1, \text{ and } \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ with eigenvalue } -4$$

a) Give a rough sketch of the vector field for this differential equation.

b) What will happen in the long run?

2. Romeo and Juliet's relationship is modeled by the equations

$$R' = 0.5R + J$$
$$J' = 2R - 0.1J$$

a) Find and classify all the equilibria for this system.

b) Use the system's eigenvectors to sketch its vector field.

3. Suppose Romeo and Juliet's love obeys the following differential equations:

$$R' = -R + 3J$$
$$J' = 3R - J$$

The matrix of this system is $\begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}$, which has the following eigenvectors:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ with eigenvalue } 2, \text{ and } \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ with eigenvalue } -4$$

We will use these two eigenvectors to define a new coordinate system, and we will use u and v to represent these coordinates. However, in this problem, we will treat u and v as *new variables*. Your goal is to rewrite this system of differential equations in terms of these new variables.

a) Starting with the definition of the coordinates u and v,

$$\begin{pmatrix} R \\ J \end{pmatrix} = u\begin{pmatrix} 1 \\ 1 \end{pmatrix} + v\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

solve for u and v in terms of R and J to get

$$u = \frac{1}{2}R + \frac{1}{2}J$$
$$v = -\frac{1}{2}R + \frac{1}{2}J$$

b) Since R and J are just functions of time, u and v are as well, and taking the derivative of both sides of the two equations above gives $u' = \frac{1}{2}R' + \frac{1}{2}J'$ and $v' = -\frac{1}{2}R' + \frac{1}{2}J'$. Substitute the original differential equations into this to get u' and v' in terms of R and J.

c) Now substitute the expressions for R and J (in terms of u and v) from part (a) into your answer from part (b) and simplify. This should give you u' and v' in terms of u and v.

d) What is the matrix of the new system of differential equations that you ended up with in part (c)? What do you notice about its form? What do you notice about the specific numbers that appear in it?