Chapter 3
Fourier Series

Principles of Fourier series go back to ancient times. The attempts of the Pythagorean school to explain musical harmony in terms of whole numbers embrace early elements of a trigonometric nature. The theory of epicycles in the *Almagest* of Ptolemy, based on work related to the circles of Appolonius, contains ideas of astronomical periodicities that we would interpret today as harmonic analysis. Early studies of acoustical and optical phenomena, as well as periodic astronomical and geophysical occurrences, provided a stimulus in the physical sciences toward the rigorous study of expansions of periodic functions. This study is carefully pursued in this chapter.

The modern theory of Fourier series begins with attempts to solve boundary value problems using trigonometric functions. The work of d'Alembert, Bernoulli, Euler, and Clairaut on the vibrating string led to the belief that it might be possible to represent arbitrary periodic functions as sums of sines and cosines. Fourier announced belief in this possibility in his solution of the problem of heat distribution in spatial bodies (in particular, for the cube $\mathbf{T}^3$) by expanding an arbitrary function of three variables as a triple sine series. Fourier's approach, although heuristic, was appealing and eventually attracted attention. It was carefully studied and further developed by many scientists, but most notably by Laplace and Dirichlet, who were the first to investigate the validity of the representation of a function in terms of its Fourier series. This is the main topic of study in this chapter.

3.1 Fourier Coefficients

We discuss some basic facts of Fourier analysis on the torus $\mathbf{T}^n$. Throughout this chapter, n denotes the dimension, i.e., a fixed positive integer.

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3.1.1 The n-Torus $\mathbf{T}^n$

The n-torus $\mathbf{T}^n$ is the cube $[0,1]^n$ with opposite sides identified. This means that the points $(x_1,\dots,0,\dots,x_n)$ and $(x_1,\dots,1,\dots,x_n)$ are identified whenever 0 and 1 appear in the same coordinate. A more precise definition can be given as follows: We say that x,y in $\mathbf{R}^n$ are equivalent and we write

$$x \equiv y \tag{3.1.1}$$

if $x-y \in \mathbf{Z}^n$. Here $\mathbf{Z}^n$ is the additive subgroup of all points in $\mathbf{R}^n$ with integer coordinates. It is a simple fact that $\equiv$ is an equivalence relation that partitions $\mathbf{R}^n$ into equivalence classes. The n-torus $\mathbf{T}^n$ is then defined as the set $\mathbf{R}^n/\mathbf{Z}^n$ of all such equivalence classes. When $n=1$, this set can be geometrically viewed as a circle by bending the line segment $[0,1]$ so that its endpoints are brought together. When $n=2$, the identification brings together the left and right sides of the unit square $[0,1]^2$ as well as the top and bottom sides. The resulting figure is a two-dimensional manifold embedded in $\mathbf{R}^3$ that looks like a donut. See Figure 3.1.

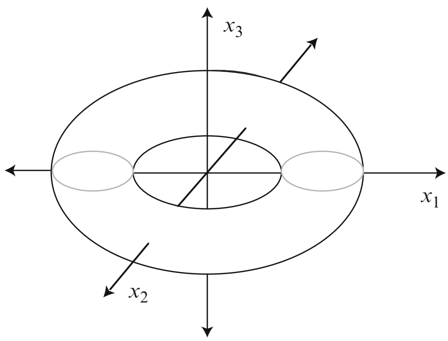

Fig. 3.1 The graph of the two-dimensional torus $\mathbf{T}^2$.

The n-torus is an additive group. The identity element of the group is 0, which of course coincides with every $e_j=(0,\dots,0,1,0,\dots,0)$. To avoid multiple appearances of the identity element in the group, we often think of the n-torus as the set $[-1/2,1/2]^n$. Since the group $\mathbf{T}^n$ is additive, the inverse of an element $x\in\mathbf{T}^n$ is denoted by $-x$. For example, $-(1/3,1/4)\equiv(2/3,3/4)$ on $\mathbf{T}^2$, or, equivalently, $-(1/3,1/4)-(2/3,3/4)\in\mathbf{Z}^2$.

The n-torus $\mathbf{T}^n$ can also be thought of as the following subset of $\mathbf{C}^n$,

$$\{(e^{2\pi i x_1},\dots,e^{2\pi i x_n})\in\mathbf{C}^n \,:\, (x_1,\dots,x_n)\in[0,1]^n\}, \tag{3.1.2}$$

in a way analogous to which the unit interval $[0,1]$ can be thought of as the unit circle in $\mathbf{C}$ once 1 and 0 are identified.

Functions on $\mathbf{T}^n$ are functions f on $\mathbf{R}^n$ that satisfy $f(x+m)=f(x)$ for all $x\in\mathbf{R}^n$ and $m\in\mathbf{Z}^n$. Such functions are called 1-*periodic* in every coordinate. Haar measure on the n-torus is the restriction of n-dimensional Lebesgue measure to the set

$\mathbf{T}^n = [0,1]^n$. This measure is still denoted by dx, while the measure of a set $A \subseteq \mathbf{T}^n$ is denoted by $|A|$. Translation invariance of Lebesgue measure and the periodicity of functions on $\mathbf{T}^n$ imply that for all integrable functions f on $\mathbf{T}^n$, we have

$$\int_{\mathbf{T}^n} f(x)\,dx = \int_{[-1/2,1/2]^n} f(x)\,dx = \int_{[a_1,1+a_1]\times\cdots\times[a_n,1+a_n]} f(x)\,dx \qquad (3.1.3)$$

for any real numbers $a_1,\dots,a_n$. In view of periodicity, integration by parts on the torus does not produce boundary terms; given f,g continuously differentiable functions on $\mathbf{T}^n$ we have

$$\int_{\mathbf{T}^n} \partial_j f(x)\,g(x)\,dx = -\int_{\mathbf{T}^n} \partial_j g(x)\,f(x)\,dx.$$

Elements of $\mathbf{Z}^n$ are denoted by $m=(m_1,\dots,m_n)$. For $m\in\mathbf{Z}^n$, we define the *total size* of m to be the number $|m| = (m_1^2+\cdots+m_n^2)^{1/2}$. Recall that for $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ in $\mathbf{R}^n$,

$$x\cdot y = x_1y_1+\cdots+x_ny_n$$

denotes the usual dot product. Finally, for $x\in\mathbf{T}^n$, $|x|$ denotes the usual Euclidean norm of x. If we identify $\mathbf{T}^n$ with $[-1/2,1/2]^n$, then $|x|$ can be interpreted as the distance of the element x from the origin, and then we have $0\le |x|\le \sqrt{n}/2$ for all $x\in\mathbf{T}^n$.

Multi-indices are elements of $(\mathbf{Z}^+\cup\{0\})^n$. For a multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$, we denote the partial derivative $\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n} f$ by $\partial^\alpha f$. The spaces $\mathscr{C}^k(\mathbf{T}^n)$ of *continuously differentiable functions of order k*, where $k\in\mathbf{Z}^+$, are defined as the sets of functions φ for which $\partial^\alpha\varphi$ exist and are continuous for all $|\alpha|\le k$. When $k=0$ we set $\mathscr{C}^0(\mathbf{T}^n)=\mathscr{C}(\mathbf{T}^n)$ to be the space of continuous functions on $\mathbf{T}^n$. The space $\mathscr{C}^\infty(\mathbf{T}^n)$ of infinitely differentiable functions on $\mathbf{T}^n$ is the union of all the $\mathscr{C}^k(\mathbf{T}^n)$. All of these spaces are contained in $L^p(\mathbf{T}^n)$, which are nested, with $L^1(\mathbf{T}^n)$ being the largest.

3.1.2 Fourier Coefficients

Definition 3.1.1. For a complex-valued function f in $L^1(\mathbf{T}^n)$ and m in $\mathbf{Z}^n$, we define

$$\widehat{f}(m) = \int_{\mathbf{T}^n} f(x)e^{-2\pi i m\cdot x}dx. \qquad (3.1.4)$$

We call $\widehat{f}(m)$ the mth *Fourier coefficient* of f. We note that $\widehat{f}(\xi)$ is not defined for $\xi\in\mathbf{R}^n\setminus\mathbf{Z}^n$, since the function $x\mapsto e^{-2\pi i\xi\cdot x}$ is not 1-periodic in any coordinate and therefore not well defined on $\mathbf{T}^n$. For a finite Borel measure μ on $\mathbf{T}^n$ and $m\in\mathbf{Z}^n$ the expression

$$\widehat{\mu}(m) = \int_{\mathbf{T}^n} e^{-2\pi i m\cdot x}d\mu \qquad (3.1.5)$$

is called the mth *Fourier coefficient* of μ.

The *Fourier series* of f at $x \in \mathbf{T}^n$ is the series

$$\sum_{m\in\mathbf{Z}^n} \widehat{f}(m)e^{2\pi i m\cdot x}. \tag{3.1.6}$$

It is not clear at present in which sense and for which $x \in \mathbf{T}^n$ (3.1.6) converges. The study of convergence of Fourier series is the main topic of study in this chapter.

We quickly recall the notation we introduced in Chapter 2. We denote by $\overline{f}$ the complex conjugate of the function f, by $\widetilde{f}$ the function $\widetilde{f}(x) = f(-x)$, and by $\tau^y(f)$ the function $\tau^y(f)(x) = f(x-y)$ for all $y \in \mathbf{T}^n$. We mention some elementary properties of Fourier coefficients.

Proposition 3.1.2. *Let f, g be in $L^1(\mathbf{T}^n)$. Then for all $m,k \in \mathbf{Z}^n$, $\lambda \in \mathbf{C}$, $y \in \mathbf{T}^n$, and all multi-indices α we have*

(1) $\widehat{f+g}(m) = \widehat{f}(m) + \widehat{g}(m)$,

(2) $\widehat{\lambda f}(m) = \lambda \widehat{f}(m)$,

(3) $\widehat{\overline{f}}(m) = \overline{\widehat{f}(-m)}$,

(4) $\widehat{\widetilde{f}}(m) = \widehat{f}(-m)$,

(5) $\widehat{\tau^y(f)}(m) = \widehat{f}(m)e^{-2\pi i m\cdot y}$,

(6) $(e^{2\pi i k(\cdot)}f)\widehat{\ }(m) = \widehat{f}(m-k)$,

(7) $\widehat{f}(0) = \int_{\mathbf{T}^n} f(x)\,dx$,

(8) $\sup_{m\in\mathbf{Z}^n} |\widehat{f}(m)| \le \|f\|_{L^1(\mathbf{T}^n)}$,

(9) $\widehat{f*g}(m) = \widehat{f}(m)\widehat{g}(m)$,

(10) $\widehat{\partial^\alpha f}(m) = (2\pi i m)^\alpha \widehat{f}(m)$*, whenever* $f \in \mathscr{C}^\alpha$.

Proof. The proof of properties (1)–(10) is rather easy and is left to the reader. We only sketch the proof of (9). We have

$$\widehat{f*g}(m) = \int_{\mathbf{T}^n}\int_{\mathbf{T}^n} f(x-y)g(y)e^{-2\pi i m\cdot(x-y)}e^{-2\pi i m\cdot y}\,dy\,dx = \widehat{f}(m)\widehat{g}(m),$$

where the interchange of integrals is justified by the absolute convergence of the integrals and Fubini's theorem. □

Remark 3.1.3. The Fourier coefficients have the following property. For a function f_1 on $\mathbf{T}^{n_1}$ and a function f_2 on $\mathbf{T}^{n_2}$, the tensor function

$$(f_1 \otimes f_2)(x_1,x_2) = f_1(x_1)f_2(x_2)$$

is a periodic function on $\mathbf{T}^{n_1+n_2}$ whose Fourier coefficients are

$$\widehat{f_1 \otimes f_2}(m_1, m_2) = \widehat{f_1}(m_1)\widehat{f_2}(m_2), \tag{3.1.7}$$

for all $m_1 \in \mathbf{Z}^{n_1}$ and $m_2 \in \mathbf{Z}^{n_2}$.

Definition 3.1.4. A *trigonometric polynomial* on $\mathbf{T}^n$ is a function of the form

$$P(x) = \sum_{m \in \mathbf{Z}^n} a_m e^{2\pi i m \cdot x}, \tag{3.1.8}$$

where $\{a_m\}_{m \in \mathbf{Z}^n}$ is a finitely supported sequence in $\mathbf{Z}^n$. The *degree* of P is the largest number $|q_1| + \cdots + |q_n|$ such that a_q is nonzero, where $q = (q_1, \dots, q_n)$. Observe that in view of the orthonormality of the exponentials we have for all $m \in \mathbf{Z}^n$

$$\widehat{P}(m) = a_m .$$

Example 3.1.5. If the sequence $\{a_m\}_m$ has only one nonzero term, then the trigonometric polynomial of Definition 3.1.4 reduces to a *trigonometric monomial*, which has the form

$$P(x) = a\, e^{2\pi i (q_1 x_1 + \cdots + q_n x_n)}$$

for some $q = (q_1, \dots, q_n) \in \mathbf{Z}^n$ and $a \in \mathbf{C}$.

Let

$$P(x) = \sum_{|m| \le N} a_m e^{2\pi i m \cdot x} = \sum_{|m| \le N} \widehat{P}(m) e^{2\pi i m \cdot x}$$

be a trigonometric polynomial on $\mathbf{T}^n$ and let μ be a finite Borel measure on $\mathbf{T}^n$. Then we have

$$(P * \mu)(x) = \int_{\mathbf{T}^n} \sum_{|m| \le N} \widehat{P}(m) e^{2\pi i m \cdot (x-y)}\, d\mu(y) = \sum_{|m| \le N} \widehat{P}(m)\widehat{\mu}(m) e^{2\pi i m \cdot x}. \tag{3.1.9}$$

In particular, if f is an integrable function on $\mathbf{T}^n$ we have

$$(P * f)(x) = \int_{\mathbf{T}^n} f(y) \sum_{|m| \le N} \widehat{P}(m) e^{2\pi i m \cdot (x-y)}\, dy = \sum_{|m| \le N} \widehat{P}(m)\widehat{f}(m) e^{2\pi i m \cdot x}. \tag{3.1.10}$$

This implies that the partial sums

$$\sum_{|m| \le N} \widehat{f}(m) e^{2\pi i m \cdot x}$$

of the Fourier series of f in (3.1.6) can be obtained by convolving f with the function

$$D_N(x) = \sum_{|m| \le N} e^{2\pi i m \cdot x}. \tag{3.1.11}$$

This function is called the *Dirichlet kernel*.

3.1.3 The Dirichlet and Fejér Kernels

Definition 3.1.6. Let $0 \le R < \infty$. The *square Dirichlet kernel* on $\mathbf{T}^n$ is the function

$$D_R^n(x) = \sum_{\substack{m\in\mathbf{Z}^n\\ |m_j|\le R}} e^{2\pi i m\cdot x}. \tag{3.1.12}$$

The *circular (or spherical) Dirichlet kernel* on $\mathbf{T}^n$ is the function

$$\overset{\circ}{D}{}_R^n(x) = \sum_{\substack{m\in\mathbf{Z}^n\\ |m|\le R}} e^{2\pi i m\cdot x}. \tag{3.1.13}$$

In dimension $n = 1$ these functions coincide and are denoted by

$$D_R(x) = D_R^1(x) = \overset{\circ}{D}{}_R^1(x).$$

This function is called the *Dirichlet kernel* and coincides with $D_N(x)$ in (3.1.11) when $N \le R < N+1$ and $N \in \mathbf{Z}^+ \cup \{0\}$; see Figure 3.2.

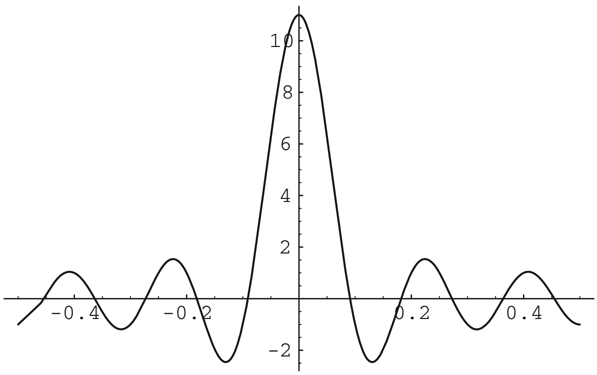

Fig. 3.2 The graph of the Dirichlet kernel D_5 plotted on the interval $[-1/2, 1/2]$.

Both the square and circular (or spherical) Dirichlet kernels are trigonometric polynomials. The square Dirichlet kernel on $\mathbf{T}^n$ is equal to a product of one-dimensional Dirichlet kernels, that is,

$$D_R^n(x_1,\dots,x_n) = D_R(x_1)\cdots D_R(x_n). \tag{3.1.14}$$

We have the following two equivalent ways to write the Dirichlet kernel D_N:

$$D_N(x) = \sum_{|m|\le N} e^{2\pi i m\cdot x} = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}, \tag{3.1.15}$$

for $x \in [0,1]$. To verify the validity of (3.1.15), we write

$$\sum_{|m|\le N} e^{2\pi i m\cdot x} = e^{-2\pi i N x}\frac{e^{2\pi i(2N+1)x}-1}{e^{2\pi i x}-1} = \frac{e^{2\pi i(N+1)x}-e^{-2\pi i N x}}{e^{\pi i x}(e^{\pi i x}-e^{-\pi i x})} = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}.$$

It follows that for $R \in \mathbf{R}^+ \cup \{0\}$ we have

$$D_R(x) = \frac{\sin(\pi x(2[R]+1))}{\sin(\pi x)}. \tag{3.1.16}$$

It is reasonable to ask whether the family $\{D_R\}_{R>0}$ forms an approximate identity as $R\to\infty$. Using (3.1.15) we see that each D_R is integrable over $[-1/2,1/2]$ and has integral equal to 1. But it follows from Exercise 3.1.5 that $\|D_R\|_{L^1} \approx \log R$ as $R\to\infty$, and therefore property (i) in Definition 1.2.15 fails for D_R. We conclude that the family $\{D_R\}_{R>0}$ is not an approximate identity on $\mathbf{T}^1$, which significantly complicates the study of Fourier series. Consequently, the family $\{D_R^n\}_{R>0}$ is not an approximate identity on $\mathbf{T}^n$, since $\|D_R^n\|_{L^1(\mathbf{T}^1)} \approx (\log R)^n$. The same is true for the family of circular (or spherical) Dirichlet kernels $\{\overset{\circ}{D}{}_R^n\}_{R>0}$. Although this is harder to prove, it will be a consequence of the results in Section 4.2.

A typical situation encountered in analysis is that the means of a sequence behave better than the original sequence. This fact led Cesàro and independently Fejér to consider the arithmetic means of the Dirichlet kernel in dimension 1, that is, the expressions

$$F_N(x) = \frac{1}{N+1}\big[D_0(x)+D_1(x)+D_2(x)+\cdots+D_N(x)\big]. \tag{3.1.17}$$

The expression in (3.1.17) is in fact equal to the Fejér kernel given in Example 1.2.18. We have the following identity concerning the kernel F_N.

Proposition 3.1.7. *For every nonnegative integer N the identity holds*

$$F_N(x) = \sum_{j=-N}^{N}\left(1-\frac{|j|}{N+1}\right)e^{2\pi i j x} = \frac{1}{N+1}\left(\frac{\sin(\pi(N+1)x)}{\sin(\pi x)}\right)^2 \tag{3.1.18}$$

for all $x\in\mathbf{T}^1$. Thus $\widehat{F_N}(m) = 1-\frac{|m|}{N+1}$ if $|m|\le N$ and zero otherwise.

Proof. The fact that the expression in (3.1.17) is equal to the middle term in (3.1.18) is a consequence of the trivial calculation:

$$\frac{1}{N+1}\sum_{k=0}^{N} D_k(x) = \frac{1}{N+1}\sum_{k=0}^{N}\sum_{|j|\le k} e^{2\pi i j x} = \sum_{|j|\le N}\frac{\#\{k\in\mathbf{Z}:\ |j|\le k\le N\}}{N+1}e^{2\pi i j x}.$$

To verify the second equality in (3.1.18) we use the simple geometric series identity $1+r+r^2+\cdots+r^N = \frac{1-r^{N+1}}{1-r}$ to write for $x\neq 0$

$$\sum_{j=1}^{N} e^{2\pi i jx} = \frac{e^{2\pi i(N+1)x}-1}{e^{2\pi ix}-1} - 1 = \frac{e^{i\pi(N+1)x}}{e^{i\pi x}} \frac{e^{\pi i(N+1)x}-e^{-\pi i(N+1)x}}{e^{\pi ix}-e^{-\pi ix}} - 1$$

from which it follows that

$$\sum_{j=1}^{N} |j| e^{2\pi i jx} = \frac{1}{2\pi i}\frac{d}{dx}\left(e^{i\pi Nx}\frac{\sin(\pi(N+1)x)}{\sin(\pi x)}\right). \tag{3.1.19}$$

Likewise we prove that

$$\sum_{j=-N}^{-1} |j| e^{2\pi i jx} = -\frac{1}{2\pi i}\frac{d}{dx}\left(e^{-i\pi Nx}\frac{\sin(\pi(N+1)x)}{\sin(\pi x)}\right). \tag{3.1.20}$$

Adding (3.1.19) and (3.1.20) we deduce

$$\sum_{|j|\le N} |j| e^{2\pi i jx} = \frac{1}{\pi}\frac{d}{dx}\left(\frac{\sin(\pi Nx)\sin(\pi(N+1)x)}{\sin(\pi x)}\right). \tag{3.1.21}$$

Multiplying (3.1.21) by $-\frac{1}{N+1}$ and adding $D_N(x)$ we obtain

$$\sum_{j=-N}^{N}\left(1-\frac{|j|}{N+1}\right)e^{2\pi i jx} = \frac{\sin(\pi(2N+1)x)}{\sin(\pi x)} - \frac{1}{\pi}\frac{d}{dx}\left(\sin(\pi Nx)\frac{\sin(\pi(N+1)x)}{(N+1)\sin(\pi x)}\right).$$

Writing the preceding expression on the right as

$$\frac{(N+1)\sin(\pi(N+1)x)\cos(\pi Nx)\sin(\pi x)+(N+1)\cos(\pi(N+1)x)\sin(\pi Nx)\sin(\pi x)}{(N+1)\sin^2(\pi x)}$$

$$-\frac{1}{\pi}\frac{\frac{d}{dx}\big\{\sin(\pi Nx)\sin(\pi(N+1)x)\big\}\sin(\pi x)-\big\{\sin(\pi Nx)\sin(\pi(N+1)x)\big\}\pi\cos(\pi x)}{(N+1)\sin^2(\pi x)},$$

computing the derivative of the expression in the curly brackets, and simplifying, we finally obtain that

$$\sum_{j=-N}^{N}\left(1-\frac{|j|}{N+1}\right)e^{2\pi i jx} = \frac{1}{N+1}\left(\frac{\sin(\pi(N+1)x)}{\sin(\pi x)}\right)^2. \tag{3.1.22}$$

This proves the second identity in (3.1.18). □

Definition 3.1.8. Let N be a nonnegative integer. The function F_N on $\mathbf{T}^1$ given by (3.1.22) is called the *Fejér kernel*.

The Fejér kernel F_N^n on $\mathbf{T}^n$ is defined as the product of the 1-dimensional Fejér kernels, or as the average of the product of the Dirichlet kernels in each variable, precisely, $F_N^1(x) = F_N(x)$ and

$$\begin{aligned} F_N^n(x_1,\dots,x_n) &= \prod_{j=1}^n F_N(x_j) \\ &= \prod_{j=1}^n \left(\frac{1}{N+1} \sum_{k=0}^N D_k(x_j) \right) \\ &= \frac{1}{(N+1)^n} \sum_{k_1=0}^N \cdots \sum_{k_n=0}^N D_{k_1}(x_1) \cdots D_{k_n}(x_n)\,. \end{aligned}$$

Note that F_N^n is a trigonometric polynomial of degree nN.

Remark 3.1.9. Using the identities for F_N in (3.1.18), we may write for all $N \ge 0$

$$F_N^n(x_1,\dots,x_n) = \sum_{\substack{m\in\mathbf{Z}^n \\ |m_j|\le N}} \left(1-\frac{|m_1|}{N+1}\right)\cdots\left(1-\frac{|m_n|}{N+1}\right) e^{2\pi i m\cdot x} \tag{3.1.23}$$

$$= \frac{1}{(N+1)^n} \prod_{j=1}^n \left(\frac{\sin(\pi(N+1)x_j)}{\sin(\pi x_j)} \right)^2, \tag{3.1.24}$$

thus $F_N^n \ge 0$. Observe that $F_0^n(x) = 1$ for all $x \in \mathbf{T}^n$ and that $F_N^n(0) = (N+1)^n$.

Proposition 3.1.10. *The family of Fejér kernels $\{F_N^n\}_{N=0}^\infty$ is an approximate identity on $\mathbf{T}^n$.*

Proof. Since $F_N^n \ge 0$ we have that $\|F_N^n\|_{L^1} = \int_{\mathbf{T}^n} F_N^n\,dx$. Also $\int_{\mathbf{T}^n} F_N^n\,dx = 1$, in view of identity (3.1.23). Thus properties (i) and (ii) of approximate identities (according to Definition 1.2.15) hold. To prove property (iii) of the definition we make use of identity (3.1.24). Using the fact that $1 \le \frac{|t|}{|\sin t|} \le \frac{\pi}{2}$ when $|t| \le \frac{\pi}{2}$, we obtain

$$F_N(x) \le \frac{1}{N+1} \min\left(\frac{(N+1)|\pi x|}{|\sin(\pi x)|}, \frac{1}{|\sin(\pi x)|} \right)^2 \le \frac{1}{N+1}\frac{\pi^2}{4} \min\left(N+1, \frac{1}{|\pi x|}\right)^2$$

when $|x| \le \frac{1}{2}$. This implies that for $\delta > 0$ we have

$$\int_{\delta\le|x|\le\frac{1}{2}} F_N(x)\,dx \le \frac{1}{N+1}\frac{\pi^2}{4} \int_{\delta\le|x|\le\frac{1}{2}} \frac{dx}{|\pi\delta|^2} \le \frac{1}{4\delta^2}\frac{1}{N+1} \to 0$$

as $N \to \infty$. In higher dimensions, given $x = (x_1,\dots,x_n) \in [-1/2,1/2]^n$ with $|x| \ge \delta$, there is a $j \in \{1,\dots,n\}$ such that $|x_j| \ge \delta/\sqrt{n}$ and thus

$$\int_{\substack{x\in\mathbf{T}^n \\ |x|\ge\delta}} F_N^n(x)\,dx \le \sum_{j=1}^n \int_{|x_j|\ge\frac{\delta}{\sqrt{n}}} F_N(x_j)\,dx_j \prod_{k\ne j} \int_{\mathbf{T}^1} F_N(x_k)\,dx_k \le \frac{n}{4(\delta/\sqrt{n})^2}\frac{1}{N+1} \to 0\,.$$

This proves the claim. □

Exercises

3.1.1. Identify $\mathbf{T}^1$ with $[-1/2, 1/2]$ and let $h(t)$ be an integrable function on $\mathbf{T}^1$.
(a) If $h(t) \geq 0$ is even, show that $\widehat{h}(m)$ is real and $|\widehat{h}(m)| \leq \widehat{h}(0)$ for all $m \in \mathbf{Z}$.
(b) If $h(t)$ is odd and $h(t) \geq 0$ on $[0, 1/2)$, then $i\widehat{h}(m)$ is real and $|\widehat{h}(m)| \leq im\widehat{h}(1)$ for all $m \in \mathbf{Z}$.

3.1.2. Suppose that h is a periodic integrable function on $[-1/2, 1/2)$ with integral zero. Define another periodic function H on $\mathbf{T}^1$ by setting

$$H(x) = \int_{-1/2}^{x} h(t)\, dt\,.$$

Compute $\widehat{H}(m)$ in terms of $\widehat{h}(m)$ for $m \in \mathbf{Z}$.

3.1.3. Suppose that $\{g_\varepsilon\}_{\varepsilon>0}$ is an approximate identity on $\mathbf{R}^n$ as $\varepsilon \to 0$ and let

$$G_\varepsilon(x) = \sum_{\ell \in \mathbf{Z}^n} g_\varepsilon(x+\ell)\,.$$

Show that the family $\{G_\varepsilon\}_{\varepsilon>0}$ is an approximate identity on $\mathbf{T}^n$.

3.1.4. On $\mathbf{T}^1$ define the *de la Vallée Poussin kernel*

$$V_N(x) = 2F_{2N+1}(x) - F_N(x)\,.$$

(a) Show that the sequence V_N is an approximate identity.
(b) Prove that $\widehat{V_N}(m) = 1$ when $|m| \leq N+1$, and $\widehat{V_N}(m) = 0$ when $|m| \geq 2N+2$.

3.1.5. (a) Show that for all $|t| \leq \frac{\pi}{2}$ we have

$$\left| \frac{1}{\sin(t)} - \frac{1}{t} \right| \leq 1 - \frac{2}{\pi}\,.$$

(b) Let D_N be the Dirichlet kernel on $\mathbf{T}^1$. Prove that for $N \in \mathbf{Z}^+$ we have

$$\frac{4}{\pi^2} \sum_{k=1}^{N} \frac{1}{k} \leq \big\|D_N\big\|_{L^1} \leq 3 - \frac{2}{\pi} + \frac{4}{\pi^2} \sum_{k=1}^{N} \frac{1}{k}\,.$$

Conclude that the numbers $\|D_N\|_{L^1}$ grow logarithmically as $N \to \infty$ and therefore the family $\{D_N\}_{N=1}^{\infty}$ is not an approximate identity on $\mathbf{T}^1$. The numbers $\|D_N\|_{L^1}$, $N = 1, 2, \dots$ are called the *Lebesgue constants*.
$\big[$*Hint:* Part (a): Show that the derivative of $\frac{1}{\sin(t)} - \frac{1}{t}$ is nonnegative on $(0, \frac{\pi}{2}]$, or equivalently prove that $\tan(t)\sin(t) \geq t^2$ on $(0, \frac{\pi}{2}]$; this is a consequence of the inequality $\sqrt{\sin(t)\tan(t)} \geq 2(\frac{1}{\sin(t)} + \frac{1}{\tan(t)})^{-1} = 2\tan(\frac{t}{2}) \geq t$. Part (b): Replace $D_N(t)$ by $\frac{\sin((2N+1)\pi t)}{\pi t}$ and estimate the difference using part (a).$\big]$

3.1.6. Let D_N be the Dirichlet kernel on $\mathbf{T}^1$. Prove that for all $1 < p < \infty$ there exist two constants $C_p, c_p > 0$ such that

$$c_p\,(2N+1)^{1/p'} \le \big\|D_N\big\|_{L^p} \le C_p\,(2N+1)^{1/p'}.$$

$\big[$*Hint:* Consider the two closest zeros of D_N near the origin and split the integral into the intervals thus obtained.$\big]$

3.1.7. The Poisson kernel on $\mathbf{T}^n$ is the function

$$P_{r_1,\dots,r_n}(x) = \sum_{m\in\mathbf{Z}^n} r_1^{|m_1|}\cdots r_n^{|m_n|} e^{2\pi i m\cdot x}$$

and is defined for $0 < r_1,\dots,r_n < 1$. Prove that $P_{r_1,\dots,r_n}$ can be written as

$$P_{r_1,\dots,r_n}(x_1,\dots,x_n) = \prod_{j=1}^{n} \mathrm{Re}\left(\frac{1+r_j e^{2\pi i x_j}}{1-r_j e^{2\pi i x_j}}\right) = \prod_{j=1}^{n} \frac{1-r_j^2}{1-2r_j\cos(2\pi x_j)+r_j^2},$$

and conclude that $P_{r,\dots,r}(x)$ is an approximate identity as $r \uparrow 1$.

3.2 Reproduction of Functions from Their Fourier Coefficients

We can obtain very interesting results using the Fejér kernel.

Proposition 3.2.1. *The set of trigonometric polynomials is dense in* $L^p(\mathbf{T}^n)$ *for* $1 \le p < \infty$.

Proof. Given f in $L^p(\mathbf{T}^n)$ for $1 \le p < \infty$, consider $f * F_N^n$. Clearly $f * F_N^n$ is also a trigonometric polynomial. In view of Theorem 1.2.19 (1), $f * F_N^n$ converges to f in L^p as $N \to \infty$. □

Corollary 3.2.2. ***(Weierstrass approximation theorem for trigonometric polynomials)*** *Every continuous function on the torus is a uniform limit of trigonometric polynomials.*

Proof. Since f is continuous on $\mathbf{T}^n$, which is a compact set, Theorem 1.2.19 (2) gives that $f * F_N^n$ converges uniformly to f as $N \to \infty$. But $f * F_N^n$ is a trigonometric polynomial, and so we conclude that every continuous function on $\mathbf{T}^n$ can be uniformly approximated by trigonometric polynomials. □

3.2.1 Partial sums and Fourier inversion

We now define the partial sums of Fourier series.

Definition 3.2.3. Let $R \geq 0$ and $N \in \mathbf{Z}^+ \cup \{0\}$. The expressions

$$(f * D_R^n)(x) = \sum_{\substack{m\in\mathbf{Z}^n\\ |m_j|\leq R}} \widehat{f}(m)e^{2\pi i m\cdot x}$$

are called the *square partial sums of the Fourier series* of f. Then the expressions

$$(f * \overset{\circ}{D}{}_R^n)(x) = \sum_{\substack{m\in\mathbf{Z}^n\\ |m|\leq R}} \widehat{f}(m)e^{2\pi i m\cdot x}$$

are called the *circular (or spherical) partial sums* of the Fourier series of f. The expressions

$$(f * F_N^n)(x) = \sum_{\substack{m\in\mathbf{Z}^n\\ |m_j|\leq N}} \left(1-\frac{|m_1|}{N+1}\right)\cdots\left(1-\frac{|m_n|}{N+1}\right)\widehat{f}(m)e^{2\pi i m\cdot x}$$

are called the *square Cesàro means* (or *square Fejér means*) of f.

Finally, for $R \geq 0$ the expressions

$$(f * \overset{\circ}{F}{}_R^n)(x) = \sum_{\substack{m\in\mathbf{Z}^n\\ |m|\leq R}} \left(1-\frac{|m|}{R}\right)\widehat{f}(m)e^{2\pi i m\cdot x}$$

are called the *circular Cesàro means* (or *circular Fejér means*) of f.

Observe that $f * \overset{\circ}{F}{}_R^n$ is equal to the average of the expressions $f * \overset{\circ}{D}{}_r^n$ in the following sense:

$$(f * \overset{\circ}{F}{}_R^n)(x) = \frac{1}{R}\int_0^R (f * \overset{\circ}{D}{}_r^n)(x)\,dr.$$

This is analogous to the fact that the Fejér kernel F_N is the average of the Dirichlet kernels $D_0, D_1, \dots, D_N$.

A fundamental problem is in what sense the partial sums of the Fourier series converge back to the function as $R \to \infty$ or $N \to \infty$. This problem is of central importance in harmonic analysis and is in part investigated in this chapter.

We now ask the question whether the Fourier coefficients uniquely determine the function. The answer is affirmative and simple.

Proposition 3.2.4. *If $f, g \in L^1(\mathbf{T}^n)$ satisfy $\widehat{f}(m) = \widehat{g}(m)$ for all m in $\mathbf{Z}^n$, then $f = g$ a.e.*

Proof. By linearity of the problem, it suffices to assume that $g = 0$. If $\widehat{f}(m) = 0$ for all $m \in \mathbf{Z}^n$, Definition 3.2.3 implies that $F_N^n * f = 0$ for all $N \in \mathbf{Z}^+$. In view of Proposition 3.1.10, the sequence $\{F_N^n\}_{N\in\mathbf{Z}^+}$ is an approximate identity as $N \to \infty$. Therefore,

$$\left\| f - F_N^n * f \right\|_{L^1} \to 0$$

as $N \to \infty$; hence $\|f\|_{L^1} = 0$, from which we conclude that $f = 0$ a.e. □

A useful consequence of the result just proved is the following.

Proposition 3.2.5. ***(Fourier inversion)*** *Suppose that $f \in L^1(\mathbf{T}^n)$ and that*

$$\sum_{m\in\mathbf{Z}^n} |\widehat{f}(m)| < \infty .$$

Then

$$f(x) = \sum_{m\in\mathbf{Z}^n} \widehat{f}(m) e^{2\pi i m\cdot x} \qquad a.e., \tag{3.2.1}$$

and therefore f is almost everywhere equal to a continuous function.

Proof. It is straightforward to check that both functions in (3.2.1) are well defined and have the same Fourier coefficients. Therefore, they must be almost everywhere equal by Proposition 3.2.4. Moreover, the function on the right in (3.2.1) is everywhere continuous. □

3.2.2 Fourier series of square summable functions

Let H be a separable Hilbert space with complex inner product $\langle\cdot\,|\,\cdot\rangle$. Recall that a subset E of H is called *orthonormal* if $\langle f\,|\,g\rangle = 0$ for all f, g in E with $f \neq g$, while $\langle f\,|\,f\rangle = 1$ for all f in E. A *complete orthonormal system* is a subset of H having the additional property that the only vector orthogonal to all of its elements is the zero vector. We summarize basic properties about orthonormal systems in the proposition below (see [307]).

Proposition 3.2.6. *Let H be a separable Hilbert space and let $\{\varphi_k\}_{k\in\mathbf{Z}}$ be an orthonormal system in H. Then the following are equivalent:*
(1) $\{\varphi_k\}_{k\in\mathbf{Z}}$ is a complete orthonormal system.
(2) For every $f \in H$ we have

$$\|f\|_H^2 = \sum_{k\in\mathbf{Z}} |\langle f\,|\,\varphi_k\rangle|^2 .$$

(3) For every $f \in H$ we have

$$f = \lim_{N\to\infty} \sum_{|k|\le N} \langle f\,|\,\varphi_k\rangle \varphi_k ,$$

where the series converges in H.

Now consider the Hilbert space space $L^2(\mathbf{T}^n)$ with inner product

$$\langle f\,|\,g\rangle = \int_{\mathbf{T}^n} f(t)\overline{g(t)}\, dt .$$

Let φ_m be the sequence of functions $\xi \mapsto e^{2\pi i m\cdot\xi}$ indexed by $m \in \mathbf{Z}^n$. The orthonormality of the sequence $\{\varphi_m\}$ is a consequence of the following simple but crucial identity:

$$\int_{[0,1]^n} e^{2\pi i m\cdot x}\,\overline{e^{2\pi i k\cdot x}}\,dx = \begin{cases} 1 & \text{when } m = k, \\ 0 & \text{when } m \neq k. \end{cases}$$

The completeness of the sequence $\{\varphi_m\}$ is also evident. Since $\langle f \,|\, \varphi_m\rangle = \widehat{f}(m)$ for all $f \in L^2(\mathbf{T}^n)$, it follows from Proposition 3.2.4 that if $\langle f \,|\, \varphi_m\rangle = 0$ for all $m \in \mathbf{Z}^n$, then $f = 0$ a.e.

The next result is a consequence of Proposition 3.2.6.

Proposition 3.2.7. *The following are valid for* $f, g \in L^2(\mathbf{T}^n)$:
*(1) (**Plancherel's identity**)*

$$\|f\|_{L^2}^2 = \sum_{m\in\mathbf{Z}^n} |\widehat{f}(m)|^2 .$$

(2) The function $f(t)$ *is a.e. equal to the* $L^2(\mathbf{T}^n)$ *limit of the sequence*

$$\lim_{M\to\infty} \sum_{|m|\le M} \widehat{f}(m) e^{2\pi i m\cdot t} .$$

*(3) (**Parseval's relation**)*

$$\int_{\mathbf{T}^n} f(t)\overline{g(t)}\,dt = \sum_{m\in\mathbf{Z}^n} \widehat{f}(m)\overline{\widehat{g}(m)} .$$

(4) The map $f \mapsto \{\widehat{f}(m)\}_{m\in\mathbf{Z}^n}$ *is an isometry from* $L^2(\mathbf{T}^n)$ *onto* ℓ^2.
(5) For all $k \in \mathbf{Z}^n$ *we have*

$$\widehat{fg}(k) = \sum_{m\in\mathbf{Z}^n} \widehat{f}(m)\widehat{g}(k-m) = \sum_{m\in\mathbf{Z}^n} \widehat{f}(k-m)\widehat{g}(m) .$$

Proof. (1) and (2) follow from the corresponding statements in Proposition 3.2.6. Notice that both sides of (3) converge by the Cauchy-Schwarz inequality. Parseval's relation (3) follows from *polarization*. By this we mean the following procedure. First replace f by $f+g$ in (1) and expand the squares to obtain

$$\begin{aligned} \|f\|_{L^2}^2 + \|g\|_{L^2}^2 + 2\mathrm{Re}\,\langle f \,|\, g\rangle &= \|f+g\|_{L^2}^2 \\ &= \sum_{m\in\mathbf{Z}^n} |\widehat{f}(m) + \widehat{g}(m)|^2 \\ &= \sum_{m\in\mathbf{Z}^n} |\widehat{f}(m)|^2 + \sum_{m\in\mathbf{Z}^n} |\widehat{g}(m)|^2 + 2\mathrm{Re} \sum_{m\in\mathbf{Z}^n} \widehat{f}(m)\overline{\widehat{g}(m)} \end{aligned}$$

and from this it follows that the real parts of the expressions in (3) are equal. Next replace f by $f+ig$ in (1) and expand the squares. Using $\mathrm{Re}\,(-iw) = \mathrm{Im}\,w$ we obtain

$$\begin{aligned}\|f\|_{L^2}^2+\|g\|_{L^2}^2+2\mathrm{Im}\,\langle f\,|\,g\rangle &=\|f+ig\|_{L^2}^2\\ &=\sum_{m\in\mathbf{Z}^n}|\widehat{f}(m)+i\widehat{g}(m)|^2\\ &=\sum_{m\in\mathbf{Z}^n}|\widehat{f}(m)|^2+\sum_{m\in\mathbf{Z}^n}|\widehat{g}(m)|^2+2\mathrm{Im}\sum_{m\in\mathbf{Z}^n}\widehat{f}(m)\overline{\widehat{g}(m)},\end{aligned}$$

and thus the imaginary parts of the expressions in (3) are equal. Thus (3) holds. Next we prove (4). We already know that the map $f\mapsto\{\widehat{f}(m)\}_{m\in\mathbf{Z}^n}$ is an injective isometry. It remains to show that it is onto. Given a square summable sequence $\{a_m\}_{m\in\mathbf{Z}^n}$ of complex numbers, define

$$f_N(t)=\sum_{|m|\le N}a_m e^{2\pi i m\cdot t}.$$

Observe that f_N is a Cauchy sequence in $L^2(\mathbf{T}^n)$ and it therefore converges to some $f\in L^2(\mathbf{T}^n)$. Then we have $\widehat{f}(m)=a_m$ for all $m\in\mathbf{Z}^n$. Finally, (5) is a consequence of (3) and Proposition 3.1.2 (6) and (3). □

3.2.3 The Poisson Summation Formula

We end this section with an important result that connects Fourier analysis on the torus with Fourier analysis on $\mathbf{R}^n$. Suppose that f is an integrable function on $\mathbf{R}^n$ and let $\widehat{f}$ be its Fourier transform. Restrict $\widehat{f}$ on $\mathbf{Z}^n$ and form the "Fourier series" (assuming that it converges)

$$\sum_{m\in\mathbf{Z}^n}\widehat{f}(m)e^{2\pi i m\cdot x}.$$

What does this series represent? Since the preceding function is 1-periodic in every variable, it follows that it cannot be equal to f, unless f is itself periodic. However, it should not come as a surprise that it is in fact equal to the periodization of f on $\mathbf{R}^n$. In other words, the Fourier expansion of a function on $\mathbf{R}^n$ reproduces the periodization of the function.

Theorem 3.2.8. ***(Poisson summation formula)*** *Let f be a continuous function on $\mathbf{R}^n$ which satisfies for some $C,\delta>0$ and for all $x\in\mathbf{R}^n$*

$$|f(x)|\le C(1+|x|)^{-n-\delta},$$

and whose Fourier transform $\widehat{f}$ restricted on $\mathbf{Z}^n$ satisfies

$$\sum_{m\in\mathbf{Z}^n}|\widehat{f}(m)|<\infty. \tag{3.2.2}$$

Then for all $x \in \mathbf{R}^n$ we have

$$\sum_{m\in\mathbf{Z}^n} \widehat{f}(m)e^{2\pi i m\cdot x} = \sum_{k\in\mathbf{Z}^n} f(x+k), \tag{3.2.3}$$

and in particular

$$\sum_{m\in\mathbf{Z}^n} \widehat{f}(m) = \sum_{k\in\mathbf{Z}^n} f(k).$$

Proof. Define a 1-periodic function on $\mathbf{T}^n$ by setting

$$F(x) = \sum_{k\in\mathbf{Z}^n} f(x+k)\,.$$

It is straightforward to verify that $\|F\|_{L^1([0,1]^n)} = \|f\|_{L^1(\mathbf{R}^n)}$, thus F lies in $L^1(\mathbf{T}^n)$. We prove that the sequence of the Fourier coefficients of F coincides with the restriction of the Fourier transform of f on $\mathbf{Z}^n$. This follows from the calculation

$$\begin{aligned}
\widehat{F}(m) &= \int_{\mathbf{T}^n} \sum_{k\in\mathbf{Z}^n} f(x+k)e^{-2\pi i m\cdot x}\,dx \\
&= \sum_{k\in\mathbf{Z}^n} \int_{\mathbf{T}^n} f(x+k)e^{-2\pi i m\cdot x}\,dx \\
&= \sum_{k\in\mathbf{Z}^n} \int_{[-\frac{1}{2},\frac{1}{2}]^n-k} f(x)e^{-2\pi i m\cdot x}\,dx \\
&= \int_{\mathbf{R}^n} f(x)e^{-2\pi i m\cdot x}\,dx \\
&= \widehat{f}(m)\,,
\end{aligned}$$

in which the interchange of the sum and the integral is justified by the Weierstrass M-test of uniform convergence of series, since

$$\sum_{k\in\mathbf{Z}^n} \frac{1}{(1+|k+x|)^{n+\delta}} \le \sum_{k\in\mathbf{Z}^n} \frac{(1+\sqrt{n})^{n+\delta}}{(1+\sqrt{n}+|k+x|)^{n+\delta}} \le \sum_{k\in\mathbf{Z}^n} \frac{C_{n,\delta}}{(1+|k|)^{n+\delta}} < \infty,$$

where we used $|k+x| \ge |k| - |x| \ge |k| - \sqrt{n}$. This calculation also shows that F is the sum of a uniformly convergent series of continuous functions on $[0,1]^n$, thus it is itself continuous. It follows that (3.2.2) holds with $|\widehat{F}(m)|$ in place of $|\widehat{f}(m)|$. Hence, Proposition 3.2.5 applies, and given the fact that F is continuous, it yields conclusion (3.2.3) for all $x \in \mathbf{T}^n$ and, by periodicity, for all $x \in \mathbf{R}^n$. □

Example 3.2.9. We have seen earlier (Exercise 2.2.11) that the following identity gives the Fourier transform of the Poisson kernel in $\mathbf{R}^n$:

$$(e^{-2\pi|x|})\widehat{\ }(\xi) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}}\,.$$

The Poisson summation formula yields the identity

$$\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\sum_{k\in\mathbf{Z}^n}\frac{\varepsilon^{-n}}{(1+\frac{|k+x|^2}{\varepsilon^2})^{\frac{n+1}{2}}}=\sum_{m\in\mathbf{Z}^n}e^{-2\pi\varepsilon|m|}e^{-2\pi i m\cdot x} \tag{3.2.4}$$

which implies

$$\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\sum_{k\in\mathbf{Z}^n}\frac{1}{(1+|k|^2)^{\frac{n+1}{2}}}=\sum_{m\in\mathbf{Z}^n}e^{-2\pi|m|}. \tag{3.2.5}$$

It follows from (3.2.4) that

$$\sum_{k\in\mathbf{Z}^n\setminus\{0\}}\frac{1}{(\varepsilon^2+|k|^2)^{\frac{n+1}{2}}}=\frac{1}{\varepsilon}\bigg(\frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}\sum_{m\in\mathbf{Z}^n}e^{-2\pi\varepsilon|m|}-\frac{1}{\varepsilon^n}\bigg),$$

from which we obtain the identity

$$\sum_{k\in\mathbf{Z}^n\setminus\{0\}}\frac{1}{|k|^{n+1}}=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\bigg(\frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}\sum_{m\in\mathbf{Z}^n}e^{-2\pi\varepsilon|m|}-\frac{1}{\varepsilon^n}\bigg). \tag{3.2.6}$$

The limit in (3.2.6) can be easily calculated in dimension 1 using that

$$\lim_{\delta\to 0}\frac{\pi^2}{\delta}\bigg(\frac{1+e^{-2\delta}}{1-e^{-2\delta}}-\frac{1}{\delta}\bigg)=\frac{\pi^2}{3},$$

and this yields

$$\sum_{k\neq 0}\frac{1}{k^2}=\frac{\pi^2}{3}.$$

Also, in dimension 1, from (3.2.5) we obtain the related identity

$$\sum_{k\in\mathbf{Z}}\frac{1}{1+k^2}=\pi\sum_{m\in\mathbf{Z}}e^{-2\pi|m|}=\pi\,\frac{1+e^{-2\pi}}{1-e^{-2\pi}}.$$

Example 3.2.10. Let $0<\operatorname{Re}\alpha<n$. We introduce a smooth function $\widehat{\Phi}(\xi)$ which is equal to 1 on the ball $|\xi|\le 1$ and vanishes outside the ball $|\xi|\le 2$. We investigate the behavior as $x\to 0$ of the expression

$$\lim_{R\to\infty}\sum_{m\in\mathbf{Z}^n\setminus\{0\}}\frac{e^{2\pi i m\cdot x}}{|m|^\alpha}\widehat{\Phi}\Big(\frac{m}{R}\Big)$$

when $x\in[-\frac{1}{2},\frac{1}{2})^n\setminus\{0\}$. As in Example 2.4.9, let η be a smooth radial function on $\mathbf{R}^n$ that is equal to 1 outside the ball $B(0,1/2)$ and vanishes on the ball $B(0,1/4)$ and define

$$g=(\eta(\xi)|\xi|^{-\alpha})\widehat{\ }.$$

Let $\Phi_\delta(x) = \delta^{-n}\Phi(x/\delta)$. The Poisson summation formula (Theorem 3.2.8) gives

$$\begin{aligned}\sum_{k\in\mathbf{Z}^n\setminus\{0\}} \frac{e^{2\pi i k\cdot x}}{|k|^\alpha}\widehat{\Phi}\left(\frac{k}{R}\right) &= \sum_{k\in\mathbf{Z}^n} \frac{\eta(k)e^{2\pi i k\cdot x}}{|k|^\alpha}\widehat{\Phi}\left(\frac{k}{R}\right)\\ &= (g*\Phi_{1/R})(x) + \sum_{m\in\mathbf{Z}^n\setminus\{0\}} (g*\Phi_{1/R})(x+m)\,.\end{aligned}$$

It was shown in Example 2.4.9 that $g(y)$ decays faster than the reciprocal of any polynomial at infinity and is equal to $\pi^{\alpha-\frac{n}{2}}\Gamma(\frac{n-\alpha}{2})\Gamma(\frac{\alpha}{2})^{-1}|y|^{\alpha-n}+h(y)$, where h is a smooth function on $\mathbf{R}^n$. Since $x\neq 0$, it follows that g is smooth in a small relatively compact neighborhood of x and, since $\{\Phi_{1/R}\}_{R>0}$ is an approximate identity, Theorem 1.2.19 (2) yields that $(g*\Phi_{1/R})(x)\to g(x)$ as $R\to\infty$. Assume for a moment that

$$\lim_{R\to\infty}\sum_{m\in\mathbf{Z}^n\setminus\{0\}}(g*\Phi_{1/R})(x+m) = \sum_{m\in\mathbf{Z}^n\setminus\{0\}}\lim_{R\to\infty}(g*\Phi_{1/R})(x+m)\,. \tag{3.2.7}$$

Since $x+m$ does not vanish for any $m\in\mathbf{Z}^n\setminus\{0\}$, the function g is smooth in a relatively compact neighborhood of $x+m$, and thus $(g*\Phi_{1/R})(x+m)\to g(x+m)$ as $R\to\infty$. Consequently, the sum on the right in (3.2.7) is equal to

$$\sum_{m\in\mathbf{Z}^n\setminus\{0\}} g(x+m)\,.$$

We conclude that

$$\lim_{R\to\infty}\sum_{m\in\mathbf{Z}^n\setminus\{0\}}\frac{e^{2\pi i m\cdot x}}{|m|^\alpha}\widehat{\Phi}\left(\frac{m}{R}\right) = \frac{\pi^{\alpha-\frac{n}{2}}\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}|x|^{\alpha-n}+h_1(x)\,,$$

where h_1 is a $\mathscr{C}^\infty$ function on $[-\frac{1}{2},\frac{1}{2})^n$ given by

$$h_1(t) = h(t) + \sum_{m\in\mathbf{Z}^n\setminus\{0\}} g(t+m)\,.$$

We now explain the passage of the limit inside the sum in (3.2.7). This is a consequence of the Lebesgue dominated convergence theorem, provided we know that

$$|(g*\Phi_{1/R})(x+m)| \le \frac{C}{|m|^{n+1}}\,, \qquad |m|>5\sqrt{n} \tag{3.2.8}$$

for some constant C independent of $R\geq 1$ and of m. Indeed, the expression on the left of this inequality is bounded by $I+II$, where

$$\begin{aligned} I &= C_{n,\alpha}\int_{|x+m-y|\le 2\sqrt{n}} |x+m-y|^{\alpha-n}\frac{R^n}{(1+R|y|)^{2n+2}}\,dy\\ II &= C_{n,\alpha}\int_{|x+m-y|\ge 2\sqrt{n}} \frac{1}{(1+|x+m-y|)^{2n+2}}\frac{R^n}{(1+R|y|)^{2n+2}}\,dy\,.\end{aligned}$$

In I we have

$$1+R|y| \geq R|x+m| - R|x+m-y| \geq R|m| - \tfrac{1}{2}R\sqrt{n} - 2R\sqrt{n} \geq \tfrac{1}{2}R|m|\,,$$

hence

$$I \leq C'_{n,\alpha} R^{-n-2} |m|^{-2n-2} \leq C'_{n,\alpha} |m|^{-2n-2}\,.$$

In II we use

$$(1+|x+m-y|)^{n+1}(1+R|y|)^{n+1} \geq (1+|m|)^{n+1}$$

while the term left produces a convergent integral, which is uniformly bounded in $R \geq 1$. This proves (3.2.8).

Exercises

3.2.1. On $\mathbf{T}^1$ let P be a trigonometric polynomial of degree $N > 0$. Show that P has at most $2N$ zeros. Construct a trigonometric polynomial with exactly $2N$ zeros.

3.2.2. (*Hausdorff–Young inequality*) Prove that when $f \in L^p(\mathbf{T}^n)$, $1 \leq p \leq 2$, the sequence of Fourier coefficients of f is in $\ell^{p'}(\mathbf{Z}^n)$ and

$$\Big(\sum_{m\in\mathbf{Z}^n} |\widehat{f}(m)|^{p'}\Big)^{1/p'} \leq \|f\|_{L^p(\mathbf{T}^n)}.$$

Also observe that 1 is the best constant in the preceding inequality.

3.2.3. Use without proof that there exists a constant $C > 0$ such that for all $t \in \mathbf{R}$ we have

$$\left|\sum_{k=2}^{N} e^{ik\log k} e^{ikt}\right| \leq C\sqrt{N}, \qquad N = 2,3,4,\dots,$$

to construct an example of a continuous function g on $\mathbf{T}^1$ with

$$\sum_{m\in\mathbf{Z}} |\widehat{g}(m)|^q = \infty$$

for all $q < 2$. Thus the Hausdorff–Young inequality of Exercise 3.2.2 fails for $p > 2$. $\big[$*Hint:* Consider $g(x) = \sum_{k=2}^{\infty} \frac{e^{ik\log k}}{k^{1/2}(\log k)^2} e^{2\pi ikx}$. For a proof of the previous estimate, see Zygmund [388, Theorem (4.7) p. 199].$\big]$

3.2.4. (*S. Bernstein*) Let $P(x)$ be a trigonometric polynomial of degree N on $\mathbf{T}^1$. Prove that $\|P'\|_{L^\infty} \leq 4\pi N\|P\|_{L^\infty}$.
$\big[$*Hint:* Prove first that $P'(x)/2\pi iN$ is equal to

$$\big((e^{-2\pi iN(\cdot)}P) * F_{N-1}\big)(x)\, e^{2\pi iNx} - \big((e^{2\pi iN(\cdot)}P) * F_{N-1}\big)(x)\, e^{-2\pi iNx}$$

and then take L^∞ norms.$\big]$

3.2.5. (*Fejér and F. Riesz*) Let $P(\xi) = \sum_{k=-N}^{N} a_k e^{2\pi i k\xi}$ be a trigonometric polynomial on $\mathbf{T}^1$ of degree N such that $P(\xi) > 0$ for all ξ. Prove that there exists a trigonometric polynomial $Q(\xi)$ of the form $\sum_{k=0}^{N} b_k e^{2\pi i k\xi}$ such that $P(\xi) = |Q(\xi)|^2$.
$\big[$*Hint:* Since $P \geq 0$ the complex-variable polynomial $R(z) = \sum_{k=-N}^{N} a_k z^{k+N}$ must satisfy $R(z) = z^{2N}\overline{R(1/\overline{z})}$, and thus it must have N zeros inside the unit circle and the other N outside. Therefore we may write $R(z) = a_N \prod_{k=1}^{s}(z-z_k)^{r_k}(z-1/\overline{z_k})^{r_k}$ for some $0 < |z_k| < 1$ and $r_k \geq 1$ with $\sum_{k=1}^{s} r_k = N$. Then take $z = e^{2\pi i\xi}$.$\big]$

3.2.6. Let g be a function on $\mathbf{R}^n$ that satisfies $|g(x)| + |\widehat{g}(x)| \leq C(1+|x|)^{-n-\delta}$ for some $C, \delta > 0$ and all $x \in \mathbf{R}^n$. Prove that

$$\lambda^n \sum_{m\in\mathbf{Z}^n} \widehat{g}(\lambda m + \alpha) e^{2\pi i x\cdot(m+\frac{\alpha}{\lambda})} = \sum_{k\in\mathbf{Z}^n} g\Big(\frac{x+k}{\lambda}\Big) e^{-2\pi i \frac{k\cdot\alpha}{\lambda}}$$

for any $x, \alpha \in \mathbf{R}^n$ and $\lambda > 0$.

3.2.7. Verify the following identity when $0 < r < 1$ and $x \in \mathbf{R}^n$

$$\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k\in\mathbf{Z}^n} \frac{\frac{1}{2\pi}\log\frac{1}{r}}{\big((\frac{1}{2\pi}\log\frac{1}{r})^2 + |x-k|^2\big)^{\frac{n+1}{2}}} = \sum_{m\in\mathbf{Z}^n} r^{|m|} e^{2\pi i m\cdot x}.$$

In the special case $n = 1$ and $x \in \mathbf{R}$ we have

$$\frac{1}{\pi} \sum_{k\in\mathbf{Z}} \frac{\frac{1}{2\pi}\log\frac{1}{r}}{(\frac{1}{2\pi}\log\frac{1}{r})^2 + |x-k|^2} = \frac{1-r^2}{1-2r\cos(2\pi x)+r^2}.$$

$\big[$*Hint:* Use identity (3.2.4) and Exercise 3.1.7 when $n = 1$.$\big]$

3.2.8. Let $\gamma \in \mathbf{R}$ and $\lambda > 0$. Show that

$$\sum_{k\in\mathbf{Z}} \frac{\cos(2\pi k\gamma)}{\lambda^2 + k^2} = \frac{\pi}{\lambda} \frac{\cosh(2\pi\lambda(\gamma - [\gamma] - \frac{1}{2}))}{\sinh(\pi\lambda)}.$$

$\big[$*Hint:* Use Exercise 3.2.6 when $n = 1$ with $x = 0$, $\alpha = -\gamma\lambda$, $g(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ and sum in m.$\big]$

3.3 Decay of Fourier Coefficients

In this section we investigate the interplay between the smoothness of a function and the decay of its Fourier coefficients.

3.3.1 Decay of Fourier Coefficients of Arbitrary Integrable Functions

We begin with the classical result asserting that the Fourier coefficients of any integrable function tend to zero at infinity. One should compare the following proposition with Proposition 2.2.17.

Proposition 3.3.1. ***(Riemann–Lebesgue lemma)*** *Given a function f in $L^1(\mathbf{T}^n)$, we have that $|\widehat{f}(m)| \to 0$ as $|m| \to \infty$.*

Proof. Given $f \in L^1(\mathbf{T}^n)$ and $\varepsilon > 0$, let P be a trigonometric polynomial such that $\|f - P\|_{L^1} < \varepsilon$. If $|m| > \text{degree}\,(P)$, then $\widehat{P}(m) = 0$ and thus

$$|\widehat{f}(m)| = |\widehat{f}(m) - \widehat{P}(m)| \le \big\|f - P\big\|_{L^1} < \varepsilon.$$

This proves that $|\widehat{f}(m)| \to 0$ as $|m| \to \infty$. □

Several questions are naturally raised. How fast can the Fourier coefficients of an L^1 function tend to zero? Does additional smoothness of the function imply faster decay of the Fourier coefficients? Can such a decay be quantitatively expressed in terms of the smoothness of the function?

We answer the first question. Fourier coefficients of an L^1 function may tend to zero arbitrarily slowly, that is, slower than any given rate of decay. To achieve this, we need the following two lemmas.

Lemma 3.3.2. *Given a sequence of positive real numbers $\{a_m\}_{m=0}^\infty$ that tends to zero as $m \to \infty$, there exists a sequence $\{c_m\}_{m=0}^\infty$ that satisfies*

$$c_m \ge a_m, \qquad c_m \downarrow 0, \quad \text{and} \quad c_{m+2} + c_m \ge 2c_{m+1} \tag{3.3.1}$$

for all $m = 0, 1, \dots$. A sequence $\{c_m\}_{m=0}^\infty$ that satisfies (3.3.1) is called convex.

Proof. Let $k_0 = 0$ and suppose that $a_m \le M$ for all $m \ge 0$. Find $k_1 > k_0$ such that for $m \ge k_1$ we have $a_m \le M/2$. Now find $k_2 > k_1 + \frac{k_1 - k_0}{2}$ such that for $m \ge k_2$ we have $a_m \le M/4$. Next find $k_3 > k_2 + \frac{k_2 - k_1}{2}$ such that for $m \ge k_3$ we have $a_m \le M/8$. Continue inductively in this way and construct a subsequence $k_0 < k_1 < k_2 < \cdots$ of the integers such that for $m \ge k_{j+1}$ we have $a_m \le 2^{-j-1}M$ and $k_{j+1} > k_j + \frac{k_j - k_{j-1}}{2}$ for $j \ge 1$. Join the points $(k_0, 2M)$, (k_1, M), $(k_2, M/2)$, $(k_3, M/4), \dots$ by straight lines and note that by the choice of the sequence $\{k_j\}_{j=0}^\infty$ the resulting piecewise linear function h is convex on $[0, \infty)$. Define $c_m = h(m)$ and observe that the sequence $\{c_m\}_{m=0}^\infty$ satisfies the required properties. Exercise 3.3.1 contains an alternative proof.

□

Lemma 3.3.3. *Given a convex decreasing sequence $\{c_m\}_{m=0}^\infty$ of positive real numbers satisfying $\lim_{m\to\infty} c_m = 0$ and a fixed integer $s \ge 0$, we have that*

$$\sum_{r=0}^{\infty}(r+1)(c_{r+s}+c_{r+s+2}-2c_{r+s+1})=c_s\,. \tag{3.3.2}$$

Proof. We begin by observing the validity of the telescoping sum

$$\begin{aligned}\sum_{r=0}^{N}&(r+1)(c_{r+s}+c_{r+s+2}-2c_{r+s+1})\\ &=c_s-(N+1)(c_{s+N+1}-c_{s+N+2})-c_{s+N+1}\,.\end{aligned} \tag{3.3.3}$$

To show that the last expression tends to c_s as $N\to\infty$, we take $M=[\frac{N}{2}]$ and we use convexity $\big(c_{s+M+j}-c_{s+M+j+1}\geq c_{s+M+j+1}-c_{s+M+j+2}\big)$ to obtain

$$\begin{aligned}c_{s+M+1}-c_{s+N+2}&=c_{s+M+1}-c_{s+M+2}\\ &\qquad+c_{s+M+2}-c_{s+M+3}\\ &\qquad+\cdots\\ &\qquad+c_{s+N+1}-c_{s+N+2}\\ &\geq(N-M+1)(c_{s+N+1}-c_{s+N+2})\\ &\geq\tfrac{N+1}{2}(c_{s+N+1}-c_{s+N+2})\geq 0\,.\end{aligned}$$

The preceding calculation implies that $(N+1)(c_{s+N+1}-c_{s+N+2})$ tends to zero as $N\to\infty$ and thus the expression in (3.3.3) converges to c_s as $N\to\infty$. □

The proof of Lemma 3.3.3 appears more natural after examining Exercise 3.3.3(a). We now state the theorem we alluded to earlier.

Theorem 3.3.4. *Let $(d_m)_{m\in\mathbf{Z}^n}$ be a sequence of positive real numbers with $d_m\to 0$ as $|m|\to\infty$. Then there exists a function $f\in L^1(\mathbf{T}^n)$ such that $\widehat{f}(m)\geq d_m$ for all $m\in\mathbf{Z}^n$. In other words, given any rate of decay, there exists an integrable function on the torus whose Fourier coefficients have slower rate of decay.*

Proof. We are given a sequence of positive numbers $\{a_m\}_{m\in\mathbf{Z}}$ that converges to zero as $|m|\to\infty$ and we would like to find an integrable function on $\mathbf{T}^1$ with $\widehat{f}(m)\geq a_m$ for all $m\in\mathbf{Z}$. Apply Lemma 3.3.2 to the sequence $\{a_m+a_{-m}\}_{m\geq 0}$ to find a convex sequence $\{c_m\}_{m\geq 0}$ that dominates $\{a_m+a_{-m}\}_{m\geq 0}$ and decreases to zero as $m\to\infty$. Extend c_m for $m<0$ by setting $c_m=c_{|m|}$. Now define

$$f(x)=\sum_{j=0}^{\infty}(j+1)(c_j+c_{j+2}-2c_{j+1})F_j(x)\,, \tag{3.3.4}$$

where F_j is the (one-dimensional) Fejér kernel. The convexity of the sequence c_m and the positivity of the Fejér kernel imply that $f\geq 0$. Lemma 3.3.3 with $s=0$ gives that

$$\sum_{j=0}^{\infty}(j+1)(c_j+c_{j+2}-2c_{j+1})\big\|F_j\big\|_{L^1}=c_0<\infty\,, \tag{3.3.5}$$

since $\|F_j\|_{L^1} = 1$ for all j. Therefore (3.3.4) defines an integrable function f on $\mathbf{T}^1$. We now compute the Fourier coefficients of f. Since the series in (3.3.4) converges in L^1, for $m \in \mathbf{Z}$ we have

$$\begin{aligned}\widehat{f}(m) &= \sum_{j=0}^{\infty}(j+1)(c_j+c_{j+2}-2c_{j+1})\widehat{F_j}(m) \\ &= \sum_{j=|m|}^{\infty}(j+1)(c_j+c_{j+2}-2c_{j+1})\left(1-\frac{|m|}{j+1}\right) \\ &= \sum_{r=0}^{\infty}(r+1)(c_{r+|m|}+c_{r+|m|+2}-2c_{r+|m|+1}) \\ &= c_{|m|} = c_m\,,\end{aligned} \tag{3.3.6}$$

where we used Lemma 3.3.3 with $s = |m|$.

Let us now extend this result on $\mathbf{T}^n$. Let $(d_m)_{m\in\mathbf{Z}^n}$ be a positive sequence with $d_m \to 0$ as $|m| \to \infty$. By Exercise 3.3.2, there exists a positive sequence $(a_j)_{j\in\mathbf{Z}}$ with $a_{m_1}\cdots a_{m_n} \ge d_{(m_1,\dots,m_n)}$ and $a_j \to 0$ as $|j| \to \infty$. Let

$$\mathbf{f}(x_1,\dots,x_n) = f(x_1)\cdots f(x_n),$$

where f is the function previously constructed when $n = 1$ so that $\widehat{f}(m) \ge a_m$. It can be seen easily using (3.1.7) that $\widehat{\mathbf{f}}(m) \ge d_m$. □

3.3.2 Decay of Fourier Coefficients of Smooth Functions

We next study the decay of the Fourier coefficients of functions that possess a certain amount of smoothness. In this section we see that the decay of the Fourier coefficients reflects the smoothness of the function in a rather precise quantitative way. Conversely, one can infer some information about the smoothness of a function from the decay of its Fourier coefficients.

Definition 3.3.5. Given $0 < \gamma < 1$ and f a function on $\mathbf{T}^n$, define the *homogeneous Lipschitz seminorm* of order γ of f by

$$\|f\|_{\dot{\Lambda}_\gamma} = \sup_{\substack{x,h\in\mathbf{T}^n\\ h\neq 0}} \frac{|f(x+h)-f(x)|}{|h|^\gamma}$$

and define the *homogeneous Lipschitz space* of order γ as

$$\dot{\Lambda}_\gamma(\mathbf{T}^n) = \{f : \mathbf{T}^n \to \mathbf{C} \text{ with } \|f\|_{\dot{\Lambda}_\gamma} < \infty\}.$$

Functions in $\dot{\Lambda}_\gamma(\mathbf{T}^n)$ are called *homogeneous Lipschitz* functions of order γ.

There is an analogous definition for the inhomogeneous norm.

Definition 3.3.6. For $0 < \gamma < 1$ and f a function on $\mathbf{T}^n$, define the *inhomogeneous Lipschitz norm* of order γ of f by

$$\|f\|_{\Lambda_\gamma} = \|f\|_{L^\infty} + \sup_{\substack{x,h\in\mathbf{T}^n\\ h\neq 0}} \frac{|f(x+h)-f(x)|}{|h|^\gamma} = \|f\|_{L^\infty} + \|f\|_{\dot{\Lambda}_\gamma}.$$

Also define the *inhomogeneous Lipschitz space* of order γ as

$$\Lambda_\gamma(\mathbf{T}^n) = \{f : \mathbf{T}^n \to \mathbf{C} \text{ with } \|f\|_{\Lambda_\gamma} < \infty\}.$$

Functions in $\Lambda_\gamma(\mathbf{T}^n)$ are called *inhomogeneous Lipschitz* functions of order γ.

Remark 3.3.7. Functions in both spaces $\Lambda_\gamma(\mathbf{T}^n)$ and $\dot{\Lambda}_\gamma(\mathbf{T}^n)$ are obviously continuous and therefore bounded. Moreover, the functional $\|\cdot\|_{\Lambda_\gamma}$ is a norm on $\Lambda_\gamma(\mathbf{T}^n)$. The positive functional $\|\cdot\|_{\dot{\Lambda}_\gamma}$ satisfies the triangle inequality, but it does not satisfy the property $\|f\|_{\dot{\Lambda}_\gamma} = 0 \implies f = 0$ required to be a norm. It is therefore a seminorm on $\dot{\Lambda}_\gamma(\mathbf{T}^n)$. However, if we identify functions whose difference is a constant, we form a space of the equivalence classes $\dot{\Lambda}_\gamma(\mathbf{T}^n)/\{\text{constants}\}$ on which $\|\cdot\|_{\dot{\Lambda}_\gamma}$ is a norm.

Remark 3.3.8. We already observed that elements of $\dot{\Lambda}_\gamma(\mathbf{T}^n)$ are continuous and thus bounded. Therefore, $\dot{\Lambda}_\gamma(\mathbf{T}^n) \subseteq L^\infty(\mathbf{T}^n)$ in the set-theoretic sense. However, the norm inequality $\|f\|_{L^\infty} \le C\|f\|_{\dot{\Lambda}_\gamma}$ for all $f \in \dot{\Lambda}_\gamma$ fails for all constants C. For example, take $f = N + \sin(2\pi x_1)$ on $\mathbf{T}^n$ and let $N \to \infty$ to see that this is the case.

The following theorem indicates how the smoothness of a function is reflected by the decay of its Fourier coefficients.

Theorem 3.3.9. *Let $s \in \mathbf{Z}$ with $s \ge 0$.*
(a) Suppose that $\partial^\alpha f$ exist and are integrable for all $|\alpha| \le s$. Then

$$|\widehat{f}(m)| \le \left(\frac{\sqrt{n}}{2\pi}\right)^s \frac{\max_{|\alpha|=s} |\widehat{\partial^\alpha f}(m)|}{|m|^s}, \qquad m \neq 0, \tag{3.3.7}$$

and thus

$$|\widehat{f}(m)|(1+|m|^s) \to 0$$

as $|m| \to \infty$. In particular this holds when f lies in $\mathscr{C}^s(\mathbf{T}^n)$.
(b) Suppose that $\partial^\alpha f$ exist for all $|\alpha| \le s$ and whenever $|\alpha| = s$, $\partial^\alpha f$ are in $\dot{\Lambda}_\gamma(\mathbf{T}^n)$ for some $0 < \gamma < 1$. Then

$$|\widehat{f}(m)| \le \frac{(\sqrt{n})^{s+\gamma}}{(2\pi)^s 2^{\gamma+1}} \frac{\max_{|\alpha|=s} \|\partial^\alpha f\|_{\dot{\Lambda}_\gamma}}{|m|^{s+\gamma}}, \qquad m \neq 0. \tag{3.3.8}$$

Proof. Fix $m \in \mathbf{Z}^n \setminus \{0\}$ and pick a j such that $|m_j| = \sup_{1\le k\le n} |m_k|$. Then clearly $m_j \neq 0$. Integrating by parts s times with respect to the variable x_j, we obtain

$$\widehat{f}(m) = \int_{\mathbf{T}^n} f(x)e^{-2\pi i x\cdot m}\,dx = (-1)^s \int_{\mathbf{T}^n} (\partial_j^s f)(x)\frac{e^{-2\pi i x\cdot m}}{(-2\pi i m_j)^s}\,dx\,, \tag{3.3.9}$$

where the boundary terms all vanish because of the periodicity of the integrand. Taking absolute values and using $|m| \le \sqrt{n}\,|m_j|$, we obtain assertion (3.3.7).

We now turn to the second part of the theorem. Let $e_j = (0,\dots,1,\dots,0)$ be the element of the torus $\mathbf{T}^n$ whose jth coordinate is one and all the others are zero. A simple change of variables together with the fact that $e^{\pi i} = -1$ gives that

$$\int_{\mathbf{T}^n} (\partial_j^s f)(x)e^{-2\pi i x\cdot m}\,dx = -\int_{\mathbf{T}^n} (\partial_j^s f)(x - \tfrac{e_j}{2m_j})e^{-2\pi i x\cdot m}\,dx\,,$$

which implies that

$$\int_{\mathbf{T}^n} (\partial_j^s f)(x)e^{-2\pi i x\cdot m}\,dx = \frac{1}{2}\int_{\mathbf{T}^n} \big[(\partial_j^s f)(x) - (\partial_j^s f)(x - \tfrac{e_j}{2m_j})\big]e^{-2\pi i x\cdot m}\,dx\,.$$

Now use the estimate

$$|(\partial_j^s f)(x) - (\partial_j^s f)(x - \tfrac{e_j}{2m_j})| \le \frac{\|\partial_j^s f\|_{\dot{\Lambda}_\gamma}}{(2|m_j|)^\gamma}$$

and identity (3.3.9) to conclude the proof of (3.3.8). □

The following is an immediate consequence.

Corollary 3.3.10. *Let $s \in \mathbf{Z}$ with $s \ge 0$.*
(a) Suppose that $\partial^\alpha f$ exist and are integrable for all $|\alpha| \le s$. Then for some constant $c_{n,s}$ we have

$$|\widehat{f}(m)| \le c_{n,s}\frac{\max\big(\|f\|_{L^1}, \max_{|\alpha|=s}\|\partial^\alpha f\|_{L^1}\big)}{(1+|m|)^s}\,. \tag{3.3.10}$$

(b) Suppose that $\partial^\alpha f$ exist for all $|\alpha| \le s$ and whenever $|\alpha| = s$, $\partial^\alpha f$ are in $\dot{\Lambda}_\gamma(\mathbf{T}^n)$ for some $0 < \gamma < 1$. Then for some constant $c'_{n,s}$ we have

$$|\widehat{f}(m)| \le c'_{n,s}\frac{\max\big(\|f\|_{L^1}, \max_{|\alpha|=s}\|\partial^\alpha f\|_{\dot{\Lambda}_\gamma}\big)}{(1+|m|)^{s+\gamma}}\,. \tag{3.3.11}$$

Remark 3.3.11. The conclusions of Theorem 3.3.9 and Corollary 3.3.10 are also valid when $\gamma = 1$. In this case the spaces $\dot{\Lambda}_\gamma$ should be replaced by the space $\operatorname{Lip} 1$ equipped with the seminorm

$$\|f\|_{\operatorname{Lip} 1} = \sup_{\substack{x,h\in\mathbf{T}^n\\ h\neq 0}} \frac{|f(x+h) - f(x)|}{|h|}\,.$$

There is a slight lack of uniformity in the notation here, since in the theory of Lipschitz spaces the notation $\dot{\Lambda}_1$ is usually reserved for the space with seminorm

$$\|f\|_{\dot{\Lambda}_1} = \sup_{\substack{x,h\in\mathbf{T}^n \\ h\neq 0}} \frac{|f(x+h)+f(x-h)-2f(x)|}{|h|}.$$

The following proposition provides a partial converse to Theorem 3.3.9. We denote below by $[[s]]$ the largest integer strictly less than a given real number s. Then $[[s]]$ is equal to the integer part $[s]$ of s, unless s is an integer, in which case $[[s]] = [s]-1$.

Proposition 3.3.12. *Let $s>0$ and suppose that f is an integrable function on the torus with*

$$|\widehat{f}(m)| \le C(1+|m|)^{-s-n} \tag{3.3.12}$$

for all $m\in\mathbf{Z}^n$. Then f has partial derivatives of all orders $|\alpha|\le[[s]]$, and for $0<\gamma<s-[[s]]$, $\partial^\alpha f\in\dot{\Lambda}_\gamma$ for all multi-indices α satisfying $|\alpha|=[[s]]$.

Proof. Since f has an absolutely convergent Fourier series, Proposition 3.2.5 gives that

$$f(x) = \sum_{m\in\mathbf{Z}^n} \widehat{f}(m) e^{2\pi i x\cdot m}, \tag{3.3.13}$$

for almost all $x\in\mathbf{T}^n$.

Suppose that a series $g=\sum_m g_m$ satisfies $\sum_m \|\partial^\beta g_m\|_{L^\infty}<\infty$ for all $|\beta|\le M$. Then the function g is in $\mathscr{C}^M$ and $\partial^\beta g = \sum_m \partial^\beta g_m$; indeed this can be proved by induction on the degree of the multi-index, since for all $|\beta|\le M-1$ we have

$$\begin{aligned}\lim_{t\to 0}\frac{\partial^\beta g(x+te_j)-\partial^\beta g(x)}{t} &= \sum_m \lim_{t\to 0}\frac{\partial^\beta g_m(x+te_j)-\partial^\beta g_m(x)}{t}\\ &= \sum_m \partial_j\partial^\beta g(x),\end{aligned}$$

where the passage of the limit inside the sum is due to the Lebesgue dominated convergence theorem, which can be applied using the uniform convergence of $\sum_m \partial_j\partial^\beta g$ via the mean value theorem.

Using the preceding observation, the function f in (3.3.13) is $\mathscr{C}^{[[s]]}(\mathbf{T}^n)$ and

$$(\partial^\alpha f)(x) = \sum_{m\in\mathbf{Z}^n} \widehat{f}(m)(2\pi i m)^\alpha e^{2\pi i x\cdot m}$$

for all multi-indices $(\alpha_1,\dots,\alpha_n)$ with $|\alpha|\le[[s]]$, since

$$\sum_{m\in\mathbf{Z}^n} \widehat{f}(m) \sup_{x\in\mathbf{T}^n} \left|(2\pi i m)^\alpha e^{2\pi i x\cdot m}\right| < \infty,$$

which holds because of (3.3.12).

Now suppose that $|\alpha| = [[s]]$ and that $0 < \gamma < s - [[s]]$. Then

$$\begin{aligned} |(\partial^\alpha f)(x+h) - (\partial^\alpha f)(x)| &= \Big| \sum_{m\in\mathbf{Z}^n} \widehat{f}(m)(2\pi i m)^\alpha e^{2\pi i x\cdot m}\big(e^{2\pi i m\cdot h} - 1\big)\Big| \\ &\le (2\pi)^{[[s]]} \sum_{m\in\mathbf{Z}^n} |m|^{[[s]]} \frac{C\,2^{1-\gamma}(2\pi)^\gamma |h|^\gamma |m|^\gamma}{(1+|m|)^{n+s}} \\ &\le C\,2^{1-\gamma}(2\pi)^s |h|^\gamma , \end{aligned}$$

where we used the relation $[[s]] + \gamma - s - n < -n$ to conclude the convergence of the series and the fact that

$$|e^{2\pi i m\cdot h} - 1| \le \min(2, 2\pi|m|\,|h|) \le 2^{1-\gamma}(2\pi)^\gamma |m|^\gamma |h|^\gamma .$$

□

Next we recall the definition of functions of bounded variation.

Definition 3.3.13. A measurable function f on $\mathbf{T}^1$ is said to be of *bounded variation* if it is defined everywhere and

$$\mathrm{Var}(f) = \sup\Big\{ \sum_{j=1}^{M} |f(x_j) - f(x_{j-1})| :\ 0 = x_0 < x_1 < \cdots < x_M = 1 \Big\} < \infty ,$$

where the supremum is taken over all partitions of the interval $[0,1]$. The expression $\mathrm{Var}(f)$ is called the *total variation* of f. The class of functions of bounded variation on $\mathbf{T}^1$ is denoted by $BV(\mathbf{T}^1)$.

Examples of functions of bounded variation can be constructed as follows: given f_1, f_2 nonnegative integrable functions on $[0,1]$ with

$$\int_0^1 f_1(t)\,dt = \int_0^1 f_2(t)\,dt ,$$

then the periodic function

$$g(x) = \int_0^x f_1(t)\,dt - \int_0^x f_2(t)\,dt ,$$

defined on$[0,1]$, is of bounded variation. Analogous examples can be constructed when f_1 and f_2 are replaced by nonnegative finite Borel measures on $[0,1]$.

Every function of bounded variation can be represented as the difference of two (not necessarily strictly) increasing functions and thus it has a finite derivative at almost every point. Moreover, for functions of bounded variation, the Lebesgue–Stieltjes integral with respect to df is well defined.

Proposition 3.3.14. *If f is in $BV(\mathbf{T}^1)$, then*

$$|\widehat{f}(m)| \le \frac{\mathrm{Var}(f)}{2\pi|m|}$$

whenever $m \neq 0$.

Proof. Integration by parts gives

$$\widehat{f}(m) = \int_{\mathbf{T}^1} f(x)e^{-2\pi imx}\,dx = \int_{\mathbf{T}^1} \frac{e^{-2\pi imx}}{-2\pi im}\,df\,,$$

where the boundary terms vanish because of periodicity. The conclusion follows from the fact that the norm of the measure df is the total variation of f. □

The following chart (Table 3.1) summarizes the decay of Fourier coefficients in terms of scales of spaces measuring the smoothness of the functions. Recall that for $q \ge 0$, $\widehat{f}(m) = o(|m|^{-q})$ means that $|\widehat{f}(m)|\,|m|^q \to 0$ as $|m| \to \infty$ and $\widehat{f}(m) = O(|m|^{-q})$ means that $|\widehat{f}(m)| \le C|m|^{-q}$ when $|m|$ is large. In this chart, we denote by $\mathscr{C}^{s,\gamma}(\mathbf{T}^n)$ the space of all $\mathscr{C}^s$ functions on $\mathbf{T}^n$, all of whose derivatives of total order s lie in $\Lambda_\gamma(\mathbf{T}^n)$, for some $0 < \gamma < 1$.

SPACE	*SEQUENCE OF FOURIER COEFFICIENTS*
$L^1(\mathbf{T}^n)$	$o(1)$
$L^p(\mathbf{T}^n)$	$\ell^{p'}(\mathbf{Z}^n)$
$L^2(\mathbf{T}^n)$	$\ell^2(\mathbf{Z}^n)$
$\dot{\Lambda}_\gamma(\mathbf{T}^n)$	$O(\lvert m\rvert^{-\gamma})$
$BV(\mathbf{T}^1)$	$O(\lvert m\rvert^{-1})$
$\mathscr{C}^1(\mathbf{T}^n)$	$o(\lvert m\rvert^{-1})$
$\mathscr{C}^{1,\gamma}(\mathbf{T}^n)$	$O(\lvert m\rvert^{-1-\gamma})$
$\mathscr{C}^2(\mathbf{T}^n)$	$o(\lvert m\rvert^{-2})$
$\mathscr{C}^{2,\gamma}(\mathbf{T}^n)$	$O(\lvert m\rvert^{-2-\gamma})$
$\mathscr{C}^3(\mathbf{T}^n)$	$o(\lvert m\rvert^{-3})$
$\cdots$	$\cdots$
$\mathscr{C}^\infty(\mathbf{T}^n)$	$o(\lvert m\rvert^{-N})$ for all $N > 0$

Table 3.1 Interconnection between smoothness of functions and decay of their Fourier coefficients. We take $0 < \gamma < 1$ and $1 < p < 2$.

3.3.3 Functions with Absolutely Summable Fourier Coefficients

Decay for the Fourier coefficients can also be indirectly deduced from knowledge about the summability of these coefficients. The simplest kind of summability is in the sense of ℓ^1. It is therefore natural to consider the class of functions on the torus whose Fourier coefficients form an absolutely summable series.

Definition 3.3.15. An integrable function f on the torus is said to have an *absolutely convergent* Fourier series if

$$\sum_{m\in\mathbf{Z}^n} |\widehat{f}(m)| < +\infty.$$

We denote by $A(\mathbf{T}^n)$ the space of all integrable functions on the torus $\mathbf{T}^n$ whose Fourier series are absolutely convergent. We then introduce a norm on $A(\mathbf{T}^n)$ by setting

$$\|f\|_{A(\mathbf{T}^n)} = \sum_{m\in\mathbf{Z}^n} |\widehat{f}(m)|.$$

In view of Proposition 3.2.5, every function f in $A(\mathbf{T}^n)$ can be changed on a set of measure zero to be made continuous and under this modification, Fourier inversion

$$f(x) = \sum_{m\in\mathbf{Z}^n} \widehat{f}(m) e^{2\pi i m\cdot x}$$

holds for all $x \in \mathbf{T}^n$. Thus functions in $A(\mathbf{T}^n)$ are continuous and bounded. Moreover, Theorem 3.3.9 yields that every function in $\mathscr{C}^n(\mathbf{T}^n)$ whose partial derivatives of order n are in $\dot{\Lambda}_\gamma$, $\gamma > 0$, must lie in $A(\mathbf{T}^n)$. The following theorem gives us a significantly better sufficient condition for a function to be in $A(\mathbf{T}^n)$.

Theorem 3.3.16. *Suppose f is a given function in $\mathscr{C}^{[n/2]}(\mathbf{T}^n)$ and that all partial derivatives of order $[\frac{n}{2}]$ of f lie in $\dot{\Lambda}_\gamma(\mathbf{T}^n)$ for some γ with $\frac{n}{2} - [\frac{n}{2}] < \gamma < 1$. Then f lies in $A(\mathbf{T}^n)$ and*

$$\|f\|_{A(\mathbf{T}^n)} \le |\widehat{f}(0)| + C(n,\gamma) \sup_{|\alpha|=[\frac{n}{2}]} \|\partial^\alpha f\|_{\dot{\Lambda}_\gamma(\mathbf{T}^n)},$$

where $C(n,\gamma)$ is a constant depending on n and γ.

Proof. For each $\ell = 0,1,2,\dots$, let

$$S_\ell = \bigg(\sum_{2^\ell \le |m| < 2^{\ell+1}} |\widehat{f}(m)|^2 \bigg)^{1/2}.$$

We begin by writing

$$\|f\|_{A(\mathbf{T}^n)} = |\widehat{f}(0)| + \sum_{\ell=0}^{\infty} \sum_{2^\ell \le |m| < 2^{\ell+1}} |\widehat{f}(m)| \le |\widehat{f}(0)| + \sqrt{c_n} \sum_{\ell=0}^{\infty} 2^{\frac{\ell n}{2}} S_\ell, \tag{3.3.14}$$

where we used the Cauchy-Schwarz inequality and the fact that there are at most $c_n 2^{\ell n}$ points in $\mathbf{Z}^n$ inside the open ball $B(0,2^{\ell+1})$, for some dimensional constant c_n.

Notice that for a multi-index $m = (m_1,\dots,m_n)$ satisfying $2^\ell \le |m| \le 2^{\ell+1}$ and for j in $\{1,\dots,n\}$ such that $|m_j| = \sup_k |m_k|$ we have

$$\frac{|m_j|}{2^\ell} \ge \frac{|m|}{2^\ell\sqrt{n}} \ge \frac{1}{\sqrt{n}}. \tag{3.3.15}$$

For $1 \le j \le n$, let e_j be the element of $\mathbf{R}^n$ with zero entries except for the jth coordinate, which is 1, and define

$$h_j^\ell = 2^{-\ell-2} e_j \,. \tag{3.3.16}$$

Using the elementary fact that $|t| \le \pi \implies |e^{it} - 1| \ge 2|t|/\pi$, we obtain

$$|e^{2\pi i m\cdot h_j^\ell} - 1| = |e^{2\pi i m_j 2^{-\ell-2}} - 1| \ge \frac{2}{\pi}\frac{|2\pi m_j|}{2^{\ell+2}} = \frac{|m_j|}{2^\ell} \ge \frac{1}{\sqrt{n}}\,, \tag{3.3.17}$$

whenever $\frac{|2\pi m_j|}{2^{\ell+2}} \le \pi$, which is always true since $\frac{|2\pi m_j|}{2^{\ell+2}} \le \frac{2\pi 2^{\ell+1}}{2^{\ell+2}} \le \pi$.

We now have

$$\begin{aligned}
S_\ell^2 &= \sum_{j=1}^n \sum_{\substack{2^\ell \le |m| < 2^{\ell+1} \\ |m_j| = \sup_k |m_k|}} |\widehat{f}(m)|^2 \\
&\le n \sum_{j=1}^n \sum_{\substack{2^\ell \le |m| < 2^{\ell+1} \\ |m_j| = \sup_k |m_k|}} |e^{2\pi i m\cdot h_j^\ell} - 1|^2 |\widehat{f}(m)|^2 \frac{|2\pi i m_j|^{2[\frac{n}{2}]}}{|2\pi m_j|^{2[\frac{n}{2}]}} \\
&\le n \frac{n^{[\frac{n}{2}]}}{(2\pi 2^\ell)^{2[\frac{n}{2}]}} \sum_{j=1}^n \sum_{m\in\mathbf{Z}^n} |e^{2\pi i m\cdot h_j^\ell} - 1|^2 |\widehat{\partial_j^{[n/2]} f}(m)|^2 \\
&= C_n 2^{-2\ell[\frac{n}{2}]} \sum_{j=1}^n \big\|\partial_j^{[n/2]} f(\cdot + h_j^\ell) - \partial_j^{[n/2]} f\big\|_{L^2}^2 \\
&\le C_n 2^{-2\ell[\frac{n}{2}]} \sum_{j=1}^n \big\|\partial_j^{[n/2]} f(\cdot + h_j^\ell) - \partial_j^{[n/2]} f\big\|_{L^\infty}^2 \\
&\le C_n' 2^{-2\ell[\frac{n}{2}]} \sup_{|\alpha| = [\frac{n}{2}]} \big\|\partial^\alpha f\big\|_{\dot{\Lambda}_\gamma}^2 \sum_{j=1}^n |h_j^\ell|^{2\gamma} \\
&= C_{n,\gamma}' 2^{-2\ell[\frac{n}{2}] - 2\ell\gamma} \sup_{|\alpha| = [\frac{n}{2}]} \big\|\partial^\alpha f\big\|_{\dot{\Lambda}_\gamma}^2\,,
\end{aligned}$$

where we used (3.3.17), (3.3.15), and (3.3.16). We conclude that

$$S_\ell \le C_{n,\gamma}'' 2^{-\ell([\frac{n}{2}]+\gamma)} \sup_{|\alpha| = [\frac{n}{2}]} \big\|\partial^\alpha f\big\|_{\dot{\Lambda}_\gamma}$$

which inserted in (3.3.14) yields the desired conclusion since $\gamma > \frac{n}{2} - [\frac{n}{2}]$. □

Exercises

3.3.1. Given a sequence $\{a_n\}_{n=0}^\infty$ of positive numbers such that $a_n \to 0$ as $n \to \infty$, find a nonnegative integrable function h on $[0,1]$ such that

$$\int_0^1 h(t)t^m \, dt \geq a_m.$$

Use this result to deduce a different proof of Lemma 3.3.2.
$\big[$*Hint:* Try $h = e \sum_{k=0}^{\infty} (\sup_{j\geq k} a_j - \sup_{j\geq k+1} a_j)(k+2)\chi_{[\frac{k+1}{k+2},1]}$.$\big]$

3.3.2. Prove that given a positive sequence $\{d_m\}_{m\in\mathbf{Z}^n}$ with $d_m \to 0$ as $|m| \to \infty$, there exists a positive sequence $\{a_j\}_{j\in\mathbf{Z}}$ with $a_{m_1}\cdots a_{m_n} \geq d_{(m_1,\dots,m_n)}$ and $a_j \to 0$ as $|j| \to \infty$.

3.3.3. (a) Use the idea of the proof of Lemma 3.3.3 to prove that if a twice continuously differentiable function $f \geq 0$ is defined on $(0,\infty)$ and satisfies $f'(x) \leq 0$ and $f''(x) \geq 0$ for all $x > 0$, then $\lim_{x\to\infty} xf'(x) = 0$.
(b) Suppose that a continuously differentiable function g is defined on $(0,\infty)$ and satisfies $g \geq 0$, $g' \leq 0$, and $\int_1^\infty g(x)\,dx < +\infty$. Prove that

$$\lim_{x\to\infty} xg(x) = 0.$$

3.3.4. Prove that for $0 < \gamma < \delta < 1$ we have $\|f\|_{\dot{\Lambda}_\gamma} \leq C_{n,\gamma,\delta}\|f\|_{\dot{\Lambda}_\delta}$ for all functions f and thus $\dot{\Lambda}_\delta$ is a subspace of $\dot{\Lambda}_\gamma$.

3.3.5. Suppose that f is a differentiable function on $\mathbf{T}^1$ whose derivative f' is in $L^2(\mathbf{T}^1)$. Prove that $f \in A(\mathbf{T}^1)$ and that

$$\|f\|_{A(\mathbf{T}^1)} \leq \|f\|_{L^1} + \frac{1}{2\pi}\Big(\sum_{j\neq 0} j^{-2}\Big)^{1/2}\|f'\|_{L^2}.$$

3.3.6. (a) Prove that the product of two functions in $A(\mathbf{T}^n)$ is also in $A(\mathbf{T}^n)$ and that

$$\|fg\|_{A(\mathbf{T}^n)} \leq \|f\|_{A(\mathbf{T}^n)}\|g\|_{A(\mathbf{T}^n)}.$$

(b) Prove that the convolution of two square integrable functions on $\mathbf{T}^n$ always gives a function in $A(\mathbf{T}^n)$.

3.3.7. Fix $0 < \alpha < 1$ and define f on $\mathbf{T}^1$ by setting

$$f(x) = \sum_{k=0}^{\infty} 2^{-\alpha k} e^{2\pi i 2^k x}.$$

Prove that the function f lies in $\dot{\Lambda}_\alpha(\mathbf{T}^1)$. Conclude that there does not exist positive $\beta > \alpha$ such that for all f in $\dot{\Lambda}_\alpha(\mathbf{T}^1)$ we have $\sup_{m\in\mathbf{Z}} |m|^\beta |\widehat{f}(m)| < \infty$.
$\big[$*Hint:* For $h \neq 0$ pick $N \in \mathbf{Z}^+$ such that $2^N|h| > 1 \geq 2^{N-1}|h|$. To estimate the difference $|f(x+h) - f(x)|$, consider the cases $k \leq N$ and $k \geq N+1$ in the sum.$\big]$

3.3.8. Use without proof that there exists a constant $C > 0$ such that

$$\sup_{t\in\mathbf{R}}\left|\sum_{k=2}^{N} e^{ik\log k} e^{ikt}\right| \le C\sqrt{N}, \qquad N = 2,3,4,\dots,$$

to prove that the function

$$g(x) = \sum_{k=2}^{\infty} \frac{e^{ik\log k}}{k} e^{2\pi ikx}$$

is in $\dot{\Lambda}_{1/2}(\mathbf{T}^1)$ but not in $A(\mathbf{T}^1)$. Conclude that the restriction $s > 1/2$ in Theorem 3.3.16 is sharp.
[*Hint:* Estimate the difference $|g(x+h) - g(x)|$ using the summation by parts identity in Appendix F, taking sums of the sequence $e^{ik\log k}e^{2\pi ikx}$ and differences of the sequence $\frac{e^{2\pi ikh}-1}{k}$.]

3.3.9. Show that there exist sequences $\{a_m\}_{m\in\mathbf{Z}^n}$ that tend to zero as $|m| \to \infty$ for which there do not exist functions f in $L^1(\mathbf{T}^n)$ with $\widehat{f}(m) = a_m$ for all m.
[*Hint:* Suppose the contrary. Then the open mapping theorem would imply the inequality $\|f\|_{L^1(\mathbf{T}^n)} \le A\|\widehat{f}\|_{\ell^\infty(\mathbf{Z}^n)}$ for some $A > 0$. To contradict it, fix a smooth nonzero function h equal to 1 on $B(0,\frac{1}{4})$ and supported in $B(0,\frac{1}{2})$. For $b > 0$ define $g_b(x) = h(x)e^{-\pi(1+ib)|x|^2}$ and extend g_b to a 1-periodic function in each variable on $\mathbf{R}^n$. Use that $\widehat{g_b}(m) = \int_{\mathbf{R}^n} \widehat{h}(y)(1+ib)^{-n/2}e^{-\frac{\pi}{1+ib}|m-y|^2}dy$, and let $b \to \infty$ in the inequality $\|g_b\|_{L^1(\mathbf{T}^n)} \le A\|\widehat{g_b}\|_{\ell^\infty(\mathbf{Z}^n)}$ to obtain a contradiction.]

3.4 Pointwise Convergence of Fourier Series

In this section we are concerned with the pointwise convergence of the square partial sums and the Fejér means of a function defined on the torus.

3.4.1 Pointwise Convergence of the Fejér Means

We saw in Section 3.1 that the Fejér kernel is an approximate identity. This implies that the Fejér (or Cesàro) means of an L^p function f on $\mathbf{T}^n$ converge to it in L^p for any $1 \le p < \infty$. Moreover, if f is continuous at x_0, then the means $(F_N^n * f)(x_0)$ converge to $f(x_0)$ as $N \to \infty$ in view of Theorem 1.2.19 (2). Although this is a satisfactory result, it is natural to ask what happens for more general functions.

Using properties of the Fejér kernel, we obtain the following one-dimensional result regarding the convergence of the Fejér means:

Theorem 3.4.1. ***(Fejér)*** *If a function f in $L^1(\mathbf{T}^1)$ has left and right limits at a point x_0, denoted by $f(x_0-)$ and $f(x_0+)$, respectively, then*

$$(F_N * f)(x_0) \to \frac{1}{2}\big(f(x_0+) + f(x_0-)\big) \qquad \text{as} \qquad N \to \infty. \tag{3.4.1}$$

In particular, this is the case for functions of bounded variation.

Proof. Let us identify $\mathbf{T}^1$ with $[-1/2, 1/2]$. Given $\varepsilon > 0$, find $\delta \in (0, 1/2)$ such that

$$0 < t < \delta \implies \left| \frac{f(x_0+t) + f(x_0-t)}{2} - \frac{f(x_0+) + f(x_0-)}{2} \right| < \varepsilon. \tag{3.4.2}$$

Using the second expression for F_N in (3.1.18), we can find an $N_0 > 0$ such that for $N \ge N_0$ we have

$$\sup_{\delta \le t \le \frac{1}{2}} F_N(t) = \frac{1}{N+1} \sup_{\delta \le t \le \frac{1}{2}} \left(\frac{\sin(\pi(N+1)t)}{\sin(\pi t)} \right)^2 \le \frac{1}{N+1} \frac{1}{\sin^2(\pi\delta)} < \varepsilon. \tag{3.4.3}$$

We now have

$$\begin{aligned} (F_N * f)(x_0) - f(x_0+) &= \int_{\mathbf{T}^1} F_N(t)\big(f(x_0+t) - f(x_0+)\big)\, dt\,, \\ (F_N * f)(x_0) - f(x_0-) &= \int_{\mathbf{T}^1} F_N(t)\big(f(x_0-t) - f(x_0-)\big)\, dt\,. \end{aligned}$$

Averaging these two identities and using that the integrand is even, we obtain

$$\begin{aligned} &(F_N * f)(x_0) - \frac{f(x_0+) + f(x_0-)}{2} \\ &\quad = 2\int_0^{1/2} F_N(t) \left(\frac{f(x_0+t) + f(x_0-t)}{2} - \frac{f(x_0+) + f(x_0-)}{2} \right) dt\,. \end{aligned} \tag{3.4.4}$$

We split the integral in (3.4.4) into two pieces, the integral over $[0, \delta)$ and the integral over $[\delta, 1/2]$. By (3.4.2), the integral over $[0, \delta)$ is controlled by $\varepsilon \int_{\mathbf{T}^1} F_N(t)\, dt = \varepsilon$. Also (3.4.3) gives that for $N \ge N_0$

$$\begin{aligned} &\left| \int_\delta^{1/2} F_N(t) \left(\frac{f(x_0-t) + f(x_0+t)}{2} - \frac{f(x_0-) + f(x_0+)}{2} \right) dt \right| \\ &\qquad\qquad \le \frac{\varepsilon}{2} \left(\big\| f - f(x_0-) \big\|_{L^1} + \big\| f - f(x_0+) \big\|_{L^1} \right) = \varepsilon\, c(f, x_0)\,, \end{aligned}$$

where $c(f, x_0)$ is a constant depending on f and x_0. We have now proved that given $\varepsilon > 0$ there exists an N_0 such that for $N \ge N_0$ the second expression in (3.4.4) is bounded by $2\varepsilon\, (c(f, x_0) + 1)$. This proves the required conclusion.

Functions of bounded variation can be written as differences of increasing functions, and since increasing functions have left and right limits everywhere, (3.4.1) holds for these functions. □

We continue with an elementary but very useful application of the preceding result.

Proposition 3.4.2. *Let $x_0 \in \mathbf{T}^1$ and let f be a complex-valued function on $\mathbf{T}^1$. Suppose that the left and right limits of f exist as $x \to x_0$ and that the partial sums (Dirichlet means) $(D_N * f)(x_0)$ converge. Then*

$$(D_N * f)(x_0) \to \frac{1}{2}\big(f(x_0+) + f(x_0-)\big)$$

as $N \to \infty$.

Proof. If $(D_N * f)(x_0) \to L(x_0)$ as $N \to \infty$, then

$$(F_N * f)(x_0) = \frac{(D_0 * f)(x_0) + (D_1 * f)(x_0) + \cdots + (D_N * f)(x_0)}{N+1} \to L(x_0)$$

as $N \to \infty$. But $(F_N * f)(x_0) \to \frac{1}{2}\big(f(x_0+) + f(x_0-)\big)$ as $N \to \infty$ in view of Theorem 3.4.1. We conclude that

$$L(x_0) = \frac{1}{2}\big(f(x_0+) + f(x_0-)\big),$$

Thus $(D_N * f)(x_0) \to \frac{1}{2}\big(f(x_0+) + f(x_0-)\big)$ as $N \to \infty$. □

This theorem is quite useful when we have a priori knowledge that the Fourier series converges. For instance, consider the following example.

Example 3.4.3. On $(-1/2, 1/2)$ let $f(t) = t$ and $f(1/2) = f(-1/2) = 1000$. Then f is discontinuous at the point $-1/2 \equiv 1/2$ but it has left and right limits at this point:

$$\lim_{t \to -\frac{1}{2}+} f(t) = -\frac{1}{2} \qquad \lim_{t \to \frac{1}{2}-} f(t) = \frac{1}{2}. \tag{3.4.5}$$

Moreover $\widehat{f}(m) = \frac{i(-1)^m}{2\pi m}$ when $m \neq 0$ and $\widehat{f}(0) = 0$ by Exercise 3.4.1 (a). It is not hard to see that the series

$$(D_N * f)(x) = \frac{i}{2\pi} \sum_{0<|m|\le N} \frac{(-1)^m}{m} e^{2\pi i m x} = \frac{i}{2\pi} \sum_{0<|m|\le N} \frac{e^{2\pi i m (x+\frac{1}{2})}}{m} \tag{3.4.6}$$

converges for every $x \in (-1/2, 1/2)$. Indeed, by Appendix F, (3.4.6) equals

$$\frac{i}{2\pi}\frac{1}{N}\sum_{m=1}^{N}\big(e^{2\pi i m(x+\frac{1}{2})} - e^{-2\pi i m(x+\frac{1}{2})}\big)$$
$$-\frac{i}{2\pi}\sum_{k=1}^{N-1}\bigg(\sum_{m=1}^{k}\big(e^{2\pi i m(x+\frac{1}{2})} - e^{-2\pi i m(x+\frac{1}{2})}\big)\bigg)\bigg(\frac{1}{k+1} - \frac{1}{k}\bigg)$$

which has a limit as $N \to \infty$, since the geometric sums

$$\sum_{m=1}^{N} e^{\pm 2\pi i m(x+\frac{1}{2})} = \frac{1 - e^{\pm 2\pi i (N+1)(x+\frac{1}{2})}}{1 - e^{\pm 2\pi i (x+\frac{1}{2})}} - 1$$

are bounded above independently of N when $x \in (-1/2, 1/2)$. We conclude that

$$f(x) = x = \lim_{N\to\infty} \frac{i}{2\pi} \sum_{0<|m|\le N} \frac{e^{2\pi i m(x+\frac{1}{2})}}{m} = -\lim_{N\to\infty} \sum_{0<|m|\le N} \frac{\sin(2\pi m(x+\frac{1}{2}))}{2\pi m}$$

whenever $|x| < 1/2$. Moreover, we have that

$$(D_N * f)(1/2) = \lim_{N\to\infty} \frac{i}{2\pi} \sum_{0<|m|\le N} \frac{0}{m} = 0,$$

which is the average of the left and right limits in (3.4.5) as Proposition 3.4.2 states. Exercise 3.4.2 contains other applications of this sort.

3.4.2 Almost Everywhere Convergence of the Fejér Means

We have seen that the Fejér means of a relatively nice function (such as of bounded variation) converge everywhere. What can we say about the Fejér means of a general integrable function? Since the Fejér kernel is an approximate identity that satisfies good estimates, the following result should not come as a surprise.

Theorem 3.4.4. *(a) For $f \in L^1(\mathbf{T}^n)$, let*

$$\mathscr{H}(f) = \sup_{N\in\mathbf{Z}^+} |f * F_N^n|.$$

Then $\mathscr{H}$ maps $L^1(\mathbf{T}^n)$ to $L^{1,\infty}(\mathbf{T}^n)$ and $L^p(\mathbf{T}^n)$ to itself for $1 < p \le \infty$.
(b) For any function $f \in L^1(\mathbf{T}^n)$, we have as $N \to \infty$

$$(F_N^n * f) \to f \qquad \text{a.e.}$$

Proof. It is an elementary fact that $|t| \le \frac{\pi}{2} \implies |\sin t| \ge \frac{2}{\pi}|t|$; see Appendix E. Using this fact and the expression (3.1.18) we obtain for all t in $[-\frac{1}{2}, \frac{1}{2}]$,

$$\begin{aligned}
|F_N(t)| &= \frac{1}{N+1}\left|\frac{\sin(\pi(N+1)t)}{\sin(\pi t)}\right|^2 \\
&\le \frac{N+1}{4}\left|\frac{\sin(\pi(N+1)t)}{(N+1)t}\right|^2 \\
&\le \frac{N+1}{4}\min\left(\pi^2, \frac{1}{(N+1)^2t^2}\right) \\
&\le \frac{\pi^2}{2}\frac{N+1}{1+(N+1)^2|t|^2}\,.
\end{aligned}$$

For $t \in \mathbf{R}$ let us set $\varphi(t) = (1+|t|^2)^{-1}$ and $\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi(\frac{t}{\varepsilon})$ for $\varepsilon > 0$. For $x = (x_1,\dots,x_n) \in \mathbf{R}^n$ and $\varepsilon > 0$ we also set

$$\Phi(x) = \varphi(x_1)\cdots\varphi(x_n)$$

and $\Phi_\varepsilon(x) = \varepsilon^{-n}\Phi(\varepsilon^{-1}x)$. Then for $|t| \le \frac{1}{2}$ we have $|F_N(t)| \le \frac{\pi^2}{2}\varphi_\varepsilon(t)$ with $\varepsilon = (N+1)^{-1}$, and for $y \in [-\frac{1}{2},\frac{1}{2}]^n$ we have

$$|F_N^n(y)| \le (\tfrac{\pi^2}{2})^n\,\Phi_\varepsilon(y), \qquad \text{with } \varepsilon = (N+1)^{-1}.$$

Now let f be an integrable function on $\mathbf{T}^n$ and let f_0 denote its periodic extension on $\mathbf{R}^n$. For $x \in [-\frac{1}{2},\frac{1}{2}]^n$ we have

$$\begin{aligned}
\mathscr{H}(f)(x) &= \sup_{N>0}\left|\int_{\mathbf{T}^n} F_N^n(y)f(x-y)\,dy\right| \\
&\le (\tfrac{\pi^2}{2})^n\sup_{\varepsilon>0}\int_{[-\frac{1}{2},\frac{1}{2}]^n}|\Phi_\varepsilon(y)|\,|f_0(x-y)|\,dy \\
&\le 5^n\sup_{\varepsilon>0}\int_{\mathbf{R}^n}|\Phi_\varepsilon(y)|\,|(f_0\chi_Q)(x-y)|\,dy \\
&= 5^n\mathscr{G}(f_0\chi_Q)(x),
\end{aligned} \tag{3.4.7}$$

where Q is the cube $[-1,1]^n$ and $\mathscr{G}$ is the operator defined on integrable functions on $\mathbf{R}^n$ by

$$\mathscr{G}(h) = \sup_{\varepsilon>0}|h| * \Phi_\varepsilon\,.$$

If we can show that $\mathscr{G}$ maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$, the corresponding conclusion for $\mathscr{H}$ on $\mathbf{T}^n$ would follow from the fact $\mathscr{H}(f) \le 5^n\mathscr{G}(f_0\chi_Q)$ proved in (3.4.7) and the sequence of inequalities

$$\big\|\mathscr{H}(f)\big\|_{L^{1,\infty}(\mathbf{T}^n)} \le 5^n\big\|\mathscr{G}(f_0\chi_Q)\big\|_{L^{1,\infty}(\mathbf{R}^n)} \le 5^nC\big\|f_0\chi_Q\big\|_{L^1(\mathbf{R}^n)} = C'\big\|f\big\|_{L^1(\mathbf{T}^n)}.$$

Moreover, the L^p conclusion about $\mathscr{H}$ follows from the weak type $(1,1)$ result and the trivial L^∞ inequality, in view of the Marcinkiewicz interpolation theorem (Theorem 1.3.2). The required weak type $(1,1)$ estimate for $\mathscr{G}$ on $\mathbf{R}^n$ is a consequence of Lemma 3.4.5. Modulo the proof of this lemma, part (a) of the theorem is proved.

To prove the statement in part (b) observe that for $f \in \mathscr{C}^\infty(\mathbf{T}^n)$, which is a dense subspace of L^1, we have $F_N^n * f \to f$ uniformly on $\mathbf{T}^n$ as $N \to \infty$, since the sequence $\{F_N\}_N$ is an approximate identity. Since by part (a), $\mathscr{H}$ maps $L^1(\mathbf{T}^n)$ to $L^{1,\infty}(\mathbf{T}^n)$, Theorem 2.1.14 yields that for $f \in L^1(\mathbf{T}^n)$, $F_N^n * f \to f$ a.e. □

We now prove the weak type $(1,1)$ boundedness of $\mathscr{G}$ used earlier.

Lemma 3.4.5. *Let $\Phi(x_1,\dots,x_n) = (1+|x_1|^2)^{-1}\cdots(1+|x_n|^2)^{-1}$ and for $\varepsilon > 0$ let $\Phi_\varepsilon(x) = \varepsilon^{-n}\Phi(\varepsilon^{-1}x)$. Then the maximal operator*

$$\mathscr{G}(f) = \sup_{\varepsilon>0} |f| * \Phi_\varepsilon$$

maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$.

Proof. Let $I_0 = [-1,1]$ and $I_k = \{t \in \mathbf{R} : 2^{k-1} \le |t| \le 2^k\}$ for $k = 1,2,\dots$. Also, let $\widetilde{I}_k$ be the convex hull of I_k, that is, the interval $[-2^k, 2^k]$. For $a_2,\dots,a_n$ fixed positive numbers, let $M_{a_2,\dots,a_n}$ be the maximal operator obtained by averaging a function on $\mathbf{R}^n$ over all products of closed intervals $J_1 \times \cdots \times J_n$ containing a given point with

$$|J_1| = 2^{a_2}|J_2| = \cdots = 2^{a_n}|J_n|.$$

In view of Exercise 2.1.9(c), we have that $M_{a_2,\dots,a_n}$ maps L^1 to $L^{1,\infty}$ with some constant independent of the a_j's. (This is due to the nice doubling property of this family of rectangles.) For a fixed $\varepsilon > 0$ we estimate the expression

$$(\Phi_\varepsilon * |f|)(0) = \int_{\mathbf{R}^n} \frac{|f(-\varepsilon y)|\,dy}{(1+y_1^2)\cdots(1+y_n^2)}.$$

Split $\mathbf{R}^n$ into $n!$ regions of the form $|y_{j_1}| \ge \cdots \ge |y_{j_n}|$, where $\{j_1,\dots,j_n\}$ is a permutation of the set $\{1,\dots,n\}$ and $y = (y_1,\dots,y_n)$. By symmetry, we examine the region $\mathscr{R}$ where $|y_1| \ge \cdots \ge |y_n|$. Then for some constant $C > 0$ we have

$$\int_{\mathscr{R}} \frac{|f(-\varepsilon y)|\,dy}{(1+y_1^2)\cdots(1+y_n^2)} \le C \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1}\cdots\sum_{k_n=0}^{k_{n-1}} 2^{-(2k_1+\cdots+2k_n)} \int_{I_{k_1}}\cdots\int_{I_{k_n}} |f(-\varepsilon y)|\,dy_{k_n}\cdots dy_1,$$

and the last expression is trivially controlled by the corresponding expression, where the I_k's are replaced by the $\widetilde{I}_k$'s. This, in turn, is controlled by

$$C' \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1}\cdots\sum_{k_n=0}^{k_{n-1}} 2^{-(k_1+\cdots+k_n)} M_{k_1-k_2,\dots,k_1-k_n}(f)(0). \tag{3.4.8}$$

Now set $s_2 = k_1 - k_2, \dots, s_n = k_1 - k_n$, observe that $s_j \geq 0$, use that

$$2^{-(k_1+\cdots+k_n)} \leq 2^{-\frac{k_1}{2}} 2^{-\frac{s_2}{2n}} \cdots 2^{-\frac{s_n}{2n}},$$

and change the indices of summation to estimate the expression in (3.4.8) by

$$C'' \sum_{k_1=0}^{\infty} \sum_{s_2=0}^{\infty} \cdots \sum_{s_n=0}^{\infty} 2^{-\frac{k_1}{2}} 2^{-\frac{s_2}{2n}} \cdots 2^{-\frac{s_n}{2n}} M_{s_2,\dots,s_n}(f)(0).$$

Argue similarly for the remaining regions $|y_{j_1}| \geq \cdots \geq |y_{j_n}|$. Finally, translate to an arbitrary point x to obtain the estimate

$$|(\Phi_\varepsilon * f)(x)| \leq C'' n! \sum_{s_2=0}^{\infty} \cdots \sum_{s_n=0}^{\infty} 2^{-\frac{s_2}{2n}} \cdots 2^{-\frac{s_n}{2n}} M_{s_2,\dots,s_n}(f)(x).$$

Now take the supremum over all $\varepsilon > 0$ and use the fact that the maximal functions $M_{s_2,\dots,s_n}$ map L^1 to $L^{1,\infty}$ uniformly in $s_2, \dots, s_n$ as well as the result of Exercise 1.4.10 to obtain the desired conclusion for $\mathscr{G}$. □

3.4.3 Pointwise Divergence of the Dirichlet Means

We now pass to the more difficult question of convergence of the square partial sums of a Fourier series. It is natural to start our investigation with the class of continuous functions. Do the partial sums of the Fourier series of continuous functions converge pointwise? The following simple proposition warns about the behavior of partial sums.

Proposition 3.4.6. *(a) (**duBois Reymond**) There exists a continuous function f on $\mathbf{T}^1$ whose partial sums diverge at a point. Precisely, for some point $x_0 \in \mathbf{T}^1$ we have*

$$\limsup_{N\to\infty} \Bigg| \sum_{\substack{m\in\mathbf{Z} \\ |m_j|\leq N}} \widehat{f}(m) e^{2\pi i x_0 m} \Bigg| = \infty.$$

(b) There exists a continuous function F on $\mathbf{T}^n$ and $x_0 \in \mathbf{T}^1$ such that the sequence

$$\limsup_{N\to\infty} \Bigg| \sum_{\substack{m\in\mathbf{Z}^n \\ |m_j|\leq N}} \widehat{F}(m) e^{2\pi i (x_0 m_1 + x_2 m_2 + \cdots + x_n m_n)} \Bigg| = \infty$$

for all $x_2, \dots, x_m$ in $\mathbf{T}^1$.

Proof. The proof of part (b) is obtained by considering the continuous function $F(x_1,\dots,x_n) = f(x_1)$, where f is as in part (a). Then we have

$$(F * D_N^n)(x_1,\dots,x_n) = (f * D_N)(x_1)$$

and thus the square partial sums of F diverge on the $(n-1)$-dimensional plane $\{(x_0,x_2,\dots,x_n) : x_2,\dots,x_n \in \mathbf{T}^1\}$.

We now prove part (a) using functional analysis. For a constructive proof, see Exercise 3.4.7. Let $C(\mathbf{T}^1)$ be the Banach space of all continuous functions on the circle equipped with the L^∞ norm. Consider the continuous linear functionals

$$f \to T_N(f) = (D_N * f)(0)$$

on $C(\mathbf{T}^1)$ for $N = 1,2,\dots$. We show that the norms of the T_N's on $C(\mathbf{T}^1)$ converge to infinity as $N \to \infty$. To see this, given any integer $N \geq 100$, let $\varphi_N(x)$ be a continuous even function on $[-\frac{1}{2},\frac{1}{2}]$ that is bounded by 1 and is equal to the sign of $D_N(x)$ except at small intervals of length $(2N+1)^{-2}$ around the $2N+1$ zeros of D_N. Call the union of all these intervals B_N and set $A_N = [-\frac{1}{2},\frac{1}{2}] \setminus B_N$. Then

$$\int_{B_N} \big|D_N(x)\big|\,dx + \left|\int_{B_N} \varphi_N(x) D_N(x)\,dx\right| \leq 2\,|B_N|(2N+1) = 2.$$

Using this estimate we obtain

$$\begin{aligned}
\big\|T_N\big\|_{C(\mathbf{T}^1)\to\mathbf{C}} &\geq |T_N(\varphi_N)| = \left|\int_{\mathbf{T}^1} D_N(-x)\varphi_N(x)\,dx\right| \\
&\geq \int_{A_N} \big|D_N(x)\big|\,dx - \left|\int_{B_N} D_N(x)\varphi_N(x)\,dx\right| \\
&= \int_{\mathbf{T}^1} \big|D_N(x)\big|\,dx - \left|\int_{B_N} D_N(x)\varphi_N(x)\,dx\right| - \int_{B_N} \big|D_N(x)\big|\,dx \\
&\geq \frac{4}{\pi^2}\sum_{k=1}^{N}\frac{1}{k} - 2\,.
\end{aligned}$$

It follows that the norms of the linear functionals T_N are not uniformly bounded. The uniform boundedness principle now implies the existence of a function $f \in C(\mathbf{T}^1)$ and of a sequence $N_j \to \infty$ such that

$$|T_{N_j}(f)| \to \infty$$

as $j \to \infty$. The Fourier series of this f diverges at $x_0 = 0$. □

3.4.4 Pointwise Convergence of the Dirichlet Means

We have seen that continuous functions may have divergent Fourier series. How about Lipschitz continuous functions? As it turns out, there is a more general condition that implies convergence for the Fourier series of functions that satisfy a certain integrability condition.

Theorem 3.4.7. ***(Dini)*** *Let f be an integrable function on $\mathbf{T}^1$, let t_0 be a point on $\mathbf{T}^1$ for which $f(t_0)$ is defined and assume that*

$$\int_{|t|\le\frac{1}{2}} \frac{|f(t+t_0)-f(t_0)|}{|t|}\,dt < \infty . \tag{3.4.9}$$

Then $(D_N^n * f)(t_0) \to f(t_0)$ as $N\to\infty$.
(Tonelli) *Let f be an integrable function on $\mathbf{T}^n$ and let $a=(a_1,\dots,a_n)\in\mathbf{T}^n$. If f is defined at a and*

$$\int_{|x_1|\le\frac{1}{2}}\cdots\int_{|x_n|\le\frac{1}{2}} \frac{|f(x+a)-f(a)|}{|x_1|\cdots|x_n|}\,dx_n\cdots dx_1 < \infty , \tag{3.4.10}$$

then we have $(D_N^n * f)(a)\to f(a)$ as $N\to\infty$.

Proof. Since the one-dimensional result is contained in the multidimensional one, we prove the latter. Replacing $f(x)$ by $f(x+a)-f(a)$, we may assume that $a=0$ and $f(a)=0$. Using identities (3.1.15) and (3.1.14), we can write

$$\begin{aligned}(D_N^n * f)(0) &= \int_{\mathbf{T}^n} f(-x)\prod_{j=1}^n \frac{\sin((2N+1)\pi x_j)}{\sin(\pi x_j)}\,dx_n\cdots dx_1 \\ &= \int_{\mathbf{T}^n} f(-x)\prod_{j=1}^n \left(\frac{\sin(2N\pi x_j)\cos(\pi x_j)}{\sin(\pi x_j)}+\cos(2N\pi x_j)\right)dx_n\cdots dx_1 .\end{aligned} \tag{3.4.11}$$

Expand out the product to express the integrand as a sum of terms of the form

$$\left\{ f(-x)\prod_{j\in I}\frac{\cos(\pi x_j)}{\sin(\pi x_j)}\right\}\prod_{j\in I}\sin(2N\pi x_j)\prod_{k\in\{1,2,\dots,n\}\setminus I}\cos(2N\pi x_k)\,, \tag{3.4.12}$$

where I is a subset of $\{1,2,\dots,n\}$; here we use the convention that the product over an empty set of indices is 1. The function f_I inside the curly brackets in (3.4.12) is integrable on $[-\frac{1}{2},\frac{1}{2})^n$ except possibly in a neighborhood of the origin, since $|\sin(\pi x_j)|\ge 2|x_j|$ when $|x_j|\le\frac{1}{2}$. But condition (3.4.10) with $a=0$ and $f(a)=0$ guarantees that f_I is also integrable in a neighborhood of the origin. Expressing the sines and cosines in (3.4.12) in terms of exponentials, we obtain that the integral of (3.4.12) over $[-\frac{1}{2},\frac{1}{2})^n$ is a finite linear combination of Fourier coefficients of f_I at the points $(\pm N,\dots,\pm N)\in\mathbf{Z}^n$. Applying Lemma 3.3.1 yields that the expression in (3.4.11) tends to zero as $N\to\infty$. □

The following are consequences of this test.

Corollary 3.4.8. *(a) (**Riemann's principle of localization**) Let f be an integrable function on $\mathbf{T}^1$ that vanishes on an open interval I. Then $D_N * f$ converges to zero on the interval I.*
(b) Let $a = (a_1, \dots, a_n) \in \mathbf{T}^n$ and suppose that an integrable function f on $\mathbf{T}^n$ is constant on the cross

$$\{x = (x_1, \dots, x_n) \in \mathbf{T}^n : \ |x_j - a_j| < \delta_j \quad \textit{for some } j\},$$

where $0 < \delta_j < 1/2$ are fixed. Then $(D_N^n * f)(a) \to f(a)$ as $N \to \infty$.

Proof. (a) Let $t_0 \in I$. If f vanishes on I, condition (3.4.9) holds, since the function $t \mapsto f(t + t_0) - f(t_0)$ vanishes on $-t_0 + I$, which is an interval containing the origin, and is integrable outside $-t_0 + I$. Thus $(D_N * f)(t_0) \to f(t_0) = 0$ for every $t_0 \in I$.
(b) We need to show that the function

$$\frac{|f(x+a) - f(a)|}{|x_1| \cdots |x_n|}$$

is integrable over $\mathbf{T}^n = [-1/2, 1/2)^n$. The integral of this function over $\mathbf{T}^n$ is equal to its integral over the region

$$S = \{(x_1, \dots, x_n) \in \mathbf{T}^n : \ |x_k| \geq \delta_k \quad \text{for all } k\},$$

since $f(x+a) - f(a)$ vanishes whenever $|x_j| < \delta_j$ for some $j \in \{1, 2, \dots, n\}$. But on S we have that

$$\frac{|f(x+a) - f(a)|}{|x_1| \cdots |x_n|} \leq \frac{|f(x+a) - f(a)|}{\delta_1 \cdots \delta_n}$$

and this function is integrable over S, since f is. We deduce that (3.4.10) holds. □

Corollary 3.4.9. *Let $a \in \mathbf{T}^n$ and suppose that $f \in L^1(\mathbf{T}^n)$ satisfies*

$$|f(x) - f(a)| \leq C|x_1 - a_1|^{\varepsilon_1} \cdots |x_n - a_n|^{\varepsilon_n}$$

for some $C, \varepsilon_j > 0$ and for all $x \in \mathbf{T}^n$. Then the square partial sums $(D_N^n * f)(a)$ converge to $f(a)$.

Proof. Note that condition (3.4.10) holds. □

Corollary 3.4.10. ***(Dirichlet)*** *If f is defined on $\mathbf{T}^1$ and is a differentiable function at a point a in $\mathbf{T}^1$, then $(D_N * f)(a) \to f(a)$.*

Proof. There exists a $\delta > 0$ (say less than $1/2$) such that $|f(x) - f(a)|/|x - a|$ is bounded by $|f'(a)| + 1$ for $|x - a| \leq \delta$. Also $|f(x) - f(a)|/|x - a|$ is bounded by $|f(x) - f(a)|/\delta$ when $|x - a| > \delta$. It follows that condition (3.4.9) holds. □

Exercises

3.4.1. Identify $\mathbf{T}^1$ with $[-1/2,1/2)$ and fix $0<b<1/2$. Prove the following:

(a) The mth Fourier coefficient of the function x is $i\frac{(-1)^m}{2\pi m}$ when $m\neq 0$ and 0 when $m=0$.
(b) The mth Fourier coefficient of the function $\chi_{[-b,b]}$ is $\frac{\sin(2\pi bm)}{m\pi}$ when $m\neq 0$ and $2b$ when $m=0$.
(c) The mth Fourier coefficient of the function $\big(1-\frac{|x|}{b}\big)_+$ is $\frac{\sin^2(\pi bm)}{bm^2\pi^2}$ when $m\neq 0$ and b when $m=0$.
(d) The mth Fourier coefficient of the function $|x|$ is $-\frac{1}{2m^2\pi^2}+\frac{(-1)^m}{2m^2\pi^2}$ when $m\neq 0$ and $\frac{1}{4}$ when $m=0$.
(e) The mth Fourier coefficient of the function x^2 is $\frac{(-1)^m}{2m^2\pi^2}$ when $m\neq 0$ and $\frac{1}{12}$ when $m=0$.
(f) The mth Fourier coefficient of the function $\cosh(2\pi x)$ is $\frac{(-1)^m}{1+m^2}\frac{\sinh\pi}{\pi}$.
(g) The mth Fourier coefficient of the function $\sinh(2\pi x)$ is $\frac{im(-1)^m}{1+m^2}\frac{\sinh\pi}{\pi}$.

3.4.2. Use Exercise 3.4.1 and Proposition 3.4.2 to prove that

$$\sum_{k\in\mathbf{Z}}\frac{1}{(2k+1)^2}=\frac{\pi^2}{4}\qquad \sum_{k\in\mathbf{Z}\setminus\{0\}}\frac{1}{k^2}=\frac{\pi^2}{3}$$
$$\sum_{k\in\mathbf{Z}\setminus\{0\}}\frac{(-1)^{k+1}}{k^2}=\frac{\pi^2}{6}\qquad \sum_{k\in\mathbf{Z}}\frac{(-1)^k}{k^2+1}=\frac{2\pi}{e^{\pi}-e^{-\pi}}\,.$$

3.4.3. Let $M>N$ be given positive integers.
(a) For $f\in L^1(\mathbf{T}^1)$, prove the following identity:

$$\begin{aligned}(D_N*f)(x) &= \frac{M+1}{M-N}(F_M*f)(x)-\frac{N+1}{M-N}(F_N*f)(x)\\ &\quad -\frac{M+1}{M-N}\sum_{N<|j|\le M}\left(1-\frac{|j|}{M+1}\right)\widehat{f}(j)e^{2\pi i jx}\,.\end{aligned}$$

(b) (*G. H. Hardy*) Suppose that a function f on $\mathbf{T}^1$ satisfies the following condition: for any $\varepsilon>0$ there exists an $a>1$ and a $k_0>0$ such that for all $k\ge k_0$ we have

$$\sum_{k<|m|\le[ak]}|\widehat{f}(m)|<\varepsilon.$$

Use part (a) to prove that if $(F_N*f)(x)$ converges (uniformly) to $A(x)$ as $N\to\infty$, then $(D_N*f)(x)$ also converges (uniformly) to $A(x)$ as $N\to\infty$.

3.4.4. Use Proposition 3.4.2 to show that for $0 < b < \frac{1}{2}$ we have

$$\lim_{N\to\infty} \sum_{\substack{m=-N \\ m\neq 0}}^{N} \frac{\sin(2\pi bm)}{m\pi} e^{2\pi i bm} = \frac{1}{2} - 2b\,.$$

[*Hint:* Use Exercise 3.4.1(b).]

3.4.5. Let f be an integrable function on $\mathbf{T}^n$ and g be a bounded function on $\mathbf{T}^n$ and let K be a compact subset of $\mathbf{T}^n$. Consider the family $\mathscr{F} = \{f_w : w \in K\}$, where $f_w(x) = f(x-w)g(x)$ for all $x \in \mathbf{T}^n$. Prove that the Riemann–Lebesgue lemma holds uniformly for the family $\mathscr{F}$. This means that given $\varepsilon > 0$ there exists an $N_0(K) > 0$ such that for $|m| \geq N_0$ we have $|\widehat{f_w}(m)| \leq \varepsilon$ for all $w \in K$.

3.4.6. Prove the following version of Corollary 3.4.8 (b). Suppose that a function f on $\mathbf{T}^n$ is constant on the cross $U = \{(x_1,\dots,x_n) \in \mathbf{T}^n : \ |x_j - a_j| < \delta \text{ for some } j\}$, for some $\delta < 1/2$. Then $D_N^n * f$ converges to $f(a)$ uniformly on compact subsets of the box $W = \{(x_1,\dots,x_n) \in \mathbf{T}^n : \ |x_j| < \delta \ \text{ for all } j\}$.
[*Hint:* Use Exercise 3.4.5.]

3.4.7. Follow the steps given to obtain a constructive proof of the existence of a continuous function whose Fourier series diverges at a point. Identify $\mathbf{T}^1$ with $[0,1)$ and define

$$g(x) = -2\pi i(x - 1/2).$$

(a) Prove that $\widehat{g}(m) = 1/m$ when $m \neq 0$ and zero otherwise.
(b) Prove that for all nonnegative integers M and N we have

$$\big((e^{2\pi i N(\cdot)}(g*D_N)) * D_M\big)(x) = e^{2\pi i N x} \sum_{1\leq |r| \leq N} \frac{1}{r} e^{2\pi i r x}$$

when $M \geq 2N$ and

$$\big((e^{2\pi i N(\cdot)}(g*D_N)) * D_M\big)(x) = e^{2\pi i N x} \sum_{\substack{-N\leq r \leq M-N \\ r\neq 0}} \frac{1}{r} e^{2\pi i r x}$$

when $M < 2N$. Conclude that there exists a constant $C > 0$ such that for all M, N, and $x \neq 0$ we have

$$|\big(e^{2\pi i N(\cdot)}(g*D_N)\big) * D_M(x)| \leq \frac{C}{|x|}\,.$$

(c) Show that there exists a constant $C_1 > 0$ such that

$$\sup_{N>0} \sup_{x\in\mathbf{T}^1} \big|(g*D_N)(x)\big| = \sup_{N>0} \sup_{x\in\mathbf{T}^1} \left| \sum_{1\leq |r| \leq N} \frac{1}{r} e^{2\pi i r x} \right| \leq C_1 < \infty\,.$$

(d) Let $\lambda_k = 1 + e^{e^k}$. Define

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} e^{2\pi i \lambda_k x} (g * D_{\lambda_k})(x)$$

and prove that f is continuous on $\mathbf{T}^1$ and that its Fourier series converges at every $x \neq 0$, but $\limsup_{M\to\infty} |(f * D_M)(0)| = \infty$.
$\big[$*Hint:* Take $M = e^{e^m}$ with $m \to \infty$. The inequality in part (b) follows by summation by parts.$\big]$

3.5 A Tauberian theorem and Functions of Bounded Variation

The relation between the partial sums of a Fourier series and the Fejér means is a particular situation of a relation between sequences of complex numbers and their arithmetic means. Given a sequence $\{a_k\}_{k=0}^{\infty}$ of complex numbers, we denote its *partial sums* by

$$s_N = a_1 + \cdots + a_N$$

for $N \geq 0$, and its arithmetic or *Cesàro means* by

$$\sigma_N = \frac{1}{N+1} \sum_{k=0}^{N} s_k = \frac{1}{N+1} \sum_{k=0}^{N} (N+1-k) a_k \,.$$

A classical result says that if $s_N \to L$ as $N \to \infty$, then $\sigma_N \to L$ as $N \to \infty$. The converse is not true, as the example $a_k = (-1)^k$ indicates. But in a particular situation the reverse implication holds.

3.5.1 A Tauberian theorem

We have the following result concerning the convergence of $\{s_k\}_{k=0}^{\infty}$ as a consequence of that of $\{\sigma_k\}_{k=0}^{\infty}$.

Theorem 3.5.1. *(a) Suppose that for a sequence $\{a_k\}_{k=0}^{\infty}$ of complex numbers we have that $\sigma_N \to L$ as $N \to \infty$ and that $|ka_k| \leq M < \infty$ for all $k = 0, 1, 2, \ldots$. Then $s_k \to L$ as $k \to \infty$.*
(b) Let X be a nonempty set. Suppose that for a sequence $\{a_k(x)\}_{k=0}^{\infty}$ of complex-valued functions on X we have that $\sigma_N(x) \to L(x)$ uniformly in $x \in X$ as $N \to \infty$ and that $\sup_{k\geq 0} \sup_{x\in X} |ka_k(x)| \leq M < \infty$. *Then $s_k(x) \to L(x)$ uniformly in $x \in X$ as $k \to \infty$.*

Proof. We prove part (b), noting that the proof of part (a) is subsumed in that of (b).

For $0 \le k < m < \infty$ we have

$$\begin{aligned}
&(m+1)\sigma_m(x) - (k+1)\sigma_k(x) - \sum_{j=k+1}^{m}(m+1-j)a_j(x)\\
=&\sum_{j=0}^{m}(m+1-j)a_j(x) - \sum_{j=0}^{k}(k+1-j)a_j(x) - \sum_{j=k+1}^{m}(m+1-j)a_j(x)\\
=&(m-k)\sum_{j=0}^{k}a_j(x)\\
=&(m-k)s_k(x)\,.
\end{aligned}$$

Therefore we have

$$\frac{m+1}{m-k}\sigma_m(x) - \frac{k+1}{m-k}\sigma_k(x) - \frac{1}{m-k}\sum_{j=k+1}^{m}(m+1-j)a_j(x) = s_k(x)$$

and thus

$$s_k(x) - \sigma_k(x) = \frac{m+1}{m-k}(\sigma_m(x) - \sigma_k(x)) - \frac{m+1}{m-k}\sum_{j=k+1}^{m}\Big(\frac{1}{j} - \frac{1}{m+1}\Big)ja_j(x)\,. \quad (3.5.1)$$

Notice that

$$\sum_{j=k+1}^{m}\Big(\frac{1}{j} - \frac{1}{m+1}\Big) \le \int_k^m \frac{dt}{t} - \sum_{j=k+1}^{m}\frac{1}{m+1} = \log\frac{m}{k} - \frac{m-k}{m+1}\,. \qquad (3.5.2)$$

Now fix $\varepsilon > 0$ such that $\varepsilon < 1$. For each $k \in \mathbf{Z}^+$ pick an $m_k \in \{k, k+1, \dots, 2k\}$ such that $\frac{m_k}{k} \to 1+\varepsilon$. Then $\frac{m_k+1}{m_k-k}$ converges to $\varepsilon^{-1}+1$ as $k \to \infty$, hence it is bounded by some constant C_ε. Then (3.5.1) and (3.5.2) with m_k in place of m yield

$$\sup_{x\in X}\big|s_k(x) - \sigma_k(x)\big| \le C_\varepsilon \sup_{x\in X}|\sigma_{m_k}(x) - \sigma_k(x)| + M\,\frac{m_k+1}{m_k-k}\Big[\log\frac{m_k}{k} - \frac{m_k-k}{m_k+1}\Big]\,.$$

Taking the $\limsup_{k\to\infty}$ in the preceding inequality and using that

$$\limsup_{k\to\infty}\sup_{x\in X}|\sigma_{m_k}(x) - \sigma_k(x)| = 0\,,$$

which is a consequence of the hypothesis that σ_k (and thus σ_{m_k}) converges to L uniformly, we obtain

$$\limsup_{k\to\infty}\sup_{x\in X}\big|s_k(x) - \sigma_k(x)\big| \le M\Big[\Big(1+\frac{1}{\varepsilon}\Big)\log(1+\varepsilon) - 1\Big]\,.$$

In view of the Taylor expansion

$$\log(1+\varepsilon) = \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 - \cdots = \varepsilon + O(\varepsilon^2),$$

which is valid for $0 < \varepsilon < 1$, we conclude that

$$\limsup_{k\to\infty} \sup_{x\in X} \big| s_k(x) - \sigma_k(x) \big| \le M\,c\,\varepsilon$$

for some absolute constant $c > 0$. Since $\varepsilon > 0$ was arbitrary, we finally deduce that $s_k(x)$ converges uniformly to the same limit as $\sigma_k(x)$, which is $L(x)$. □

Corollary 3.5.2. *Suppose that a function f on $\mathbf{T}^1$ is continuous and there is a constant $M > 0$ such that $|\widehat{f}(m)| \le M|m|^{-1}$ for all $m \in \mathbf{Z}^+ \setminus \{0\}$. Then the Fourier series of f converges uniformly to f. In particular, if f is a continuous function of bounded variation on the circle, then $f * D_N \to f$ uniformly on $\mathbf{T}^1$ as $N \to \infty$.*

Proof. The Fejér means $\{F_N\}_{N=0}^{\infty}$ are an approximate identity on $\mathbf{T}^n$ (Proposition 3.1.10) and so $F_N * f$ converge uniformly to f on $\mathbf{T}^1$ as $N \to \infty$ in view of Theorem 1.2.19 (2). Moreover, we have $|m|\,|\widehat{f}(m)| \le M$ for all $m \in \mathbf{Z}$. It follows from Theorem 3.5.1 that $D_N * f$ converges uniformly to f.

If, additionally, f is a function of bounded variation, then $|m|\,|\widehat{f}(m)| \le \frac{1}{2\pi}\mathrm{Var}(f)$, as shown in Proposition 3.3.14. Then the claimed conclusion follows. □

3.5.2 *The sine integral function*

We examine a few useful properties of the antiderivative of $\sin(t)/t$.

Definition 3.5.3. For $0 \le x < \infty$ define the *sine integral function*

$$Si(x) = \int_0^x \frac{\sin(t)}{t}\,dt. \tag{3.5.3}$$

Integrating by parts we write

$$Si(x) = \int_0^1 \frac{\sin(t)}{t}\,dt + \frac{-\cos(x)}{x} + \cos(1) - \int_1^x \frac{\cos(t)}{t^2}\,dt,$$

from which it follows that the limit of $Si(x)$ as $x \to \infty$ exists and is equal to

$$\lim_{x\to\infty} Si(x) = \int_0^1 \frac{\sin(t)}{t}\,dt + \cos(1) - \int_1^\infty \frac{\cos(t)}{t^2}\,dt.$$

To precisely evaluate the limit of $Si(x)$ as $x \to \infty$ we write

$$
\begin{aligned}
Si((N+\tfrac{1}{2})\pi) =& \pi \int_0^{\frac{1}{2}} \frac{\sin((2N+1)\pi t)}{\pi t}\, dt \\
=& \pi \int_0^{\frac{1}{2}} \frac{\sin((2N+1)\pi t)}{\sin(\pi t)}\, dt + \pi \int_0^{\frac{1}{2}} \sin((2N+1)\pi t)\left\{\frac{1}{\pi t} - \frac{1}{\sin(\pi t)}\right\} dt \\
=& \frac{\pi}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} D_N(t)\, dt + \frac{\pi}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{(2N+1)\pi i t} - e^{-(2N+1)\pi i t}}{2i} \left\{\frac{1}{\pi t} - \frac{1}{\sin(\pi t)}\right\} dt,
\end{aligned}
$$

which converges to $\pi/2 + 0$ as $N \to \infty$, in view of the Riemann-Lebesgue lemma (Proposition 3.3.1), since the function inside the curly brackets is integrable over the circle. We conclude that $\lim_{x\to\infty} Si(x) = \pi/2$.

Note that Si' vanishes at $n\pi$, $n = 0,1,2,\dots$ and $Si''(n\pi) = (-1)^n/n\pi$. Consequently, $Si(x)$ has local maxima at the points π, 3π, 5π, $\dots$ and local minima at the points 2π, 4π, 6π, $\dots$. Moreover, it is increasing on the intervals $[2k\pi,(2k+1)\pi]$ and decreasing on $[(2k+1)\pi,(2k+2)\pi]$, $k = 0,1,2,\dots$. Also, observe that

$$
Si(3\pi) - Si(\pi) = \int_\pi^{2\pi} \frac{\sin(t)}{t}\, dt + \int_\pi^{2\pi} \frac{\sin(t+\pi)}{t+\pi}\, dt = \int_\pi^{2\pi} \sin(t)\left(\frac{1}{t} - \frac{1}{t+\pi}\right) dt < 0
$$

and likewise we can prove the remaining inequalities in the sequence

$$
Si(\pi) > Si(3\pi) > Si(5\pi) > Si(7\pi) > \cdots > \frac{\pi}{2}.
$$

Similarly, one can show that

$$
Si(2\pi) < Si(4\pi) < Si(6\pi) < \cdots < \frac{\pi}{2}.
$$

Hence $Si(\pi)$ is the absolute maximum of $Si(x)$ on $[0,\infty)$, while 0 is the absolute minimum of $Si(x)$ on $[0,\infty)$; $Si(\pi)$ is the absolute minimum of $Si(x)$ on $[\pi,\infty)$.

3.5.3 Further properties of functions of bounded variation

Next we have the following theorem concerning functions of bounded variation. Recall that functions of bounded variation are differences of increasing functions and thus have left and right limits at every point.

Theorem 3.5.4. *Let $0 < \delta \le 1/2$. Suppose that f is an integrable function on $\mathbf{T}^1$ which is of bounded variation on the neighborhood $[t_0 - \delta, t_0 + \delta]$ of the point $t_0 \in \mathbf{T}^1$. Then*

$$
\lim_{N\to\infty} (f * D_N)(t_0) = \frac{f(t_0+) + f(t_0-)}{2}.
$$

Proof. We write W for the neighborhood $(-\delta,\delta)$ of 0, $F_{t_0}(t)=\frac{f(t_0-t)+f(t_0+t)}{2}$, and $L_{t_0}=\frac{f(t_0+)+f(t_0-)}{2}$. We have

$$(f*D_N)(t_0)=\int_{\mathbf{T}^1} f(t_0-t)D_N(t)\,dt=\int_{\mathbf{T}^1} f(t_0+t)D_N(t)\,dt\,,$$

hence, averaging yields

$$(f*D_N)(t_0)=\int_{\mathbf{T}^1}\frac{f(t_0-t)+f(t_0+t)}{2}D_N(t)\,dt=\int_{\mathbf{T}^1}F_{t_0}(t)D_N(t)\,dt\,.$$

Therefore we have

$$(f*D_N)(t_0)-L_{t_0}=\int_W\big(F_{t_0}(t)-L_{t_0}\big)D_N(t)\,dt+\int_{\mathbf{T}^1\setminus W}\big(F_{t_0}(t)-L_{t_0}\big)D_N(t)\,dt$$

and since in the second integral $|t|\geq\delta$, the Riemann-Lebesgue lemma shows that the second term is $o(1)$, i.e., it tends to zero as $N\to\infty$. We now show that the first integral also goes to zero. We write

$$\begin{aligned}\int_W\big(F_{t_0}(t)-L_{t_0}\big)D_N(t)\,dt&=\int_W\big(F_{t_0}(t)-L_{t_0}\big)\frac{\sin((2N+1)\pi t)}{\pi t}\,dt \qquad (3.5.4)\\&\quad+\int_W\big(F_{t_0}(t)-L_{t_0}\big)\bigg(\frac{1}{\sin(\pi t)}-\frac{1}{\pi t}\bigg)\sin((2N+1)\pi t)\,dt\,,\end{aligned}$$

but since the function $\frac{1}{\pi t}-\frac{1}{\sin(\pi t)}$ remains bounded on $[-\frac12,\frac12]$ (Exercise 3.1.5 (a)), it follows from the Riemann-Lebesgue lemma that the second term is $o(1)$ as $N\to\infty$. Consequently,

$$(f*D_N)(t_0)-L_{t_0}=\frac{1}{\pi}\int_W\big(F_{t_0}(t)-L_{t_0}\big)\frac{\sin((2N+1)\pi t)}{t}\,dt+o(1)$$

as $N\to\infty$. To prove the required conclusion, it will suffice to show that

$$\frac{2}{\pi}\int_0^\delta\big(F_{t_0}(t)-L_{t_0}\big)\frac{\sin((2N+1)\pi t)}{t}\,dt\to 0 \qquad (3.5.5)$$

as $N\to\infty$. Let $Si(t)$ be as defined in (3.5.3). We express the integral in (3.5.5) as

$$\int_0^\delta\big(F_{t_0}(t)-L_{t_0}\big)Si'((2N+1)\pi t)\,dt \qquad (3.5.6)$$

Integrating by parts we obtain that (3.5.6) is equal to

$$(F_{t_0}(\delta-)-L_{t_0})Si((2N+1)\pi\delta)-\int_0^\delta Si((2N+1)\pi t)dF_{t_0}(t)\,. \qquad (3.5.7)$$

Letting $N \to \infty$ and using the Lebesgue dominated convergence theorem, we conclude that (3.5.7) converges to

$$(F_{t_0}(\delta-)-L_{t_0})\frac{\pi}{2}-\int_0^\delta \frac{\pi}{2}dF_{t_0}(t)=(F_{t_0}(\delta-)-L_{t_0})\frac{\pi}{2}-(F_{t_0}(\delta-)-F_{t_0}(0+))\frac{\pi}{2}=0$$

noticing that $L_{t_0}=F_{t_0}(0+)$. □

Next, we obtain an explicit bound for the partial sums of functions of bounded variation. Let $Si(t)$ be as in (3.5.3).

Theorem 3.5.5. *Suppose that f is a function of bounded variation on the circle $\mathbf{T}^1$. Then the partial sums of the Fourier series of f are uniformly bounded, in particular, we have*

$$\sup_{t_0\in\mathbf{T}^1}\sup_{N\in\mathbf{Z}^+}|(f*D_N)(t_0)|\le\Big(1-\frac{2}{\pi}+Si(\pi)\Big)\|f\|_{L^\infty}+Si(\pi)\mathrm{Var}(f)\,. \tag{3.5.8}$$

Proof. We take $\delta=1/2$ in the proof of the preceding theorem. For a point $t_0\in\mathbf{T}^1$, let $F_{t_0}(t)=\frac{f(t_0-t)+f(t_0+t)}{2}$. We have that

$$\begin{aligned}(f*D_N)(t_0)=&\int_{\mathbf{T}^1}F_{t_0}(t)\frac{\sin((2N+1)\pi t)}{\pi t}\,dt\\&+\int_{\mathbf{T}^1}F_{t_0}(t)\Big(\frac{1}{\sin(\pi t)}-\frac{1}{\pi t}\Big)\sin((2N+1)\pi t)\,dt\,.\end{aligned} \tag{3.5.9}$$

Using that $\left|\frac{1}{\sin(\pi t)}-\frac{1}{\pi t}\right|\le 1-\frac{2}{\pi}$ when $|t|\le\frac{1}{2}$ (Exercise 3.1.5 (a)), we obtain that the second integral in (3.5.9) is bounded by $(1-\frac{2}{\pi})\|f\|_{L^\infty}$. Integrating by parts as in the proof of the preceding theorem, we express the first integral in (3.5.9) as

$$F_{t_0}(\tfrac{1}{2}-)Si((2N+1)\pi\tfrac{1}{2})-\int_0^{\frac{1}{2}}Si((2N+1)\pi t)\,dF_{t_0}(t)\,, \tag{3.5.10}$$

which is bounded (in absolute value) by $\|f\|_{L^\infty}Si(\pi)+Si(\pi)\mathrm{Var}(f)$. Assertion (3.5.8) now follows. □

3.5.4 Gibbs phenomenon

It is not reasonable to expect that the Fourier series of a discontinuous function converges uniformly in a neighborhood of a discontinuity. The lack of uniformity in the convergence can be measured in terms of the worst jump, called the *overshoot*. The exact form of nonuniform convergence is illustrated in the following example:

Example 3.5.6. Consider the function

$$h(t)=\begin{cases}\frac{1}{2}-t & \text{when } 0<t\le\frac{1}{2}\\ 0 & \text{when } t=0\\ -\frac{1}{2}-t & \text{when } -\frac{1}{2}<t<0.\end{cases} \tag{3.5.11}$$

Clearly $h(t)$ is a function of bounded variation and is continuous except at the point $t=0$ at which it has a jump discontinuity. Since h is an odd function, its Fourier coefficients are

$$\widehat{h}(m)=\int_{-1/2}^{1/2}h(t)e^{-2\pi imt}\,dt=-2i\int_0^{1/2}(\tfrac{1}{2}-t)\sin(2\pi mt)\,dt=-\frac{i}{2m\pi}$$

when $m\neq 0$ and $\widehat{h}(0)=0$. The partial sums of the Fourier series of h are

$$(h*D_N)(t)=-\frac{i}{2\pi}\sum_{\substack{|m|\le N\\ m\neq 0}}\frac{e^{2\pi imt}}{m}.$$

Notice that

$$\frac{d}{dt}(h*D_N)(t)=\sum_{\substack{|m|\le N\\ m\neq 0}}e^{2\pi imt}=D_N(t)-1.$$

Then, if we define $d(s)=\frac{1}{\sin(\pi s)}-\frac{1}{\pi s}$, we can write

$$\begin{aligned}(h*D_N)(t)&=\int_0^t D_N(s)-1\,ds\\ &=-t+\int_0^t\frac{\sin((2N+1)\pi s)}{\sin(\pi s)}\,ds\\ &=-t+\int_0^t d(s)\sin((2N+1)\pi s)\,ds+\int_0^t\frac{\sin((2N+1)\pi s)}{\pi s}\,ds.\end{aligned}$$

Notice that $d(s)$ is continuous at zero and $d(0)=0$, while $\lim_{s\to 0}\frac{d(s)}{s}=\frac{\pi}{6}$; thus d is a differentiable function on $[0,\frac{1}{2}]$ and $d'(0)=\frac{\pi}{6}$. Moreover, $\lim_{s\to 0}d'(s)=d'(0)$, thus d is continuously differentiable on $[0,\frac{1}{2}]$. Additionally both d and d' are nonnegative and increasing on $[0,\frac{1}{2}]$, thus $d'\le\frac{4}{\pi}$; see the hint of Exercise 3.1.5. It follows that

$$\int_0^t d(s)\sin((2N+1)\pi s)ds=-\frac{\cos((2N+1)\pi t)}{(2N+1)\pi}d(t)+\int_0^t d'(s)\frac{\cos((2N+1)\pi s)}{(2N+1)\pi}ds$$

and the preceding expression is bounded in absolute value by $\big(\frac{d(\frac{1}{2})}{\pi}+\frac{1}{2}\frac{d'(\frac{1}{2})}{\pi}\big)\frac{1}{2N+1}$.

We deduce that

$$(h*D_N)(t)=-t+\frac{1}{\pi}\int_0^t\frac{\sin((2N+1)\pi s)}{s}\,ds+O\Big(\frac{1}{2N+1}\Big),$$

where $O\big(\frac{1}{2N+1}\big)$ is a function bounded by $\frac{1}{\pi}\frac{1}{2N+1}$. Consequently,

$$(h*D_N)(t) = -t + \frac{1}{\pi}\int_0^{(2N+1)\pi t} \frac{\sin(s)}{s}\,ds + O\Big(\frac{1}{2N+1}\Big).$$

Hence for $t \in (0,-\frac{1}{2}]$ we have $\lim_{N\to\infty}(h*D_N)(t) = -t + \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2} - t$ as expected. Analogously for $t \in [-\frac{1}{2},0)$ we have $\lim_{N\to\infty}(h*D_N)(t) = -\frac{1}{2} - t$. Also for $t = 0$, $\lim_{N\to\infty}(h*D_N)(0) = 0$. Thus the Fourier series of h at zero converges to the "fair" value of the average of $h(0+)$ and $h(0-)$ which happens to be $h(0) = 0$.

To quantitatively estimate the nonuniformity of the convergence of $(h*D_N)(t)$ we note that

$$(h*D_N)(t) - (\tfrac{1}{2} - t) = \frac{1}{\pi}\int_0^{(2N+1)\pi t} \frac{\sin(s)}{s}\,ds - \frac{1}{2} + O\Big(\frac{1}{2N+1}\Big).$$

Thus for all $N = 1,2,\dots$ and $t \in (0,\frac{1}{2}]$ we have

$$(h*D_N)(t) - h(t) \le \frac{Si(\pi)}{\pi} - \frac{1}{2} + \frac{1}{\pi}\frac{1}{2N+1} \le .08949\dots + \frac{\pi^{-1}}{2N+1}.$$

Also, for any sequence $t_N \to 0+$ we have

$$\limsup_{N\to\infty}\big[(h*D_N)(t_N) - h(t_N)\big] \le \frac{Si(\pi)}{\pi} - \frac{1}{2} = .08949\dots, \tag{3.5.12}$$

while if for each N we consider the value $t_N = 1/(2N+1)$, we obtain that

$$\limsup_{N\to\infty}\big[(h*D_N)(t_N) - h(t_N)\big] = \frac{Si(\pi)}{\pi} - \frac{1}{2} = .08949\dots. \tag{3.5.13}$$

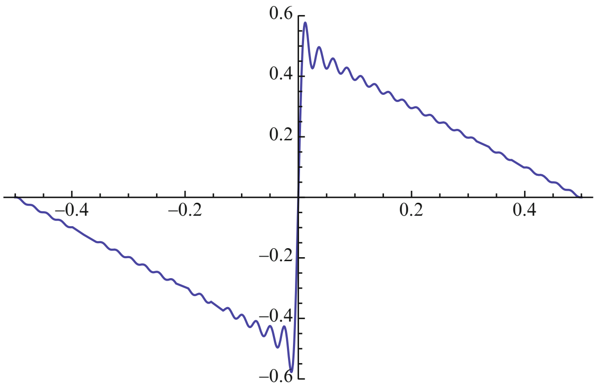

Fig. 3.3 The partial sums $(h*D_{40})(t)$ showing the overshoot of approximately 9% of the jump of h at zero.

The quantity .08949... is called the *overshoot* of the partial sums of the Fourier series of the function h in a neighborhood of zero. See Figure 3.3.

We now examine the preceding phenomenon in the setting of functions of bounded variation. These functions can be written as a differences of two increasing functions, so they have countable sets of discontinuities. Suppose we are given a function $f \in L^1(\mathbf{T}^1)$ of bounded variation, and for the sake of simplicity, let us consider the situation where it has exactly one discontinuity, say at the point $t_0 \in \mathbf{T}^1$. Consider the function h defined in (3.5.11) and define

$$f_0(t) = \begin{cases} \big(f(t_0+) - f(t_0-)\big)h(t-t_0) + \frac{f(t_0+)+f(t_0-)}{2} & \text{when } t \neq t_0, \\ f(t_0) & \text{when } t = t_0. \end{cases} \tag{3.5.14}$$

Now the function $f - f_0$ is of bounded variation and is also continuous and satisfies $(f - f_0)(t_0) = 0$. In view of Corollary 3.5.2, the Fourier series of $f - f_0$ converges uniformly to $f - f_0$ and so the lack of uniformity of the convergence of the partial sums of f is due to the presence of f_0.

We express these observations as a theorem.

Theorem 3.5.7. *(a) Let h be defined in* (3.5.11)*. Then the set of accumulation points of sets of the form $\{(h*D_N)(t_N)\}_{N\in\mathbf{Z}^+}$, where $t_N \in [0,1/2]$, is the interval*

$$\big[0, \tfrac{Si(\pi)}{\pi}\big] = \big[0, 0.58949\dots\big].$$

In particular if $t_N \to 0$ such that $Nt_N \to \frac{1}{2}$, then

$$\lim_{N\to\infty}(h*D_N)(t_N) = \frac{Si(\pi)}{\pi} = 0.58949\dots.$$

(b) Let f be a function of bounded variation on the circle with a single discontinuity at the point t_0, such that $f(t_0+) - f(t_0-) > 0$. Then the set of accumulation points of sets of the form $\{(f*D_N)(t_N)\}_{N\in\mathbf{Z}^+}$, where $t_N \in [t_0, t_0+\delta]$, for some $\delta > 0$, is the interval

$$\Big[\frac{f(t_0+)+f(t_0-)}{2}, \frac{f(t_0+)+f(t_0-)}{2} + \frac{Si(\pi)}{\pi}\big(f(t_0+) - f(t_0-)\big)\Big].$$

In particular if $t_N \to t_0+$ such that $N(t_N - t_0) \to \frac{1}{2}$, then

$$\lim_{N\to\infty}(f*D_N)(t_N) = \frac{f(t_0+)+f(t_0-)}{2} + \frac{Si(\pi)}{\pi}\big(f(t_0+) - f(t_0-)\big).$$

Proof. (a) Since $h \geq 0$ on $(0, \frac{1}{2}]$ and $(h*D_N)(t) \to \frac{1}{2} - t$ for $0 < t \leq \frac{1}{2}$ we have that all accumulation points of sequences $(h*D_N)(t_N)$ are nonnegative. We showed in (3.5.12) that all accumulation points of sequences $(h*D_N)(t_N) - h(t_N)$ are at most $\frac{Si(\pi)}{\pi} - \frac{1}{2}$; but $h(t_N) \leq \frac{1}{2}$, when $t_N \in [0, \frac{1}{2}]$, hence all accumulation points of sequences $(h*D_N)(t_N)$ are at most $\frac{Si(\pi)}{\pi}$ and thus contained in $\big[0, \frac{Si(\pi)}{\pi}\big]$. Also, 0 is attained as the accumulation point of $(h*D_N)(0)$ and the number $\frac{Si(\pi)}{\pi}$ is attained as the accumulation point of the sequence $(h*D_N)(\frac{1}{2N+1})$ as shown in (3.5.13); notice that the same assertion is valid for any other sequence $t_N \to 0$ such that $Nt_N \to \frac{1}{2}$. Now, since the functions $(h * D_N)(t)$ are continuous, given any c in $\left[0, \frac{Si(\pi)}{\pi}\right]$, there is a t'_N between 0 and $\frac{1}{2N+1}$ such that $(h * D_N)(t'_N) = c$ for all N; this shows that the set of accumulation points of sequences of the form $\{(f * D_N)(t_N)\}_{N \in \mathbf{Z}^+}$ is the interval $\left[0, \frac{Si(\pi)}{\pi}\right]$.

(b) To examine the behavior of $f * D_N$ near the point of a single jump discontinuity t_0 of f, we reduce matters to the preceding situation as alluded earlier, by introducing the function f_0 defined in (3.5.14). Then for a sequence t_N converging to t_0 from the right we have

$$\begin{aligned}
&(f * D_N)(t_N)\\
&= (f - f_0) * D_N(t_N) + \big(f(t_0+) - f(t_0-)\big)(h * D_N)(t_N - t_0) + \tfrac{f(t_0+)+f(t_0-)}{2}\\
&= \big(f(t_0+) - f(t_0-)\big)\Big[(h * D_N)(t_N - t_0) - h(t_N - t_0)\Big]\\
&\qquad + \big(f(t_0+) - f(t_0-)\big)h(t_N - t_0) + (f - f_0) * D_N(t_N) + \tfrac{f(t_0+)+f(t_0-)}{2}\,.
\end{aligned}$$

Applying $\limsup_{N\to\infty}$ and using (3.5.12), we obtain as $N \to \infty$

$$\limsup_{N\to\infty}(f * D_N)(t_N) \le \big(f(t_0+) - f(t_0-)\big)\left(\tfrac{Si(\pi)}{\pi} - \tfrac{1}{2} + \tfrac{1}{2}\right) + \tfrac{f(t_0+)+f(t_0-)}{2}\,,$$

where we used the following consequence of Theorem 3.5.4

$$\limsup_{N\to\infty}\,(f - f_0) * D_N(t_N) = (f - f_0)(t_0) = 0\,.$$

This shows that all accumulation points of sequences of the form $\{(f * D_N)(t_N)\}_{N \in \mathbf{Z}^+}$ are at most $\big(f(t_0+) - f(t_0-)\big)\frac{Si(\pi)}{\pi} + \frac{1}{2}\big(f(t_0+) + f(t_0-)\big)$. Also, since all accumulation points of sequences $(h * D_N)(t_N)$ are nonnegative, when t_N lies to the right of t_0, it follows from the identity $f = (f - f_0) + f_0$ and (3.5.14) that all accumulation points of $\{(f * D_N)(t_N)\}_{N \in \mathbf{Z}^+}$ are at least $\frac{1}{2}\big(f(t_0+) + f(t_0-)\big)$. As in part (a) the intermediate value theorem implies that every point in the interval having the aforementioned endpoints is also an accumulation point for the sequence at hand. □

Exercises

3.5.1. Let $Si(t)$ be the sine function as defined in (3.5.3).
(a) Prove that $|\frac{\pi}{2} - Si(t)| \le \frac{2}{t}$.
(b) Show that $Si(Nt) \to \frac{\pi}{2}$ uniformly in $t \in [\delta, \infty)$ for any $\delta > 0$.

3.5.2. Show that the sine integral function has the following expansion

$$Si(x) = \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)\,(2k+1)!}\,.$$

3.5.3. Let $L^1_1(\mathbf{T}^1)$ be the space of all differentiable functions on $\mathbf{T}^1$ whose derivatives are integrable. Obtain the inclusions $L^1_1(\mathbf{T}^1) \subseteq BV(\mathbf{T}^1) \subseteq L^\infty(\mathbf{T}^1)$ as follows:
(a) If $f \in L^1_1(\mathbf{T}^1)$, then $\mathrm{Var}(f) \le \|f'\|_{L^1}$.
(b) If $f \in BV(\mathbf{T}^1)$, then $\|f\|_{L^\infty} \le \mathrm{Var}(f) + |f(0)|$.

3.5.4. (a) Let $a_k \ge 0$, $s_N = \sum_{k=-N}^N a_k$, and $\sigma_N = \frac{1}{N+1}(\sigma_0 + \cdots + \sigma_N)$. Suppose that $\sigma_N \to L < \infty$ as $N \to \infty$. Prove that $s_N \to L$ as $N \to \infty$.
(b) Apply the preceding result to show that if a complex-valued function h on $\mathbf{T}^1$ is continuous in a neighborhood of 0 and $\widehat{h}(m) \ge 0$ for all $m \in \mathbf{Z}$, then $h(0) \ge 0$ and $\sum_{m\in\mathbf{Z}} \widehat{h}(m) = h(0) < \infty$; i.e., the partial sums of the Fourier series of h converge at zero.

3.5.5. Let $h \in L^1(\mathbf{T}^1)$, $t_0 \in \mathbf{T}^1$, and $0 < \delta < 1/2$.
(a) Show that $(h * D_N)(t_0) \to L$ as $N \to \infty$ if and only if

$$\lim_{M\to\infty} \int_0^\delta \left(\frac{h(t_0 - t) + h(t_0 + t)}{2} - L \right) \frac{\sin(Mt)}{t} \, dt = 0 .$$

(b) Conclude that if an integrable function h on $\mathbf{T}^1$ satisfies

$$\int_0^\delta \frac{|h(t_0 - t) + h(t_0 + t) - 2L|}{t} \, dt < \infty ,$$

then $(h * D_N)(t_0) \to L$ as $N \to \infty$.
(c) In particular, if there are constants $C, \beta > 0$ with $\beta < 1$ such that for all t with $0 < t < \delta$ we have

$$|h(t_0 - t) + h(t_0 + t) - 2h(t_0)| \le C t^\beta ,$$

then $(h * D_N)(t_0) \to h(t_0)$ as $N \to \infty$.
(d) If h is an odd function, then $(h * D_N)(0) \to 0$ as $N \to \infty$.

3.5.6. Let $f \in L^1(\mathbf{T}^1)$ and suppose that (a, b) is an interval in $\mathbf{T}^1$. Then we have

$$\lim_{N\to\infty} \int_a^b (f * D_N)(t) \, dt = \int_a^b f(t) \, dt .$$

[*Hint:* Use Theorems 3.5.4 and 3.5.5 and the fact that the operator $f \mapsto f * D_N$ is self-adjoint.]

3.6 Lacunary Series and Sidon Sets

Lacunary series provide examples of 1-periodic functions on the line that possess certain remarkable properties.

3.6.1 Definition and Basic Properties of Lacunary Series

We begin by defining lacunary sequences.

Definition 3.6.1. A sequence of positive integers $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ is called *lacunary* if there exists a constant $A > 1$ such that $\lambda_{k+1} \geq A\lambda_k$ for all $k \in \mathbf{Z}^+$.

Examples of lacunary sequences are provided by exponential sequences, such as $\lambda_k = 2^k, 3^k, 4^k, \dots$. Observe that polynomial sequences such as $\lambda_k = 1 + k^2$ are not lacunary. Note that lacunary sequences tend to infinity as $k \to \infty$.

An important observation about lacunary sequences is the following: for any $m, k_0 \in \mathbf{Z}^+$ we have

$$1 \leq |m - \lambda_{k_0}| < (1 - A^{-1})\lambda_{k_0} \implies m \notin \Lambda\,. \tag{3.6.1}$$

Indeed, to prove this assertion, notice that the closest numbers to λ_{k_0} among of the terms of the sequence $\{\lambda_k\}_{k=1}^{\infty}$ are λ_{k_0+1} and λ_{k_0-1} (the latter only if $k_0 > 1$) and thus if $j > k_0$ we have

$$|\lambda_j - \lambda_{k_0}| \geq \lambda_{k_0+1} - \lambda_{k_0} \geq A\lambda_{k_0} - \lambda_{k_0} = (A-1)\lambda_{k_0} \geq (1 - A^{-1})\lambda_{k_0}\,,$$

while if $j < k_0$

$$|\lambda_j - \lambda_{k_0}| \geq \lambda_{k_0} - \lambda_{k_0-1} \geq \lambda_{k_0} - \frac{1}{A}\lambda_{k_0} = (1 - A^{-1})\lambda_{k_0}\,.$$

Thus (3.6.1) follows.

We begin with the following result.

Proposition 3.6.2. *Let $\{\lambda_k\}_{k=1}^{\infty}$ be a lacunary sequence and let f be an integrable function on the circle that is differentiable at a point and has Fourier coefficients*

$$\widehat{f}(m) = \begin{cases} a_m & \textit{when } m = \lambda_k\,, \\ 0 & \textit{when } m \neq \lambda_k\,. \end{cases} \tag{3.6.2}$$

Then we have

$$\lim_{k \to +\infty} \widehat{f}(\lambda_k)\lambda_k = 0\,.$$

Proof. Applying translation, we may assume that the point at which f is differentiable is the origin. Replacing f by the 1-periodic function

$$g(t) = f(t) - f(0)\cos(2\pi t) - f'(0)\frac{\sin(2\pi t)}{2\pi}$$

we may assume that $f(0) = f'(0) = 0$. (We have $\widehat{g}(m) = \widehat{f}(m)$ for $|m| \geq 2$ and thus the final conclusion for f is equivalent to that for g.)

Using (3.6.1) and (3.6.2), we obtain that for any $m \in \mathbf{Z}$ we have

$$1 \le |m-\lambda_k| < (1-A^{-1})\lambda_k \implies \widehat{f}(m) = 0\,. \tag{3.6.3}$$

Let $[t]$ denote the integer part of t. Given $\varepsilon > 0$, pick a positive integer k_0 such that if $[(1-A^{-1})\lambda_{k_0}] = 2N_0$, then $N_0^{-2} < \varepsilon$, and

$$\sup_{|x|<N_0^{-\frac{1}{4}}} \left|\frac{f(x)}{x}\right| < \varepsilon\,. \tag{3.6.4}$$

The expression in (3.6.4) can be made arbitrarily small, since f is differentiable at the origin. Now take an integer k with $k \ge k_0$ and set $2N = [\min(A-1, 1-A^{-1})\lambda_k]$, which is of course at least $2N_0$. Using (3.6.3), we obtain that for any trigonometric polynomial K_N of degree $2N$ with $\widehat{K_N}(0) = 1$ we have

$$\widehat{f}(\lambda_k) = \int_{|x|\le \frac{1}{2}} f(x)K_N(x)e^{-2\pi i\lambda_k x}\,dx\,. \tag{3.6.5}$$

We take $K_N = (F_N/\|F_N\|_{L^2})^2$, where F_N is the Fejér kernel. Using (3.1.18), we obtain first the identity

$$\|F_N\|_{L^2}^2 = \sum_{j=-N}^{N}\left(1-\frac{|j|}{N+1}\right)^2 = 1+\frac{1}{3}\frac{N(2N+1)}{N+1} > \frac{N}{3} \tag{3.6.6}$$

and also the estimate

$$F_N(x)^2 \le \left(\frac{1}{N+1}\frac{1}{4x^2}\right)^2, \tag{3.6.7}$$

which is valid for $|x| \le 1/2$. In view of (3.6.6) and (3.6.7), we have the estimate

$$K_N(x) \le \frac{3}{16}\frac{1}{N^3}\frac{1}{x^4}\,. \tag{3.6.8}$$

We now use (3.6.5) to obtain

$$\lambda_k\widehat{f}(\lambda_k) = \lambda_k\int_{|x|\le\frac{1}{2}} f(x)K_N(x)e^{-2\pi i\lambda_k x}\,dx = I_k^1 + I_k^2 + I_k^3,$$

where

$$\begin{aligned}
I_k^1 &= \lambda_k\int_{|x|\le N^{-1}} f(x)K_N(x)e^{-2\pi i\lambda_k x}\,dx,\\
I_k^2 &= \lambda_k\int_{N^{-1}<|x|\le N^{-\frac{1}{4}}} f(x)K_N(x)e^{-2\pi i\lambda_k x}\,dx,\\
I_k^3 &= \lambda_k\int_{N^{-\frac{1}{4}}<|x|\le\frac{1}{2}} f(x)K_N(x)e^{-2\pi i\lambda_k x}\,dx.
\end{aligned}$$

Since $\|K_N\|_{L^1} = 1$, it follows that

$$|I_k^1| \le \frac{\lambda_k}{N} \sup_{|x|<N^{-1}} \left|\frac{f(x)}{x}\right| \le \frac{(2N+1)\,\varepsilon}{\min(A-1, 1-A^{-1})N},$$

which can be made arbitrarily small if ε is small. Also, using (3.6.8), we obtain

$$|I_k^2| \le \frac{3\lambda_k}{16N^3} \sup_{|x|<N^{-\frac{1}{4}}} \left|\frac{f(x)}{x}\right| \int_{N^{-1}<|x|\le N^{-\frac{1}{4}}} \frac{dx}{|x|^3} \le \frac{3\lambda_k}{16N} \sup_{|x|<N^{-\frac{1}{4}}} \left|\frac{f(x)}{x}\right|,$$

which, as observed, is bounded by a constant multiple of ε. Finally, using again (3.6.8), we obtain

$$|I_k^3| \le \frac{3}{16N^3} N\lambda_k \int_{N^{-\frac{1}{4}}<|x|\le\frac{1}{2}} |f(x)|\,dx \le \frac{3}{16N^2}\|f\|_{L^1} < \frac{3\varepsilon}{16}\|f\|_{L^1}.$$

It follows that for all $k \ge k_0$ we have

$$|\lambda_k \widehat{f}(\lambda_k)| \le |I_k^1| + |I_k^2| + |I_k^3| \le C(f)\,\varepsilon$$

for some fixed constant $C(f)$. This proves the required conclusion. □

Corollary 3.6.3. *(**Weierstrass**) There exists a continuous function on the circle that is nowhere differentiable.*

Proof. Consider the 1-periodic function

$$f(t) = \sum_{k=0}^{\infty} 2^{-k} e^{2\pi i 3^k t}.$$

Since this series converges absolutely and uniformly, f is a continuous function. If f were differentiable at a point, then by Proposition 3.6.2 we would have that $3^k \widehat{f}(3^k)$ tends to zero as $k \to \infty$. Since $\widehat{f}(3^k) = 2^{-k}$ for $k \ge 0$, this is not the case. Therefore, f is nowhere differentiable. The real and imaginary parts of this function are displayed in Figure 3.4. □

3.6.2 Equivalence of L^p Norms of Lacunary Series

We now turn to one of the most important properties of lacunary series, equivalence of their norms. It is a remarkable result that lacunary Fourier series have comparable L^p norms for $1 \le p < \infty$. More precisely, we have the following theorem:

Theorem 3.6.4. *Let $1 \le \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ be a lacunary sequence with constant $A > 1$. Set $\Lambda = \{\lambda_k : k \in \mathbf{Z}^+\}$. Then for all $1 \le p < \infty$, there exists a constant $C_p(A)$ such that for all $f \in L^1(\mathbf{T}^1)$, with $\widehat{f}(k) = 0$, when $k \in \mathbf{Z} \setminus \Lambda$ we have*

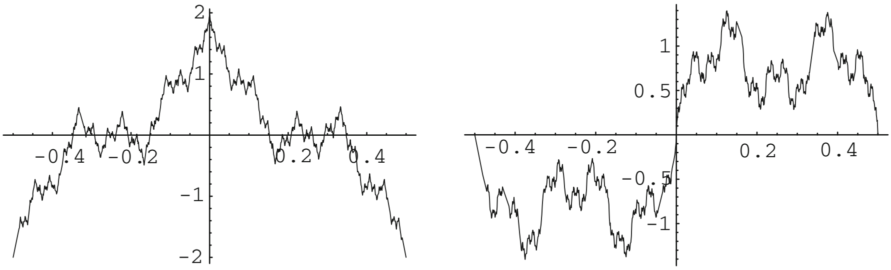

Fig. 3.4 The graph of the real and imaginary parts of the function $f(t) = \sum_{k=0}^{\infty} 2^{-k} e^{2\pi i 3^k t}$.

$$\|f\|_{L^p(\mathbf{T}^1)} \le C_p(A)\|f\|_{L^1(\mathbf{T}^1)}. \tag{3.6.9}$$

Moreover, the converse inequality to (3.6.9) is valid, and thus all L^p norms of lacunary Fourier series are equivalent for $1 \le p < \infty$.

Proof. We suppose initially that $f \in L^2(\mathbf{T}^1)$ and f is nonzero. We define

$$f_N(x) = \sum_{j=1}^{N} \widehat{f}(\lambda_j) e^{2\pi i \lambda_j x}. \tag{3.6.10}$$

Given $2 \le p < \infty$, we pick an integer m with $2m > p$ and we also pick a positive integer r such that $A^r > m$. Then we can write f_N as a sum of r functions φ_s, $s = 1, 2, \dots, r$, where each φ_s has Fourier coefficients that vanish except possibly on the lacunary set

$$\{\lambda_{kr+s} : k \in \mathbf{Z}^+ \cup \{0\}\} = \{\mu_1, \mu_2, \mu_3, \dots\}.$$

It is a simple fact that the sequence $\{\mu_k\}_k$ is lacunary with constant A^r. Then we have

$$\int_0^1 |\varphi_s(x)|^{2m}\, dx = \sum_{\substack{1 \le j_1, \dots, j_m, k_1, \dots, k_m \le N \\ \mu_{j_1} + \cdots + \mu_{j_m} = \mu_{k_1} + \cdots + \mu_{k_m}}} \widehat{\varphi_s}(\mu_{j_1}) \cdots \widehat{\varphi_s}(\mu_{j_m}) \overline{\widehat{\varphi_s}(\mu_{k_1})} \cdots \overline{\widehat{\varphi_s}(\mu_{k_m})}.$$

We claim that if $\mu_{j_1} + \cdots + \mu_{j_m} = \mu_{k_1} + \cdots + \mu_{k_m}$, then

$$\max(\mu_{j_1}, \dots, \mu_{j_m}) = \max(\mu_{k_1}, \dots, \mu_{k_m}).$$

Indeed, if $\max(\mu_{j_1}, \dots, \mu_{j_m}) > \max(\mu_{k_1}, \dots, \mu_{k_m})$, then

$$\max(\mu_{j_1}, \dots, \mu_{j_m}) \le \mu_{k_1} + \cdots + \mu_{k_m} \le m \max(\mu_{k_1}, \dots, \mu_{k_m}).$$

But since

$$A^r \max(\mu_{k_1}, \dots, \mu_{k_m}) \le \max(\mu_{j_1}, \dots, \mu_{j_m}),$$

it would follow that $A^r \le m$, which contradicts our choice of r. Likewise, we eliminate the case $\max(\mu_{j_1},\dots,\mu_{j_m}) < \max(\mu_{k_1},\dots,\mu_{k_m})$. We conclude that these numbers are equal. We can now continue the same reasoning using induction to conclude that if $\mu_{j_1}+\cdots+\mu_{j_m} = \mu_{k_1}+\cdots+\mu_{k_m}$, then

$$\{\mu_{k_1},\dots,\mu_{k_m}\} = \{\mu_{j_1},\dots,\mu_{j_m}\}.$$

Using this fact in the evaluation of the previous multiple sum, we obtain

$$\int_0^1 |\varphi_s(x)|^{2m}\,dx = \sum_{j_1=1}^{N}\cdots\sum_{j_m=1}^{N} |\widehat{\varphi_s}(\mu_{j_1})|^2\cdots|\widehat{\varphi_s}(\mu_{j_m})|^2 = \left(\|\varphi_s\|_{L^2}^2\right)^m,$$

which implies that $\|\varphi_s\|_{L^{2m}} = \|\varphi_s\|_{L^2}$ for all $s\in\{1,2,\dots,r\}$. Then we have

$$\|f_N\|_{L^p} \le \|f_N\|_{L^{2m}} \le \sqrt{r}\left(\sum_{s=1}^{r}\|\varphi_s\|_{L^{2m}}^2\right)^{\frac12} = \sqrt{r}\left(\sum_{s=1}^{r}\|\varphi_s\|_{L^2}^2\right)^{\frac12} = \sqrt{r}\,\|f_N\|_{L^2},$$

since the functions φ_s are orthogonal in L^2. Since r can be chosen to be $[\log_A m]+1$ and m can be taken to be $[\frac{p}{2}]+1$, we have now established the inequality

$$\|f_N\|_{L^p(\mathbf{T}^1)} \le c_p(A)\|f_N\|_{L^2(\mathbf{T}^1)}, \qquad p\ge 2, \tag{3.6.11}$$

with $c_p(A) = \sqrt{1+\left[\log_A\left([\frac{p}{2}]+1\right)\right]}$ for every f_N of the form (3.6.10).

To replace f_N by f in (3.6.11), we recall our assumption that $f\in L^2(\mathbf{T}^1)$. We observe that $f_N\to f$ in L^2 and thus f_{N_j} tends to f a.e. for some subsequence. Then Fatou's lemma and (3.6.11) imply for $1<p<\infty$

$$\begin{aligned}\int_0^1 |f(x)|^p\,dx &= \int_0^1 \liminf_{j\to\infty}|f_{N_j}(x)|^p\,dx\\ &\le \liminf_{j\to\infty}\int_0^1 |f_{N_j}(x)|^p\,dx\\ &\le c_p(A)^p \liminf_{j\to\infty}\|f_{N_j}\|_{L^2}^p\\ &= c_p(A)^p\|f\|_{L^2}^p.\end{aligned}$$

We conclude that

$$\|f\|_{L^p(\mathbf{T}^1)} \le c_p(A)\|f\|_{L^2(\mathbf{T}^1)}, \qquad p\ge 2. \tag{3.6.12}$$

By interpolation we obtain

$$\|f\|_{L^2} \le \|f\|_{L^4}^{\frac23}\|f\|_{L^1}^{\frac13} \le \left([\log_A 3]+1\right)^{\frac12\cdot\frac23}\|f\|_{L^2}^{\frac23}\|f\|_{L^1}^{\frac13}.$$

We are assuming that $0 < \|f\|_{L^2} < \infty$ and the preceding inequality implies that

$$\|f\|_{L^2(\mathbf{T}^1)} \le \big([\log_A 3]+1\big)\|f\|_{L^1(\mathbf{T}^1)}. \tag{3.6.13}$$

Finally an easy consequence of Hölder's inequality is that

$$\|f\|_{L^p(\mathbf{T}^1)} \le \|f\|_{L^2(\mathbf{T}^1)}, \qquad 1 \le p < 2. \tag{3.6.14}$$

Combining (3.6.12) and (3.6.14) with (3.6.13) yields (3.6.9) with

$$C_p(A) = c_p(A)\big([\log_A 3]+1\big)$$

for all $1 \le p < \infty$ under the hypothesis that $\widehat{f}(k) = 0$ for all $k \in \mathbf{Z} \setminus \Lambda$ and the additional assumption that $f \in L^2$.

We now extend the result to $f \in L^1(\mathbf{T}^1)$. Given $f \in L^1(\mathbf{T}^1)$ with $\widehat{f}(k) = 0$ when $k \in \mathbf{Z} \setminus \Lambda$, consider the functions $f * F_M$, where F_M is the Fejér kernel and $M \in \mathbf{Z}^+$. Then $f * F_M$ lie in L^2, $f * F_M$ converge to f in L^1 and in L^p, and $\widehat{f * F_M}(k) = 0$ when $k \in \mathbf{Z} \setminus \Lambda$. The inequality

$$\|f * F_M\|_{L^p(\mathbf{T}^1)} \le C_p(A)\|f * F_M\|_{L^1(\mathbf{T}^1)} \tag{3.6.15}$$

holds since $f * F_M$ lie in L^2, so letting $M \to \infty$ yields (3.6.9). □

Theorem 3.6.4 describes the equivalence of the L^p norms of lacunary Fourier series for $p < \infty$. The question that remains is whether there is a similar characterization for the L^∞ norms of lacunary Fourier series. Such a characterization is investigated below. Before we state and prove this theorem, we need a classical tool, referred to as a Riesz product.

Definition 3.6.5. A *Riesz product* is a function of the form

$$P_N(x) = \prod_{j=1}^{N} \big(1 + a_j \cos(2\pi\lambda_j x + 2\pi\gamma_j)\big), \tag{3.6.16}$$

where N is a positive integer, $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ is a lacunary sequence of positive integers, a_j are real numbers in $[-1,1]$, and $\gamma_j \in [0,1]$.

We make a few observations about Riesz products. A simple calculation gives that if $P_{N,j}(x) = 1 + a_j \cos(2\pi\lambda_j x + 2\pi\gamma_j)$, then

$$\widehat{P_{N,j}}(m) = \begin{cases} 1 & \text{when } m = 0, \\ \frac{1}{2}a_j e^{2\pi i\gamma_j} & \text{when } m = \lambda_j, \\ \frac{1}{2}a_j e^{-2\pi i\gamma_j} & \text{when } m = -\lambda_j, \\ 0 & \text{when } m \notin \{0\}\bigcup_{j=1}^{\infty}\{\lambda_j, -\lambda_j\}. \end{cases} \tag{3.6.17}$$

Assume that the constant A associated with the lacunary sequence $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ satisfies $A \ge 3$. Then each integer m has *at most one* representation as a sum

$$m = \varepsilon_1\lambda_1 + \cdots + \varepsilon_N\lambda_N\,,$$

where $\varepsilon_j \in \{-1,1,0\}$; see Exercise 3.6.1. We now calculate the Fourier coefficients of the Riesz product defined in (3.6.16). For a fixed integer b, let us denote by δ_b the sequence of integers that is equal to 1 at b and zero otherwise. Then, using (3.6.17), we obtain that

$$\widehat{P_{N,j}} = \delta_0 + \tfrac{1}{2}a_j e^{2\pi i\gamma_j}\delta_{\lambda_j} + \tfrac{1}{2}a_j e^{-2\pi i\gamma_j}\delta_{-\lambda_j}\,,$$

and thus $\widehat{P_N}$ is the N-fold convolution of these functions. Using that $\delta_a * \delta_b = \delta_{a+b}$, we obtain

$$\widehat{P_N}(m) = \begin{cases} 1 & \text{when } m = 0,\\ \prod_{j=1}^N \frac{1}{2}a_j e^{2\pi i\varepsilon_j\gamma_j} & \text{when } m = \sum_{j=1}^N \varepsilon_j\lambda_j \text{ and } \sum_{j=1}^N |\varepsilon_j| > 0,\\ 0 & \text{otherwise.}\end{cases}$$

It follows that $\widehat{P_N}(\lambda_j) = 0$ since for $j \ge N+1$, λ_j cannot be expressed as a linear combination of $\lambda_1,\dots,\lambda_N$ with coefficients in $\{\pm 1, 0\}$. Also $\widehat{P_N}(\lambda_k) = \frac{1}{2}a_j e^{2\pi i\gamma_k}$ for $1 \le k \le N$, since each λ_k is written uniquely as $0\cdot\lambda_1 + \cdots + 0\cdot\lambda_{k-1} + 1\cdot\lambda_k$. Hence when $A \ge 3$ we have that $\widehat{P_N}(\lambda_k) = \frac{1}{2}a_j e^{2\pi i\gamma_k}$ when $1 \le k \le N$ and $\widehat{P_N}(\lambda_k) = 0$ for $k \ge N+1$.

Next, we discuss an important property of Riesz products. Suppose that for some $m \in \mathbf{Z}$ we have $\widehat{P_N}(m) \ne 0$. We write $m = \sum_{j=1}^N \varepsilon_j\lambda_j$ uniquely with $\varepsilon_j \in \{-1,0,1\}$. Let k be the largest integer less than or equal to N such that $\varepsilon_k \ne 0$. Then we have

$$\big||m| - \lambda_k\big| \le \lambda_1 + \cdots + \lambda_{k-1} \le \frac{\lambda_k}{A^{k-1}} + \cdots + \frac{\lambda_k}{A} \le \frac{\lambda_k}{A}\frac{1}{1-\frac{1}{A}} = \frac{\lambda_k}{A-1}\,. \tag{3.6.18}$$

Another important property of the Riesz product is that since $P_N \ge 0$ we have

$$\|P_N\|_{L^1} = \int_{\mathbf{T}^1} P_N(t)\,dt = \widehat{P_N}(0) = 1\,.$$

We recall the space $A(\mathbf{T}^1)$ of all functions with absolutely summable Fourier coefficients normed with the ℓ^1 norm of the coefficients.

Theorem 3.6.6. *Let $1 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ be a lacunary sequence of integers with constant $A > 1$. Set $\Lambda = \{\lambda_k :\ k \in \mathbf{Z}^+\}$. Then there exists a constant $C(A)$ such that for all $f \in L^\infty(\mathbf{T}^1)$ with $\widehat{f}(k) = 0$ when $k \in \mathbf{Z}\setminus\Lambda$ we have*

$$\big\|f\big\|_{A(\mathbf{T}^1)} = \sum_{k\in\Lambda} |\widehat{f}(k)| \le C(A)\big\|f\big\|_{L^\infty(\mathbf{T}^1)}\,. \tag{3.6.19}$$

Proof. Let us assume first that $A \ge 3$. Also fix $f \in L^\infty(\mathbf{T}^1)$. We consider the Riesz product

$$P_N(x) = \prod_{j=1}^N \big(1 + \cos(2\pi\lambda_j x + 2\pi\gamma_j)\big)\,,$$

where γ_j is chosen to satisfy the identity $|\widehat{f}(\lambda_j)| = e^{2\pi i\gamma_j}\overline{\widehat{f}(\lambda_j)}$. In view of Parseval's relation and of the fact that $\|P_N\|_{L^1} = 1$ we obtain

$$\left|\sum_{m\in\mathbf{Z}} \widehat{P_N}(m)\overline{\widehat{f}(m)}\right| = \left|\int_0^1 P_N(x)\overline{f(x)}\,dx\right| \le \|f\|_{L^\infty}, \tag{3.6.20}$$

and the sum in (3.6.20) is finite, since the Fourier coefficients of $\widehat{P_N}$ form a finitely supported sequence. But $\widehat{f}(m) = 0$ for $m \notin \Lambda$, while $\widehat{P_N}(\lambda_j) = \frac{1}{2}e^{2\pi i\gamma_j}$ for $1 \le j \le N$ since $A \ge 3$, and moreover, $\widehat{P_N}(\lambda_j) = 0$ for $j \ge N+1$, as observed earlier. Thus (3.6.20) reduces to

$$\frac{1}{2}\sum_{j=1}^{N} |\widehat{f}(\lambda_j)| = \left|\sum_{j=1}^{N} \frac{1}{2}e^{2\pi i\gamma_j}\overline{\widehat{f}(\lambda_j)}\right| \le \|f\|_{L^\infty}.$$

Letting $N \to \infty$, we deduce that $\sum_{j=1}^{\infty} |\widehat{f}(\lambda_j)| \le 2\|f\|_{L^\infty}$, which proves (3.6.19) when $A \ge 3$.

We now consider the case $A < 3$. We fix $1 < A < 3$ and we pick a positive integer r such that

$$A^r > 3 \qquad \text{and} \qquad \frac{1}{A^r - 1} < 1 - \frac{1}{A}. \tag{3.6.21}$$

This is possible, since $(A^r - 1)^{-1} \to 0$ as $r \to \infty$.

For each $s \in \{1, \dots, r\}$, define the sequences $\lambda_k^s = \lambda_{s+(k-1)r}$ indexed by $k = 1, 2, 3, \dots$ and observe that $\lambda_{(k+1)}^s > A^r\lambda_k^s$ for all $k = 1, 2, \dots$; i.e., each such sequence is lacunary with constant A^r. We consider the Riesz product

$$P_N^s(x) = \prod_{k=1}^{N}\left(1 + \cos(2\pi\lambda_k^s x + 2\pi\gamma_k^s)\right),$$

where γ_k^s is defined via the identity $|\widehat{f}(\lambda_k^s)| = e^{2\pi i\gamma_k^s}\overline{\widehat{f}(\lambda_k^s)}$.

Using (3.6.18) we obtain that, if $m \in \mathbf{Z}$ is such that $\widehat{P_N^s}(m) \ne 0$, then there exists a $k \in \{1, 2 \dots, N\}$ such that

$$\big||m| - \lambda_k^s\big| < \frac{\lambda_k^s}{A^r - 1}.$$

This combined with (3.6.21) yields

$$\big||m| - \lambda_k^s\big| < \Big(1 - \frac{1}{A}\Big)\lambda_k^s.$$

Using (3.6.1) we obtain that either $m = \pm\lambda_k^s$ or $|m| \notin \Lambda$. Thus we have

$$\{m \in \mathbf{Z}^+ : \widehat{P_N^s}(m) \ne 0\} \subseteq \{\lambda_1^s, \lambda_2^s, \dots, \lambda_N^s\} \cup \Lambda^c.$$

This observation, the fact that $\widehat{f}$ is supported in Λ, and Parseval's relation yield

$$\left|\sum_{k=1}^{N}\widehat{P_N^s}(\lambda_k^s)\overline{\widehat{f}(\lambda_k^s)}\right| = \left|\sum_{m\in\mathbf{Z}}\widehat{P_N^s}(m)\overline{\widehat{f}(m)}\right| = \left|\int_0^1 P_N^s(x)\overline{f(x)}\,dx\right| \le \big\|f\big\|_{L^\infty}. \quad (3.6.22)$$

Since $\widehat{P_N}(\lambda_k^s) = \frac{1}{2}e^{2\pi i\gamma_k^s}$ for $1 \le k \le N$, (3.6.22) reduces to

$$\frac{1}{2}\sum_{k=1}^{N}\big|\widehat{f}(\lambda_k^s)\big| \le \big\|f\big\|_{L^\infty}.$$

Letting $N \to \infty$ gives

$$\sum_{j=1}^{\infty}\big|\widehat{f}(\lambda_j^s)\big| \le 2\big\|f\big\|_{L^\infty}.$$

Summing over s in the set $\{1,2,\dots,r\}$, we obtain the required conclusion with $C(A) = 2r$ and note that r can be taken to be $[\max(\log_A \frac{2A-1}{A-1}, \log_3 A)] + 2$. □

Corollary 3.6.7. *Let $\Lambda = \{\lambda_k : k \in \mathbf{Z}^+\}$ be a lacunary set and let f be a bounded function on the circle that satisfies $\widehat{f}(k) = 0$ when $k \in \mathbf{Z}\setminus\Lambda$. Then f is almost everywhere equal to the absolutely (and uniformly) convergent series*

$$f(x) = \sum_{k\in\Lambda}\widehat{f}(k)e^{2\pi ikx} \qquad \text{a.e.} \quad (3.6.23)$$

and thus it is almost everywhere equal to a continuous function.

Proof. It follows from Theorem 3.6.6 that if $\widehat{f}(k) = 0$ when $k \in \mathbf{Z}\setminus\Lambda$, then we have that $f \in A(\mathbf{T}^1)$. Applying the inversion result in Proposition 3.2.5 we obtain that f is almost everywhere equal to a continuous function and that (3.6.23) holds for almost all $x \in \mathbf{T}^1$. □

3.6.3 Sidon sets

Given a subset E of the integers, we denote by $\mathscr{C}_E$ the space of all continuous functions on $\mathbf{T}^1$ such that

$$m \in \mathbf{Z}\setminus E \implies \widehat{f}(m) = 0. \quad (3.6.24)$$

It is straightforward that $\mathscr{C}_E$ is a closed subspace of all bounded functions on the circle $\mathbf{T}^1$ with the standard L^∞ norm.

Definition 3.6.8. A set of integers E is called a *Sidon set* if every function in $\mathscr{C}_E$ has an absolutely convergent Fourier series.

There are several characterizations of Sidon sets. We state them below.

Proposition 3.6.9. *The following assertions are equivalent for a subset E of $\mathbf{Z}$.*

(1) There is a constant K such that for all trigonometric polynomials P with $\widehat{P}$ supported in E we have

$$\sum_{m\in\mathbf{Z}} |\widehat{P}(m)| \le K\big\|P\big\|_{L^\infty}$$

(2) There exists a constant K such that

$$\big\|\widehat{f}\big\|_{\ell^1(\mathbf{Z})} \le K\big\|f\big\|_{L^\infty(\mathbf{T}^1)}$$

for every bounded function f on $\mathbf{T}^1$ with $\widehat{f}$ supported in E.

(3) Every function f in $\mathscr{C}_E$ has an absolutely convergent Fourier series; i.e., E is a Sidon set.

(4) For every bounded function b on E there is a finite Borel measure μ on $\mathbf{T}^1$ such that $\widehat{\mu}(m) = b(m)$ for all $m \in E$.

(5) For every function b on $\mathbf{Z}$ with the property $b(m) \to 0$ as $m \to \infty$, there is a function $g \in L^1(\mathbf{T}^1)$ such that $\widehat{g}(m) = b(m)$ for all $m \in E$.

Proof. Suppose that (1) holds. Given f in $L^\infty(\mathbf{T}^1)$ with $\widehat{f}$ is supported in E, write

$$(f * F_N)(x) = \sum_{m=-N}^{N}\left(1 - \frac{|m|}{N+1}\right)\widehat{f}(m)e^{2\pi imx},$$

where F_N is the Fejér kernel. These are trigonometric polynomials whose Fourier coefficients vanish on $\mathbf{Z}\setminus E$. Applying (1) we obtain

$$\sum_{k\in\mathbf{Z}}\left(1 - \frac{|m|}{N+1}\right)|\widehat{f}(m)| \le K\big\|f * F_N\big\|_{L^\infty}.$$

Letting $N \to \infty$ we obtain (2).

It is trivial that (2) implies (3).

If (3) holds, then the map $f \mapsto \widehat{f}$ is a linear bijection from $\mathscr{C}_E$ to $\ell^1(E)$. Moreover its inverse mapping $\widehat{f} \mapsto f$ is continuous, since

$$\big\|f\big\|_{L^\infty(\mathbf{T}^1)} \le \sup_{t\in[0,1]}\Big|\sum_{k\in\mathbf{Z}}\widehat{f}(k)e^{2\pi ikt}\Big| \le \sum_{k\in\mathbf{Z}}|\widehat{f}(k)| = \big\|\widehat{f}\big\|_{\ell^1(\mathbf{Z})}.$$

By the open mapping theorem, it follows that $f \mapsto \widehat{f}$ is a continuous mapping, which proves the existence of a constant K such that (1) holds.

We have now proved the equivalence of (1), (2), and (3).

We show that (2) implies (4). If E is a Sidon set and if b is a bounded function on E, say $\|b\|_{\ell^\infty} \le 1$, then the mapping

$$f \mapsto \sum_{m\in E}\widehat{f}(m)\widehat{b}(m)$$

is a bounded linear functional on $\mathscr{C}_E$ with norm at most K. By the Hanh-Banach theorem this functional admits an extension to $\mathscr{C}(\mathbf{T}^1)$ with the same norm. Hence there is a measure μ, whose total variation $\|\mu\|$ does not exceed K, such that

$$\sum_{m\in E} \widehat{f}(m)\widehat{b}(m) = \int_{\mathbf{T}^1} f(t)\,d\mu(t)\,.$$

Taking $f(t) = e^{2\pi imt}$ in (2) we obtain $\widehat{\mu}(m) = b(m)$ for all $m \in E$.

If (4) holds and $b(m) \to 0$ as $|m| \to \infty$, using Lemma 3.3.2 there is a convex sequence $c(m)$ such that $c(m) > 0$, $c(m) \to 0$ as $|m| \to \infty$, $c(-m) = c(m)$, and $|b(m)| \le c(m)$ for all $m \in \mathbf{Z}$. By (4), there is a finite Borel measure μ with $\widehat{\mu}(m) = b(m)/c(m)$ for all $m \in E$.

By Theorem 3.3.4, there is a function g in $L^1(\mathbf{T}^1)$ such that $\widehat{g}(m) = c(m)$ for all $m \in \mathbf{Z}$. Then $b(m) = \widehat{g}(m)\widehat{\mu}(m)$ for all $m \in E$. Since $f = g * \mu$ is in L^1, we have $b(m) = \widehat{f}(m)$ for all $m \in E$, and thus (4) implies (5).

Finally, if (5) holds, we show (3). Given $f \in \mathscr{C}_E$, we show that for an arbitrary sequence d_m tending to zero, we have $\sum_{m\in\mathbf{Z}} |\widehat{f}(m)d_m| < \infty$; this implies that $\sum_{m\in\mathbf{Z}} |\widehat{f}(m)| < \infty$. Given a sequence $d_m \to 0$, pick a function g in L^1 such that $\widehat{g}(m)\widehat{f}(m) = |\widehat{f}(m)|\,|d_m|$ for all $m \in E$ by assumption (5). Then the series

$$\sum_{m\in\mathbf{Z}} \widehat{g}(m)\widehat{f}(m) = \sum_{m\in\mathbf{Z}} \widehat{f*g}(m) \tag{3.6.25}$$

has nonnegative terms and the function $f*g$ is continuous, thus $F_N*(f*g)(0) \to (f*g)(0)$ as $N \to \infty$. It follows that $D_N*(f*g)(0) \to (f*g)(0)$, thus the series in (3.6.25) converges (see Exercise 3.5.4) and hence $\sum_{m\in\mathbf{Z}} |\widehat{f}(m)d_m| < \infty$. □

Example 3.6.10. Every lacunary set is a Sidon set. Indeed, suppose that E is a lacunary set with constant A. If f is a continuous function which satisfies (3.6.24), then Theorem 3.6.6 gives that

$$\sum_{m\in\Lambda} |\widehat{f}(m)| \le C(A)\|f\|_{L^\infty} < \infty;$$

hence f has an absolutely convergent Fourier series.

Example 3.6.11. There exist subsets of $\mathbf{Z}$ that are not Sidon. For example, $\mathbf{Z}\setminus\{0\}$ is not a Sidon set. See Exercise 3.6.7.

Exercises

3.6.1. Suppose that $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N$ is a lacunary sequence of integers with constant $A \ge 3$. Prove that for every integer m there exists at most one N-tuple $(\varepsilon_1,\dots,\varepsilon_N)$ with each $\varepsilon_j \in \{-1,1,0\}$ such that

$$m = \varepsilon_1\lambda_1 + \cdots + \varepsilon_N\lambda_N\,.$$

[*Hint:* Suppose there exist two such N-tuples. Pick the largest k such that the coefficients of λ_k are different.]

3.6.2. Is the sequence $\lambda_k = \left[e^{(\log k)^2}\right]$, $k = 2,3,4,\dots$ lacunary?

3.6.3. Let $a_k \geq 0$ for all $k \in \mathbf{Z}^+$ and $1 \leq p < \infty$. Show that there exist constants C_p, c_p such that for all $N \in \mathbf{Z}^+$ we have

$$c_p\Big(\sum_{k=1}^{N}|a_k|^2\Big)^{\frac{1}{2}} \leq \Big(\int_0^1 \Big|\sum_{k=1}^{N} a_k e^{2\pi i 2^k x}\Big|^p dx\Big)^{\frac{1}{p}} \leq C_p\Big(\sum_{k=1}^{N}|a_k|^2\Big)^{\frac{1}{2}},$$

while

$$\sup_{x\in[0,1]}\Big|\sum_{k=1}^{N} a_k e^{2\pi i 2^k x}\Big| = \sum_{k=1}^{N}|a_k|.$$

3.6.4. Suppose that $0 < \lambda_1 < \lambda_2 < \cdots$ is a lacunary sequence and let f be a bounded function on the circle that satisfies $\widehat{f}(m) = 0$ whenever $m \in \mathbf{Z}\setminus\{\lambda_1, \lambda_2, \dots\}$. Suppose also that

$$\sup_{t\neq 0}\frac{|f(t)-f(0)|}{|t|^\alpha} = B < \infty$$

for some $0 < \alpha < 1$.
(a) Prove that there is a constant C such that $|\widehat{f}(\lambda_k)| \leq CB\lambda_k^{-\alpha}$ for all $k \geq 1$.
(b) Prove that $f \in \dot{\Lambda}_\alpha(\mathbf{T}^1)$.
[*Hint:* Let $2N = [(1-A^{-1})\lambda_k]$ and let K_N be as in the proof of Proposition 3.6.2. Write

$$\begin{aligned}\widehat{f}(\lambda_k) &= \int_{|x|\leq N^{-1}}(f(x)-f(0))e^{-2\pi i\lambda_k x}K_N(x)\,dx \\ &\quad + \int_{N^{-1}\leq|x|\leq\frac{1}{2}}(f(x)-f(0))e^{-2\pi i\lambda_k x}K_N(x)\,dx\,.\end{aligned}$$

Use that $\|K_N\|_{L^1} = 1$ and also the estimate (3.6.7). Part (b): Use the estimate in part (a).]

3.6.5. Let f be an integrable function on the circle whose Fourier coefficients vanish outside a lacunary set $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \dots\}$. Suppose that f vanishes identically in a small neighborhood of the origin. Show that f is in $\mathscr{C}^\infty(\mathbf{T}^1)$.
[*Hint:* Let $2N = [(1-A^{-1})\lambda_k]$ and let K_N be as in the proof of Proposition 3.6.2. Write

$$\widehat{f}(\lambda_k) = \int_{|x|\leq\frac{1}{2}} f(x)e^{-2\pi i\lambda_k x}K_N(x)\,dx$$

and use estimate (3.6.7) to obtain that f is in $\mathscr{C}^2$. Continue by induction.]

3.6.6. Let $1 < a, b < \infty$. Consider the 1-periodic function

$$f(x) = \sum_{k=0}^{\infty} a^{-k}e^{2\pi i b^k x}.$$

Prove that the following statements are equivalent:
(a) f is differentiable at a point.
(b) $b < a$.
(c) f is differentiable everywhere.

3.6.7. Use the example in Proposition 3.4.6 (a) to show that $\mathbf{Z}\setminus\{q_1,\dots,q_L\}$ is not a Sidon set for any finite subset $\{q_1,\dots,q_L\}$ of the integers.

3.6.8. Let $0<\delta<1$. Let E be a subset of the integers such that for any sequence of complex numbers $\{d_m\}_{m\in E}$ with $|d_m|=1$ there is a finite Borel measure μ on $\mathbf{T}^1$ such that

$$|\widehat{\mu}(m)-d_m|<1-\delta$$

for all $m\in E$. Show that E is a Sidon set.
$\big[$*Hint:* Given f be in $\mathscr{C}_E$ define d_m via the identity $d_m\widehat{f}(m)=|\widehat{f}(m)|$ if $\widehat{f}(m)\neq 0$, otherwise set $d_m=1$. For the measure μ given by the hypothesis, notice that $\operatorname{Re}(\widehat{\mu}(m)\widehat{f}(m))\ge\delta|\widehat{f}(m)|$ for all $m\in\mathbf{Z}$.$\big]$

HISTORICAL NOTES

Trigonometric series in one dimension were first considered in the study of the vibrating string problem and are implicitly contained in the work of d'Alembert, D. Bernoulli, Clairaut, and Euler. The analogous problem for vibrating higher-dimensional bodies naturally suggested the use of multiple trigonometric series. However, it was the work of Fourier on steady-state heat conduction that inspired the subsequent systematic development of such series. Fourier announced his results in 1811, although his classical book *Théorie de la chaleur* was published in 1822. This book contains several examples of heuristic use of trigonometric expansions and motivated other mathematicians to carefully study such expansions. The systematic development of the theory of Fourier series began by Dirichlet [96], who studied the pointwise convergence of the Fourier series of piecewise monotonic functions via the use of the kernel D_N, today called the Dirichlet kernel.

The fact that the Fourier series of a continuous function can diverge was first observed by DuBois Reymond in 1873. The Riemann–Lebesgue lemma was first proved by Riemann in his memoir on trigonometric series (appeared between 1850 and 1860). It carries Lebesgue's name today because Lebesgue later extended it to his notion of integral. The rebuilding of the theory of Fourier series based on Lebesgue's integral was mainly achieved by de la Vallée-Poussin and Fatou.

Theorem 3.3.16 was obtained by Bernstein [26] in dimension $n=1$. Higher-dimensional analogues of the Hardy–Littlewood series of Exercise 3.3.8 were studied by Wainger [370]. These series can be used to produce examples indicating that the restriction $s>\alpha+n/2$ in Bernstein's theorem is sharp even in higher dimensions. Part (b) of Theorem 3.4.4 is due to Lebesgue when $n=1$ and Marcinkiewicz and Zygmund [243] when $n=2$. Marcinkiewicz and Zygmund's proof also extends to higher dimensions. The proof given here is based on Lemma 3.4.5 proved by Stein [342] in a different context. The proof of Lemma 3.4.5 presented here was suggested by T. Tao.

Abel proved that if an infinite series $\sum_{k=0}^{\infty}a_k$ converges and has sum L, then the power series $f(x)=\sum_{k=0}^{\infty}a_kx^k$ converges for $|x|<1$ and tends to L as $x\to 1-$. The converse of this theorem under the additional assumption that $ka_k\to 0$ as $k\to\infty$ was proved by Tauber [358]. Hardy [143] extended Tauber's result (Theorem 3.5.1) for Cesàro summability under the weaker assumption that the sequence ka_k is bounded. Jordan [180] studied of functions of bounded variation and proved Theorem 3.5.4. The existence of a continuous function which is nowhere differentiable (Corollary 3.6.3) was first published in 1872 by K. Weierstrass, although earlier findings of such functions were published later. The exposition on Sidon sets is taken from the classical article of Rudin [305], which also contains Exercise 3.6.8.

The Gibbs phenomenon (a version of Theorem 3.5.7) was discovered by Wilbraham [376] and rediscovered by Gibbs [128]; this phenomenon describes the particular way in which the Fourier sums of a piecewise continuously differentiable periodic function have large oscillations and overshoot at the jump discontinuity of the function. Bôcher [29] gave a detailed mathematical analysis of that overshoot, which he called the "Gibbs phenomenon".

The main references for trigonometric series are the books of Bary [20] and Zygmund [388], [389]. Other references for one-dimensional Fourier series include the books of Edwards [106], Dym and McKean [105], Katznelson [190], Körner [202], Pinsky [283], and the first eight chapters in Torchinsky [363]. The reader may also consult the book of Krantz [203] for a historical introduction to the subject of Fourier series. A review of the heritage and continuing significance of Fourier Analysis is written by Kahane [182].

A classical treatment of multiple Fourier series can be found in the last chapter of Bochner's book [32] and in parts of his other book [31]. Other references include the last chapter in Zygmund [389], the books of Yanushauskas [381] (in Russian) and Zhizhiashvili [384], the last chapter in Stein and Weiss [348], and the article of Alimov, Ashurov, and Pulatov in [3]. A brief survey article on the subject was written by Ash [11]. More extensive expositions were written by Shapiro [320], Igari [171], and Zhizhiashvili [383]. A short note on the history of Fourier series was written by Zygmund [390]. The book of Shapiro [321] contains a very detailed study of Fourier series in several variables as well as applications of this theory.