Chapter 1
L^p Spaces and Interpolation

Many quantitative properties of functions are expressed in terms of their integrability to a power. For this reason it is desirable to acquire a good understanding of spaces of functions whose modulus to a power p is integrable. These are called Lebesgue spaces and are denoted by L^p. Although an in-depth study of Lebesgue spaces falls outside the scope of this book, it seems appropriate to devote a chapter to reviewing some of their fundamental properties.

The emphasis of this review is basic interpolation between Lebesgue spaces. Many problems in Fourier analysis concern boundedness of operators on Lebesgue spaces, and interpolation provides a framework that often simplifies this study. For instance, in order to show that a linear operator maps L^p to itself for all $1 < p < \infty$, it is sufficient to show that it maps the (smaller) Lorentz space $L^{p,1}$ into the (larger) Lorentz space $L^{p,\infty}$ for the same range of p's. Moreover, some further reductions can be made in terms of the Lorentz space $L^{p,1}$. This and other considerations indicate that interpolation is a powerful tool in the study of boundedness of operators.

Although we are mainly concerned with L^p subspaces of Euclidean spaces, we discuss in this chapter L^p spaces of arbitrary measure spaces, since they represent a useful general setting. Many results in the text require working with general measures instead of Lebesgue measure.

1.1 L^p and Weak L^p

A *measure space* is a set X equipped with a σ-algebra of subsets of it and a function μ from the σ-algebra to $[0,\infty]$ that satisfies $\mu(\emptyset) = 0$ and

$$\mu\Big(\bigcup_{j=1}^{\infty} B_j\Big) = \sum_{j=1}^{\infty} \mu(B_j)$$

for any sequence B_j of pairwise disjoint elements of the σ-algebra. The function μ is called a (positive) measure on X and elements of the σ-algebra of X are called

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measurable sets. Measure spaces will be assumed to be complete, i.e., subsets of the σ-algebra of measure zero also belong to the σ-algebra. A measure space X is called *σ-finite* if there is a sequence of measurable subsets X_n of it such that

$$X = \bigcup_{n=1}^{\infty} X_n$$

and $\mu(X_n) < \infty$. A real-valued function f on a measure space is called *measurable* if the set $\{x \in X : f(x) > \lambda\}$ is measurable for all real numbers λ. A complex-valued function is measurable if and only if its real and imaginary parts are measurable. A *simple function* is a finite linear combination of characteristic functions of measurable subsets of X; these subsets may have infinite measure. A *finitely simple* function has the form

$$\sum_{j=1}^{N} c_j \chi_{B_j}$$

where $N < \infty$, $c_j \in \mathbf{C}$, and B_j are pairwise disjoint measurable sets with $\mu(B_j) < \infty$. If $N = \infty$, this function will be called *countably simple*. Finitely simple functions are exactly the integrable simple functions. Every nonnegative measurable function is the pointwise limit of an increasing sequence of simple functions; if the space is σ-finite, these simple functions can be chosen to be finitely simple.

For $0 < p < \infty$, $L^p(X,\mu)$ denotes the set of all complex-valued μ-measurable functions on X whose modulus to the pth power is integrable. $L^\infty(X,\mu)$ is the set of all complex-valued μ-measurable functions f on X such that for some $B > 0$, the set $\{x : |f(x)| > B\}$ has μ-measure zero. Two functions in $L^p(X,\mu)$ are considered equal if they are equal μ-almost everywhere. When $0 < p < \infty$ finitely simple functions are dense in $L^p(X,\mu)$. Within context and in the absence of ambiguity, $L^p(X,\mu)$ is simply written as L^p.

The notation $L^p(\mathbf{R}^n)$ is reserved for the space $L^p(\mathbf{R}^n, |\cdot|)$, where $|\cdot|$ denotes n-dimensional Lebesgue measure. Lebesgue measure on $\mathbf{R}^n$ is also denoted by dx. Other measures will be considered on the *Borel σ-algebra* of $\mathbf{R}^n$, i.e., is the smallest σ-algebra that contains the closed subsets of $\mathbf{R}^n$. Measures on the σ-algebra of Borel measurable subsets are called *Borel measures*; such measures will be assumed to be finite on compact subsets of $\mathbf{R}^n$. A Borel measure μ with $\mu(\mathbf{R}^n) < \infty$ is called a *finite Borel measure*. A Borel measure on $\mathbf{R}^n$ is called *regular* for all Borel measurable sets E we have

$$\mu(E) = \inf\{\mu(O) : E \subseteqq O,\ O \text{ open}\} = \sup\{\mu(K) : K \subseteqq E,\ K \text{ compact}\}.$$

The space $L^p(\mathbf{Z})$ equipped with counting measure is denoted by $\ell^p(\mathbf{Z})$ or simply ℓ^p.

For $0 < p < \infty$, we define the L^p norm of a function f (or quasi-norm if $p < 1$) by

$$\|f\|_{L^p(X,\mu)} = \left(\int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \tag{1.1.1}$$

and for $p = \infty$ by

$$\|f\|_{L^\infty(X,\mu)} = \text{ess.sup}\,|f| = \inf\big\{B > 0 : \ \mu(\{x : |f(x)| > B\}) = 0\big\}. \tag{1.1.2}$$

It is well known that Minkowski's (or the triangle) inequality

$$\|f+g\|_{L^p(X,\mu)} \le \|f\|_{L^p(X,\mu)} + \|g\|_{L^p(X,\mu)} \tag{1.1.3}$$

holds for all f, g in $L^p = L^p(X,\mu)$, whenever $1 \le p \le \infty$. Since in addition $\|f\|_{L^p(X,\mu)} = 0$ implies that $f = 0$ (μ-a.e.), the L^p spaces are normed linear spaces for $1 \le p \le \infty$. For $0 < p < 1$, inequality (1.1.3) is reversed when $f, g \ge 0$. However, the following substitute of (1.1.3) holds:

$$\|f+g\|_{L^p(X,\mu)} \le 2^{\frac{1-p}{p}} \big(\|f\|_{L^p(X,\mu)} + \|g\|_{L^p(X,\mu)}\big), \tag{1.1.4}$$

and thus $L^p(X,\mu)$ is a quasi-normed linear space. See also Exercise 1.1.5. For all $0 < p \le \infty$, it can be shown that every Cauchy sequence in $L^p(X,\mu)$ is convergent, and hence the spaces $L^p(X,\mu)$ are complete. For the case $0 < p < 1$ we refer to Exercise 1.1.8. Therefore, the L^p spaces are Banach spaces for $1 \le p \le \infty$ and quasi-Banach spaces for $0 < p < 1$. For any $p \in (0,\infty) \setminus \{1\}$ we use the notation $p' = \frac{p}{p-1}$. Moreover, we set $1' = \infty$ and $\infty' = 1$, so that $p'' = p$ for all $p \in (0,\infty]$. Hölder's inequality says that for all $p \in [1,\infty]$ and all measurable functions f, g on (X,μ) we have

$$\|fg\|_{L^1} \le \|f\|_{L^p} \|g\|_{L^{p'}}.$$

It is a well-known fact that the dual $(L^p)^*$ of L^p is isometric to $L^{p'}$ for all $1 \le p < \infty$. Furthermore, the L^p norm of a function can be obtained via duality when $1 \le p \le \infty$ as follows:

$$\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}} = 1} \left| \int_X f\,g\,d\mu \right|.$$

For the endpoint cases $p = 1$, $p = \infty$, see Exercise 1.4.12 (a), (b).

1.1.1 The Distribution Function

Definition 1.1.1. For f a measurable function on X, the *distribution function* of f is the function d_f defined on $[0,\infty)$ as follows:

$$d_f(\alpha) = \mu(\{x \in X : |f(x)| > \alpha\}). \tag{1.1.5}$$

The distribution function d_f provides information about the size of f but not about the behavior of f itself near any given point. For instance, a function on $\mathbf{R}^n$ and each of its translates have the same distribution function. It follows from Definition 1.1.1 that d_f is a decreasing function of α (not necessarily strictly).

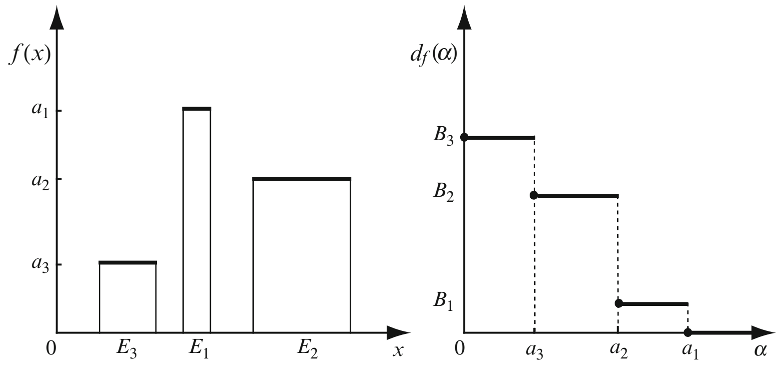

Fig. 1.1 The graph of a simple function $f = \sum_{k=1}^{3} a_k \chi_{E_k}$ and its distribution function $d_f(\alpha)$. Here $B_j = \sum_{k=1}^{j} \mu(E_k)$.

Example 1.1.2. For pedagogical reasons we compute the distribution function d_f of a nonnegative simple function

$$f(x) = \sum_{j=1}^{N} a_j \chi_{E_j}(x),$$

where the sets E_j are pairwise disjoint and $a_1 > \cdots > a_N > 0$. If $\alpha \geq a_1$, then clearly $d_f(\alpha) = 0$. However, if $a_2 \leq \alpha < a_1$ then $|f(x)| > \alpha$ precisely when $x \in E_1$, and in general, if $a_{j+1} \leq \alpha < a_j$, then $|f(x)| > \alpha$ precisely when $x \in E_1 \cup \cdots \cup E_j$. Setting

$$B_j = \sum_{k=1}^{j} \mu(E_k),$$

for $j \in \{1, \ldots, N\}$, $B_0 = a_{N+1} = 0$, and $a_0 = \infty$, we have

$$d_f(\alpha) = \sum_{j=0}^{N} B_j \chi_{[a_{j+1}, a_j)}(\alpha).$$

Note that these formulas are valid even when $\mu(E_i) = \infty$ for some i. Figure 1.1 presents an illustration of this example when $N = 3$ and $\mu(E_j) < \infty$ for all j.

Proposition 1.1.3. *Let f and g be measurable functions on (X, μ). Then for all $\alpha, \beta > 0$ we have*

(1) $|g| \leq |f|$ μ-a.e. implies that $d_g \leq d_f$;

(2) $d_{cf}(\alpha) = d_f(\alpha/|c|)$, for all $c \in \mathbf{C} \setminus \{0\}$;

(3) $d_{f+g}(\alpha + \beta) \leq d_f(\alpha) + d_g(\beta)$;

(4) $d_{fg}(\alpha\beta) \leq d_f(\alpha) + d_g(\beta)$.

Proof. The simple proofs are left to the reader. □

Knowledge of the distribution function d_f provides sufficient information to evaluate the L^p norm of a function f precisely. We state and prove the following important description of the L^p norm in terms of the distribution function.

Proposition 1.1.4. *Let (X,μ) be a σ-finite measure space. Then for f in $L^p(X,\mu)$, $0<p<\infty$, we have*

$$\|f\|_{L^p}^p = p\int_0^\infty \alpha^{p-1} d_f(\alpha)\,d\alpha\,. \tag{1.1.6}$$

Moreover, for any increasing continuously differentiable function φ on $[0,\infty)$ with $\varphi(0)=0$ and every measurable function f on X with $\varphi(|f|)$ integrable on X, we have

$$\int_X \varphi(|f|)\,d\mu = \int_0^\infty \varphi'(\alpha) d_f(\alpha)\,d\alpha\,. \tag{1.1.7}$$

Proof. Indeed, we have

$$\begin{aligned} p\int_0^\infty \alpha^{p-1} d_f(\alpha)\,d\alpha &= p\int_0^\infty \alpha^{p-1}\int_X \chi_{\{x:\,|f(x)|>\alpha\}}\,d\mu(x)\,d\alpha \\ &= \int_X \int_0^{|f(x)|} p\alpha^{p-1}\,d\alpha\,d\mu(x) \\ &= \int_X |f(x)|^p\,d\mu(x) \\ &= \|f\|_{L^p}^p\,, \end{aligned}$$

where in the second equality we used Fubini's theorem, which requires the measure space to be σ-finite. This proves (1.1.6). Identity (1.1.7) follows similarly, replacing the function α^p by the more general function $\varphi(\alpha)$ which has similar properties. □

Definition 1.1.5. For $0<p<\infty$, the space *weak* $L^p(X,\mu)$ is defined as the set of all μ-measurable functions f such that

$$\begin{aligned} \|f\|_{L^{p,\infty}} &= \inf\Big\{C>0:\, d_f(\alpha)\le \frac{C^p}{\alpha^p} \quad \text{for all} \quad \alpha>0\Big\} & (1.1.8) \\ &= \sup\big\{\gamma d_f(\gamma)^{1/p}:\, \gamma>0\big\} & (1.1.9) \end{aligned}$$

is finite. The space *weak* $L^\infty(X,\mu)$ is by definition $L^\infty(X,\mu)$.

One should check that (1.1.9) and (1.1.8) are in fact equal. The weak L^p spaces are denoted by $L^{p,\infty}(X,\mu)$. Two functions in $L^{p,\infty}(X,\mu)$ are considered equal if they are equal μ-a.e. The notation $L^{p,\infty}(\mathbf{R}^n)$ is reserved for $L^{p,\infty}(\mathbf{R}^n,|\cdot|)$. Using Proposition 1.1.3 (2), we can easily show that

$$\|kf\|_{L^{p,\infty}} = |k|\|f\|_{L^{p,\infty}}\,, \tag{1.1.10}$$

for any complex constant k. The analogue of (1.1.3) is

$$\big\|f+g\big\|_{L^{p,\infty}} \le c_p\big(\big\|f\big\|_{L^{p,\infty}}+\big\|g\big\|_{L^{p,\infty}}\big)\,, \tag{1.1.11}$$

where $c_p = \max(2, 2^{1/p})$, a fact that follows from Proposition 1.1.3 (3), taking both α and β equal to $\alpha/2$. We also have that

$$\big\|f\big\|_{L^{p,\infty}(X,\mu)} = 0 \Rightarrow f = 0 \qquad \mu\text{-a.e.} \tag{1.1.12}$$

In view of (1.1.10), (1.1.11), and (1.1.12), $L^{p,\infty}$ is a *quasi-normed linear space* for $0 < p < \infty$.

The weak L^p spaces are larger than the usual L^p spaces. We have the following:

Proposition 1.1.6. *For any $0 < p < \infty$ and any f in $L^p(X,\mu)$ we have*

$$\big\|f\big\|_{L^{p,\infty}} \le \big\|f\big\|_{L^p}\,.$$

Hence the embedding $L^p(X,\mu) \subseteq L^{p,\infty}(X,\mu)$ holds.

Proof. This is just a trivial consequence of Chebyshev's inequality:

$$\alpha^p d_f(\alpha) \le \int_{\{x:\,|f(x)|>\alpha\}} |f(x)|^p\, d\mu(x) \le \|f\|_{L^p}^p\,.$$

Using (1.1.9) we obtain that $\|f\|_{L^{p,\infty}} \le \|f\|_{L^p}$. □

The inclusion $L^p \subseteq L^{p,\infty}$ is strict. For example, on $\mathbf{R}^n$ with the usual Lebesgue measure, let $h(x) = |x|^{-\frac{n}{p}}$. Obviously, h is not in $L^p(\mathbf{R}^n)$ but h is in $L^{p,\infty}(\mathbf{R}^n)$ with $\|h\|_{L^{p,\infty}(\mathbf{R}^n)} = v_n^{1/p}$, where v_n is the measure of the unit ball of $\mathbf{R}^n$.

It is not immediate from their definition that the weak L^p spaces are complete with respect to the quasi-norm $\|\cdot\|_{L^{p,\infty}}$. The completeness of these spaces is proved in Theorem 1.4.11, but it is also a consequence of Theorem 1.1.13, proved in this section.

1.1.2 Convergence in Measure

Next we discuss some convergence notions. The following notion is important in probability theory.

Definition 1.1.7. Let f, f_n, $n = 1,2,\dots$, be measurable functions on the measure space (X,μ). The sequence f_n is said to *converge in measure* to f if for all $\varepsilon > 0$ there exists an $n_0 \in \mathbf{Z}^+$ such that

$$n > n_0 \implies \mu(\{x \in X:\ |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon\,. \tag{1.1.13}$$

Remark 1.1.8. The preceding definition is equivalent to the following statement:

$$\text{For all } \varepsilon>0 \qquad \lim_{n\to\infty}\mu(\{x\in X:\ |f_n(x)-f(x)|>\varepsilon\})=0. \tag{1.1.14}$$

Clearly (1.1.14) implies (1.1.13). To see the converse given $\varepsilon>0$, pick $0<\delta<\varepsilon$ and apply (1.1.13) for this δ. There exists an $n_0\in\mathbf{Z}^+$ such that

$$\mu(\{x\in X:\ |f_n(x)-f(x)|>\delta\})<\delta$$

holds for $n>n_0$. Since

$$\mu(\{x\in X:\ |f_n(x)-f(x)|>\varepsilon\})\le\mu(\{x\in X:\ |f_n(x)-f(x)|>\delta\}),$$

we conclude that

$$\mu(\{x\in X:\ |f_n(x)-f(x)|>\varepsilon\})<\delta$$

for all $n>n_0$. Let $n\to\infty$ to deduce that

$$\limsup_{n\to\infty}\mu(\{x\in X:\ |f_n(x)-f(x)|>\varepsilon\})\le\delta. \tag{1.1.15}$$

Since (1.1.15) holds for all $0<\delta<\varepsilon$, (1.1.14) follows by letting $\delta\to 0$.

Convergence in measure is a weaker notion than convergence in either L^p or $L^{p,\infty}$, $0<p\le\infty$, as the following proposition indicates:

Proposition 1.1.9. *Let $0<p\le\infty$ and f_n, f be in $L^{p,\infty}(X,\mu)$.*

(1) If f_n, f are in L^p and $f_n\to f$ in L^p, then $f_n\to f$ in $L^{p,\infty}$.
(2) If $f_n\to f$ in $L^{p,\infty}$, then f_n converges to f in measure.

Proof. Fix $0<p<\infty$. Proposition 1.1.6 gives that for all $\varepsilon>0$ we have

$$\mu(\{x\in X:\ |f_n(x)-f(x)|>\varepsilon\})\le\frac{1}{\varepsilon^p}\int_X|f_n-f|^p\,d\mu.$$

This shows that convergence in L^p implies convergence in weak L^p. The case $p=\infty$ is tautological.

Given $\varepsilon>0$ find an n_0 such that for $n>n_0$, we have

$$\big\|f_n-f\big\|_{L^{p,\infty}}=\sup_{\alpha>0}\alpha\mu(\{x\in X:\ |f_n(x)-f(x)|>\alpha\})^{\frac{1}{p}}<\varepsilon^{\frac{1}{p}+1}.$$

Taking $\alpha=\varepsilon$, we conclude that convergence in $L^{p,\infty}$ implies convergence in measure. □

Example 1.1.10. Note that there is no general converse of statement (2) in the preceding proposition. Fix $0<p<\infty$ and on $[0,1]$ define the functions

$$f_{k,j}=k^{1/p}\chi_{(\frac{j-1}{k},\frac{j}{k})},\qquad k\ge1,\ 1\le j\le k.$$

Consider the sequence $\{f_{1,1}, f_{2,1}, f_{2,2}, f_{3,1}, f_{3,2}, f_{3,3}, \dots\}$. Observe that

$$|\{x: \ f_{k,j}(x) > 0\}| = 1/k.$$

Therefore, $f_{k,j}$ converges to 0 in measure. Likewise, observe that

$$\big\|f_{k,j}\big\|_{L^{p,\infty}} = \sup_{\alpha>0} \alpha|\{x: \ f_{k,j}(x) > \alpha\}|^{1/p} \geq \sup_{k\geq 1} \frac{(k-1/k)^{1/p}}{k^{1/p}} = 1,$$

which implies that $f_{k,j}$ does not converge to 0 in $L^{p,\infty}$.

It turns out that every sequence convergent in $L^p(X,\mu)$ or in $L^{p,\infty}(X,\mu)$ has a subsequence that converges a.e. to the same limit.

Theorem 1.1.11. *Let f_n and f be complex-valued measurable functions on a measure space (X,μ) and suppose that f_n converges to f in measure. Then some subsequence of f_n converges to f μ-a.e.*

Proof. For all $k = 1,2,\dots$ choose inductively n_k such that

$$\mu(\{x \in X: \ |f_{n_k}(x) - f(x)| > 2^{-k}\}) < 2^{-k} \tag{1.1.16}$$

and such that $n_1 < n_2 < \cdots < n_k < \cdots$. Define the sets

$$A_k = \{x \in X: \ |f_{n_k}(x) - f(x)| > 2^{-k}\}.$$

Equation (1.1.16) implies that

$$\mu\Big(\bigcup_{k=m}^{\infty} A_k\Big) \leq \sum_{k=m}^{\infty} \mu(A_k) \leq \sum_{k=m}^{\infty} 2^{-k} = 2^{1-m} \tag{1.1.17}$$

for all $m = 1,2,3,\dots$. It follows from (1.1.17) that

$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) \leq 1 < \infty. \tag{1.1.18}$$

Using (1.1.17) and (1.1.18), we conclude that the sequence of the measures of the sets $\{\bigcup_{k=m}^{\infty} A_k\}_{m=1}^{\infty}$ converges as $m \to \infty$ to

$$\mu\Big(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty} A_k\Big) = 0. \tag{1.1.19}$$

To finish the proof, observe that the null set in (1.1.19) contains the set of all $x \in X$ for which $f_{n_k}(x)$ does not converge to $f(x)$. □

In many situations we are given a sequence of functions and we would like to extract a convergent subsequence. One way to achieve this is via the next theorem, which is a useful variant of Theorem 1.1.11. We first give a relevant definition.

Definition 1.1.12. We say that a sequence of measurable functions $\{f_n\}$ on the measure space (X,μ) is *Cauchy in measure* if for every $\varepsilon > 0$, there exists an $n_0 \in \mathbf{Z}^+$ such that for $n,m > n_0$ we have

$$\mu(\{x \in X : |f_m(x) - f_n(x)| > \varepsilon\}) < \varepsilon.$$

Theorem 1.1.13. *Let (X,μ) be a measure space and let $\{f_n\}$ be a complex-valued sequence on X that is Cauchy in measure. Then some subsequence of f_n converges μ-a.e.*

Proof. The proof is very similar to that of Theorem 1.1.11. For all $k = 1,2,\dots$ choose n_k inductively such that

$$\mu(\{x \in X : |f_{n_k}(x) - f_{n_{k+1}}(x)| > 2^{-k}\}) < 2^{-k} \tag{1.1.20}$$

and such that $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$. Define

$$A_k = \{x \in X : |f_{n_k}(x) - f_{n_{k+1}}(x)| > 2^{-k}\}.$$

As shown in the proof of Theorem 1.1.11, (1.1.20) implies that

$$\mu\Big(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty} A_k\Big) = 0. \tag{1.1.21}$$

For $x \notin \bigcup_{k=m}^{\infty} A_k$ and $i \ge j \ge j_0 \ge m$ (and j_0 large enough) we have

$$|f_{n_i}(x) - f_{n_j}(x)| \le \sum_{l=j}^{i-1} |f_{n_l}(x) - f_{n_{l+1}}(x)| \le \sum_{l=j}^{i-1} 2^{-l} \le 2^{1-j} \le 2^{1-j_0}.$$

This implies that the sequence $\{f_{n_i}(x)\}_i$ is Cauchy for every x in the set $(\bigcup_{k=m}^{\infty} A_k)^c$ and therefore converges for all such x. We define a function

$$f(x) = \begin{cases} \lim_{j\to\infty} f_{n_j}(x) & \text{when } x \notin \bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty} A_k\,, \\ 0 & \text{when } x \in \bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty} A_k\,. \end{cases}$$

Then $f_{n_j} \to f$ almost everywhere. □

1.1.3 A First Glimpse at Interpolation

It is a useful fact that if a function f is in $L^p(X,\mu)$ and in $L^q(X,\mu)$, then it also lies in $L^r(X,\mu)$ for all $p < r < q$. The usefulness of the spaces $L^{p,\infty}$ can be seen from the following sharpening of this statement:

Proposition 1.1.14. *Let $0 < p < q \le \infty$ and let f in $L^{p,\infty}(X,\mu) \cap L^{q,\infty}(X,\mu)$, where X is a σ-finite measure space. Then f is in $L^r(X,\mu)$ for all $p < r < q$ and*

$$\|f\|_{L^r} \le \left(\frac{r}{r-p} + \frac{r}{q-r}\right)^{\frac{1}{r}} \|f\|_{L^{p,\infty}}^{\frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}} \|f\|_{L^{q,\infty}}^{\frac{\frac{1}{p}-\frac{1}{r}}{\frac{1}{p}-\frac{1}{q}}}, \tag{1.1.22}$$

with the interpretation that $1/\infty = 0$.

Proof. Let us take first $q < \infty$. We know that

$$d_f(\alpha) \le \min\left(\frac{\|f\|_{L^{p,\infty}}^p}{\alpha^p}, \frac{\|f\|_{L^{q,\infty}}^q}{\alpha^q}\right). \tag{1.1.23}$$

Set

$$B = \left(\frac{\|f\|_{L^{q,\infty}}^q}{\|f\|_{L^{p,\infty}}^p}\right)^{\frac{1}{q-p}}. \tag{1.1.24}$$

We now estimate the L^r norm of f. By (1.1.23), (1.1.24), and Proposition 1.1.4 we have

$$\begin{aligned} \|f\|_{L^r(X,\mu)}^r &= r\int_0^\infty \alpha^{r-1} d_f(\alpha)\,d\alpha \\ &\le r\int_0^\infty \alpha^{r-1}\min\left(\frac{\|f\|_{L^{p,\infty}}^p}{\alpha^p}, \frac{\|f\|_{L^{q,\infty}}^q}{\alpha^q}\right)d\alpha \\ &= r\int_0^B \alpha^{r-1-p}\|f\|_{L^{p,\infty}}^p\,d\alpha + r\int_B^\infty \alpha^{r-1-q}\|f\|_{L^{q,\infty}}^q\,d\alpha \\ &= \frac{r}{r-p}\|f\|_{L^{p,\infty}}^p B^{r-p} + \frac{r}{q-r}\|f\|_{L^{q,\infty}}^q B^{r-q} \\ &= \left(\frac{r}{r-p} + \frac{r}{q-r}\right)\left(\|f\|_{L^{p,\infty}}^p\right)^{\frac{q-r}{q-p}}\left(\|f\|_{L^{q,\infty}}^q\right)^{\frac{r-p}{q-p}}. \end{aligned} \tag{1.1.25}$$

Observe that the integrals converge, since $r-p>0$ and $r-q<0$.

The case $q = \infty$ is easier. Since $d_f(\alpha) = 0$ for $\alpha > \|f\|_{L^\infty}$ we need to use only the inequality $d_f(\alpha) \le \alpha^{-p}\|f\|_{L^{p,\infty}}^p$ for $\alpha \le \|f\|_{L^\infty}$ in estimating the first integral in (1.1.25). We obtain

$$\|f\|_{L^r}^r \le \frac{r}{r-p}\|f\|_{L^{p,\infty}}^p \|f\|_{L^\infty}^{r-p},$$

which is nothing other than (1.1.22) when $q = \infty$. This completes the proof. □

Note that (1.1.22) holds with constant 1 if $L^{p,\infty}$ and $L^{q,\infty}$ are replaced by L^p and L^q, respectively. It is often convenient to work with functions that are only locally in some L^p space. This leads to the following definition.

Definition 1.1.15. For $0 < p < \infty$, the space $L^p_{\text{loc}}(\mathbf{R}^n, |\cdot|)$ or simply $L^p_{\text{loc}}(\mathbf{R}^n)$ is the set of all Lebesgue-measurable functions f on $\mathbf{R}^n$ that satisfy

$$\int_K |f(x)|^p\,dx < \infty \tag{1.1.26}$$

for any compact subset K of $\mathbf{R}^n$. Functions that satisfy (1.1.26) with $p = 1$ are called *locally integrable* functions on $\mathbf{R}^n$.

The union of all $L^p(\mathbf{R}^n)$ spaces for $1 \le p \le \infty$ is contained in $L^1_{\text{loc}}(\mathbf{R}^n)$. More generally, for $0 < p < q < \infty$ we have the following:

$$L^q(\mathbf{R}^n) \subsetneqq L^q_{\text{loc}}(\mathbf{R}^n) \subsetneqq L^p_{\text{loc}}(\mathbf{R}^n).$$

Functions in $L^p(\mathbf{R}^n)$ for $0 < p < 1$ may not be locally integrable. For example, take $f(x) = |x|^{-n-\alpha}\chi_{|x|\le 1}$, which is in $L^p(\mathbf{R}^n)$ when $\alpha > 0$ and $p < n/(n+\alpha)$, and observe that f is not integrable over any open set in $\mathbf{R}^n$ containing the origin.

Exercises

1.1.1. Suppose f and f_n are measurable functions on (X,μ). Prove that
(a) d_f is right continuous on $[0,\infty)$.
(b) If $|f| \le \liminf_{n\to\infty} |f_n|$ μ-a.e., then $d_f \le \liminf_{n\to\infty} d_{f_n}$.
(c) If $|f_n| \uparrow |f|$, then $d_{f_n} \uparrow d_f$.
[*Hint:* Part (a): Let t_n be a decreasing sequence of positive numbers that tends to zero. Show that $d_f(\alpha_0 + t_n) \uparrow d_f(\alpha_0)$ using a convergence theorem. Part (b): Let $E = \{x \in X : |f(x)| > \alpha\}$ and $E_n = \{x \in X : |f_n(x)| > \alpha\}$. Use that $\mu\big(\bigcap_{n=m}^\infty E_n\big) \le \liminf_{n\to\infty} \mu(E_n)$ and $E \subsetneqq \bigcup_{m=1}^\infty \bigcap_{n=m}^\infty E_n$ μ-a.e.]

1.1.2. (*Hölder's inequality*) Let $0 < p, p_1, \dots, p_k \le \infty$, where $k \ge 2$, and let f_j be in $L^{p_j} = L^{p_j}(X,\mu)$. Assume that

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_k}.$$

(a) Show that the product $f_1 \cdots f_k$ is in L^p and that

$$\big\|f_1 \cdots f_k\big\|_{L^p} \le \big\|f_1\big\|_{L^{p_1}} \cdots \big\|f_k\big\|_{L^{p_k}}.$$

(b) When no p_j is infinite, show that if equality holds in part (a), then it must be the case that $c_1|f_1|^{p_1} = \cdots = c_k|f_k|^{p_k}$ μ-a.e. for some $c_j \ge 0$.
(c) Let $0 < q < 1$ and $q' = \frac{q}{q-1}$. For $r < 0$ and $g > 0$ almost everywhere, define $\|g\|_{L^r} = \|g^{-1}\|^{-1}_{L^{|r|}}$. Show that if g is strictly positive μ-a.e. and lies in $L^{q'}$ and f is measurable such that fg belongs to L^1, we have

$$\big\|fg\big\|_{L^1} \ge \big\|f\big\|_{L^q} \big\|g\big\|_{L^{q'}}.$$

1.1.3. Let (X,μ) be a measure space.
(a) If f is in $L^{p_0}(X,\mu)$ for some $p_0 < \infty$, prove that

$$\lim_{p\to\infty} \big\|f\big\|_{L^p} = \big\|f\big\|_{L^\infty}.$$

(b) (*Jensen's inequality*) Suppose that $\mu(X) = 1$. Show that

$$\|f\|_{L^p} \geq \exp\left(\int_X \log|f(x)|\,d\mu(x) \right)$$

for all $0 < p < \infty$.
(c) If $\mu(X) = 1$ and f is in some $L^{p_0}(X,\mu)$ for some $p_0 > 0$, then

$$\lim_{p\to 0} \|f\|_{L^p} = \exp\left(\int_X \log|f(x)|\,d\mu(x) \right)$$

with the interpretation $e^{-\infty} = 0$.
[*Hint:* Part (a): If $0 < \|f\|_{L^\infty} < \infty$, use that $\|f\|_{L^p} \leq \|f\|_{L^\infty}^{(p-p_0)/p} \|f\|_{L^{p_0}}^{p_0/p}$ to obtain $\limsup_{p\to\infty} \|f\|_{L^p} \leq \|f\|_{L^\infty}$. Conversely, let $E_\gamma = \{x \in X: \ |f(x)| > \gamma\|f\|_{L^\infty}\}$ for γ in $(0,1)$. Then $\mu(E_\gamma) > 0$, $\|f\|_{L^{p_0}(E_\gamma)} > 0$, and $\|f\|_{L^p} \geq \big(\gamma\|f\|_{L^\infty}\big)^{(p-p_0)/p} \|f\|_{L^{p_0}(E_\gamma)}^{p_0/p}$, hence $\liminf_{p\to\infty} \|f\|_{L^p} \geq \gamma\|f\|_{L^\infty}$. If $\|f\|_{L^\infty} = \infty$, set $G_n = \{|f| > n\}$ and use that $\|f\|_{L^p} \geq \|f\|_{L^p(G_n)} \geq n\mu(G_n)^{\frac{1}{p}}$ to obtain $\liminf_{p\to\infty} \|f\|_{L^p} \geq n$. Part (b) is a direct consequence of Jensen's inequality $\int_X \log|h|\,d\mu \leq \log\big(\int_X |h|\,d\mu\big)$. Part (c): Fix a sequence $0 < p_n < p_0$ such that $p_n \downarrow 0$ and define

$$h_n(x) = \frac{1}{p_0}(|f(x)|^{p_0} - 1) - \frac{1}{p_n}(|f(x)|^{p_n} - 1).$$

Use that $\frac{1}{p}(t^p - 1) \downarrow \log t$ as $p \downarrow 0$ for all $t > 0$. The Lebesgue monotone convergence theorem yields $\int_X h_n\,d\mu \uparrow \int_X h\,d\mu$, hence $\int_X \frac{1}{p_n}(|f|^{p_n} - 1)\,d\mu \downarrow \int_X \log|f|\,d\mu$, where the latter could be $-\infty$. Use

$$\exp\left(\int_X \log|f|\,d\mu \right) \leq \left(\int_X |f|^{p_n}\,d\mu \right)^{\frac{1}{p_n}} \leq \exp\left(\int_X \frac{1}{p_n}(|f|^{p_n} - 1)\,d\mu \right)$$

to complete the proof.]

1.1.4. Let a_j be a sequence of positive reals. Show that
(a) $\big(\sum_{j=1}^\infty a_j\big)^\theta \leq \sum_{j=1}^\infty a_j^\theta$, for any $0 \leq \theta \leq 1$.
(b) $\sum_{j=1}^\infty a_j^\theta \leq \big(\sum_{j=1}^\infty a_j\big)^\theta$, for any $1 \leq \theta < \infty$.
(c) $\big(\sum_{j=1}^N a_j\big)^\theta \leq N^{\theta-1} \sum_{j=1}^N a_j^\theta$, when $1 \leq \theta < \infty$.
(d) $\sum_{j=1}^N a_j^\theta \leq N^{1-\theta} \big(\sum_{j=1}^N a_j\big)^\theta$, when $0 \leq \theta \leq 1$.

1.1.5. Let $\{f_j\}_{j=1}^N$ be a sequence of $L^p(X,\mu)$ functions.
(a) (*Minkowski's inequality*) For $1 \leq p \leq \infty$ show that

$$\Big\| \sum_{j=1}^N f_j \Big\|_{L^p} \leq \sum_{j=1}^N \|f_j\|_{L^p}\,.$$

(b) (*Reverse Minkowski inequality*) For $0 < p < 1$ and $f_j \geq 0$ prove that

$$\sum_{j=1}^{N} \|f_j\|_{L^p} \leq \Big\| \sum_{j=1}^{N} f_j \Big\|_{L^p}.$$

(c) For $0 < p < 1$ show that

$$\Big\| \sum_{j=1}^{N} f_j \Big\|_{L^p} \leq N^{\frac{1-p}{p}} \sum_{j=1}^{N} \|f_j\|_{L^p}.$$

(d) The constant $N^{\frac{1-p}{p}}$ in part (c) is best possible.
[*Hint:* Part (c): Use Exercise 1.1.4 (c). Part (d): Take $\{f_j\}_{j=1}^N$ to be characteristic functions of disjoint sets with the same measure.]

1.1.6. (a) (*Minkowski's integral inequality*) Let (X,μ) and (T,ν) be two σ-finite measure spaces and let $1 \leq p < \infty$. Show that for every nonnegative measurable function F on the product space $(X,\mu)\times(T,\nu)$ we have

$$\left[\int_T \left(\int_X F(x,t)\,d\mu(x)\right)^p d\nu(t)\right]^{\frac{1}{p}} \leq \int_X \left[\int_T F(x,t)^p\,d\nu(t)\right]^{\frac{1}{p}} d\mu(x),$$

(b) State and prove an analogous inequality when $p = \infty$.
(c) Prove that when $0 < p < 1$, then the preceding inequality is reversed.
(d) (Y. Sawano) Consider the example $X = T = [0,1]$, μ is counting measure, ν is Lebesgue measure, $F(x,t) = 1$ when $x = t$ and zero otherwise. What is the relevance of this example with the inequalities in (a) and (b)?
[*Hint:* Part (a) Split the power p as $1 + (p-1)$ and apply Hölder's inequality with exponents p and p'. Part (b) Let $p \to \infty$ on subsets of X with finite measure.]

1.1.7. Let $f_1,\dots,f_N$ be in $L^{p,\infty}(X,\mu)$.
(a) Prove that for $1 \leq p < \infty$ we have

$$\Big\| \sum_{j=1}^{N} f_j \Big\|_{L^{p,\infty}} \leq N \sum_{j=1}^{N} \|f_j\|_{L^{p,\infty}}.$$

(b) Show that for $0 < p < 1$ we have

$$\Big\| \sum_{j=1}^{N} f_j \Big\|_{L^{p,\infty}} \leq N^{\frac{1}{p}} \sum_{j=1}^{N} \|f_j\|_{L^{p,\infty}}.$$

[*Hint:* Use that $\mu(\{|f_1+\cdots+f_N| > \alpha\}) \leq \sum_{j=1}^N \mu(\{|f_j| > \alpha/N\})$ and Exercise 1.1.4 (a) and (c).]

1.1.8. Let $0 < p < \infty$. Prove that $L^p(X,\mu)$ is a complete quasi-normed space. This means that every quasi-norm Cauchy sequence is quasi-norm convergent.

[*Hint:* Let f_n be a Cauchy sequence in L^p. Pass to a subsequence $\{n_i\}_i$ such that $\|f_{n_{i+1}} - f_{n_i}\|_{L^p} \le 2^{-i}$. Then the series $f = f_{n_1} + \sum_{i=1}^{\infty}(f_{n_{i+1}} - f_{n_i})$ converges in L^p.]

1.1.9. Let (X,μ) be a measure space with $\mu(X) < \infty$. Suppose that a sequence of measurable functions f_n on X converges to f μ-a.e. Prove that f_n converges to f in measure.
[*Hint:* For $\varepsilon > 0$, $\{x \in X : f_n(x) \to f(x)\} \subseteq \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{x \in X : |f_n(x) - f(x)| < \varepsilon\}$.]

1.1.10. Let f be a measurable function on (X,μ) such such $d_f(\alpha) < \infty$ for all $\alpha > 0$. Fix $\gamma > 0$ and define $f_\gamma = f\chi_{|f|>\gamma}$ and $f^\gamma = f - f_\gamma = f\chi_{|f|\le\gamma}$.
(a) Prove that

$$d_{f_\gamma}(\alpha) = \begin{cases} d_f(\alpha) & \text{when} \quad \alpha > \gamma, \\ d_f(\gamma) & \text{when} \quad \alpha \le \gamma, \end{cases}$$

$$d_{f^\gamma}(\alpha) = \begin{cases} 0 & \text{when} \quad \alpha \ge \gamma, \\ d_f(\alpha) - d_f(\gamma) & \text{when} \quad \alpha < \gamma. \end{cases}$$

(b) If $f \in L^p(X,\mu)$ then

$$\|f_\gamma\|_{L^p}^p = p\int_\gamma^\infty \alpha^{p-1} d_f(\alpha)\,d\alpha + \gamma^p d_f(\gamma),$$

$$\|f^\gamma\|_{L^p}^p = p\int_0^\gamma \alpha^{p-1} d_f(\alpha)\,d\alpha - \gamma^p d_f(\gamma),$$

$$\int_{\gamma<|f|\le\delta} |f|^p\,d\mu = p\int_\gamma^\delta d_f(\alpha)\alpha^{p-1}\,d\alpha - \delta^p d_f(\delta) + \gamma^p d_f(\gamma).$$

(c) If f is in $L^{p,\infty}(X,\mu)$ prove that f^γ is in $L^q(X,\mu)$ for any $q > p$ and f_γ is in $L^q(X,\mu)$ for any $q < p$. Thus $L^{p,\infty} \subseteq L^{p_0} + L^{p_1}$ when $0 < p_0 < p < p_1 \le \infty$.

1.1.11. Let (X,μ) be a measure space and let E be a subset of X with $\mu(E) < \infty$. Assume that f is in $L^{p,\infty}(X,\mu)$ for some $0 < p < \infty$.
(a) Show that for $0 < q < p$ we have

$$\int_E |f(x)|^q\,d\mu(x) \le \frac{p}{p-q}\mu(E)^{1-\frac{q}{p}} \|f\|_{L^{p,\infty}}^q .$$

(b) Conclude that if $\mu(X) < \infty$ and $0 < q < p$, then

$$L^p(X,\mu) \subseteq L^{p,\infty}(X,\mu) \subseteq L^q(X,\mu).$$

[*Hint:* Part (a): Use $\mu\big(E \cap \{|f| > \alpha\}\big) \le \min\big(\mu(E), \alpha^{-p}\|f\|_{L^{p,\infty}}^p\big)$.]

1.1.12. (*Normability of weak L^p for $p > 1$*) Let (X,μ) be a σ-finite measure space and let $0 < p < \infty$. Pick $0 < r < p$ and define

$$|||\, f \,|||_{L^{p,\infty}} = \sup_{0<\mu(E)<\infty} \mu(E)^{-\frac{1}{r}+\frac{1}{p}} \left(\int_E |f|^r\,d\mu \right)^{\frac{1}{r}},$$

where the supremum is taken over all measurable subsets E of X of finite measure.
(a) Use Exercise 1.1.11 with $q = r$ to conclude that

$$|||\, f\, |||_{L^{p,\infty}} \le \left(\frac{p}{p-r}\right)^{\frac{1}{r}} \|f\|_{L^{p,\infty}}$$

for all f in $L^{p,\infty}(X,\mu)$. (It is not needed that X be σ-finite here).
(b) Prove that for all f in $L^{p,\infty}(X,\mu)$ we have

$$\|f\|_{L^{p,\infty}} \le |||\, f\, |||_{L^{p,\infty}}\,.$$

(Y. Oi) Notice that if $X = \{1,2\}$, $\mu(\{1\}) = 1$, $\mu(\{2\}) = \infty$, then X is not σ-finite, and verify that for the function $f = 1$ the preceding inequality fails.
(c) Show that $L^{p,\infty}(X,\mu)$ is metrizable for all $0 < p < \infty$, i.e., there is a metric on the space that generates the same topology as the quasi-norm. Also show that $L^{p,\infty}(X,\mu)$ is *normable* when $p > 1$, i.e., there is a norm on the space equivalent to $\|\cdot\|_{L^{p,\infty}}$.
(d) Use the characterization of the weak L^p quasi-norm obtained in parts (a) and (b) to prove Fatou's lemma for this space: For all measurable functions g_n on X we have

$$\Big\|\liminf_{n\to\infty}|g_n|\Big\|_{L^{p,\infty}} \le C_p \liminf_{n\to\infty}\big\|g_n\big\|_{L^{p,\infty}}$$

for some constant C_p that depends only on $p \in (0,\infty)$.
$\big[$*Hint:* Part (b): Write $X = \bigcup_{k=1}^{\infty} X_k$ with $\mu(X_k) < \infty$ and take $E = \{|f| > \alpha\} \cap X_k$.$\big]$

1.1.13. Consider the $N!$ functions on the line

$$f_\sigma = \sum_{j=1}^{N} \frac{N}{\sigma(j)} \chi_{[\frac{j-1}{N},\frac{j}{N})}\,,$$

where σ is a permutation of the set $\{1,2,\dots,N\}$.
(a) Show that each f_σ satisfies $\|f_\sigma\|_{L^{1,\infty}} = 1$.
(b) Show that $\|\sum_{\sigma\in S_N} f_\sigma\|_{L^{1,\infty}} = N!\big(1+\frac{1}{2}+\cdots+\frac{1}{N}\big)$.
(c) Conclude that the space $L^{1,\infty}(\mathbf{R})$ is not normable (this means that $\|\cdot\|_{L^{1,\infty}}$ is not equivalent to a norm).
(d) Use a similar argument to prove that $L^{1,\infty}(\mathbf{R}^n)$ is not normable by considering the functions

$$f_\sigma(x_1,\dots,x_n) = \sum_{j_1=1}^{N}\cdots\sum_{j_n=1}^{N} \frac{N^n}{\sigma(\tau(j_1,\dots,j_n))} \chi_{[\frac{j_1-1}{N},\frac{j_1}{N})}(x_1)\cdots\chi_{[\frac{j_n-1}{N},\frac{j_n}{N})}(x_n)\,,$$

where σ is a permutation of the set $\{1,2,\dots,N^n\}$ and τ is a fixed injective map from the set of all n-tuples of integers with coordinates $1 \le j \le N$ onto the set $\{1,2,\dots,N^n\}$. One may take

$$\tau(j_1,\dots,j_n) = j_1 + N(j_2-1) + N^2(j_3-1) + \cdots + N^{n-1}(j_n-1),$$

for instance.

1.1.14. Let (X,μ) be a measure space and let $s>0$.
(a) Let f be a measurable function on X. Show that if $0<p<q<\infty$ we have

$$\int_{|f|\le s} |f|^q\,d\mu \le \frac{q}{q-p}\,s^{q-p}\,\big\|f\big\|_{L^{p,\infty}}^p .$$

(b) Let f_j, $1\le j\le m$, be measurable functions on X and let $0<p<\infty$. Show that

$$\Big\|\max_{1\le j\le m}|f_j|\Big\|_{L^{p,\infty}}^p \le \sum_{j=1}^m \big\|f_j\big\|_{L^{p,\infty}}^p .$$

(c) Conclude from part (b) that for $0<p<1$ we have

$$\big\|f_1+\cdots+f_m\big\|_{L^{p,\infty}}^p \le \frac{2-p}{1-p}\sum_{j=1}^m \big\|f_j\big\|_{L^{p,\infty}}^p .$$

The latter estimate is referred to as the *p-normability* of weak L^p.
[*Hint:* Part (a): Use the distribution function. Part (c): First obtain the estimate

$$d_{f_1+\cdots+f_m}(\alpha) \le \mu(\{|f_1+\cdots+f_m|>\alpha,\max|f_j|\le\alpha\})+d_{\max_j|f_j|}(\alpha)$$

for all $\alpha>0$ and then use part (b).]

1.1.15. (*Hölder's inequality for weak spaces*) Let f_j be in $L^{p_j,\infty}$ of a measure space X where $0<p_j<\infty$ and $1\le j\le k$. Let

$$\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_k}.$$

Prove that

$$\big\|f_1\cdots f_k\big\|_{L^{p,\infty}} \le p^{-\frac{1}{p}}\prod_{j=1}^k p_j^{\frac{1}{p_j}}\prod_{j=1}^k\big\|f_j\big\|_{L^{p_j,\infty}}.$$

[*Hint:* Take $\|f_j\|_{L^{p_j,\infty}}=1$ for all j. Control $d_{f_1\cdots f_k}(\alpha)$ by

$$\begin{aligned}&\mu(\{|f_1|>\alpha/s_1\})+\cdots+\mu(\{|f_{k-1}|>s_{k-2}/s_{k-1}\})+\mu(\{|f_k|>s_{k-1}\})\\ &\quad\le (s_1/\alpha)^{p_1}+(s_2/s_1)^{p_2}+\cdots+(s_{k-1}/s_{k-2})^{p_{k-1}}+(1/s_{k-1})^{p_k}.\end{aligned}$$

Set $x_1=s_1/\alpha$, $x_2=s_2/s_1,\dots,x_k=1/s_{k-1}$. Minimize $x_1^{p_1}+\cdots+x_k^{p_k}$ subject to the constraint $x_1\cdots x_k=1/\alpha$.]

1.1.16. Let $0<p_0<p<p_1\le\infty$ and let $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ for some $\theta\in[0,1]$. Prove the following:

$$\begin{aligned}\big\|f\big\|_{L^p} &\le \big\|f\big\|_{L^{p_0}}^{1-\theta}\big\|f\big\|_{L^{p_1}}^{\theta},\\ \big\|f\big\|_{L^{p,\infty}} &\le \big\|f\big\|_{L^{p_0,\infty}}^{1-\theta}\big\|f\big\|_{L^{p_1,\infty}}^{\theta}.\end{aligned}$$

1.1.17. ([231]) Follow the steps below to prove the *isoperimetric inequality*. For $n \ge 2$ and $1 \le j \le n$ define the projection maps $\pi_j : \mathbf{R}^n \to \mathbf{R}^{n-1}$ by setting for $x = (x_1, \dots, x_n)$,

$$\pi_j(x) = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n),$$

with the obvious interpretations when $j = 1$ or $j = n$.
(a) For maps $f_j : \mathbf{R}^{n-1} \to \mathbf{C}$ prove that

$$\Lambda(f_1, \dots, f_n) = \int_{\mathbf{R}^n} \prod_{j=1}^n \left| f_j \circ \pi_j \right| dx \le \prod_{j=1}^n \left\| f_j \right\|_{L^{n-1}(\mathbf{R}^{n-1})} .$$

(b) Let Ω be a compact set with a rectifiable boundary in $\mathbf{R}^n$ where $n \ge 2$. Show that there is a constant c_n independent of Ω such that

$$|\Omega| \le c_n |\partial\Omega|^{\frac{n}{n-1}} ,$$

where the expression $|\partial\Omega|$ denotes the $(n-1)$-dimensional surface measure of the boundary of Ω.
[*Hint:* Part (a): Use induction starting with $n = 2$. For $n \ge 3$ write

$$\begin{aligned} \Lambda(f_1, \dots, f_n) &\le \int_{\mathbf{R}^{n-1}} P(x_1, \dots, x_{n-1}) |f_n(\pi_n(x))| \, dx_1 \cdots dx_{n-1} \\ &\le \|P\|_{L^{\frac{n-1}{n-2}}(\mathbf{R}^{n-1})} \left\| f_n \circ \pi_n \right\|_{L^{n-1}(\mathbf{R}^{n-1})} , \end{aligned}$$

where $P(x_1, \dots, x_{n-1}) = \int_{\mathbf{R}} |f_1(\pi_1(x)) \cdots f_{n-1}(\pi_{n-1}(x))| \, dx_n$, and apply the induction hypothesis to the $n-1$ functions

$$\left[\int_{\mathbf{R}} f_j(\pi_j(x))^{n-1} \, dx_n \right]^{\frac{1}{n-2}} ,$$

for $j = 1, \dots, n-1$, to obtain the required conclusion. Part (b): Specialize part (a) to the case $f_j = \chi_{\pi_j[\Omega]}$ to obtain

$$|\Omega| \le |\pi_1[\Omega]|^{\frac{1}{n-1}} \cdots |\pi_n[\Omega]|^{\frac{1}{n-1}}$$

and then use that $|\pi_j[\Omega]| \le \frac{1}{2}|\partial\Omega|$.]

1.2 Convolution and Approximate Identities

The notion of convolution can be defined on measure spaces endowed with a group structure. It turns out that the most natural environment to define convolution is the context of topological groups. Although the focus of this book is harmonic analysis on Euclidean spaces, we develop the notion of convolution on general groups. This allows us to study this concept on $\mathbf{R}^n$, $\mathbf{Z}^n$, and $\mathbf{T}^n$, in a unified way. Moreover, since the basic properties of convolutions and approximate identities do not require commutativity of the group operation, we may assume that the underlying groups are not necessarily abelian. Thus, the results in this section can be also applied to nonabelian structures such as the Heisenberg group.

1.2.1 Examples of Topological Groups

A *topological group* G is a Hausdorff topological space that is also a group with law

$$(x,y) \mapsto xy \tag{1.2.1}$$

such that the maps $(x,y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous. The identity element of the group is the unique element e with the property $xe = ex = x$ for all $x \in G$. We adopt the standard notation

$$AB = \{ab : a \in A, b \in B\}, \qquad A^{-1} = \{a^{-1} : a \in A\}$$

for subsets A and B of G. Note that $(AB)^{-1} = B^{-1}A^{-1}$. Every topological group G has an open basis at e consisting of symmetric neighborhoods, i.e., open sets U satisfying $U = U^{-1}$. A topological group is called *locally compact* if there is an open set U containing the identity element such that $\overline{U}$ is compact. Then every point in the group has an open neighborhood with compact closure.

Let G be a locally compact group. It is known that G possesses a positive measure λ on the Borel sets that is nonzero on all nonempty open sets, finite on compact sets, and is left invariant, meaning that

$$\lambda(tA) = \lambda(A), \tag{1.2.2}$$

for all measurable sets A and all $t \in G$. Such a measure λ is called a (left) *Haar measure* on G. Similarly, G possesses a *right Haar measure* which is right invariant, i.e., $\lambda(At) = \lambda(A)$ for all measurable $A \subseteq G$ and all $t \in G$. For the existence of Haar measure we refer to [152, §15] or [213, §16.3]. Furthermore, Haar measure is unique up to positive multiplicative constants. If G is abelian then any left Haar measure on G is a constant multiple of any given right Haar measure on G. A locally compact group which is a countable union of compact subsets is a σ-finite measure space under left or right Haar measure. This is case for connected locally compact groups.

Example 1.2.1. The standard examples are provided by the spaces $\mathbf{R}^n$ and $\mathbf{Z}^n$ with the usual topology and the usual addition of n-tuples. Another example is the space $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ defined as follows:

$$\mathbf{T}^n = \underbrace{[0,1) \times \cdots \times [0,1)}_{n \text{ times}}$$

with the usual topology and group law:

$$(x_1,\dots,x_n)+(y_1,\dots,y_n)=((x_1+y_1) \bmod 1,\dots,(x_n+y_n) \bmod 1).$$

Example 1.2.2. Let $G=\mathbf{R}^*=\mathbf{R}\setminus\{0\}$ with group law the usual multiplication. It is easy to verify that the measure $\lambda = dx/|x|$ is invariant under multiplicative translations, that is,

$$\int_{-\infty}^{\infty} f(tx)\,\frac{dx}{|x|}=\int_{-\infty}^{\infty} f(x)\,\frac{dx}{|x|},$$

for all f in $L^1(G,\mu)$ and all $t\in\mathbf{R}^*$. Therefore, $dx/|x|$ is a Haar measure. [Taking $f=\chi_A$ gives $\lambda(tA)=\lambda(A)$.]

Example 1.2.3. Similarly, on the multiplicative group $G=\mathbf{R}^+$, a Haar measure is dx/x.

Example 1.2.4. Counting measure is a Haar measure on the group $\mathbf{Z}^n$ with the usual addition as group operation.

Example 1.2.5. The *Heisenberg group* $\mathbf{H}^n$ is the set $\mathbf{C}^n\times\mathbf{R}$ with the group operation

$$(z_1,\dots,z_n,t)(w_1,\dots,w_n,s)=\Big(z_1+w_1,\dots,z_n+w_n,t+s+2\operatorname{Im}\sum_{j=1}^{n} z_j\overline{w_j}\Big).$$

It can easily be seen that the identity element e of this group is $0\in\mathbf{C}^n\times\mathbf{R}$ and $(z_1,\dots,z_n,t)^{-1}=(-z_1,\dots,-z_n,-t)$. Topologically the Heisenberg group is identified with $\mathbf{C}^n\times\mathbf{R}$, and both left and right Haar measure on $\mathbf{H}^n$ is Lebesgue measure. The norm

$$|(z_1,\dots,z_n,t)|=\left[\Big(\sum_{j=1}^{n}|z_j|^2\Big)^2+t^2\right]^{\frac{1}{4}}$$

introduces balls $B_r(x)=\{y\in\mathbf{H}^n:\,|y^{-1}x|<r\}$ on the Heisenberg group that are quite different from Euclidean balls. For x close to the origin, the balls $B_r(x)$ are not far from being Euclidean, but for x far away from $e=0$ they look like slanted truncated cylinders. The Heisenberg group can be naturally identified as the boundary of the unit ball in $\mathbf{C}^n$ and plays an important role in quantum mechanics.

1.2.2 Convolution

Throughout the rest of this section, we fix a locally compact group G and a left invariant Haar measure λ on G. We assume that G is a countable union of compact subsets, hence the pair (G,λ) forms a σ-finite measure space. The spaces $L^p(G,\lambda)$ and $L^{p,\infty}(G,\lambda)$ are simply denoted by $L^p(G)$ and $L^{p,\infty}(G)$.

Left invariance of λ is equivalent to the fact that for all $t \in G$ and all nonnegative measurable functions f on G we have

$$\int_G f(tx)\,d\lambda(x) = \int_G f(x)\,d\lambda(x)\,. \tag{1.2.3}$$

Equation (1.2.3) is a restatement of (1.2.2) if f is a characteristic function. Obviously (1.2.3) also holds for $f \in L^1(G)$ by linearity and approximation.

We are now ready to define the operation of convolution.

Definition 1.2.6. Let f, g be in $L^1(G)$. Define the *convolution* $f * g$ by

$$(f*g)(x) = \int_G f(y)g(y^{-1}x)\,d\lambda(y)\,. \tag{1.2.4}$$

For instance, if $G = \mathbf{R}^n$ with the usual additive structure, then $y^{-1} = -y$ and the integral in (1.2.4) is written as

$$(f*g)(x) = \int_{\mathbf{R}^n} f(y)g(x-y)\,dy\,.$$

Remark 1.2.7. The right-hand side of (1.2.4) is defined a.e., since the following double integral converges absolutely:

$$\begin{aligned}
&\int_G\int_G |f(y)||g(y^{-1}x)|\,d\lambda(y)\,d\lambda(x)\\
&\qquad= \int_G\int_G |f(y)||g(y^{-1}x)|\,d\lambda(x)\,d\lambda(y)\\
&\qquad= \int_G |f(y)|\int_G |g(y^{-1}x)|\,d\lambda(x)\,d\lambda(y)\\
&\qquad= \int_G |f(y)|\int_G |g(x)|\,d\lambda(x)\,d\lambda(y) \qquad\qquad \text{by (1.2.2)}\\
&\qquad= \|f\|_{L^1(G)}\|g\|_{L^1(G)} < +\infty\,.
\end{aligned}$$

The change of variables $z = x^{-1}y$ yields that (1.2.4) is in fact equal to

$$(f*g)(x) = \int_G f(xz)g(z^{-1})\,d\lambda(z)\,, \tag{1.2.5}$$

where the substitution of $d\lambda(y)$ by $d\lambda(z)$ is justified by left invariance.

Example 1.2.8. On $\mathbf{R}$ let $f(x) = 1$ when $-1 \le x \le 1$ and zero otherwise. We see that $(f*f)(x)$ is equal to the length of the intersection of the intervals $[-1,1]$ and $[x-1,x+1]$. It follows that $(f*f)(x) = 2-|x|$ for $|x| \le 2$ and zero otherwise. Observe that $f*f$ is a smoother function than f. Similarly, we obtain that $f*f*f$ is a smoother function than $f*f$.

There is an analogous calculation when g is the characteristic function of the unit disk $B(0,1)$ in $\mathbf{R}^2$. A simple computation gives

$$(g*g)(x) = \left|B(0,1)\cap B(x,1)\right| = \int_{-\sqrt{1-\frac{1}{4}|x|^2}}^{+\sqrt{1-\frac{1}{4}|x|^2}} \left(2\sqrt{1-t^2}-|x|\right) dt$$
$$= 2\arcsin\left(\sqrt{1-\tfrac{1}{4}|x|^2}\right) - |x|\sqrt{1-\tfrac{1}{4}|x|^2}$$

when $x=(x_1,x_2)$ in $\mathbf{R}^2$ satisfies $|x|\le 2$, while $(g*g)(x)=0$ if $|x|\ge 2$.

A calculation similar to that in Remark 1.2.7 yields that

$$\big\|f*g\big\|_{L^1(G)} \le \big\|f\big\|_{L^1(G)}\big\|g\big\|_{L^1(G)}, \tag{1.2.6}$$

that is, the convolution of two integrable functions is also an integrable function with L^1 norm less than or equal to the product of the L^1 norms.

Proposition 1.2.9. *For all f, g, h in $L^1(G)$, the following properties are valid:*

(1) $f*(g*h)=(f*g)*h$ (associativity)
(2) $f*(g+h)=f*g+f*h$ and $(f+g)*h=f*h+g*h$ (distributivity)

Proof. The easy proofs are omitted. □

Proposition 1.2.9 implies that $L^1(G)$ is a (not necessarily commutative) Banach algebra under the convolution product.

1.2.3 Basic Convolution Inequalities

The most fundamental inequality involving convolutions is the following.

Theorem 1.2.10. ***(Minkowski's inequality)*** *Let $1\le p\le\infty$. For f in $L^p(G)$ and g in $L^1(G)$ we have that $g*f$ exists λ-a.e. and satisfies*

$$\big\|g*f\big\|_{L^p(G)} \le \big\|g\big\|_{L^1(G)}\big\|f\big\|_{L^p(G)}. \tag{1.2.7}$$

Proof. Estimate (1.2.7) follows directly from Exercise 1.1.6. Here we give a direct proof. We may assume that $1<p<\infty$, since the cases $p=1$ and $p=\infty$ are simple. We first show that the convolution $|g|*|f|$ exists λ-a.e. Indeed,

$$(|g|*|f|)(x) = \int_G |f(y^{-1}x)|\,|g(y)|\,d\lambda(y). \tag{1.2.8}$$

Apply Hölder's inequality in (1.2.8) with respect to the measure $|g(y)|\,d\lambda(y)$ to the functions $y\mapsto f(y^{-1}x)$ and 1 with exponents p and $p'=p/(p-1)$, respectively. We obtain

$$(|g|*|f|)(x) \le \left(\int_G |f(y^{-1}x)|^p|g(y)|\,d\lambda(y)\right)^{\frac{1}{p}}\left(\int_G |g(y)|\,d\lambda(y)\right)^{\frac{1}{p'}}. \tag{1.2.9}$$

Taking L^p norms of both sides of (1.2.9) we deduce

$$\begin{aligned}\big\||g|*|f|\big\|_{L^p} &\le \left(\|g\|_{L^1}^{p-1}\int_G\int_G |f(y^{-1}x)|^p|g(y)|\,d\lambda(y)\,d\lambda(x)\right)^{\frac{1}{p}}\\ &= \left(\|g\|_{L^1}^{p-1}\int_G\int_G |f(y^{-1}x)|^p\,d\lambda(x)|g(y)|\,d\lambda(y)\right)^{\frac{1}{p}}\\ &= \left(\|g\|_{L^1}^{p-1}\int_G\int_G |f(x)|^p\,d\lambda(x)|g(y)|\,d\lambda(y)\right)^{\frac{1}{p}} \qquad \text{by (1.2.3)}\\ &= \left(\|f\|_{L^p}^p\|g\|_{L^1}\|g\|_{L^1}^{p-1}\right)^{\frac{1}{p}}\\ &= \|f\|_{L^p}\|g\|_{L^1}<\infty,\end{aligned}$$

where the second equality follows by Fubini's theorem. This shows that $|g|*|f|$ is finite λ-a.e. and satisfies (1.2.7); then $g*f$ exists λ-a.e. and also satisfies (1.2.7), since $|g*f|\le|g|*|f|$. □

Remark 1.2.11. Theorem 1.2.10 may fail for nonabelian groups if $g*f$ is replaced by $f*g$ in (1.2.7). Note, however, that if for all $h\in L^1(G)$ we have

$$\|h\|_{L^1}=\|\widetilde{h}\|_{L^1}, \tag{1.2.10}$$

where $\widetilde{h}(x)=h(x^{-1})$, then (1.2.7) holds when the quantity $\|g*f\|_{L^p(G)}$ is replaced by $\|f*g\|_{L^p(G)}$. To see this, observe that if (1.2.10) holds, then we can use (1.2.5) to conclude that if f in $L^p(G)$ and g in $L^1(G)$, then

$$\|f*g\|_{L^p(G)}\le\|g\|_{L^1(G)}\|f\|_{L^p(G)}. \tag{1.2.11}$$

If the left Haar measure satisfies

$$\lambda(A)=\lambda(A^{-1}) \tag{1.2.12}$$

for all measurable $A\subseteq G$, then (1.2.10) holds and thus (1.2.11) is satisfied for all g in $L^1(G)$ and $f\in L^p(G)$. This is, for instance, the case for the Heisenberg group $\mathbf{H}^n$.

Minkowski's inequality (1.2.11) is only a special case of Young's inequality in which the function g can be in any space $L^r(G)$ for $1\le r\le\infty$.

Theorem 1.2.12. *(**Young's inequality**) Let $1\le p,q,r\le\infty$ satisfy*

$$\frac{1}{q}+1=\frac{1}{p}+\frac{1}{r}. \tag{1.2.13}$$

Then for all f in $L^p(G)$ and all g in $L^r(G)$ satisfying $\|g\|_{L^r(G)}=\|\widetilde{g}\|_{L^r(G)}$ we have $f*g$ exists λ-a.e. and satisfies

$$\|f*g\|_{L^q(G)}\le\|g\|_{L^r(G)}\|f\|_{L^p(G)}. \tag{1.2.14}$$

Proof. Young's inequality is proved in a way similar to Minkowski's inequality. We do a suitable splitting of the product $|f(y)||g(y^{-1}x)|$ and apply Hölder's inequality. Observe that when $r<\infty$, the hypotheses on the indices imply that

$$\frac{1}{r'}+\frac{1}{q}+\frac{1}{p'}=1\,,\qquad \frac{p}{q}+\frac{p}{r'}=1\,,\qquad \frac{r}{q}+\frac{r}{p'}=1\,.$$

Using Hölder's inequality with exponents r', q, and p', we obtain

$$\begin{aligned}
|(|f|*|g|)(x)| &\le \int_G |f(y)|\,|g(y^{-1}x)|\,d\lambda(y)\\
&= \int_G |f(y)|^{\frac{p}{r'}}\big(|f(y)|^{\frac{p}{q}}|g(y^{-1}x)|^{\frac{r}{q}}\big)|g(y^{-1}x)|^{\frac{r}{p'}}\,d\lambda(y)\\
&\le \|f\|_{L^p}^{\frac{p}{r'}}\Big(\int_G |f(y)|^p|g(y^{-1}x)|^r\,d\lambda(y)\Big)^{\frac{1}{q}}\Big(\int_G |g(y^{-1}x)|^r\,d\lambda(y)\Big)^{\frac{1}{p'}}\\
&= \|f\|_{L^p}^{\frac{p}{r'}}\Big(\int_G |f(y)|^p|g(y^{-1}x)|^r\,d\lambda(y)\Big)^{\frac{1}{q}}\Big(\int_G |\widetilde{g}(x^{-1}y)|^r\,d\lambda(y)\Big)^{\frac{1}{p'}}\\
&= \Big(\int_G |f(y)|^p|g(y^{-1}x)|^r\,d\lambda(y)\Big)^{\frac{1}{q}}\|f\|_{L^p}^{\frac{p}{r'}}\|\widetilde{g}\|_{L^r}^{\frac{r}{p'}}\,,
\end{aligned}$$

where we used left invariance. Now take L^q norms (in x) and apply Fubini's theorem to deduce that

$$\begin{aligned}
\big\||f|*|g|\big\|_{L^q} &\le \|f\|_{L^p}^{\frac{p}{r'}}\|\widetilde{g}\|_{L^r}^{\frac{r}{p'}}\Big(\int_G\int_G |f(y)|^p|g(y^{-1}x)|^r\,d\lambda(x)\,d\lambda(y)\Big)^{\frac{1}{q}}\\
&= \|f\|_{L^p}^{\frac{p}{r'}}\|\widetilde{g}\|_{L^r}^{\frac{r}{p'}}\|f\|_{L^p}^{\frac{p}{q}}\|g\|_{L^r}^{\frac{r}{q}}\\
&= \|g\|_{L^r}\|f\|_{L^p}<\infty,
\end{aligned}$$

using the hypothesis on g. This implies that $|f|*|g|$ is finite λ-a.e. and satisfies (1.2.14); then $f*g$ exists λ-a.e. and also satisfies (1.2.14).

Finally, note that if $r=\infty$, the assumptions on p and q imply that $p=1$ and $q=\infty$, in which case the required inequality trivially holds. □

We now give a version of Theorem 1.2.12 for weak L^p spaces. Theorem 1.2.13 is improved in Section 1.4.

Theorem 1.2.13. ***(Young's inequality for weak type spaces)*** *Let G be a locally compact group with left Haar measure λ that satisfies (1.2.12). Let $1\le p<\infty$ and $1<q,r<\infty$ satisfy*

$$\frac{1}{q}+1=\frac{1}{p}+\frac{1}{r}\,. \tag{1.2.15}$$

Then there exists a constant $C_{p,q,r}>0$ such that for all f in $L^p(G)$ and g in $L^{r,\infty}(G)$, the convolution $f*g$ exists λ-a.e. and satisfies

$$\|f*g\|_{L^{q,\infty}(G)}\le C_{p,q,r}\|g\|_{L^{r,\infty}(G)}\|f\|_{L^p(G)}\,. \tag{1.2.16}$$

Proof. As in the proofs of Theorems 1.2.10 and 1.2.12, we first obtain (1.2.16) for the convolution of the absolute values of the functions. This implies that $|f| * |g| < \infty$ λ-a.e., and thus $f * g$ exists λ-a.e. and satisfies $|f * g| \le |f| * |g|$. We may therefore assume that $f, g \ge 0$ λ-a.e. The proof is based on a suitable splitting of the function g. Let M be a positive real number to be chosen later. Define $g_1 = g\chi_{|g| \le M}$ and $g_2 = g\chi_{|g|>M}$. In view of Exercise 1.1.10 (a) we have

$$d_{g_1}(\alpha) = \begin{cases} 0 & \text{if } \alpha \ge M, \\ d_g(\alpha) - d_g(M) & \text{if } \alpha < M, \end{cases} \tag{1.2.17}$$

$$d_{g_2}(\alpha) = \begin{cases} d_g(\alpha) & \text{if } \alpha > M, \\ d_g(M) & \text{if } \alpha \le M. \end{cases} \tag{1.2.18}$$

Proposition 1.1.3 gives for all $\beta > 0$

$$d_{f*g}(\beta) \le d_{f*g_1}(\beta/2) + d_{f*g_2}(\beta/2), \tag{1.2.19}$$

and thus it suffices to estimate the distribution functions of $f * g_1$ and $f * g_2$. Since g_1 is the "small" part of g, it is in L^s for any $s > r$. In fact, we have

$$\begin{aligned} \int_G g_1(x)^s \, d\lambda(x) &= s\int_0^\infty \alpha^{s-1} d_{g_1}(\alpha)\, d\alpha \\ &= s\int_0^M \alpha^{s-1} (d_g(\alpha) - d_g(M))\, d\alpha \\ &\le s\int_0^M \alpha^{s-1-r} \|g\|_{L^{r,\infty}}^r \, d\alpha - s\int_0^M \alpha^{s-1} d_g(M)\, d\alpha \\ &= \frac{s}{s-r} M^{s-r} \|g\|_{L^{r,\infty}}^r - M^s d_g(M), \end{aligned} \tag{1.2.20}$$

when $s < \infty$.

Similarly, since g_2 is the "large" part of g, it is in L^t for any $t < r$, and

$$\begin{aligned} \int_G g_2(x)^t \, d\lambda(x) &= t\int_0^\infty \alpha^{t-1} d_{g_2}(\alpha)\, d\alpha \\ &= t\int_0^M \alpha^{t-1} d_g(M)\, d\alpha + t\int_M^\infty \alpha^{t-1} d_g(\alpha)\, d\alpha \\ &\le M^t d_g(M) + t\int_M^\infty \alpha^{t-1-r} \|g\|_{L^{r,\infty}}^r \, d\alpha \\ &\le M^{t-r} \|g\|_{L^{r,\infty}}^r + \frac{t}{r-t} M^{t-r} \|g\|_{L^{r,\infty}}^r \\ &= \frac{r}{r-t} M^{t-r} \|g\|_{L^{r,\infty}}^r . \end{aligned} \tag{1.2.21}$$

Since $1/r = 1/p' + 1/q$, it follows that $1 < r < p'$. Select $t = 1$ and $s = p'$. Hölder's inequality and (1.2.20) give when $p' < \infty$

$$|(f * g_1)(x)| \le \|f\|_{L^p} \|g_1\|_{L^{p'}} \le \|f\|_{L^p} \left(\frac{p'}{p'-r} M^{p'-r} \|g\|_{L^{r,\infty}}^r \right)^{\frac{1}{p'}} \tag{1.2.22}$$

and

$$|(f*g_1)(x)| \le \|f\|_{L^p} M \tag{1.2.23}$$

when $p' = \infty$. If $p' < \infty$ choose an M such that the right-hand side of (1.2.22) is equal to $\beta/2$. If $p' = \infty$ choose M such that the right-hand side of (1.2.23) is also equal to $\beta/2$. That is, choose

$$M = (\beta^{p'} 2^{-p'} r q^{-1} \|f\|_{L^p}^{-p'} \|g\|_{L^{r,\infty}}^{-r})^{1/(p'-r)}$$

if $p' < \infty$ and $M = \beta/(2\|f\|_{L^1})$ if $p' = \infty$. For these choices of M we have that

$$d_{f*g_1}(\beta/2) = 0.$$

Next by Theorem 1.2.10 and (1.2.21) with $t = 1$ we obtain

$$\|f*g_2\|_{L^p} \le \|f\|_{L^p} \|g_2\|_{L^1} \le \|f\|_{L^p} \frac{r}{r-1} M^{1-r} \|g\|_{L^{r,\infty}}^r . \tag{1.2.24}$$

For the value of M chosen, using (1.2.24) and Chebyshev's inequality, we obtain

$$\begin{aligned} d_{f*g}(\beta) &\le d_{f*g_2}(\beta/2) \\ &\le (2\|f*g_2\|_{L^p} \beta^{-1})^p \\ &\le (2r\|f\|_{L^p} M^{1-r} \|g\|_{L^{r,\infty}}^r (r-1)^{-1} \beta^{-1})^p \\ &= C_{p,q,r}^q \beta^{-q} \|f\|_{L^p}^q \|g\|_{L^{r,\infty}}^q , \end{aligned} \tag{1.2.25}$$

which is the required inequality. This proof gives that the constant $C_{p,q,r}$ blows up like $(r-1)^{-p/q}$ as $r \to 1$. □

Example 1.2.14. Theorem 1.2.13 may fail at some endpoints:

(1) $r = 1$ and $1 \le p = q \le \infty$. On **R** take $g(x) = 1/|x|$ and $f = \chi_{[0,1]}$. Clearly, g is in $L^{1,\infty}$ and f in L^p for all $1 \le p \le \infty$, but the convolution of f and g is identically equal to infinity on the interval $[0,1]$. Therefore, (1.2.16) fails in this case.
(2) $q = \infty$ and $1 < r = p' < \infty$. On **R** let $f(x) = (|x|^{1/p} \log|x|)^{-1}$ for $|x| \ge 2$ and zero otherwise, and also let $g(x) = |x|^{-1/r}$. We see that $(f*g)(x) = \infty$ for $|x| \le 1$. Thus (1.2.16) fails in this case also.
(3) $r = q = \infty$ and $p = 1$. Then inequality (1.2.16) trivially holds.

1.2.4 Approximate Identities

We now introduce the notion of approximate identities. The Banach algebra $L^1(G)$ may not have a unit element, that is, an element f_0 such that

$$f_0 * f = f = f * f_0 \tag{1.2.26}$$

for all $f \in L^1(G)$. In particular, this is the case when $G = \mathbf{R}$; in fact, the only f_0 that satisfies (1.2.26) for all $f \in L^1(\mathbf{R})$ is not a function but the Dirac delta distribution, introduced in Chapter 2. It is reasonable therefore to introduce the notion of approximate unit or identity, a family of functions k_ε with the property $k_\varepsilon * f \to f$ in L^1 as $\varepsilon \to 0$.

Definition 1.2.15. An *approximate identity* (as $\varepsilon \to 0$) is a family of $L^1(G)$ functions k_ε with the following three properties:

(i) There exists a constant $c > 0$ such that $\|k_\varepsilon\|_{L^1(G)} \le c$ for all $\varepsilon > 0$.

(ii) $\int_G k_\varepsilon(x)\, d\lambda(x) = 1$ for all $\varepsilon > 0$.

(iii) For any neighborhood V of the identity element e of the group G we have $\int_{V^c} |k_\varepsilon(x)|\, d\lambda(x) \to 0$ as $\varepsilon \to 0$.

The construction of approximate identities on general locally compact groups G is beyond the scope of this book and is omitted; see [152] for details. In this book we are interested only in groups with Euclidean structure, where approximate identities exist in abundance.

Sometimes we think of approximate identities as sequences $\{k_n\}_n$. In this case property (iii) holds as $n \to \infty$. It is best to visualize approximate identities as sequences of positive functions k_n that spike near 0 in such a way that the signed area under the graph of each function remains constant (equal to one) but the support shrinks to zero. See Figure 1.2.

Example 1.2.16. On $\mathbf{R}$ let $P(x) = (\pi(x^2+1))^{-1}$ and $P_\varepsilon(x) = \varepsilon^{-1}P(\varepsilon^{-1}x)$ for $\varepsilon > 0$. Since P_ε and P have the same L^1 norm and

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\, dx = \lim_{x\to+\infty} \big[\arctan(x) - \arctan(-x)\big] = (\pi/2) - (-\pi/2) = \pi\,,$$

property (ii) is satisfied. Property (iii) follows from the fact that

$$\frac{1}{\pi}\int_{|x|\ge\delta} \frac{1}{\varepsilon}\,\frac{1}{(x/\varepsilon)^2+1}\, dx = 1 - \frac{2}{\pi}\arctan(\delta/\varepsilon) \to 0 \qquad \text{as } \varepsilon \to 0,$$

for all $\delta > 0$. The function P_ε is called the *Poisson kernel*.

The Poisson kernel may be replaced by any integrable function of integral 1 as the following example indicates.

Example 1.2.17. On $\mathbf{R}^n$ let $k(x)$ be an integrable function with integral one. Let $k_\varepsilon(x) = \varepsilon^{-n}k(\varepsilon^{-1}x)$. It is straightforward to see that $k_\varepsilon(x)$ is an approximate identity. Property (iii) follows from the fact that

$$\int_{|x|\ge\delta/\varepsilon} |k(x)|\, dx \to 0$$

as $\varepsilon \to 0$ for δ fixed.

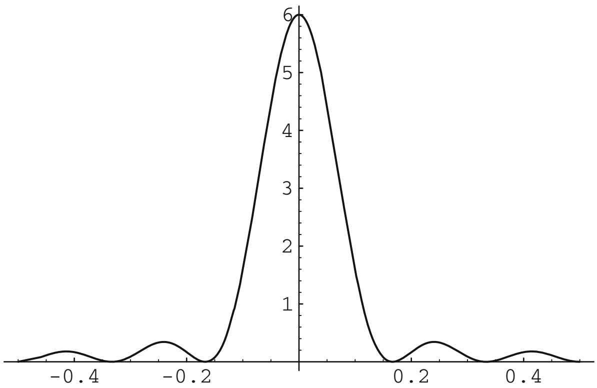

Fig. 1.2 The Fejér kernel F_5 plotted on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

Example 1.2.18. On the circle group $\mathbf{T}^1$ let

$$F_N(t) = \sum_{j=-N}^{N} \left(1 - \frac{|j|}{N+1}\right) e^{2\pi i j t} = \frac{1}{N+1} \left(\frac{\sin(\pi(N+1)t)}{\sin(\pi t)}\right)^2 . \tag{1.2.27}$$

To check the previous equality we use that

$$\sin^2(x) = (2 - e^{2ix} - e^{-2ix})/4 ,$$

and we carry out the calculation. F_N is called the *Fejér kernel*. See Figure 1.2. To see that the sequence $\{F_N\}_N$ is an approximate identity, we check conditions (i), (ii), and (iii) in Definition 1.2.15. Property (iii) follows from the expression giving F_N in terms of sines, while property (i) follows from the expression giving F_N in terms of exponentials. Property (ii) is identical to property (i), since F_N is nonnegative.

Next comes the basic theorem concerning approximate identities.

Theorem 1.2.19. *Let k_ε be an approximate identity on a locally compact group G with left Haar measure λ.*

(1) If f lies in $L^p(G)$ for $1 \le p < \infty$, then $\|k_\varepsilon * f - f\|_{L^p(G)} \to 0$ as $\varepsilon \to 0$.
(2) Let f be a function in $L^\infty(G)$ that is uniformly continuous on a subset K of G, in the sense that for all $\delta > 0$ there is a neighborhood V of the identity element such that for all $x \in K$ and $y \in V$ we have $|f(y^{-1}x) - f(x)| < \delta$. Then we have that $\|k_\varepsilon * f - f\|_{L^\infty(K)} \to 0$ as $\varepsilon \to 0$. In particular, if f is bounded and continuous at a point $x_0 \in G$, then $(k_\varepsilon * f)(x_0) \to f(x_0)$ as $\varepsilon \to 0$.

Proof. We start with the case $1 \le p < \infty$. We recall that continuous functions with compact support are dense in L^p of locally compact Hausdorff spaces equipped with measures arising from nonnegative linear functionals; see [152, Theorem 12.10]. For a continuous function g supported in a compact set L we have we have $|g(h^{-1}x) - g(x)|^p \le (2\|g\|_{L^\infty})^p \chi_{W^{-1}L}$ for h in a relatively compact neighborhood

W of the identity element e. By the Lebesgue dominated convergence theorem we obtain

$$\int_G |g(h^{-1}x) - g(x)|^p \, d\lambda(x) \to 0 \tag{1.2.28}$$

as $h \to e$. Now approximate a given f in $L^p(G)$ by a continuous function with compact support g to deduce that

$$\int_G |f(h^{-1}x) - f(x)|^p \, d\lambda(x) \to 0 \qquad \text{as} \qquad h \to e\,. \tag{1.2.29}$$

Because of (1.2.29), given a $\delta > 0$ there exists a neighborhood V of e such that

$$h \in V \implies \int_G |f(h^{-1}x) - f(x)|^p \, d\lambda(x) < \left(\frac{\delta}{2c}\right)^p, \tag{1.2.30}$$

where c is the constant that appears in Definition 1.2.15 (i). Since k_ε has integral one for all $\varepsilon > 0$, we have

$$\begin{aligned} (k_\varepsilon * f)(x) - f(x) &= (k_\varepsilon * f)(x) - f(x) \int_G k_\varepsilon(y)\, d\lambda(y) \\ &= \int_G (f(y^{-1}x) - f(x)) k_\varepsilon(y)\, d\lambda(y) \\ &= \int_V (f(y^{-1}x) - f(x)) k_\varepsilon(y)\, d\lambda(y) \\ &\qquad + \int_{V^c} (f(y^{-1}x) - f(x)) k_\varepsilon(y)\, d\lambda(y)\,. \end{aligned} \tag{1.2.31}$$

Now take L^p norms in x in (1.2.31). In view of (1.2.30),

$$\begin{aligned} &\left\| \int_V (f(y^{-1}x) - f(x)) k_\varepsilon(y)\, d\lambda(y) \right\|_{L^p(G, d\lambda(x))} \\ &\qquad \le \int_V \left\| f(y^{-1}x) - f(x) \right\|_{L^p(G, d\lambda(x))} |k_\varepsilon(y)|\, d\lambda(y) \\ &\qquad \le \int_V \frac{\delta}{2c} |k_\varepsilon(y)|\, d\lambda(y) < \frac{\delta}{2}, \end{aligned} \tag{1.2.32}$$

while

$$\begin{aligned} &\left\| \int_{V^c} (f(y^{-1}x) - f(x)) k_\varepsilon(y)\, d\lambda(y) \right\|_{L^p(G, d\lambda(x))} \\ &\qquad \le \int_{V^c} 2\|f\|_{L^p(G)} |k_\varepsilon(y)|\, d\lambda(y) < \frac{\delta}{2}, \end{aligned} \tag{1.2.33}$$

provided we have that

$$\int_{V^c} |k_\varepsilon(x)|\, d\lambda(x) < \frac{\delta}{4(\|f\|_{L^p} + 1)}\,. \tag{1.2.34}$$

Choose $\varepsilon_0 > 0$ such that (1.2.34) is valid for $\varepsilon < \varepsilon_0$ by property (iii). Now (1.2.32) and (1.2.33) imply the required conclusion.

The case $p = \infty$ follows similarly. Let f be a bounded function on G that is uniformly continuous on K. Given $\delta > 0$, there is a neighborhood V of e such that, whenever $y \in V$ and $x \in K$ we have

$$|f(y^{-1}x) - f(x)| < \frac{\delta}{2c}, \tag{1.2.35}$$

where c is as in Definition 1.2.15 (i). By property (iii) in Definition 1.2.15, there is an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ we have

$$\int_{V^c} |k_\varepsilon(y)|\, d\lambda(y) < \frac{\delta}{4\big(\|f\|_{L^\infty(G)} + 1\big)}. \tag{1.2.36}$$

Using (1.2.35) and (1.2.36), we deduce that

$$\begin{aligned}
&\sup_{x\in K} |(k_\varepsilon * f)(x) - f(x)| \\
&\le \int_V |k_\varepsilon(y)| \sup_{x\in K} |f(y^{-1}x) - f(x)|\, d\lambda(y) + \int_{V^c} |k_\varepsilon(y)| \sup_{x\in K} |f(y^{-1}x) - f(x)|\, d\lambda(y) \\
&\le c\,\frac{\delta}{2c} + \frac{\delta}{4\big(\|f\|_{L^\infty(G)} + 1\big)}\, 2\|f\|_{L^\infty(G)} \le \delta.
\end{aligned}$$

This shows that $k_\varepsilon * f$ converge uniformly to f on K as $\varepsilon \to 0$. In particular, if $K = \{x_0\}$ and f is bounded and continuous at x_0, we have $(k_\varepsilon * f)(x_0) \to f(x_0)$. □

Remark 1.2.20. Observe that if Haar measure satisfies (1.2.12), then the conclusion of Theorem 1.2.19 also holds for $f * k_\varepsilon$.

A simple modification in the proof of Theorem 1.2.19 yields the following variant, which presents a significant difference only when $a = 0$.

Theorem 1.2.21. *Let k_ε be a family of functions on a locally compact group G that satisfies properties (i) and (iii) of Definition 1.2.15 and also*

$$\int_G k_\varepsilon(x)\, d\lambda(x) = a$$

for some fixed $a \in \mathbf{C}$ and for all $\varepsilon > 0$. Let $f \in L^p(G)$ for some $1 \le p \le \infty$.

(a) If $1 \le p < \infty$, then $\|k_\varepsilon * f - af\|_{L^p(G)} \to 0$ as $\varepsilon \to 0$.
(b) If $p = \infty$ and f is uniformly continuous on a subset K of G, in the sense that for any $\delta > 0$ there is a neighborhood V of the identity element of G such that $\sup_{x\in G} \sup_{y\in V} |f(y^{-1}x) - f(x)| \le \delta$, then we have that $\|k_\varepsilon * f - af\|_{L^\infty(K)} \to 0$ as $\varepsilon \to 0$.

Exercises

1.2.1. Let G be a locally compact group and let f,g in $L^1(G)$ be supported in the subsets A and B of G, respectively. Prove that $f*g$ is supported in the algebraic product set AB.

1.2.2. For a function f on a locally compact group G and $t\in G$, let ${}^tf(x)=f(tx)$ and $f^t(x)=f(xt)$. Show that

$$ {}^tf*g={}^t(f*g) \qquad \text{and} \qquad f*g^t=(f*g)^t $$

whenever $f,g\in L^1(G)$, equipped with left Haar measure.

1.2.3. Let G be a locally compact group with left Haar measure. Let $f\in L^p(G)$ and $\tilde{g}\in L^{p'}(G)$, where $1<p<\infty$; recall that $\widetilde{g}(x)=g(x^{-1})$. For $t,x\in G$, let ${}^tg(x)=g(tx)$. Show that for any $\varepsilon>0$ there exists a relatively compact symmetric neighborhood of the origin U such that $u\in U$ implies $\|{}^u\widetilde{g}-\widetilde{g}\|_{L^{p'}(G)}<\varepsilon$ and therefore

$$ |(f*g)(v)-(f*g)(w)|<\|f\|_{L^p}\,\varepsilon $$

whenever $v^{-1}w\in U$.

1.2.4. (a) Prove that compactly supported functions are dense in $L^p(\mathbf{R}^n)$ for all $0<p<\infty$.
(b) Show that smooth functions with compact support are dense in $L^p(\mathbf{R}^n)$ for all $1\le p<\infty$.
[*Hint:* Part (b): Use Theorem 1.2.19 with $k_\varepsilon(x)=\varepsilon^{-n}k(\varepsilon^{-1}x)$ and k smooth and compactly supported function.]

1.2.5. Show that a Haar measure λ for the multiplicative group of all positive real numbers is

$$ \lambda(A)=\int_0^\infty \chi_A(t)\,\frac{dt}{t}\,. $$

1.2.6. Let $G=\mathbf{R}^2\setminus\{(0,y):\ y\in\mathbf{R}\}$ with group operation $(x,y)(z,w)=(xz,xw+y)$. [Think of G as the group of all 2×2 matrices with bottom row $(0,1)$ and nonzero top left entry.] Show that a left Haar measure on G is

$$ \lambda(A)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\chi_A(x,y)\,\frac{dx\,dy}{x^2}\,, $$

while a right Haar measure on G is

$$ \rho(A)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\chi_A(x,y)\,\frac{dx\,dy}{|x|}\,. $$

1.2.7. ([144], [145]) Use Theorem 1.2.10 to prove that

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)^p dx\right)^{\frac{1}{p}} \le \frac{p}{p-1}\left\|f\right\|_{L^p(0,\infty)},$$

$$\left(\int_0^\infty \left(\int_x^\infty |f(t)|\,dt\right)^p dx\right)^{\frac{1}{p}} \le p\left(\int_0^\infty |f(t)|^p t^p\,dt\right)^{\frac{1}{p}},$$

when $1 < p < \infty$.

[*Hint:* On the multiplicative group $(\mathbf{R}^+, \frac{dt}{t})$ consider the convolution of the function $|f(x)|x^{\frac{1}{p}}$ with the function $x^{-\frac{1}{p'}}\chi_{[1,\infty)}$ and the convolution of the function $|f(x)|x^{1+\frac{1}{p}}$ with $x^{\frac{1}{p}}\chi_{(0,1]}$.]

1.2.8. (*G. H. Hardy*) Let $0 < b < \infty$ and $1 \le p < \infty$. Prove that

$$\left(\int_0^\infty \left(\int_0^x |f(t)|\,dt\right)^p x^{-b-1}\,dx\right)^{\frac{1}{p}} \le \frac{p}{b}\left(\int_0^\infty |f(t)|^p t^{p-b-1}\,dt\right)^{\frac{1}{p}},$$

$$\left(\int_0^\infty \left(\int_x^\infty |f(t)|\,dt\right)^p x^{b-1}\,dx\right)^{\frac{1}{p}} \le \frac{p}{b}\left(\int_0^\infty |f(t)|^p t^{p+b-1}\,dt\right)^{\frac{1}{p}}.$$

[*Hint:* On the multiplicative group $(\mathbf{R}^+, \frac{dt}{t})$ consider the convolution of the function $|f(x)|x^{1-\frac{b}{p}}$ with $x^{-\frac{b}{p}}\chi_{[1,\infty)}$ and of the function $|f(x)|x^{1+\frac{b}{p}}$ with $x^{\frac{b}{p}}\chi_{(0,1]}$.]

1.2.9. On $\mathbf{R}^n$ let $T(f) = f * K$, where K is a positive L^1 function and f is in L^p, $1 \le p \le \infty$. Prove that the operator norm of $T : L^p \to L^p$ is equal to $\left\|K\right\|_{L^1}$.

[*Hint:* Clearly, $\|T\|_{L^p\to L^p} \le \|K\|_{L^1}$. Conversely, fix $0 < \varepsilon < 1$ and let N be a positive integer. Let $\chi_N = \chi_{B(0,N)}$ and for any $R > 0$ let $K_R = K\chi_{B(0,R)}$, where $B(x,R)$ is the ball of radius R centered at x. Observe that for $|x| \le (1-\varepsilon)N$, we have $B(0,N\varepsilon) \subsetneqq B(x,N)$; thus $\int_{\mathbf{R}^n} \chi_N(x-y)K_{N\varepsilon}(y)\,dy = \int_{\mathbf{R}^n} K_{N\varepsilon}(y)\,dy = \left\|K_{N\varepsilon}\right\|_{L^1}$. Then

$$\frac{\left\|K * \chi_N\right\|_{L^p}^p}{\left\|\chi_N\right\|_{L^p}^p} \ge \frac{\left\|K_{N\varepsilon} * \chi_N\right\|_{L^p(B(0,(1-\varepsilon)N)}^p}{\left\|\chi_N\right\|_{L^p}^p} \ge \left\|K_{N\varepsilon}\right\|_{L^1}^p (1-\varepsilon)^n.$$

Let $N \to \infty$ first and then $\varepsilon \to 0$.]

1.2.10. On the multiplicative group $(\mathbf{R}^+, \frac{dt}{t})$ let $T(f) = f * K$, where K is a positive L^1 function and f is in L^p, $1 \le p \le \infty$. Prove that the operator norm of $T : L^p \to L^p$ is equal to the L^1 norm of K. Deduce that the constants $p/(p-1)$ and p/b are sharp in Exercises 1.2.7 and 1.2.8.

[*Hint:* Adapt the idea of Exercise 1.2.9 to this setting.]

1.2.11. Let $Q_k(t) = c_k(1-t^2)^k$ for $t \in [-1,1]$ and zero elsewhere, where c_k is chosen such that $\int_{-1}^1 Q_k(t)\,dt = 1$ for all $k = 1,2,\dots$.

(a) Show that $c_k < \sqrt{k}$.

(b) Use part (a) to show that $\{Q_k\}_k$ is an approximate identity on $\mathbf{R}$ as $k \to \infty$.
(c) Given a continuous function f on $\mathbf{R}$ that vanishes outside the interval $[-1,1]$, show that $f * Q_k$ converges to f uniformly on $[-1,1]$ as $k \to \infty$.
(d) (*Weierstrass*) Prove that every continuous function on $[-1,1]$ can be approximated uniformly by polynomials.
$\big[$*Hint:* Part (a): Estimate the integral $\int_{|t|\le k^{-1/2}} Q_k(t)\,dt$ from below using the inequality $(1-t^2)^k \ge 1-kt^2$ for $|t| \le 1$. Part (d): Consider the function $g(t) = f(t) - f(-1) - \frac{t+1}{2}(f(1)-f(-1))$.$\big]$

1.2.12. Show that the *Laplace transform* $L(f)(x) = \int_0^\infty f(t)e^{-xt}dt$ maps $L^2(0,\infty)$ to itself with norm at most $\sqrt{\pi}$.
$\big[$*Hint:* Consider convolution with the kernel $\sqrt{t}\,e^{-t}$ on the group $L^2((0,\infty), \frac{dt}{t})$.$\big]$

1.2.13. ([62]) Let $F \ge 0$, $G \ge 0$ be measurable functions on the sphere $\mathbf{S}^{n-1}$ and let $K \ge 0$ be a measurable function on $[-1,1]$. Prove that

$$\int_{\mathbf{S}^{n-1}}\int_{\mathbf{S}^{n-1}} F(\theta)G(\varphi)K(\theta\cdot\varphi)\,d\varphi\,d\theta \le C\,\|F\|_{L^p(\mathbf{S}^{n-1})}\|G\|_{L^{p'}(\mathbf{S}^{n-1})},$$

where $1 \le p \le \infty$, $\theta\cdot\varphi = \sum_{j=1}^n \theta_j\varphi_j$ and $C = \int_{\mathbf{S}^{n-1}} K(\theta\cdot\varphi)\,d\varphi$, which is independent of θ. Moreover, show that C is the best possible constant in the preceding inequality. Using duality, compute the norm of the linear operator

$$F(\theta) \mapsto \int_{\mathbf{S}^{n-1}} F(\theta)K(\theta\cdot\varphi)\,d\varphi$$

from $L^p(\mathbf{S}^{n-1})$ to itself.
$\big[$*Hint:* Observe that $\int_{\mathbf{S}^{n-1}}\int_{\mathbf{S}^{n-1}} F(\theta)G(\varphi)K(\theta\cdot\varphi)\,d\varphi\,d\theta$ is bounded by the quantity

$$\left\{\int_{\mathbf{S}^{n-1}}\left[\int_{\mathbf{S}^{n-1}} F(\theta)K(\theta\cdot\varphi)\,d\theta\right]^p d\varphi\right\}^{\frac{1}{p}} \|G\|_{L^{p'}(\mathbf{S}^{n-1})}.$$

Apply Hölder's inequality to the functions F and 1 with respect to the measure $K(\theta\cdot\varphi)\,d\theta$ to deduce that $\int_{\mathbf{S}^{n-1}} F(\theta)K(\theta\cdot\varphi)\,d\theta$ is controlled by

$$\left(\int_{\mathbf{S}^{n-1}} F(\theta)^p K(\theta\cdot\varphi)\,d\theta\right)^{1/p}\left(\int_{\mathbf{S}^{n-1}} K(\theta\cdot\varphi)\,d\theta\right)^{1/p'}.$$

Use Fubini's theorem to bound the latter by

$$\|F\|_{L^p(\mathbf{S}^{n-1})}\|G\|_{L^{p'}(\mathbf{S}^{n-1})}\int_{\mathbf{S}^{n-1}} K(\theta\cdot\varphi)\,d\varphi.$$

Note that equality is attained if and only if both F and G are constants.$\big]$

1.3 Interpolation

The theory of interpolation of operators is vast and extensive. In this section we are mainly concerned with a couple of basic interpolation results that appear in a variety of applications and constitute the foundation of the field. These results are the *Marcinkiewicz interpolation theorem* and the *Riesz–Thorin interpolation theorem*. These theorems are traditionally proved using real and complex variables techniques, respectively. A byproduct of the Riesz–Thorin interpolation theorem, *Stein's theorem on interpolation of analytic families of operators*, has also proved to be an important and useful tool in many applications and is presented at the end of the section.

We begin by setting up the background required to formulate the results of this section. Let (X,μ) and (Y,ν) be two measure spaces. Suppose we are given a linear operator T, initially defined on the set of simple functions on X, such that for all f simple on X, $T(f)$ is a ν-measurable function on Y. Let $0<p<\infty$ and $0<q<\infty$. If there exists a constant $C_{p,q}>0$ such that for all simple functions f on X we have

$$\big\|T(f)\big\|_{L^q(Y,\nu)} \le C_{p,q}\big\|f\big\|_{L^p(X,\mu)}, \tag{1.3.1}$$

then by density, T admits a unique bounded extension from $L^p(X,\mu)$ to $L^q(Y,\nu)$. This extension is also denoted by T. Operators that map L^p to L^q are called of *strong type* (p,q) and operators that map L^p to $L^{q,\infty}$ are called *weak type* (p,q).

1.3.1 Real Method: The Marcinkiewicz Interpolation Theorem

Definition 1.3.1. Let T be an operator defined on a linear space of complex-valued measurable functions on a measure space (X,μ) and taking values in the set of all complex-valued finite almost everywhere measurable functions on a measure space (Y,ν). Then T is called *linear* if for all f, g in the domain of T and all $\lambda\in\mathbf{C}$ we have

$$T(f+g)=T(f)+T(g) \qquad \text{and} \qquad T(\lambda f)=\lambda T(f). \tag{1.3.2}$$

T is called *sublinear* if for all f, g in the domain of T and all $\lambda\in\mathbf{C}$ we have

$$|T(f+g)|\le|T(f)|+|T(g)| \qquad \text{and} \qquad |T(\lambda f)|=|\lambda||T(f)|. \tag{1.3.3}$$

T is called *quasi-linear* if for all f, g in the domain of T and all $\lambda\in\mathbf{C}$ we have

$$|T(f+g)|\le K(|T(f)|+|T(g)|) \qquad \text{and} \qquad |T(\lambda f)|=|\lambda||T(f)| \tag{1.3.4}$$

for some constant $K>0$. Sublinearity is a special case of quasi-linearity.

For instance, T_1 and T_2 are linear operators, then $(|T_1|^p+|T_2|^p)^{1/p}$ is sublinear if $p\ge 1$ and quasi-linear if $0<p<1$.

Theorem 1.3.2. *Let (X,μ) be a σ-finite measure space, let (Y,ν) be another measure space, and let $0 < p_0 < p_1 \le \infty$. Let T be a sublinear operator defined on $L^{p_0}(X)+L^{p_1}(X) = \{f_0+f_1 :\ f_j \in L^{p_j}(X_j), j=0,1\}$ and taking values in the space of measurable functions on Y. Assume that there exist $A_0, A_1 < \infty$ such that*

$$\big\|T(f)\big\|_{L^{p_0,\infty}(Y)} \le A_0\big\|f\big\|_{L^{p_0}(X)} \qquad \textit{for all } f \in L^{p_0}(X), \tag{1.3.5}$$

$$\big\|T(f)\big\|_{L^{p_1,\infty}(Y)} \le A_1\big\|f\big\|_{L^{p_1}(X)} \qquad \textit{for all } f \in L^{p_1}(X). \tag{1.3.6}$$

Then for all $p_0 < p < p_1$ and for all f in $L^p(X)$ we have the estimate

$$\big\|T(f)\big\|_{L^p(Y)} \le A\big\|f\big\|_{L^p(X)}, \tag{1.3.7}$$

where

$$A = 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{\frac{1}{p}} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}. \tag{1.3.8}$$

Proof. Assume first that $p_1 < \infty$. Fix f a function in $L^p(X)$ and $\alpha > 0$. We split $f = f_0^\alpha + f_1^\alpha$, where f_0^α is in L^{p_0} and f_1^α is in L^{p_1}. The splitting is obtained by cutting $|f|$ at height $\delta\alpha$ for some $\delta > 0$ to be determined later. Set

$$f_0^\alpha(x) = \begin{cases} f(x) & \text{for } |f(x)| > \delta\alpha, \\ 0 & \text{for } |f(x)| \le \delta\alpha, \end{cases}$$

$$f_1^\alpha(x) = \begin{cases} f(x) & \text{for } |f(x)| \le \delta\alpha, \\ 0 & \text{for } |f(x)| > \delta\alpha. \end{cases}$$

It can be checked easily that f_0^α (the unbounded part of f) is an L^{p_0} function and that f_1^α (the bounded part of f) is an L^{p_1} function. Indeed, since $p_0 < p$, we have

$$\big\|f_0^\alpha\big\|_{L^{p_0}}^{p_0} = \int_{|f|>\delta\alpha} |f(x)|^p |f(x)|^{p_0-p}\, d\mu(x) \le (\delta\alpha)^{p_0-p}\big\|f\big\|_{L^p}^p$$

and similarly, since $p < p_1$,

$$\big\|f_1^\alpha\big\|_{L^{p_1}}^{p_1} \le (\delta\alpha)^{p_1-p}\big\|f\big\|_{L^p}^p.$$

In view of the subadditivity property of T contained in (1.3.3) we obtain that

$$|T(f)| \le |T(f_0^\alpha)| + |T(f_1^\alpha)|,$$

which implies

$$\{y\in Y\colon |T(f)(y)| > \alpha\} \subseteqq \{y\in Y\colon |T(f_0^\alpha)(y)| > \alpha/2\} \cup \{y\in Y\colon |T(f_1^\alpha)(y)| > \alpha/2\},$$

and therefore

$$d_{T(f)}(\alpha) \le d_{T(f_0^\alpha)}(\alpha/2) + d_{T(f_1^\alpha)}(\alpha/2). \tag{1.3.9}$$

Hypotheses (1.3.5) and (1.3.6) together with (1.3.9) now give

$$d_{T(f)}(\alpha) \le \frac{A_0^{p_0}}{(\alpha/2)^{p_0}} \int_{|f|>\delta\alpha} |f(x)|^{p_0}\, d\mu(x) + \frac{A_1^{p_1}}{(\alpha/2)^{p_1}} \int_{|f|\le\delta\alpha} |f(x)|^{p_1}\, d\mu(x).$$

In view of the last estimate and Proposition 1.1.4, we obtain that

$$\begin{aligned}
\big\|T(f)\big\|_{L^p}^p &\le p(2A_0)^{p_0} \int_0^\infty \alpha^{p-1}\alpha^{-p_0} \int_{|f|>\delta\alpha} |f(x)|^{p_0}\, d\mu(x)\, d\alpha \\
&\qquad + p(2A_1)^{p_1} \int_0^\infty \alpha^{p-1}\alpha^{-p_1} \int_{|f|\le\delta\alpha} |f(x)|^{p_1}\, d\mu(x)\, d\alpha \\
&= p(2A_0)^{p_0} \int_X |f(x)|^{p_0} \int_0^{\frac{1}{\delta}|f(x)|} \alpha^{p-1-p_0}\, d\alpha\, d\mu(x) \\
&\qquad + p(2A_1)^{p_1} \int_X |f(x)|^{p_1} \int_{\frac{1}{\delta}|f(x)|}^\infty \alpha^{p-1-p_1}\, d\alpha\, d\mu(x) \\
&= \frac{p(2A_0)^{p_0}}{p-p_0} \frac{1}{\delta^{p-p_0}} \int_X |f(x)|^{p_0} |f(x)|^{p-p_0}\, d\mu(x) \\
&\qquad + \frac{p(2A_1)^{p_1}}{p_1-p} \frac{1}{\delta^{p-p_1}} \int_X |f(x)|^{p_1} |f(x)|^{p-p_1}\, d\mu(x) \\
&= p\left(\frac{(2A_0)^{p_0}}{p-p_0}\frac{1}{\delta^{p-p_0}} + \frac{(2A_1)^{p_1}}{p_1-p}\delta^{p_1-p}\right)\big\|f\big\|_{L^p}^p,
\end{aligned}$$

and the convergence of the integrals in α is justified from $p_0 < p < p_1$, while the interchange of the integrals (Fubini's theorem) uses the hypothesis that (X,μ) is a σ-finite measure space. We pick $\delta > 0$ such that

$$(2A_0)^{p_0} \frac{1}{\delta^{p-p_0}} = (2A_1)^{p_1} \delta^{p_1-p},$$

and observe that the last displayed constant is equal to the pth power of the constant in (1.3.8). We have therefore proved the theorem when $p_1 < \infty$.

We now consider the case $p_1 = \infty$. Write $f = f_0^\alpha + f_1^\alpha$, where

$$\begin{aligned}
f_0^\alpha(x) &= \begin{cases} f(x) & \text{for} \quad |f(x)| > \gamma\alpha, \\ 0 & \text{for} \quad |f(x)| \le \gamma\alpha, \end{cases} \\
f_1^\alpha(x) &= \begin{cases} f(x) & \text{for} \quad |f(x)| \le \gamma\alpha, \\ 0 & \text{for} \quad |f(x)| > \gamma\alpha. \end{cases}
\end{aligned}$$

We have

$$\big\|T(f_1^\alpha)\big\|_{L^\infty} \le A_1 \big\|f_1^\alpha\big\|_{L^\infty} \le A_1\gamma\alpha = \alpha/2,$$

provided we choose $\gamma = (2A_1)^{-1}$. It follows that the set $\{y \in Y : |T(f_1^\alpha)(y)| > \alpha/2\}$ has measure zero. Therefore,

$$d_{T(f)}(\alpha) \le d_{T(f_0^\alpha)}(\alpha/2).$$

Since T maps L^{p_0} to $L^{p_0,\infty}$ with norm at most A_0, it follows that

$$d_{T(f_0^\alpha)}(\alpha/2) \le \frac{(2A_0)^{p_0}\|f_0^\alpha\|_{L^{p_0}}^{p_0}}{\alpha^{p_0}} = \frac{(2A_0)^{p_0}}{\alpha^{p_0}} \int_{|f|>\gamma\alpha} |f(x)|^{p_0}\, d\mu(x). \tag{1.3.10}$$

Using (1.3.10) and Proposition 1.1.4, we obtain

$$\begin{aligned}
\|T(f)\|_{L^p}^p &= p\int_0^\infty \alpha^{p-1} d_{T(f)}(\alpha)\, d\alpha \\
&\le p\int_0^\infty \alpha^{p-1} d_{T(f_0^\alpha)}(\alpha/2)\, d\alpha \\
&\le p\int_0^\infty \alpha^{p-1} \frac{(2A_0)^{p_0}}{\alpha^{p_0}} \int_{|f|>\alpha/(2A_1)} |f(x)|^{p_0}\, d\mu(x)\, d\alpha \\
&= p(2A_0)^{p_0} \int_X |f(x)|^{p_0} \int_0^{2A_1|f(x)|} \alpha^{p-p_0-1}\, d\alpha\, d\mu(x) \\
&= \frac{p(2A_1)^{p-p_0}(2A_0)^{p_0}}{p-p_0} \int_X |f(x)|^p\, d\mu(x)\,.
\end{aligned}$$

This proves the theorem with constant

$$A = 2\left(\frac{p}{p-p_0}\right)^{\frac{1}{p}} A_1^{1-\frac{p_0}{p}} A_0^{\frac{p_0}{p}}\,. \tag{1.3.11}$$

Observe that when $p_1 = \infty$, the constant in (1.3.11) coincides with that in (1.3.8). $\square$

Remark 1.3.3. Notice that the proof of Theorem 1.3.2 only makes use of the *subadditivity* property $|T(f+g)| \le |T(f)| + |T(g)|$ of T in hypothesis (1.3.3).

If T is a linear operator (instead of sublinear), then we can relax the hypotheses of Theorem 1.3.2 by assuming that (1.3.5) and (1.3.6) hold for all simple functions f on X. Then the functions f_0^α and f_1^α constructed in the proof are also simple, and we conclude that (1.3.7) holds for all simple functions f on X. By density, T has a unique extension on $L^p(X)$ that also satisfies (1.3.7).

1.3.2 Complex Method: The Riesz–Thorin Interpolation Theorem

The next interpolation theorem assumes stronger endpoint estimates, but yields a more natural bound on the norm of the operator on the intermediate spaces. Unfortunately, it is mostly applicable for linear operators and in some cases for sublinear operators (often via a linearization process) but it does not apply to quasi-linear operators without some loss in the constant.

Recall that a simple function is called finitely simple if it is supported in a set of finite measure. Finitely simple functions are dense in $L^p(X,\mu)$ for $0 < p < \infty$, whenever (X,μ) is a σ-finite measure space.

Theorem 1.3.4. *Let (X,μ) and (Y,ν) be two σ-finite measure spaces. Let T be a linear operator defined on the set of all finitely simple functions on X and taking values in the set of measurable functions on Y. Let $1 \le p_0, p_1, q_0, q_1 \le \infty$ and assume that*

$$\begin{aligned} \big\|T(f)\big\|_{L^{q_0}} &\le M_0 \big\|f\big\|_{L^{p_0}}\,, \\ \big\|T(f)\big\|_{L^{q_1}} &\le M_1 \big\|f\big\|_{L^{p_1}}\,, \end{aligned} \tag{1.3.12}$$

for all finitely simple functions f on X. Then for all $0 < \theta < 1$ we have

$$\big\|T(f)\big\|_{L^q} \le M_0^{1-\theta} M_1^{\theta} \big\|f\big\|_{L^p} \tag{1.3.13}$$

for all finitely simple functions f on X, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \textit{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}\,. \tag{1.3.14}$$

Consequently, when $p < \infty$, by density, T has a unique bounded extension from $L^p(X,\mu)$ to $L^q(Y,\nu)$ when p and q are as in (1.3.14).

Proof. Let

$$f = \sum_{k=1}^{m} a_k e^{i\alpha_k} \chi_{A_k}$$

be a finitely simple function on X, where $a_k > 0$, α_k are real, and A_k are pairwise disjoint subsets of X with finite measure.

We need to control

$$\big\|T(f)\big\|_{L^q(Y,\nu)} = \sup_g \left| \int_Y T(f)(y) g(y)\, d\nu(y) \right|,$$

where the supremum is taken over all finitely simple functions g on Y with $L^{q'}$ norm less than or equal to 1. Write

$$g = \sum_{j=1}^{n} b_j e^{i\beta_j} \chi_{B_j}\,,$$

where $b_j > 0$, β_j are real, and B_j are pairwise disjoint subsets of Y with finite ν-measure. Let

$$P(z) = \frac{p}{p_0}(1-z) + \frac{p}{p_1} z \qquad \text{and} \qquad Q(z) = \frac{q'}{q'_0}(1-z) + \frac{q'}{q'_1} z\,. \tag{1.3.15}$$

For z in the closed strip $\overline{S} = \{z \in \mathbf{C} : 0 \le \operatorname{Re} z \le 1\}$, define

$$f_z = \sum_{k=1}^{m} a_k^{P(z)} e^{i\alpha_k} \chi_{A_k}, \quad g_z = \sum_{j=1}^{n} b_j^{Q(z)} e^{i\beta_j} \chi_{B_j}, \tag{1.3.16}$$

and

$$F(z) = \int_Y T(f_z)(y)\, g_z(y)\, d\nu(y).$$

Notice that $f_\theta = f$ and $g_\theta = f$. By linearity we have

$$F(z) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_k^{P(z)} b_j^{Q(z)} e^{i\alpha_k} e^{i\beta_j} \int_Y T(\chi_{A_k})(y)\, \chi_{B_j}(y)\, d\nu(y).$$

Since $a_k, b_j > 0$, F is analytic in z, and the expression

$$\int_Y T(\chi_{A_k})(y)\, \chi_{B_j}(y)\, d\nu(y)$$

is a finite constant, being an absolutely convergent integral; this is seen by Hölder's inequality with exponents q_0 and q_0' (or q_1 and q_1') and (1.3.12).

By the disjointness of the sets A_k we have (even when $p_0 = \infty$)

$$\big\|f_{it}\big\|_{L^{p_0}} = \big\|f\big\|_{L^p}^{\frac{p}{p_0}},$$

since $|a_k^{P(it)}| = a_k^{\frac{p}{p_0}}$, and by the disjointness of the B_j's we have (even when $q_0 = 1$)

$$\big\|g_{it}\big\|_{L^{q_0'}} = \big\|g\big\|_{L^{q'}}^{\frac{q'}{q_0'}},$$

since $|b_j^{Q(it)}| = b_j^{\frac{q'}{q_0'}}$. Thus Hölder's inequality and the hypothesis give

$$\begin{aligned} |F(it)| &\le \big\|T(f_{it})\big\|_{L^{q_0}} \big\|g_{it}\big\|_{L^{q_0'}} \\ &\le M_0 \big\|f_{it}\big\|_{L^{p_0}} \big\|g_{it}\big\|_{L^{q_0'}} \\ &= M_0 \big\|f\big\|_{L^p}^{\frac{p}{p_0}} \big\|g\big\|_{L^{q'}}^{\frac{q'}{q_0'}}. \end{aligned} \tag{1.3.17}$$

By similar calculations, which are valid even when $p_1 = \infty$ and $q_1 = 1$, we have

$$\big\|f_{1+it}\big\|_{L^{p_1}} = \big\|f\big\|_{L^p}^{\frac{p}{p_1}}$$

and

$$\big\|g_{1+it}\big\|_{L^{q_1'}} = \big\|g\big\|_{L^{q'}}^{\frac{q'}{q_1'}}.$$

Also, in a way analogous to that we obtained (1.3.17) we deduce that

$$|F(1+it)| \le M_1 \big\|f\big\|_{L^p}^{\frac{p}{p_1}} \big\|g\big\|_{L^{q'}}^{\frac{q'}{q'_1}}. \tag{1.3.18}$$

To finish the proof we will need the following lemma, known as *Hadamard's three lines lemma*.

Lemma 1.3.5. *Let F be analytic in the open strip $S = \{z \in \mathbf{C} : 0 < \text{Re } z < 1\}$, continuous and bounded on its closure, such that $|F(z)| \le B_0$ when* Re $z = 0$ *and $|F(z)| \le B_1$ when* Re $z = 1$*, for some $0 < B_0, B_1 < \infty$. Then $|F(z)| \le B_0^{1-\theta} B_1^{\theta}$ when* Re $z = \theta$*, for any $0 \le \theta \le 1$.*

To prove the lemma we define analytic functions

$$G(z) = F(z)(B_0^{1-z}B_1^z)^{-1} \qquad \text{and} \qquad G_n(z) = G(z)e^{(z^2-1)/n}$$

for z in the unit strip S, for $n = 1, 2, \dots$. Since F is bounded on $\overline{S}$ and

$$|B_0^{1-z}B_1^z| \ge \min(1, B_0)\min(1, B_1) > 0$$

for all $z \in \overline{S}$, we conclude that G is bounded by some constant M on $\overline{S}$. Since

$$|G_n(x+iy)| \le Me^{-y^2/n}e^{(x^2-1)/n} \le Me^{-y^2/n},$$

we deduce that $G_n(x+iy)$ converges to zero uniformly in $0 \le x \le 1$ as $|y| \to \infty$. Select $y(n) > 0$ such that for $|y| \ge y(n)$, we have $|G_n(x+iy)| \le 1$ for all $x \in [0,1]$. Also, the assumptions on F imply that G is bounded by one on the two lines forming the boundary of $\overline{S}$. By the maximum principle we obtain that $|G_n(z)| \le 1$ for all z in the rectangle $[0,1] \times [-y(n), y(n)]$; hence $|G_n(z)| \le 1$ everywhere in the closed strip. Letting $n \to \infty$, we conclude that $|G(z)| \le 1$ in the closed strip. Taking $z = \theta + it$ we deduce that

$$|F(\theta + it)| \le |B_0^{1-\theta-it}B_1^{\theta+it}| = B_0^{1-\theta}B_1^{\theta}$$

whenever t is real. This proves the required conclusion. □

Returning to the proof of Theorem 1.3.4, we observe that F is analytic in the open strip S and continuous on its closure. Also, F is bounded on the closed unit strip (by some constant that depends on f and g). Therefore, (1.3.17), (1.3.18), and Lemma 1.3.5 give

$$|F(z)| \le \Big(M_0\big\|f\big\|_{L^p}^{\frac{p}{p_0}}\big\|g\big\|_{L^{q'}}^{\frac{q'}{q'_0}}\Big)^{1-\theta}\Big(M_1\big\|f\big\|_{L^p}^{\frac{p}{p_1}}\big\|g\big\|_{L^{q'}}^{\frac{q'}{q'_1}}\Big)^{\theta} = M_0^{1-\theta}M_1^{\theta}\big\|f\big\|_{L^p}\big\|g\big\|_{L^{q'}},$$

when $\text{Re}\, z = \theta$. Observe that $P(\theta) = Q(\theta) = 1$ and hence

$$F(\theta) = \int_Y T(f)\, g\, d\nu.$$

Taking the supremum over all finitely simple functions g on Y with $L^{q'}$ norm less than or equal to one, we conclude the proof of the theorem. □

We now give an application of Theorem 1.3.4.

Example 1.3.6. One may prove Young's inequality (Theorem 1.2.12) using the Riesz–Thorin interpolation theorem (Theorem 1.3.4). Fix a function g in L^r and let $T(f) = f * g$. Since $T : L^1 \to L^r$ with norm at most $\|g\|_{L^r}$ and $T : L^{r'} \to L^\infty$ with norm at most $\|g\|_{L^r}$, Theorem 1.3.4 gives that T maps L^p to L^q with norm at most the quantity $\|g\|_{L^r}^{\theta}\|g\|_{L^r}^{1-\theta} = \|g\|_{L^r}$, where

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{r'} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{r} + \frac{\theta}{\infty}. \tag{1.3.19}$$

Finally, observe that equations (1.3.19) give (1.2.13).

1.3.3 Interpolation of Analytic Families of Operators

Theorem 1.3.4 can be extended to the case in which the interpolated operators are allowed to vary. In particular, if a family of operators depends analytically on a parameter z, then the proof of this theorem can be adapted to work in this setting.

We describe the setup for this theorem. Let (X,μ) and (Y,ν) be σ-finite measure spaces. Suppose that for every z in the closed strip $\overline{S} = \{z \in \mathbf{C} : 0 \le \operatorname{Re} z \le 1\}$ there is an associated linear operator T_z defined on the space of finitely simple functions on X and taking values in the space of measurable functions on Y such that

$$\int_Y |T_z(\chi_A)\,\chi_B|\,d\nu < \infty \tag{1.3.20}$$

whenever A and B are subsets of finite measure of X and Y, respectively. The family $\{T_z\}_z$ is said to be *analytic* if for all f,g finitely simple functions we have that the function

$$z \mapsto \int_Y T_z(f)\,g\,d\nu \tag{1.3.21}$$

is analytic in the open strip $S = \{z \in \mathbf{C} : 0 < \operatorname{Re} z < 1\}$ and continuous on its closure. The analytic family $\{T_z\}_z$ is called of *admissible growth* if there is a constant τ_0 with $0 \le \tau_0 < \pi$ such that for finitely simple functions f on X and g on Y there is constant $C(f,g)$ such that

$$\log\left|\int_Y T_z(f)\,g\,d\nu\right| \le C(f,g)\,e^{\tau_0|\operatorname{Im} z|} \tag{1.3.22}$$

for all z satisfying $0 \le \operatorname{Re} z \le 1$. Note that if there is $\tau_0 \in (0,\pi)$ such that for all measurable subsets A of X and B of Y of finite measure there is a constant $c(A,B)$ such that

$$\log\left|\int_B T_z(\chi_A)\,d\nu\right| \le c(A,B)\,e^{\tau_0|\operatorname{Im} z|}, \tag{1.3.23}$$

then (1.3.22) holds for $f=\sum_{k=1}^{M}a_k\chi_{A_k}$ and $g=\sum_{j=1}^{N}b_j\chi_{B_j}$ and

$$C(f,g)=\log(MN)+\sum_{k=1}^{M}\sum_{j=1}^{N}\Big(c(A_k,B_j)+\big|\log|a_k b_j|\big|\Big)\,.$$

The extension of the Riesz–Thorin interpolation theorem is as follows.

Theorem 1.3.7. *Let T_z be an analytic family of linear operators of admissible growth defined on the space of finitely simple functions of a σ-finite measure space (X,μ) and taking values in the set of measurable functions of another σ-finite measure space (Y,ν). Let $1\le p_0,p_1,q_0,q_1\le\infty$ and suppose that M_0 and M_1 are positive functions on the real line such that for some τ_1 with $0\le\tau_1<\pi$ we have*

$$\sup_{-\infty<y<+\infty} e^{-\tau_1|y|}\log M_j(y)<\infty \tag{1.3.24}$$

for $j=0,1$. Fix $0<\theta<1$ and define p,q by the equations

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\qquad \textit{and}\qquad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}\,. \tag{1.3.25}$$

Suppose that for all finitely simple functions f on X we have

$$\big\|T_{iy}(f)\big\|_{L^{q_0}}\le M_0(y)\big\|f\big\|_{L^{p_0}}\,, \tag{1.3.26}$$

$$\big\|T_{1+iy}(f)\big\|_{L^{q_1}}\le M_1(y)\big\|f\big\|_{L^{p_1}}\,. \tag{1.3.27}$$

Then for all finitely simple functions f on X we have

$$\big\|T_\theta(f)\big\|_{L^{q}}\le M(\theta)\big\|f\big\|_{L^{p}} \tag{1.3.28}$$

where for $0<x<1$

$$M(x)=\exp\left\{\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log M_0(t)}{\cosh(\pi t)-\cos(\pi x)}+\frac{\log M_1(t)}{\cosh(\pi t)+\cos(\pi x)}\right]dt\right\}.$$

Thus, by density, T_θ has a unique bounded extension from $L^p(X,\mu)$ to $L^q(Y,\nu)$ when p and q are as in (1.3.25).

Note that in view of (1.3.24), the integral defining $M(t)$ converges absolutely. The proof of the previous theorem is based on an extension of Lemma 1.3.5.

Lemma 1.3.8. *Let F be analytic on the open strip $S=\{z\in\mathbf{C}:\,0<\operatorname{Re} z<1\}$ and continuous on its closure such that for some $A<\infty$ and $0\le\tau_0<\pi$ we have*

$$\log|F(z)|\le A\,e^{\tau_0|\operatorname{Im} z|} \tag{1.3.29}$$

for all $z \in \overline{S}$. *Then*

$$|F(x+iy)| \le \exp\left\{ \frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log|F(it+iy)|}{\cosh(\pi t)-\cos(\pi x)} + \frac{\log|F(1+it+iy)|}{\cosh(\pi t)+\cos(\pi x)} \right] dt \right\}$$

whenever $0<x<1$, *and* y *is real.*

Assuming Lemma 1.3.8, we prove Theorem 1.3.7.

Proof. Fix $0<\theta<1$ and finitely simple functions f on X and g on Y such that $\|f\|_{L^p} = \|g\|_{L^{q'}} = 1$. Note that since $0<\theta<1$ we must have $1<p,q<\infty$. Let

$$f = \sum_{k=1}^{m} a_k e^{i\alpha_k} \chi_{A_k} \quad \text{and} \quad g = \sum_{j=1}^{n} b_j e^{i\beta_j} \chi_{B_j},$$

where $a_k>0$, $b_j>0$, α_k, β_j are real, A_k are pairwise disjoint subsets of X with finite measure, and B_j are pairwise disjoint subsets of Y with finite measure for all k,j. Let $P(z)$, $Q(z)$ be as in (1.3.15) and f_z, g_z as in (1.3.16). Define for $z \in \overline{S}$

$$F(z) = \int_Y T_z(f_z)\, g_z \, d\nu . \tag{1.3.30}$$

Linearity gives that

$$F(z) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_k^{P(z)} b_j^{Q(z)} e^{i\alpha_k} e^{i\beta_j} \int_Y T_z(\chi_{A_k})(x)\, \chi_{B_j}(x)\, d\nu(x),$$

and conditions (1.3.20) together with the fact that $\{T_z\}_z$ is an analytic family imply that $F(z)$ is a well-defined analytic function on the unit strip that extends continuously to its boundary.

Since $\{T_z\}_z$ is a family of admissible growth, (1.3.23) holds for some $c(A_k,B_j)$ and $\tau_0 \in (0,\pi)$ and this combined with the facts that

$$|a_k^{P(z)}| \le a_k^{\frac{p}{p_0}+\frac{p}{p_1}} \qquad \text{and} \qquad |b_j^{Q(z)}| \le b_j^{\frac{q'}{q'_0}+\frac{q'}{q'_1}}$$

for all z with $0<\operatorname{Re} z<1$, implies (1.3.29) with τ_0 as in (1.3.23) and

$$A = \log(mn) + \sum_{k=1}^{m}\sum_{j=1}^{n} \left(c(A_k,B_j) + \left(\frac{p}{p_0}+\frac{p}{p_1}\right) \big|\log a_k\big| + \left(\frac{q'}{q'_0}+\frac{q'}{q'_1}\right) \big|\log b_j\big| \right).$$

Thus F satisfies the hypotheses of Lemma 1.3.8. Moreover, the calculations in the proof of Theorem 1.3.4 show that (even when $p_0=\infty$, $q_0=1$, $p_1=\infty$, $q_1=1$)

$$\big\|f_{iy}\big\|_{L^{p_0}} = \big\|f\big\|_{L^p}^{\frac{p}{p_0}} = 1 = \big\|g\big\|_{L^{q'}}^{\frac{q'}{q'_0}} = \big\|g_{iy}\big\|_{L^{q'_0}} \quad \text{when } y \in \mathbf{R}, \tag{1.3.31}$$

$$\big\|f_{1+iy}\big\|_{L^{p_1}} = \big\|f\big\|_{L^p}^{\frac{p}{p_1}} = 1 = \big\|g\big\|_{L^{q'}}^{\frac{q'}{q'_1}} = \big\|g_{1+iy}\big\|_{L^{q'_1}} \quad \text{when } y \in \mathbf{R}. \tag{1.3.32}$$

Hölder's inequality, (1.3.31), and the hypothesis (1.3.26) now give

$$|F(iy)| \leq \left\|T_{iy}(f_{iy})\right\|_{L^{q_0}} \left\|g_{iy}\right\|_{L^{q_0'}} \leq M_0(y) \left\|f_{iy}\right\|_{L^{p_0}} \left\|g_{iy}\right\|_{L^{q_0'}} = M_0(y)$$

for all y real. Similarly, (1.3.32), and (1.3.27) imply

$$|F(1+iy)| \leq \left\|T_{1+iy}(f_{1+iy})\right\|_{L^{q_1}} \left\|g_{1+iy}\right\|_{L^{q_1'}} \leq M_1(y) \left\|f_{1+iy}\right\|_{L^{p_1}} \left\|g_{1+iy}\right\|_{L^{q_1'}} = M_1(y)$$

for all $y \in \mathbf{R}$. These inequalities and the conclusion of Lemma 1.3.8 yield

$$|F(x)| \leq \exp\left\{\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log M_0(t)}{\cosh(\pi t)-\cos(\pi x)}+\frac{\log M_1(t)}{\cosh(\pi t)+\cos(\pi x)}\right]dt\right\} = M(x)$$

for all $0 < x < 1$. But notice that

$$F(\theta) = \int_Y T_\theta(f)\, g\, d\nu\,. \tag{1.3.33}$$

Taking absolute values and the supremum over all finitely simple functions g on Y with $L^{q'}$ norm equal to one, we conclude the proof of (1.3.28) for finitely simple functions f with L^p norm one. Then (1.3.28) follows by replacing f by $f/\|f\|_{L^p}$. □

We end this section with the proof of Lemma 1.3.8.

Proof of Lemma 1.3.8. Recall the Poisson integral formula

$$U(z) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} U(Re^{i\varphi})\frac{R^2-\rho^2}{|Re^{i\varphi}-\rho e^{i\theta}|^2}d\varphi\,, \qquad z = \rho e^{i\theta}, \tag{1.3.34}$$

which is valid for a harmonic function U defined on the unit disk $D = \{z : |z| < 1\}$ when $|z| < R < 1$. See [307, p. 258].

Consider now a subharmonic function u on D that is continuous on the circle $|\zeta| = R < 1$. When $U = u$, the right side of (1.3.34) defines a harmonic function on the set $\{z \in \mathbf{C} : |z| < R\}$ that coincides with u on the circle $|\zeta| = R$. The maximum principle for subharmonic functions ([307, p. 362]) implies that for $|z| < R < 1$ we have

$$u(z) \leq \frac{1}{2\pi}\int_{-\pi}^{+\pi} u(Re^{i\varphi})\frac{R^2-\rho^2}{|Re^{i\varphi}-\rho e^{i\theta}|^2}d\varphi\,, \quad z = \rho e^{i\theta}. \tag{1.3.35}$$

This is valid for all subharmonic functions u on D that are continuous on the circle $|\zeta| = R$ when $\rho < R < 1$.

It is not difficult to verify that

$$h(\zeta) = \frac{1}{\pi i}\log\left(i\frac{1+\zeta}{1-\zeta}\right)$$

is a conformal map from D onto the strip $S = (0,1)\times\mathbf{R}$. Indeed, $i(1+\zeta)/(1-\zeta)$ lies in the upper half-plane and the preceding complex logarithm is a well defined holomorphic function that takes the upper half-plane onto the strip $\mathbf{R}\times(0,\pi)$. Since

$F \circ h$ is a holomorphic function on D, $\log |F \circ h|$ is a subharmonic function on D. Applying (1.3.35) to the function $z \mapsto \log |F(h(z))|$, we obtain

$$\log |F(h(z))| \le \frac{1}{2\pi} \int_{-\pi}^{+\pi} \log |F(h(Re^{i\varphi}))| \frac{R^2 - \rho^2}{R^2 - 2\rho R \cos(\theta - \varphi) + \rho^2} \, d\varphi \tag{1.3.36}$$

when $z = \rho e^{i\theta}$ and $|z| = \rho < R$. Observe that when $|\zeta| = 1$ and $\zeta \neq \pm 1$, $h(\zeta)$ has real part zero or one. It follows from the hypothesis that

$$\log |F(h(\zeta))| \le A e^{\tau_0 |\mathrm{Im}\, h(\zeta)|} = A e^{\tau_0 \left|\mathrm{Im} \frac{1}{\pi i} \log\left(i \frac{1+\zeta}{1-\zeta}\right)\right|} = A e^{\frac{\tau_0}{\pi} \left|\log \left|\frac{1+\zeta}{1-\zeta}\right|\right|}.$$

Therefore, $\log |F(h(\zeta))|$ is bounded by a multiple of $|1+\zeta|^{-\tau_0/\pi} |1-\zeta|^{-\tau_0/\pi}$, which is integrable over the set $|\zeta| = 1$, since $\tau_0 < \pi$. Fix now $z = \rho e^{i\theta}$ with $\rho < R$ and let $R \to 1$ in (1.3.36). The Lebesgue dominated convergence theorem gives that

$$\log |F(h(\rho e^{i\theta}))| \le \frac{1}{2\pi} \int_{-\pi}^{+\pi} \log |F(h(e^{i\varphi}))| \frac{1 - \rho^2}{1 - 2\rho \cos(\theta - \varphi) + \rho^2} \, d\varphi \, . \tag{1.3.37}$$

Setting $x = h(\rho e^{i\theta})$, we obtain that

$$\rho e^{i\theta} = h^{-1}(x) = \frac{e^{\pi i x} - i}{e^{\pi i x} + i} = -i \frac{\cos(\pi x)}{1 + \sin(\pi x)} = \left(\frac{\cos(\pi x)}{1 + \sin(\pi x)}\right) e^{-i(\pi/2)} \, ,$$

from which it follows that $\rho = (\cos(\pi x))/(1 + \sin(\pi x))$ and $\theta = -\pi/2$ when $0 < x \le \frac{1}{2}$, while $\rho = -(\cos(\pi x))/(1 + \sin(\pi x))$ and $\theta = \pi/2$ when $\frac{1}{2} \le x < 1$. In either case we easily deduce that

$$\frac{1 - \rho^2}{1 - 2\rho \cos(\theta - \varphi) + \rho^2} = \frac{\sin(\pi x)}{1 + \cos(\pi x) \sin(\varphi)} \, .$$

Using this we write (1.3.37) as

$$\log |F(x)| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(\pi x)}{1 + \cos(\pi x) \sin(\varphi)} \log |F(h(e^{i\varphi}))| \, d\varphi \, . \tag{1.3.38}$$

We now change variables. On the interval $[-\pi, 0)$ we use the change of variables $it = h(e^{i\varphi})$ or, equivalently, $e^{i\varphi} = -\tanh(\pi t) - i \,\mathrm{sech}(\pi t)$. Observe that as φ ranges from $-\pi$ to 0, t ranges from $+\infty$ to $-\infty$. Furthermore, $d\varphi = -\pi \,\mathrm{sech}(\pi t) \, dt$. We have

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\pi}^{0} \frac{\sin(\pi x)}{1 + \cos(\pi x) \sin(\varphi)} \log |F(h(e^{i\varphi}))| \, d\varphi \\ &\qquad = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin(\pi x)}{\cosh(\pi t) - \cos(\pi x)} \log |F(it)| \, dt \, . \end{aligned} \tag{1.3.39}$$

On the interval $(0,\pi]$ we use the change of variables $1+it = h(e^{i\varphi})$ or, equivalently, $e^{i\varphi} = -\tanh(\pi t) + i\,\mathrm{sech}(\pi t)$. Observe that as φ ranges from 0 to π, t ranges from $-\infty$ to $+\infty$. Furthermore, $d\varphi = \pi\,\mathrm{sech}(\pi t)\,dt$. Similarly, we obtain

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{\pi} \frac{\sin(\pi t)}{1+\cos(\pi t)\sin(\varphi)} \log|F(h(e^{i\varphi}))|\,d\varphi \\ &\qquad = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\sin(\pi x)}{\cosh(\pi t)+\cos(\pi x)} \log|F(1+it)|\,dt.\end{aligned} \tag{1.3.40}$$

Adding (1.3.39) and (1.3.40) and using (1.3.38) we conclude the proof when $y = 0$.

We now consider the case where $y \neq 0$. Fix $y \neq 0$ and define the function $G(z) = F(z+iy)$. Then G is analytic on the open strip $S = \{z \in \mathbf{C} : 0 < \mathrm{Re}\, z < 1\}$ and continuous on its closure. Moreover, for some $A < \infty$ and $0 \le \tau_0 < \pi$ we have

$$\log|G(z)| = \log|F(z+iy)| \le A\, e^{\tau_0 |\mathrm{Im}\, z + y|} \le A e^{\tau_0|y|}\, e^{\tau_0 |\mathrm{Im}\, z|}$$

for all $z \in \overline{S}$. Then the case $y = 0$ for G (with A replaced by $Ae^{\tau_0|y|}$) yields

$$|G(x)| \le \exp\left\{ \frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log|G(it)|}{\cosh(\pi t)-\cos(\pi x)} + \frac{\log|G(1+it)|}{\cosh(\pi t)+\cos(\pi x)} \right] dt \right\},$$

which yields the required conclusion for any real y, since $G(x) = F(x+iy)$, $G(it) = F(it+iy)$, and $G(1+it) = F(1+it+iy)$. □

Exercises

1.3.1. Generalize Theorem 1.3.2 to the situation in which T is *quasi-subadditive*, that is, it satisfies for some $K > 0$,

$$|T(f+g)| \le K(|T(f)| + |T(g)|),$$

for all f, g in the domain of T. Prove that in this case, the constant A in (1.3.7) can be taken to be K times the constant in (1.3.8).

1.3.2. Let (X,μ), (Y,ν) be two σ-finite measure spaces. Let $1 < p < r \le \infty$ and suppose that T be a sublinear operator defined on the space $L^{p_0}(X) + L^{p_1}(X)$ and taking values in the space of measurable functions on Y. Assume that T maps $L^1(X)$ to $L^{1,\infty}(Y)$ with norm A_0 and $L^r(X)$ to $L^r(Y)$ with norm A_1. Let $0 < p_0 < p_1 \le \infty$. Prove that T maps L^p to L^p with norm at most

$$8\,(p-1)^{-\frac{1}{p}} A_0^{\frac{\frac{1}{p}-\frac{1}{r}}{1-\frac{1}{r}}} A_1^{\frac{1-\frac{1}{p}}{1-\frac{1}{r}}}.$$

[*Hint:* First interpolate between L^1 and L^r using Theorem 1.3.2 and then interpolate between $L^{\frac{p+1}{2}}$ and L^r using Theorem 1.3.4.]

1.3.3. Let $0 < p_0 < p < p_1 \le \infty$ and let T be an operator as in Theorem 1.3.2 that also satisfies

$$|T(f)| \le T(|f|),$$

for all $f \in L^{p_0} + L^{p_1}$.
(a) If $p_0 = 1$ and $p_1 = \infty$, prove that T maps L^p to L^p with norm at most

$$\frac{p}{p-1} A_0^{\frac{1}{p}} A_1^{1-\frac{1}{p}}.$$

(b) More generally, if $p_0 < p < \infty$, prove that the norm of T from L^p to L^p is at most

$$p^{1+\frac{1}{p}} \left[\frac{B(p_0+1, p-p_0)}{p_0^{p_0}(p-p_0)^{p-p_0}} \right]^{\frac{1}{p}} A_0^{\frac{p_0}{p}} A_1^{1-\frac{p_0}{p}},$$

where $B(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}\, dx$ is the usual Beta function.
(c) When $0 < p_0 < p_1 < \infty$, then the norm of T from L^p to L^p is at most

$$\min_{0<\lambda<1} p^{\frac{1}{p}} \left(\frac{B(p-p_0, p_0+1)}{(1-\lambda)^{p_0}} + \frac{\frac{p_1-p+1}{p_1-p}}{\lambda^{p_1}} \right)^{\frac{1}{p}} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}.$$

[*Hint:* The hypothesis $|T(f)| \le T(|f|)$ reduces matters to nonnegative functions. Parts (a), (b): Given $f \ge 0$ and $\alpha > 0$ write $f = f_0 + f_1$, where $f_0 = f - \lambda\alpha/A_1$ when $f \ge \lambda\alpha/A_1$ and zero otherwise. Here $0 < \lambda < 1$ to be chosen later. Then we have that $|\{|T(f)| > \alpha\}| \le |\{|T(f_0)| > (1-\lambda)\alpha\}|$. Part (c): Write $f = f_0 + f_1$, where $f_0 = f - \delta\alpha$ when $f \ge \delta\alpha$ and zero otherwise. Use that

$$|\{|T(f)| > \alpha\}| \le |\{|T(f_0)| > (1-\lambda)\alpha\}| + |\{|T(f_1)| > \lambda\alpha\}|$$

and optimize over $\delta > 0$.]

1.3.4. Let $0 \le \gamma, \delta < \pi$. For every $z \in S_{a,b} = \{z \in \mathbf{C} : a < \operatorname{Re} z < b\}$, let T_z be a family of linear operators defined on finetely simple functions on a σ-finite measure space (X,μ) and taking values in another σ-finite measure space (Y,ν). Assume that $\{T_z\}_z$ is an analytic on of $S_{a,b}$, in the sense of (1.3.21), continuous on its closure, and that for all simple functions f on X and g on Y there is a constant $C_{f,g} < \infty$ such that for all $z \in S_{a,b}$,

$$\log \left| \int_Y T_z(f) g \, d\nu \right| \le C_{f,g}\, e^{\gamma |\operatorname{Im} z|/(b-a)}.$$

Let $1 \le p_0, q_0, p_1, q_1 \le \infty$. Suppose that T_{a+iy} maps $L^{p_0}(X)$ to $L^{q_0}(Y)$ with bound $M_0(y)$ and T_{b+iy} maps $L^{p_1}(X)$ to $L^{q_1}(Y)$ with bound $M_1(y)$, where

$$\sup_{-\infty<y<\infty} e^{-\delta|y|/(b-a)} \log M_j(y) < \infty, \quad j = 0, 1.$$

Then for $a < t < b$, T_t maps $L^p(X)$ to $L^q(Y)$, where

$$\frac{1}{p} = \frac{\frac{b-t}{b-a}}{p_0} + \frac{\frac{t-a}{b-a}}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{\frac{b-t}{b-a}}{q_0} + \frac{\frac{t-a}{b-a}}{q_1}.$$

1.3.5. ([331]) On $\mathbf{R}^n$ let $x = (x_1, \dots, x_n)$ and $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$. Let

$$K_\lambda(x) = \frac{\pi^{\frac{n-1}{2}} \Gamma(\lambda+1)}{\Gamma(\lambda + \frac{n+1}{2})} \int_{-1}^{+1} e^{2\pi i s|x|}(1-s^2)^{\lambda+\frac{n-1}{2}}\, ds = \frac{\Gamma(\lambda+1)}{\pi^\lambda |x|^{\lambda+\frac{n}{2}}} J_{\lambda+\frac{n}{2}}(2\pi|x|),$$

where λ is a complex number and $J_{\lambda+\frac{n}{2}}$ is the Bessel function of order $\lambda + \frac{n}{2}$. Let T_λ be the operator given by convolution with K_λ. Show that T_λ maps $L^p(\mathbf{R}^n)$ to itself for $\operatorname{Re}\lambda > (n-1)|\frac{1}{2} - \frac{1}{p}|$.
$\big[$*Hint:* In view of the calculation of the Fourier transform of K_λ contained in Appendix B.5, we have that when $\operatorname{Re}\lambda = 0$, T_λ maps $L^2(\mathbf{R}^n)$ to itself with norm 1. Using the estimates in Appendices B.6 and B.7, conclude that K_λ is integrable and thus T_λ maps $L^1(\mathbf{R}^n)$ to itself with an appropriate constant when $\operatorname{Re}\lambda = (n-1)/2 + \delta$ (for $\delta > 0$). Then use Exercise 1.3.4.$\big]$

1.3.6. Observe that Theorem 1.3.7 yields the stronger conclusion

$$\big\|T_z(f)\big\|_{L^q} \le M(z)\,\big\|f\big\|_{L^p}$$

for $z \in S = \{z \in \mathbf{C} : 0 < \operatorname{Re} z < 1\}$, where for $z = x + iy$

$$M(z) = \exp\left\{\frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log M_0(t+y)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t+y)}{\cosh(\pi t) + \cos(\pi x)}\right] dt\right\}.$$

1.3.7. ([380]) Let (X,μ) and (Y,ν) be two measure spaces with $\mu(X) < \infty$ and $\nu(Y) < \infty$. Let T be a *countably subadditive* operator that maps $L^p(X)$ to $L^p(Y)$ for every $1 < p \le 2$ with norm $\|T\|_{L^p \to L^p} \le A(p-1)^{-\alpha}$ for some fixed $A, \alpha > 0$. (Countably subadditive means that $|T(\sum_j f_j)| \le \sum_j |T(f_j)|$ for all f_j in $L^p(X)$ with $\sum_j f_j \in L^p$.) Prove that for all f measurable on X we have

$$\int_Y |T(f)|\, d\nu \le 6A(1+\nu(Y))^{\frac{1}{2}} \left[\int_X |f|(\log_2^+|f|)^\alpha\, d\mu + C_\alpha + \mu(X)^{\frac{1}{2}}\right],$$

where $C_\alpha = \sum_{k=1}^\infty k^\alpha (2/3)^k$. This result provides an example of *extrapolation*.
$\big[$*Hint:* Write

$$f = \sum_{k=0}^{\infty} f\chi_{S_k},$$

where $S_k = \{2^k \le |f| < 2^{k+1}\}$ when $k \ge 1$ and $S_0 = \{|f| < 2\}$. Using Hölder's inequality and the hypotheses on T, obtain that

$$\int_Y |T(f\chi_{S_k})|\, d\nu \le 2A\nu(Y)^{\frac{1}{k+1}} 2^k k^\alpha \mu(S_k)^{\frac{k}{k+1}}$$

for $k \geq 1$. Note that for $k \geq 1$ we have $\nu(Y)^{\frac{1}{k+1}} \leq \max(1, \nu(Y))^{\frac{1}{2}}$ and consider the cases $\mu(S_k) \geq 3^{-k-1}$ and $\mu(S_k) \leq 3^{-k-1}$ when summing in $k \geq 1$. The term with $k = 0$ is easier.$\big]$

1.3.8. Prove that for $0 < x < 1$ we have

$$\begin{aligned}\frac{\sin(\pi x)}{2} \int_{-\infty}^{+\infty} \frac{1}{\cosh(\pi t) + \cos(\pi x)}\, dt &= x\,,\\ \frac{\sin(\pi x)}{2} \int_{-\infty}^{+\infty} \frac{1}{\cosh(\pi t) - \cos(\pi x)}\, dt &= 1 - x\,,\end{aligned}$$

and conclude that Lemma 1.3.8 reduces to Lemma 1.3.5 when the functions $M_0(y)$ and $M_1(y)$ are constant and assumption (1.3.29) is replaced by the stronger assumption that F is bounded on $\overline{S}$.
$\big[$*Hint:* In the first integral write $\cosh(\pi t) = \frac{1}{2}(e^{\pi t} + e^{-\pi t})$. Then use the change of variables $s = e^{\pi t}$.$\big]$

1.3.9. Let (X, μ), (Y, ν) be σ-finite measure spaces, and let $0 < p_0 < p_1 \leq \infty$. Let T be a sublinear operator defined on the space $L^{p_0}(X) + L^{p_1}(X)$ and taking values in the space of measurable functions on Y. Suppose T is a sublinear operator such that maps L^{p_0} to L^∞ with constant A_0 and L^{p_1} to L^∞ with constant A_1. Prove T maps L^p to L^∞ with constant $2A_0^{1-\theta} A_1^\theta$ where

$$\frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p}\,.$$

1.4 Lorentz Spaces

Suppose that f is a measurable function on a measure space (X, μ). It would be desirable to have another function f^* defined on $[0, \infty)$ that is decreasing and *equidistributed* with f. By this we mean

$$d_f(\alpha) = d_{f^*}(\alpha) \tag{1.4.1}$$

for all $\alpha \geq 0$. This is achieved via a simple construction discussed in this section.

1.4.1 Decreasing Rearrangements

Definition 1.4.1. Let f be a complex-valued function defined on X. The *decreasing rearrangement* of f is the function f^* defined on $[0, \infty)$ by

$$f^*(t) = \inf\{s > 0 : \, d_f(s) \leq t\} = \inf\{s \geq 0 : \, d_f(s) \leq t\}\,. \tag{1.4.2}$$

We adopt the convention $\inf\emptyset = \infty$, thus having $f^*(t) = \infty$ whenever $d_f(\alpha) > t$ for all $\alpha \geq 0$. Observe that f^* is decreasing and supported in $[0, \mu(X)]$.

Before we proceed with properties of the function f^*, we work out three examples.

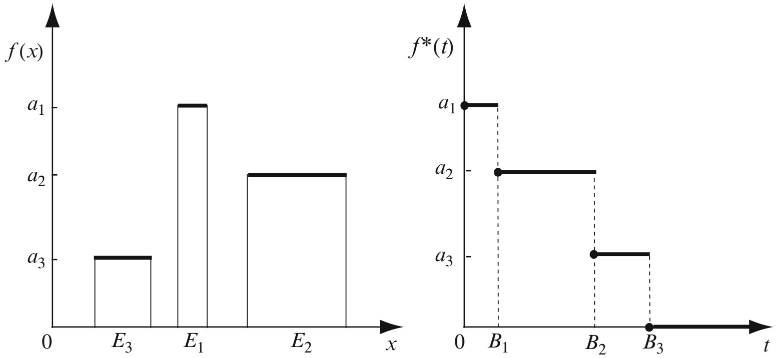

Fig. 1.3 The graph of a simple function $f(x)$ and its decreasing rearrangement $f^*(t)$.

Example 1.4.2. Consider the simple function of Example 1.1.2,

$$f(x) = \sum_{j=1}^{N} a_j \chi_{E_j}(x),$$

where E_j are pairwise disjoint sets of finite measure and $a_1 > \cdots > a_N > 0$. We saw in Example 1.1.2 that

$$d_f(\alpha) = \sum_{j=0}^{N} B_j \chi_{[a_{j+1}, a_j)}(\alpha),$$

where

$$B_j = \sum_{i=1}^{j} \mu(E_i)$$

and $a_{N+1} = B_0 = 0$ and $a_0 = \infty$. Observe that for $B_0 \leq t < B_1$, the smallest $s > 0$ with $d_f(s) \leq t$ is a_1. Similarly, for $B_1 \leq t < B_2$, the smallest $s > 0$ with $d_f(s) \leq t$ is a_2. Arguing this way, it is not difficult to see that

$$f^*(t) = \sum_{j=1}^{N} a_j \chi_{[B_{j-1}, B_j)}(t).$$

See Figure 1.3.

Example 1.4.3. On $(\mathbf{R}^n, dx)$ let

$$f(x) = \frac{1}{1+|x|^p}, \qquad 0 < p < \infty.$$

A computation shows that

$$d_f(\alpha) = \begin{cases} v_n(\frac{1}{\alpha} - 1)^{\frac{n}{p}} & \text{if } \alpha < 1\,, \\ 0 & \text{if } \alpha \geq 1\,, \end{cases}$$

and therefore

$$f^*(t) = \frac{1}{(t/v_n)^{p/n} + 1}\,,$$

where v_n is the volume of the unit ball in $\mathbf{R}^n$.

Example 1.4.4. Again on $(\mathbf{R}^n, dx)$ let $g(x) = 1 - e^{-|x|^2}$. We can easily see that $d_g(\alpha) = 0$ if $\alpha \geq 1$ and $d_g(\alpha) = \infty$ if $\alpha < 1$. We conclude that $g^*(t) = 1$ for all $t \geq 0$. This example indicates that although quantitative information is preserved, significant qualitative information is lost in passing from a function to its decreasing rearrangement.

It is clear from the previous examples that f^* is continuous from the right and decreasing. The following are some properties of the function f^*.

Proposition 1.4.5. *For f, g, f_n μ-measurable, $k \in \mathbf{C}$, and $0 \leq t, s, t_1, t_2 < \infty$ we have*

(1) $f^*(d_f(\alpha)) \leq \alpha$ *whenever* $\alpha > 0$.

(2) $d_f(f^*(t)) \leq t$.

(3) $f^*(t) > s$ *if and only if* $t < d_f(s)$*; that is,* $\{t \geq 0:\ f^*(t) > s\} = [0, d_f(s))$.

(4) $|g| \leq |f|$ μ*-a.e. implies that* $g^* \leq f^*$ *and* $|f|^* = f^*$.

(5) $(kf)^* = |k| f^*$.

(6) $(f+g)^*(t_1+t_2) \leq f^*(t_1) + g^*(t_2)$.

(7) $(fg)^*(t_1+t_2) \leq f^*(t_1) g^*(t_2)$.

(8) $|f_n| \uparrow |f|$ μ*-a.e. implies* $f_n^* \uparrow f^*$.

(9) $|f| \leq \liminf_{n\to\infty} |f_n|$ μ*-a.e. implies* $f^* \leq \liminf_{n\to\infty} f_n^*$.

(10) f^* *is right continuous on* $[0, \infty)$.

(11) *If* $f^*(t) < \infty$, $c > 0$, *and* $\mu(\{|f| \geq f^*(t) - c\}) < \infty$, *then* $t \leq \mu(\{|f| \geq f^*(t)\})$.

(12) $d_f = d_{f^*}$.

(13) $(|f|^p)^* = (f^*)^p$ *when* $0 < p < \infty$.

(14) $\displaystyle\int_X |f|^p \, d\mu = \int_0^\infty f^*(t)^p \, dt$ *when* $0 < p < \infty$.

(15) $\big\|f\big\|_{L^\infty} = f^*(0)$.

(16) $\displaystyle\sup_{t>0} t^q f^*(t) = \sup_{\alpha>0} \alpha \left(d_f(\alpha)\right)^q$ *for* $0 < q < \infty$.

Proof. Property (1): The set $A = \{s > 0 : d_f(s) \le d_f(\alpha)\}$ contains α and thus $f^*(d_f(\alpha)) = \inf A \le \alpha$.

Property (2): Let $s_n \in \{s > 0 : d_f(s) \le t\}$ be such that $s_n \downarrow f^*(t)$. Then $d_f(s_n) \le t$, and the right continuity of d_f (Exercise 1.1.1 (a)) implies that $d_f(f^*(t)) \le t$.

Property (3): If $s < f^*(t) = \inf\{u > 0 : d_f(u) \le t\}$, then $s \notin \{u > 0 : d_f(u) \le t\}$ which gives $d_f(s) > t$. Conversely, if for some $t < d_f(s)$ we had $f^*(t) \le s$, applying d_f and using property (2) would yield the contradiction $d_f(s) \le d_f(f^*(t)) \le t$.

Properties (4) and (5) are left to the reader.

Properties (6) and (7): Let $A = \{s_1 > 0 : d_f(s_1) \le t_1\}$, $B = \{s_2 > 0 : d_g(s_2) \le t_2\}$, $P = \{s > 0 : d_{fg}(s) \le t_1 + t_2\}$, and $S = \{s > 0 : d_{f+g}(s) \le t_1 + t_2\}$. Then $A + B \subseteq S$ and $A \cdot B \subseteq P$; thus $(f+g)^*(t_1+t_2) = \inf S \le s_1 + s_2$ and $(fg)^*(t_1+t_2) = \inf P \le s_1 s_2$ are valid for all $s_1 \in A$ and $s_2 \in B$. Taking the infimum over all $s_1 \in A$ and $s_2 \in B$ yields the conclusions.

Property (8): It follows from the definition of decreasing rearrangements that $f_n^* \le f_{n+1}^* \le f^*$ for all n. Let $h = \lim_{n\to\infty} f_n^*$; then obviously $h \le f^*$. Since $f_n^* \le h$, we have $d_{f_n}(h(t)) \le d_{f_n}(f_n^*(t)) \le t$, which implies, in view of Exercise 1.1.1 (c), that $d_f(h(t)) \le t$ by letting $n \to \infty$. It follows that $f^* \le h$, hence $h = f^*$.

Property (9): Set $F_n = \inf_{m\ge n} |f_m|$ and $h = \liminf_{n\to\infty} |f_n| = \sup_{n\ge1} F_n$. Since $F_n \uparrow h$, property (8) yields that $F_n^* \uparrow h^*$ as $n \to \infty$. By hypothesis we have $|f| \le h$, hence $f^* \le h^* = \sup_n F_n^*$. Since $F_n \le |f_m|$ for $m \ge n$, it follows that $F_n^* \le f_m^*$ for $m \ge n$; thus $F_n^* \le \inf_{m\ge n} f_m^*$. Putting these facts together, we obtain $f^* \le h^* \le \sup_n \inf_{m\ge n} f_m^* = \liminf_{n\to\infty} f_n^*$.

Property (10): If $f^*(t_0) = 0$, then $f^*(t) = 0$ for all $t > t_0$ and thus f^* is right continuous at t_0. Suppose $f^*(t_0) > 0$. Pick α such that $0 < \alpha < f^*(t_0)$ and let $\{t_n\}_{n=1}^\infty$ be a sequence of real numbers decreasing to zero. The definition of f^* yields that $d_f(f^*(t_0) - \alpha) > t_0$. Since $t_n \downarrow 0$, there is an $n_0 \in \mathbf{Z}^+$ such that $d_f(f^*(t_0) - \alpha) > t_0 + t_n$ for all $n \ge n_0$. Property (3) yields that for all $n \ge n_0$ we have $f^*(t_0) - \alpha < f^*(t_0 + t_n)$, and since the latter is at most $f^*(t_0)$, the right continuity of f^* follows.

Property (11): The definition of f^* yields that the set $A_n = \{|f| > f^*(t) - c/n\}$ has measure $\mu(A_n) > t$. The sets A_n form a decreasing sequence as n increases and $\mu(A_1) < \infty$ by assumption. Consequently, $\{|f| \ge f^*(t)\} = \bigcap_{n=1}^\infty A_n$ has measure greater than or equal to t.

Property (12): This is immediate for nonnegative simple functions in view of Examples 1.1.2 and 1.4.2. For an arbitrary measurable function f, find a sequence of nonnegative simple functions f_n such that $f_n \uparrow |f|$ and apply (9).

Property (13): It follows from $d_{|f|^p}(\alpha) = d_f(\alpha^{1/p}) = d_{f^*}(\alpha^{1/p}) = d_{(f^*)^p}(\alpha)$ for all $\alpha > 0$.

Property (14): This is a consequence of property (12) and of Proposition 1.1.4.

Property (15): This is a restatement of (1.1.2).

Property (16): Given $\alpha > 0$, without loss of generality we may assume $d_f(\alpha) > 0$. Pick ε satisfying $0 < \varepsilon < d_f(\alpha)$. Property (3) yields $f^*(d_f(\alpha) - \varepsilon) > \alpha$, which implies that

$$\sup_{t>0} t^q f^*(t) \ge (d_f(\alpha) - \varepsilon)^q f^*(d_f(\alpha) - \varepsilon) > (d_f(\alpha) - \varepsilon)^q \alpha\,.$$

We first let $\varepsilon \to 0$ and then take the supremum over all $\alpha > 0$ to obtain one direction. Conversely, given $t > 0$, assume without loss of generality that $f^*(t) > 0$, and pick ε such that $0 < \varepsilon < f^*(t)$. Property (3) yields $d_f(f^*(t)-\varepsilon) > t$. This implies that $\sup_{\alpha>0}\alpha(d_f(\alpha))^q \geq (f^*(t)-\varepsilon)(d_f(f^*(t)-\varepsilon))^q > (f^*(t)-\varepsilon)t^q$. We first let $\varepsilon \to 0$ and then take the supremum over all $t > 0$ to obtain the opposite direction of the claimed equality. □

1.4.2 Lorentz Spaces

Having disposed of the basic properties of decreasing rearrangements of functions, we proceed with the definition of the Lorentz spaces.

Definition 1.4.6. Given f a measurable function on a measure space (X,μ) and $0 < p,q \leq \infty$, define

$$\|f\|_{L^{p,q}} = \begin{cases} \left(\displaystyle\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } q < \infty, \\ \displaystyle\sup_{t>0} t^{\frac{1}{p}} f^*(t) & \text{if } q = \infty. \end{cases}$$

The set of all f with $\|f\|_{L^{p,q}} < \infty$ is denoted by $L^{p,q}(X,\mu)$ and is called the *Lorentz space* with indices p and q.

As in L^p and in weak L^p, two functions in $L^{p,q}(X,\mu)$ are considered equal if they are equal μ-almost everywhere. Observe that the previous definition implies that $L^{\infty,\infty} = L^\infty$, $L^{p,\infty} =$ weak L^p in view of Proposition 1.4.5 (16) and that $L^{p,p} = L^p$.

Remark 1.4.7. Observe that for all $0 < p,r < \infty$ and $0 < q \leq \infty$ we have

$$\big\||g|^r\big\|_{L^{p,q}} = \|g\|_{L^{pr,qr}}^r . \tag{1.4.3}$$

On $\mathbf{R}^n$ let $\delta^\varepsilon(f)(x) = f(\varepsilon x)$, $\varepsilon > 0$, be the dilation operator. It is straightforward that $d_{\delta^\varepsilon(f)}(\alpha) = \varepsilon^{-n} d_f(\alpha)$ and $(\delta^\varepsilon(f))^*(t) = f^*(\varepsilon^n t)$. It follows that Lorentz norms satisfy the following dilation identity:

$$\|\delta^\varepsilon(f)\|_{L^{p,q}} = \varepsilon^{-n/p}\|f\|_{L^{p,q}} . \tag{1.4.4}$$

Next, we calculate the Lorentz norms of a finitely simple function.

Example 1.4.8. Using the notation of Example 1.4.2, when $0 < p,q < \infty$ we have

$$\|f\|_{L^{p,q}} = \left(\frac{p}{q}\right)^{\frac{1}{q}} \left[a_1^q B_1^{\frac{q}{p}} + a_2^q\left(B_2^{\frac{q}{p}} - B_1^{\frac{q}{p}}\right) + \cdots + a_N^q\left(B_N^{\frac{q}{p}} - B_{N-1}^{\frac{q}{p}}\right)\right]^{\frac{1}{q}} ,$$

and also

$$\|f\|_{L^{p,\infty}} = \sup_{1\le j\le N} a_j B_j^{\frac{1}{p}} .$$

Next, we calculate $\|f\|_{L^{\infty,q}}$ for the simple function f of Example 1.4.2. when $q<\infty$. It turns out that

$$\|f\|_{L^{\infty,q}} = \left[a_1^q \log\left(\frac{B_1}{B_0}\right) + a_2^q \log\left(\frac{B_2}{B_1}\right) + \cdots + a_N^q \log\left(\frac{B_N}{B_{N-1}}\right)\right]^{\frac{1}{q}} = \infty ,$$

since $B_0 = 0$. We conclude that the only nonnegative simple function with finite $L^{\infty,q}$ norm is the zero function. Given a general nonzero function $g \in L^{\infty,q}$ with $0<q<\infty$, there is a nonzero simple function s with $0 \le s \le g$. Then s has infinite norm, and therefore so does g. We deduce that $L^{\infty,q}(X) = \{0\}$ when $0<q<\infty$.

Proposition 1.4.9. *For $0<p<\infty$ and $0<q\le\infty$, we have the identity*

$$\|f\|_{L^{p,q}} = \begin{cases} p^{\frac{1}{q}}\left(\displaystyle\int_0^\infty \left[d_f(s)^{\frac{1}{p}} s\right]^q \frac{ds}{s}\right)^{\frac{1}{q}} & \text{when } q<\infty \\ \sup_{s>0} s d_f(s)^{\frac{1}{p}} & \text{when } q=\infty . \end{cases} \tag{1.4.5}$$

Proof. The case $q=\infty$ is statement (16) in Proposition 1.4.5, and we may therefore concentrate on the case $q<\infty$. If f is the simple function of Example 1.1.2, then

$$d_f(s) = \sum_{j=1}^N B_j \chi_{[a_{j+1},a_j)}(s)$$

with the understanding that $a_{N+1}=0$. Using the this formula and identity in Example 1.4.8, we obtain the validity of (1.4.5) for simple functions. In general, given a measurable function f, find a sequence of nonnegative simple functions such that $f_n \uparrow |f|$ a.e. Then $d_{f_n} \uparrow d_f$ (Exercise 1.1.1 (c)) and $f_n^* \uparrow f^*$ (Proposition 1.4.5 (8)). Using the Lebesgue monotone convergence theorem we deduce (1.4.5). □

Since $L^{p,p} \subsetneqq L^{p,\infty}$, one may wonder whether these spaces are nested. The next result shows that for any fixed p, the Lorentz spaces $L^{p,q}$ increase as the exponent q increases.

Proposition 1.4.10. *Suppose $0<p\le\infty$ and $0<q<r\le\infty$. Then there exists a constant $c_{p,q,r}$ (which depends on p, q, and r) such that*

$$\|f\|_{L^{p,r}} \le c_{p,q,r}\|f\|_{L^{p,q}} . \tag{1.4.6}$$

In other words, $L^{p,q}$ is a subspace of $L^{p,r}$.

Proof. We may assume $p<\infty$, since the case $p=\infty$ is trivial. We have

$$\begin{aligned} t^{1/p} f^*(t) &= \left\{ \frac{q}{p} \int_0^t [s^{1/p} f^*(t)]^q \frac{ds}{s} \right\}^{1/q} \\ &\le \left\{ \frac{q}{p} \int_0^t [s^{1/p} f^*(s)]^q \frac{ds}{s} \right\}^{1/q} \qquad \text{since } f^* \text{ is decreasing,} \\ &\le \left(\frac{q}{p} \right)^{1/q} \| f \|_{L^{p,q}} . \end{aligned}$$

Hence, taking the supremum over all $t>0$, we obtain

$$\| f \|_{L^{p,\infty}} \le \left(\frac{q}{p} \right)^{1/q} \| f \|_{L^{p,q}} . \tag{1.4.7}$$

This establishes (1.4.6) in the case $r=\infty$. Finally, when $r<\infty$, we have

$$\| f \|_{L^{p,r}} = \left\{ \int_0^\infty [t^{1/p} f^*(t)]^{r-q+q} \frac{dt}{t} \right\}^{1/r} \le \| f \|_{L^{p,\infty}}^{(r-q)/r} \| f \|_{L^{p,q}}^{q/r} . \tag{1.4.8}$$

Inequality (1.4.7) combined with (1.4.8) gives (1.4.6) with $c_{p,q,r} = (q/p)^{(r-q)/rq}$. □

Unfortunately, the functionals $\| \cdot \|_{L^{p,q}}$ do not satisfy the triangle inequality. For instance, consider the functions $f(t)=t$ and $g(t)=1-t$ defined on $[0,1]$. Then $f^*(\alpha) = g^*(\alpha) = (1-\alpha)\chi_{[0,1]}(\alpha)$. A simple calculation shows that the inequality $\| f+g \|_{L^{p,q}} \le \| f \|_{L^{p,q}} + \| g \|_{L^{p,q}}$ would be equivalent to

$$\frac{p}{q} \le 2^q \, \frac{\Gamma(q+1)\Gamma(q/p)}{\Gamma(q+1+q/p)} ,$$

which fails in general. However, since for all $t>0$ we have

$$(f+g)^*(t) \le f^*(t/2) + g^*(t/2) ,$$

the estimate

$$\| f+g \|_{L^{p,q}} \le c_{p,q} \big(\| f \|_{L^{p,q}} + \| g \|_{L^{p,q}} \big) , \tag{1.4.9}$$

where $c_{p,q} = 2^{1/p} \max(1, 2^{(1-q)/q})$, is a consequence of (1.1.4). Also, if $\| f \|_{L^{p,q}} = 0$ then we must have $f=0$ μ-a.e. Therefore, $L^{p,q}$ is a quasi-normed space for all p,q with $0<p,q\le\infty$. Is this space complete with respect to its quasi-norm? The next theorem answers this question.

Theorem 1.4.11. *Let (X,μ) be a measure space. Then for all $0<p,q\le\infty$, the spaces $L^{p,q}(X,\mu)$ are complete with respect to their quasi-norm and they are therefore quasi-Banach spaces.*

Proof. We consider only the case $p < \infty$. First we note that convergence in $L^{p,q}$ implies convergence in measure. When $q = \infty$, this is proved in Proposition 1.1.9. When $q < \infty$, in view of Proposition 1.4.5 (16) and (1.4.7), it follows that

$$\sup_{t>0} t^{1/p} f^*(t) = \sup_{\alpha>0} \alpha d_f(\alpha)^{1/p} \le \left(\frac{q}{p}\right)^{1/q} \|f\|_{L^{p,q}}$$

for all $f \in L^{p,q}$, and from this it follows that convergence in $L^{p,q}$ implies convergence in measure.

Now let $\{f_n\}$ be a Cauchy sequence in $L^{p,q}$. Then $\{f_n\}$ is Cauchy in measure, and hence it has a subsequence $\{f_{n_k}\}$ that converges almost everywhere to some f by Theorem 1.1.13. Fix k_0 and apply property (9) in Proposition 1.4.5. Since $|f - f_{n_{k_0}}| = \lim_{k\to\infty} |f_{n_k} - f_{n_{k_0}}|$, it follows that

$$(f - f_{n_{k_0}})^*(t) \le \liminf_{k\to\infty} (f_{n_k} - f_{n_{k_0}})^*(t). \tag{1.4.10}$$

Raise (1.4.10) to the power q, multiply by $t^{q/p}$, integrate with respect to dt/t over $(0,\infty)$, and apply Fatou's lemma to obtain

$$\big\|f - f_{n_{k_0}}\big\|_{L^{p,q}}^q \le \liminf_{k\to\infty} \big\|f_{n_k} - f_{n_{k_0}}\big\|_{L^{p,q}}^q. \tag{1.4.11}$$

Now let $k_0 \to \infty$ in (1.4.11) and use the fact that $\{f_n\}$ is Cauchy to conclude that f_{n_k} converges to f in $L^{p,q}$. It is a general fact that if a Cauchy sequence has a convergent subsequence in a quasi-normed space, then the sequence is convergent to the same limit. It follows that f_n converges to f in $L^{p,q}$. □

Remark 1.4.12. It can be shown that the spaces $L^{p,q}$ are normable when p, q are bigger than 1; see Exercise 1.4.3. Therefore, these spaces can be normed to become Banach spaces.

It is well known that finitely simple functions are dense in L^p of any measure space, when $0 < p < \infty$. It is natural to ask whether finitely simple functions are also dense in $L^{p,q}$. This is in fact the case when $q \neq \infty$.

Theorem 1.4.13. *Finitely simple functions are dense in $L^{p,q}(X,\mu)$ when $0 < q < \infty$.*

Proof. Let $f \in L^{p,q}(X,\mu)$. Assume without loss of generality that $f \ge 0$. Since f lies in $L^{p,q} \subsetneq L^{p,\infty}$ we have $\mu(\{f > \varepsilon\})^{1/p}\varepsilon \le \|f\|_{L^{p,q}} < \infty$ for every $\varepsilon > 0$ and consequently for any $A > 0$, $\mu(\{f > A\})$ is finite and tends to zero as $A \to \infty$. Thus for every $n = 1,2,3,\dots$, there is an $A_n > 0$ such that $\mu(\{f > A_n\}) < 2^{-n}$.

For each $n = 1,2,3,\dots$ define the function

$$\varphi_n(x) = \sum_{k=0}^{1+2^n A_n} \frac{k}{2^n} \chi_{\{k2^{-n} < f \le (k+1)2^{-n}\}} \chi_{\{2^{-n} < f \le A_n\}}.$$

Then φ_n is supported in the set $\{2^{-n} < f \le A_n\}$ which has finite μ measure, thus φ_n is finitely simple and satisfies

$$f(x) - 2^{-n} \le \varphi_n(x) \le f(x)$$

for every $x \in \{x \in X : \ 2^{-n} < f(x) \le A_n\}$. It follows that

$$\mu(\{x \in X : \ |f(x) - \varphi_n(x)| > 2^{-n}\}) < 2^{-n}$$

which implies that $(f - \varphi_n)^*(t) \le 2^{-n}$ for $t \ge 2^{-n}$. Thus

$$(f - \varphi_n)^*(t) \to 0 \quad \text{as } n \to \infty \text{ and } \quad \varphi_n^*(t) \le f^*(t) \quad \text{for all } t > 0.$$

Since $(f - \varphi_n)^*(t) \le f^*(t)$, an application of the Lebesgue dominated convergence theorem gives that $\|\varphi_n - f\|_{L^{p,q}} \to 0$ as $n \to \infty$. □

Remark 1.4.14. One may wonder whether simple functions are dense in $L^{p,\infty}$. This turns out to be false for all $0 < p \le \infty$. However, countable linear combinations of characteristic functions of sets with finite measure are dense in $L^{p,\infty}(X,\mu)$. We call such functions *countably simple*. See Exercise 1.4.4 for details.

1.4.3 Duals of Lorentz Spaces

Given a quasi-Banach space Z with norm $\|\cdot\|_Z$, its dual Z^* is defined as the space of all continuous linear functionals T on Z equipped with the norm

$$\big\|T\big\|_{Z^*} = \sup_{\|x\|_Z=1} |T(x)|\,.$$

Observe that the dual of a quasi-Banach space is always a Banach space.

We are now considering the following question: What are the dual spaces $(L^{p,q})^*$ of $L^{p,q}$? The answer to this question presents some technical difficulties for general measure spaces. In this exposition we restrict our attention to σ-finite nonatomic measure spaces, where the situation is simpler.

Definition 1.4.15. A measurable subset A of a measure space (X,μ) is called an *atom* if $\mu(A) > 0$ and every measurable subset B of A has measure either equal to zero or equal to $\mu(A)$. A measure space (X,μ) is called *nonatomic* if it contains no atoms. In other words, X is nonatomic if and only if for any $A \subseteq X$ with $\mu(A) > 0$, there exists a proper subset $B \subsetneqq A$ with $\mu(B) > 0$ and $\mu(A \setminus B) > 0$.

For instance, **R** with Lebesgue measure is nonatomic, but any measure space with counting measure is atomic. Nonatomic spaces have the property that every measurable subset of them with strictly positive measure contains subsets of any given measure smaller than the measure of the original subset. See Exercise 1.4.5.

Theorem 1.4.16. *Suppose that* (X,μ) *is a nonatomic* σ*-finite measure space. Then*

(i)	$(L^{p,q})^* = \{0\}$,	*when* $0<p<1$, $0<q\le\infty$,
(ii)	$(L^{p,q})^* = L^\infty$,	*when* $p=1$, $0<q\le 1$,
(iii)	$(L^{p,q})^* = \{0\}$,	*when* $p=1$, $1<q<\infty$,
(iv)	$(L^{p,q})^* \neq \{0\}$,	*when* $p=1$, $q=\infty$,
(v)	$(L^{p,q})^* = L^{p',\infty}$,	*when* $1<p<\infty$, $0<q\le 1$,
(vi)	$(L^{p,q})^* = L^{p',q'}$,	*when* $1<p<\infty$, $1<q<\infty$,
(vii)	$(L^{p,q})^* \neq \{0\}$,	*when* $1<p<\infty$, $q=\infty$,
(viii)	$(L^{p,q})^* \neq \{0\}$,	*when* $p=q=\infty$.

Proof. Since X is σ-finite, we have $X=\bigcup_{N=1}^\infty K_N$, where K_N is an increasing sequence of sets with $\mu(K_N)<\infty$. Let $\mathscr{A}$ be the σ-algebra on which μ is defined and define $\mathscr{A}_N=\{A\cap K_N : A\in\mathscr{A}\}$. Given $T\in(L^{p,q})^*$, where $0<p,q<\infty$, for each $N=1,2,\dots$, consider the measure $\sigma_N(E)=T(\chi_E)$ defined on $\mathscr{A}_N$. Since σ_N satisfies $|\sigma_N(E)|\le (p/q)^{1/q}\|T\|\mu(E)^{1/p}$, it follows that σ_N is absolutely continuous with respect to μ restricted on $\mathscr{A}_N$. By the Radon–Nikodym theorem (see [153] (19.36)), there exists a unique (up to a set of μ-measure zero) complex-valued measurable function g_N which satisfies $\int_{K_N}|g_N|\,d\mu<\infty$ such that

$$\int_{K_N} f\,d\sigma_N = \int_{K_N} g_N f\,d\mu \tag{1.4.12}$$

for all f in $L^1(K_N,\mathscr{A}_N,\sigma_N)$. Since $\sigma_N=\sigma_{N+1}$ on $\mathscr{A}_N$, it follows that $g_N=g_{N+1}$ μ-a.e. on K_N and hence there is a well-defined measurable function g on X that coincides with each g_N on K_N. But the linear functionals $f\mapsto T(f)$ and $f\mapsto\int_{K_N} f\,d\sigma_N$ coincide on simple functions supported in K_N and therefore they must be equal on $L^1(K_N,\mathscr{A}_N,\sigma_N)\cap L^{p,q}(X,\mu)$ by density; consequently, (1.4.12) is also equal to $T(f)$ for f in $L^1(K_N,\mathscr{A}_N,\sigma_N)\cap L^{p,q}(X,\mu)$.

Note that if $f\in L^\infty(K_N,\mu)$, then $f\in L^{p,q}(K_N,\mu)$ and also in $L^\infty(K_N,\sigma_N)$, which is contained in $L^1(K_N,\mathscr{A}_N,\sigma_N)$. It follows from (1.4.12) and the preceding discussion that

$$T(f)=\int_X g f\,d\mu \tag{1.4.13}$$

for every $f\in L^\infty(K_N)$. We have now proved that for every linear functional T on $L^{p,q}(X,\mu)$ with $0<p,q<\infty$ there is a function g satisfying $\int_{K_N}|g|\,d\mu<\infty$ for all $N=1,2,\dots$ such that (1.4.13) holds for all $f\in L^\infty(K_N)$.

We now examine each case (i)–(viii) separately.

(i) We consider the case $0<p<1$. Let $f=\sum_n a_n\chi_{E_n}$ be a finitely simple function on X (which is taken to be countably simple when $q=\infty$). Since X is nonatomic, we split each E_n as a union of m disjoint sets $E_{j,n}$, $j=1,2,\dots,m$, each having measure $m^{-1}\mu(E_n)$. Let $f_j=\sum_n a_n\chi_{E_{j,n}}$. We see that $\|f_j\|_{L^{p,q}}=m^{-1/p}\|f\|_{L^{p,q}}$. Now if $T\in(L^{p,q})^*$, it follows that

$$|T(f)| \le \sum_{j=1}^{m} |T(f_j)| \le \|T\| \sum_{j=1}^{m} \|f_j\|_{L^{p,q}} = \|T\| m^{1-1/p} \|f\|_{L^{p,q}} .$$

Let $m \to \infty$ and use that $p < 1$ to obtain that $T = 0$.

(ii) We now consider the case $p = 1$ and $0 < q \le 1$. Clearly, every $h \in L^\infty$ gives a bounded linear functional on $L^{1,q}$, since

$$\left| \int_X f h \, d\mu \right| \le \|h\|_{L^\infty} \|f\|_{L^1} \le C_q \|h\|_{L^\infty} \|f\|_{L^{1,q}} .$$

Conversely, suppose that $T \in (L^{1,q})^*$ where $q \le 1$. The function g given in (1.4.13) satisfies

$$\left| \int_E g \, d\mu \right| = |T(\chi_E)| \le \|T\| q^{-1/q} \mu(E)$$

for all $E \subsetneqq K_N$, and hence $|g| \le q^{-1/q} \|T\|$ μ-a.e. on every K_N; see [307, Theorem 1.40 on p. 31] for a proof of this fact. It follows that $\|g\|_{L^\infty} \le q^{-1/q} \|T\|$ and hence $(L^{1,q})^* = L^\infty$.

(iii) Let us now take $p = 1$, $1 < q < \infty$, and suppose that $T \in (L^{1,q})^*$. Then

$$\left| \int_X f g \, d\mu \right| \le \|T\| \, \|f\|_{L^{1,q}}, \tag{1.4.14}$$

where g is the function in (1.4.13) and $f \in L^\infty(K_N)$. We will show that $g = 0$ a.e. Suppose that $|g| \ge \delta$ on some set E_0 with $\mu(E_0) > 0$. Then there exists N such that $\mu(E_0 \cap K_N) > 0$. Let $f = \overline{g} |g|^{-2} \chi_{E_0 \cap K_N} h \chi_{h \le M}$, where h is a nonnegative function. Then (1.4.14) implies for all $h \ge 0$ that

$$\|h \chi_{h \le M}\|_{L^1(E_0 \cap K_N)} \le \|T\| \, \|h \chi_{h \le M}\|_{L^{1,q}(E_0 \cap K_N)} .$$

Letting $M \to \infty$, we obtain that $L^{1,q}(E_0 \cap K_N)$ is contained in $L^1(E_0 \cap K_N)$, but since the reverse inclusion is always valid, these spaces must be equal. Since X is nonatomic, this can't happen; see Exercise 1.4.8 (d). Thus $g = 0$ μ-a.e. and $T = 0$.

(iv) In the case $p = 1$, $q = \infty$ an interesting phenomenon appears. Since every continuous linear functional on $L^{1,\infty}$ extends to a continuous linear functional on $L^{1,q}$ for $1 < q < \infty$, it must necessarily vanish on all simple functions by part (iii). However, $(L^{1,\infty})^*$ contains nontrivial linear functionals; see [84], [85].

(v) We now take up the case $1 < p < \infty$ and $0 < q \le 1$. Using Exercise 1.4.1 (b) and Proposition 1.4.10, we see that if $f \in L^{p,q}$ and $h \in L^{p',\infty}$, then

$$\begin{aligned} \int_X |f h| \, d\mu &\le \int_0^\infty t^{\frac{1}{p}} f^*(t) t^{\frac{1}{p'}} h^*(t) \, \frac{dt}{t} \\ &\le \|f\|_{L^{p,1}} \|h\|_{L^{p',\infty}} \\ &\le C_{p,q} \|f\|_{L^{p,q}} \|h\|_{L^{p',\infty}} ; \end{aligned}$$

thus every $h \in L^{p',\infty}$ gives rise to a bounded linear functional $f \mapsto \int h f \, d\mu$ on $L^{p,q}$ with norm at most $C_{p,q}\|h\|_{L^{p',\infty}}$. Conversely, let $T \in (L^{p,q})^*$ where $1 < p < \infty$ and $0 < q \le 1$. Let g satisfy (1.4.13) for all $f \in L^\infty(K_N)$. Taking $f = \overline{g}|g|^{-1}\chi_{K_N \cap \{|g|>\alpha\}}$ for $\alpha > 0$ and using that

$$\left| \int_X f g \, d\mu \right| \le \|T\| \|f\|_{L^{p,q}} ,$$

we obtain that

$$\alpha \mu(K_N \cap \{|g| > \alpha\}) \le (p/q)^{1/q} \|T\| \, \mu(K_N \cap \{|g| > \alpha\})^{\frac{1}{p}} .$$

Divide by $\mu(K_N \cap \{|g| > \alpha\})^{\frac{1}{p}}$, let $N \to \infty$, and take the supremum over $\alpha > 0$ to obtain that $\|g\|_{L^{p',\infty}} \le (p/q)^{1/q}\|T\|$.

(vi) Using Exercise 1.4.1 (b) and Hölder's inequality, we obtain

$$\left| \int_X f \varphi \, d\mu \right| \le \int_0^\infty t^{\frac{1}{p}} f^*(t) \, t^{\frac{1}{p'}} \varphi^*(t) \, \frac{dt}{t} \le \|f\|_{L^{p,q}} \|\varphi\|_{L^{p',q'}} ;$$

thus every $\varphi \in L^{p',q'}$ gives a bounded linear functional on $L^{p,q}$ with norm at most $\|\varphi\|_{L^{p',q'}}$. Conversely, let T be in $(L^{p,q})^*$. By (1.4.13), T is given by integration against a locally integrable function g. It remains to prove that $g \in L^{p',q'}$. We let $g_{N,M} = g \chi_{K_N} \chi_{|g| \le M}$. Then $(g_{N,M})^* \le g^*$ for all $M, N = 1, 2, \dots$ and $(g_{N,M})^* \uparrow g^*$ as $M, N \to \infty$ by Proposition 1.4.5 (4), (8).

For a bounded function f in $L^{p,q}(X)$ we have

$$\begin{aligned}
\int_0^\infty f^*(t)(g_{N,M})^*(t) \, dt &= \sup_{h:\, d_h = d_f} \left| \int_X h \, g_{N,M} \, d\mu \right| \\
&= \sup_{h:\, d_h = d_f} \left| \int_{K_N} h \chi_{|g| \le M} \, g \, d\mu \right| \\
&= \sup_{h:\, d_h = d_f} \left| T(h \chi_{K_N} \chi_{|g| \le M}) \right| \\
&\le \sup_{h:\, d_h = d_f} \|T\| \, \|h \chi_{K_N} \chi_{|g| \le M}\|_{L^{p,q}} \\
&\le \sup_{h:\, d_h = d_f} \|T\| \, \|h\|_{L^{p,q}} \\
&= \|T\| \, \|f\|_{L^{p,q}} ,
\end{aligned} \tag{1.4.15}$$

where the first equality is a consequence of the fact that X is nonatomic (see Exercise 1.4.5 (d)). Using the result of Exercise 1.4.5 (b), pick a function f on X such that

$$f^*(t) = \int_{t/2}^\infty s^{\frac{q'}{p'} - 1} (g_{N,M})^*(s)^{q'-1} \, \frac{ds}{s} , \tag{1.4.16}$$

noting that the preceding integral converges since $(g_{N,M})^*(s) \le M\chi_{[0,\mu(K_N)]}(s)$. It follows that $f^* \le c_{p,q}M^{q'-1}$, which implies that f is bounded, and also that $f^*(t) = 0$ when $t > 2\mu(K_N)$, which implies that f is supported in a set of measure at most $2\mu(K_N)$; thus the function f defined in (1.4.16) is bounded and lies in $L^{p,q}(X)$.

We have the following calculation regarding the $L^{p,q}$ norm of f:

$$\begin{aligned}\|f\|_{L^{p,q}} &= \left(\int_0^\infty t^{\frac{q}{p}}\left[\int_{t/2}^\infty s^{\frac{q'}{p'}-1}(g_{N,M})^*(s)^{q'-1}\frac{ds}{s}\right]^q\frac{dt}{t}\right)^{\frac{1}{q}} \\ &\le C_1(p,q)\left(\int_0^\infty (t^{\frac{1}{p'}}(g_{N,M})^*(t))^{q'}\frac{dt}{t}\right)^{\frac{1}{q}} \\ &= C_1(p,q)\|g_{N,M}\|_{L^{p',q'}}^{q'/q} < \infty,\end{aligned} \tag{1.4.17}$$

which is a consequence of Hardy's second inequality in Exercise 1.2.8 with $b = q/p$.

Using (1.4.15) and (1.4.17) we deduce that

$$\int_0^\infty f^*(t)(g_{N,M})^*(t)\,dt \le \|T\|\,\|f\|_{L^{p,q}} \le C_1(p,q)\|T\|\,\|g_{N,M}\|_{L^{p',q'}}^{q'-1}. \tag{1.4.18}$$

On the other hand, we have

$$\begin{aligned}\int_0^\infty f^*(t)(g_{N,M})^*(t)\,dt &\ge \int_0^\infty\int_{t/2}^t s^{\frac{q'}{p'}-1}(g_{N,M})^*(s)^{q'-1}\frac{ds}{s}\,(g_{N,M})^*(t)\,dt \\ &\ge \int_0^\infty (g_{N,M})^*(t)^{q'}\int_{t/2}^t s^{\frac{q'}{p'}-1}\frac{ds}{s}\,dt \\ &= C_2(p,q)\|g_{N,M}\|_{L^{p',q'}}^{q'}.\end{aligned} \tag{1.4.19}$$

Combining (1.4.18) and (1.4.19), and using the fact that $\|g_{N,M}\|_{L^{p',q'}} < \infty$, we obtain $\|g_{N,M}\|_{L^{p',q'}} \le C(p,q)\|T\|$. Letting $N,M \to \infty$ we deduce $\|g\|_{L^{p',q'}} \le C(p,q)\|T\|$ and this proves the reverse inequality required to complete case (vi).

(vii) For a complete characterization of this space, we refer to [83].

(viii) The dual of $L^\infty = L^{\infty,\infty}$ can be identified with the set of all bounded finitely additive set functions; see [99]. □

Remark 1.4.17. Some parts of Theorem 1.4.16 are false if X is atomic. For instance, the dual of $\ell^p(\mathbf{Z})$ contains ℓ^∞ when $0 < p < 1$ and thus it is not equal to $\{0\}$.

1.4.4 The Off-Diagonal Marcinkiewicz Interpolation Theorem

We now present the main result of this section, the off-diagonal extension of Marcinkiewicz's interpolation theorem (Theorem 1.3.2). For a measure space (X,μ), let $S(X)$ be the space of finitely simple functions on X and $S_0^+(X)$ be the subset of $S(X)$ of functions of the form

$$\sum_{i=m}^{n} 2^{-i}\chi_{A_i}$$

where $m < n$ are integers and A_i are subsets of X of finite measure. The sets A_i are not required to be different nor disjoint; consequently, the sum of two elements in $S_0^+(X)$ also belong to $S_0^+(X)$. We define $S_0^{real}(X) = S_0^+(X) - S_0^+(X)$ be the space of all functions of the form $f_1 - f_2$, where f_1, f_2 lie in $S_0^+(X)$ and $S_0(X)$ be the space of functions of the form $h_1 + ih_2$, where h_1, h_2 lie in $S_0^{real}(X)$.

An operator T defined on $S_0(X)$ is called *quasi-linear* if there is a $K \geq 1$ such that

$$|T(\lambda f)| = |\lambda|\,|T(f)| \qquad \text{and} \qquad |T(f+g)| \leq K(|T(f)| + |T(g)|),$$

for all $\lambda \in \mathbf{C}$ and all functions f, g in $S_0(X)$. If $K = 1$, then T is called *sublinear*.

Definition 1.4.18. Let T be a linear operator defined on the space of finitely simple functions $S(X)$ on a measure space (X, μ) and let $0 < p, q \leq \infty$. We say that T is of *restricted weak type* (p, q) if

$$\big\|T(\chi_A)\big\|_{L^{q,\infty}} \leq C\,\mu(A)^{1/p} \tag{1.4.20}$$

for all measurable subsets A of X with finite measure. Estimates of the form (1.4.20) are called *restricted weak type* estimates.

It is important to observe that if an operator is of restricted weak type (p_0, q_0) and of restricted weak type (p_1, q_1), then it is of restricted weak type (p, q), where the indices are as in (1.4.23). It will be a considerable effort to extend the latter estimate to all functions in $S_0(X)$. The next theorem addresses this extension.

Theorem 1.4.19. *Let $0 < r \leq \infty$, $0 < p_0 \neq p_1 \leq \infty$, and $0 < q_0 \neq q_1 \leq \infty$ and let (X, μ), (Y, ν) be σ-finite measure spaces. Let T be a quasi-linear operator defined on the space of simple functions on X and taking values in the set of measurable functions on Y. Assume that for some $M_0, M_1 < \infty$ the following restricted weak type estimates hold:*

$$\big\|T(\chi_A)\big\|_{L^{q_0,\infty}} \leq M_0\,\mu(A)^{1/p_0}\,, \tag{1.4.21}$$

$$\big\|T(\chi_A)\big\|_{L^{q_1,\infty}} \leq M_1\,\mu(A)^{1/p_1}\,, \tag{1.4.22}$$

for all measurable subsets A of X with $\mu(A) < \infty$. Fix $0 < \theta < 1$ and let

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}\,. \tag{1.4.23}$$

Then there exists a constant $C_*(p_0, q_0, p_1, q_1, K, r, \theta) < \infty$ such that for all functions f in $S_0(X)$ we have

$$\big\|T(f)\big\|_{L^{q,r}} \leq C_*(p_0, q_0, p_1, q_1, K, r, \theta) M_0^{1-\theta} M_1^{\theta} \big\|f\big\|_{L^{p,r}}\,. \tag{1.4.24}$$

Additionally, if $0 < p, r < \infty$ *and if* T *is linear (or sublinear with nonnegative values), then it admits a unique bounded extension from* $L^{p,r}(X)$ *to* $L^{q,r}(Y,\nu)$ *such that* (1.4.24) *holds for all* f *in* $L^{p,r}$.

Before we give the proof of Theorem 1.4.19, we state and prove a lemma that is interesting on its own.

Lemma 1.4.20. *Let* $0 < p < \infty$ *and* $0 < q \le \infty$ *and let* (X,μ), (Y,ν) *be* σ*-finite measure spaces. Let* T *be a quasi-linear operator defined on* $S(X)$ *and taking values in the set of measurable functions on* Y. *Suppose that there exists a constant* $M > 0$ *such that for all measurable subsets* A *of* X *of finite measure we have*

$$\big\|T(\chi_A)\big\|_{L^{q,\infty}} \le M\,\mu(A)^{\frac{1}{p}}. \tag{1.4.25}$$

Then for all α *with* $0 < \alpha < \min(q, \frac{\log 2}{\log 2K})$ *there exists a constant* $C(p,q,K,\alpha) > 0$ *such that for all functions* f *in* $S_0(X)$ *we have the estimate*

$$\big\|T(f)\big\|_{L^{q,\infty}} \le C(p,q,K,\alpha) M \big\|f\big\|_{L^{p,\alpha}} \tag{1.4.26}$$

where

$$C(p,q,K,\alpha) = 2^{8+\frac{2}{p}+\frac{2}{q}} K^3 \left(\frac{q}{q-\alpha}\right)^{\frac{2}{\alpha}} (1-2^{-\alpha})^{-\frac{1}{\alpha}} (\log 2)^{-\frac{1}{\alpha}}.$$

Proof. A function f in $S_0(X)$ can be written as $f = h_1 - h_2 + i(h_3 - h_4)$, where h_j are in $S_0^+(X)$. We write $f = f_1 - f_2 + i(f_3 - f_4)$, where $f_1 = \max(h_1 - h_2, 0)$, $f_2 = \max(-(h_1 - h_2), 0)$, $f_3 = \max(h_3 - h_4, 0)$, and $f_4 = \max(-(h_3 - h_4), 0)$. We note that f_j lie in $S_0^+(X)$; indeed, if $h_1 = \sum_\ell 2^{-\ell}\chi_{A_\ell}$ and $h_2 = \sum_k 2^{-k}\chi_{B_k}$, where both sums are finite, then

$$f_1 = \sum_{\ell:\, A_\ell \cap (\cup_k B_k) = \emptyset} 2^{-\ell}\chi_{A_\ell} + \sum_{(\ell,k):\, \ell<k,\, A_\ell \cap B_k \ne \emptyset} (2^{-\ell} - 2^{-k})\chi_{A_\ell \cap B_k}.$$

Since the second sum is equal to $\sum_{s=\ell+1}^{k} 2^{-s}\chi_{A_\ell \cap B_k}$, we obtain that $h_1 \in S_0^+(X)$. Likewise we can show that f_2, f_3, f_4 lie in $S_0^+(X)$. Moreover, we have $f_j \le |f|$ and Proposition 1.4.5(4), yields

$$\|f_j\|_{L^{p,\alpha}(X)} \le \|f\|_{L^{p,\alpha}(X)}$$

for all $j = 1,2,3,4$. Suppose now that (1.4.26) holds for functions in $S_0^+(X)$ with constant $C'(p,q,\alpha)$ in place of $C(p,q,K,\alpha)$. By the quasi-linearity of T we have

$$\begin{aligned}\|T(f)\|_{L^{q,\infty}(Y)} &\leq K^3 \left\| \sum_{j=1}^{4} |T(f_j)| \right\|_{L^{q,\infty}(Y)} \\ &\leq K^3 4^{1+\frac{1}{q}} \sum_{j=1}^{4} \|T(f_j)\|_{L^{q,\infty}(Y)} \\ &\leq K^3 4^{2+\frac{2}{q}} M C'(p,q,K,\alpha) \, \|f\|_{L^{p,\alpha}(X)}\end{aligned}$$

which proves (1.4.26) for all f in $S_0(X)$ with constant

$$C(p,q,K,\alpha) = 2^{4+\frac{2}{q}} K^3 C'(p,q,\alpha) \, .$$

We now prove (1.4.26) for functions in $S_0^+(X)$ with constant $C'(p,q,\alpha)$ in place of $C(p,q,K,\alpha)$. It follows from the Aoki–Rolewicz theorem (Exercise 1.4.6) that for all $N \in \mathbf{Z}^+$ and for all $f_1, \dots, f_N$ in $S_0^+(X)$ we have the pointwise inequality

$$|T(f_1 + \cdots + f_N)| \leq 4 \left(\sum_{j=1}^{N} |T(f_j)|^{\alpha_1} \right)^{\frac{1}{\alpha_1}} \leq 4 \left(\sum_{j=1}^{N} |T(f_j)|^{\alpha} \right)^{\frac{1}{\alpha}} , \qquad (1.4.27)$$

where $0 < \alpha \leq \alpha_1$ and α_1 satisfies the equation $(2K)^{\alpha_1} = 2$. The second inequality in (1.4.27) is a simple consequence of the fact that $\alpha \leq \alpha_1$. Fix α_0 with

$$0 < \alpha_0 \leq \alpha_1 = \frac{\log 2}{\log 2K} \qquad \text{and} \qquad \alpha_0 < q \, .$$

This ensures that the quasi-normed space $L^{q/\alpha,\infty}$ is normable when $\alpha \leq \alpha_0$. In fact, since Y is σ-finite, Exercise 1.1.12 gives that the space $L^{s,\infty}$ is normable as long as $s > 1$ and there is an equivalent norm $|||f|||_{L^{s,\infty}}$ such that

$$\big\|f\big\|_{L^{s,\infty}} \leq \big|\big|\big| \, f \, \big|\big|\big|_{L^{s,\infty}} \leq \frac{s}{s-1} \big\|f\big\|_{L^{s,\infty}} .$$

Next we claim that for any nonnegative function f in $S_0^+(X)$ we have

$$\big\|T(f\chi_A)\big\|_{L^{q,\infty}} \leq 4 \left(\frac{q}{q-\alpha}\right)^{\frac{1}{\alpha}} (1-2^{-\alpha})^{-\frac{1}{\alpha}} M \mu(A)^{\frac{1}{p}} \big\|f\chi_A\big\|_{L^\infty} . \qquad (1.4.28)$$

To show this, we write $f = \sum_{j=m}^{n} 2^{-j} \chi_{S_j}$, where $m < n$ are integers, S_j are subsets of X of finite measure for all $j \in \{m, m+1, \dots, n\}$, $\mu(S_m) \neq 0$ and $\mu(S_n) \neq 0$. Setting $B_j = S_j \cap A$ we have

$$f\chi_A = \sum_{j=m}^{n} 2^{-j} \chi_{B_j}$$

and $2^{-m} \leq \|f\chi_A\|_{L^\infty(X)}$.

We use (1.4.27) once and (1.4.3) twice in the following argument. We have

$$\begin{aligned}
\big\|T(f\chi_A)\big\|_{L^{q,\infty}} &\le 4\bigg\|\Big(\sum_{j=m}^{n} 2^{-j\alpha}|T(\chi_{B_j})|^{\alpha}\Big)^{\frac{1}{\alpha}}\bigg\|_{L^{q,\infty}} \\
&= 4\bigg\|\sum_{j=m}^{n} 2^{-j\alpha}|T(\chi_{B_j})|^{\alpha}\bigg\|_{L^{q/\alpha,\infty}}^{\frac{1}{\alpha}} \\
&\le 4\,\bigg|\bigg|\bigg|\sum_{j=m}^{n} 2^{-j\alpha}|T(\chi_{B_j})|^{\alpha}\bigg|\bigg|\bigg|_{L^{q/\alpha,\infty}}^{\frac{1}{\alpha}} \\
&\le 4\bigg(\sum_{j=m}^{n} 2^{-j\alpha}\,\Big|\Big|\Big|\,|T(\chi_{B_j})|^{\alpha}\,\Big|\Big|\Big|_{L^{q/\alpha,\infty}}\bigg)^{\frac{1}{\alpha}} \\
&\le 4\Big(\frac{q}{q-\alpha}\Big)^{\frac{1}{\alpha}}\bigg(\sum_{j=m}^{n} 2^{-j\alpha}\Big\||T(\chi_{B_j})|^{\alpha}\Big\|_{L^{q/\alpha,\infty}}\bigg)^{\frac{1}{\alpha}} \\
&= 4\Big(\frac{q}{q-\alpha}\Big)^{\frac{1}{\alpha}}\bigg(\sum_{j=m}^{n} 2^{-j\alpha}\big\|T(\chi_{B_j})\big\|_{L^{q,\infty}}^{\alpha}\bigg)^{\frac{1}{\alpha}} \\
&\le 4\Big(\frac{q}{q-\alpha}\Big)^{\frac{1}{\alpha}} M\bigg(\sum_{j=m}^{n} 2^{-j\alpha}\mu(B_j)^{\frac{\alpha}{p}}\bigg)^{\frac{1}{\alpha}} \\
&\le 4\Big(\frac{q}{q-\alpha}\Big)^{\frac{1}{\alpha}}(1-2^{-\alpha})^{-\frac{1}{\alpha}} M\mu(A)^{\frac{1}{p}}2^{-m},
\end{aligned}$$

using $B_j \subseteqq A$. Using that $2^{-m} \le \|f\chi_A\|_{L^\infty}$ establishes (1.4.28).

We now apply (1.4.28) to obtain (1.4.26). For any $f \in S_0^+(X)$ we define measurable sets

$$A_k = \{x\in X:\ f^*(2^{k+1}) < |f(x)| \le f^*(2^k)\} \tag{1.4.29}$$

and we note that these sets are pairwise disjoint. We may write the finitely simple function f as $\sum_{j=1}^{n} a_j\chi_{E_j}$, where $0<a_j<\infty$, $E_1 \subseteqq E_2 \subseteqq \cdots \subseteqq E_n$ and $0<\mu(E_j)<\infty$ for $j\in\{1,2,\dots,n\}$. Clearly, we have

$$f^* = \sum_{j=1}^{n} a_j\chi_{[0,\mu(E_j))}.$$

Thus, when $t\in(\mu(E_n),\infty)$, $f^*(t)$ vanishes, and when $t\in(0,\mu(E_1))$, $f^*(t)=\sum_{j=1}^{n} a_j$ is a positive constant. So there exists $N\in\mathbf{Z}^+$ such that $f^*(2^k)=0$ when $k>N$, and that $f^*(2^k)$ is a positive constant when $k<-N$. This also implies that $A_k=\emptyset$ if $|k|>N$ and thus we express

$$f = \sum_{k=-N}^{N} f\chi_{A_k}.$$

Proposition 1.4.5(2) implies $\mu(A_k) \le d_f(f^*(2^{k+1})) \le 2^{k+1}$. Using (1.4.27) we obtain

$$\begin{aligned}
\big\|T(f)\big\|_{L^{q,\infty}(Y)} &\le 4\Big\|\Big(\sum_{k=-N}^{N}|T(f\chi_{A_k})|^\alpha\Big)^{\frac{1}{\alpha}}\Big\|_{L^{q,\infty}(Y)} \\
&= 4\Big\|\sum_{k=-N}^{N}|T(f\chi_{A_k})|^\alpha\Big\|_{L^{q/\alpha,\infty}(Y)}^{\frac{1}{\alpha}} \\
&\le 4\,\Big|\Big|\Big|\sum_{k=-N}^{N}|T(f\chi_{A_k})|^\alpha\Big|\Big|\Big|_{L^{q/\alpha,\infty}(Y)}^{\frac{1}{\alpha}} \\
&\le 4\Big(\sum_{k=-N}^{N}\Big|\Big|\Big|\,|T(f\chi_{A_k})|^\alpha\Big|\Big|\Big|_{L^{q/\alpha,\infty}(Y)}\Big)^{\frac{1}{\alpha}} \\
&\le 4\Big(\frac{q}{q-\alpha}\Big)^{\frac{1}{\alpha}}\Big(\sum_{k=-N}^{N}\Big\||T(f\chi_{A_k})|^\alpha\Big\|_{L^{q/\alpha,\infty}(Y)}\Big)^{\frac{1}{\alpha}} \\
&\le 4\left(\frac{q}{q-\alpha}\right)^{\frac{1}{\alpha}}\left(\sum_{k=-N}^{N}\|T(f\chi_{A_k})\|_{L^{q,\infty}(Y)}^{\alpha}\right)^{\frac{1}{\alpha}} \\
&\le 16\left(\frac{q}{q-\alpha}\right)^{\frac{2}{\alpha}}(1-2^{-\alpha})^{-\frac{1}{\alpha}}M\left(\sum_{k=-N}^{N}\mu(A_k)^{\frac{\alpha}{p}}\big\|f\chi_{A_k}\big\|_{L^\infty}^{\alpha}\right)^{\frac{1}{\alpha}} \\
&\le 16\left(\frac{q}{q-\alpha}\right)^{\frac{2}{\alpha}}(1-2^{-\alpha})^{-\frac{1}{\alpha}}2^{\frac{1}{p}}M\left(\sum_{k=-\infty}^{\infty}[f^*(2^k)]^\alpha 2^{\frac{k\alpha}{p}}\right)^{\frac{1}{\alpha}} \\
&\le 16\left(\frac{q}{q-\alpha}\right)^{\frac{2}{\alpha}}(1-2^{-\alpha})^{-\frac{1}{\alpha}}2^{\frac{2}{p}}(\log 2)^{-\frac{1}{\alpha}}M\|f\|_{L^{p,\alpha}(X)},
\end{aligned}$$

where we made use of (1.4.28) and in the last inequality, we used

$$\begin{aligned}
\|f\|_{L^{p,\alpha}(X)}^{\alpha} &= \sum_{k=-\infty}^{\infty}\int_{2^{k-1}}^{2^k} t^{\frac{\alpha}{p}}[f^*(t)]^\alpha\,\frac{dt}{t} \\
&\ge \sum_{k=-\infty}^{\infty}(2^{k-1})^{\frac{\alpha}{p}}[f^*(2^k)]^\alpha\int_{2^{k-1}}^{2^k}\frac{dt}{t} \\
&= 2^{-\frac{\alpha}{p}}\log 2\sum_{k=-\infty}^{\infty}[f^*(2^k)]^\alpha 2^{\frac{k\alpha}{p}}.
\end{aligned}$$

This completes the proof of the required inequality for nonnegative functions in $S_0^+(X)$ with constant

$$C'(p,q,\alpha) = 16\left(\frac{q}{q-\alpha}\right)^{\frac{2}{\alpha}}(1-2^{-\alpha})^{-\frac{1}{\alpha}}2^{\frac{2}{p}}(\log 2)^{-\frac{1}{\alpha}}$$

As noted, the constant in general is $C(p,q,K,\alpha) = 2^{4+\frac{2}{p}}K^3C'(p,q,\alpha)$. $\square$

We now proceed with the proof of Theorem 1.4.19.

Proof. We assume that $p_0 < p_1$, since if $p_0 > p_1$ we may simply reverse the roles of p_0 and p_1. We first consider the case $p_1, r < \infty$. Lemma 1.4.20 implies that

$$\begin{aligned} \big\|T(f)\big\|_{L^{q_0,\infty}} &\le M_0' \big\|f\big\|_{L^{p_0,m}}\,, \\ \big\|T(f)\big\|_{L^{q_1,\infty}} &\le M_1' \big\|f\big\|_{L^{p_1,m}}\,, \end{aligned} \tag{1.4.30}$$

for all f in $S_0(X)$, where $m = \frac{1}{2}\min\big(q_0, q_1, \frac{\log 2}{\log 2K}, 2r\big)$, $M_0' = C(p_0, q_0, K, m)M_0$, $M_1' = C(p_1, q_1, K, m)M_1$, and $C(p, q, K, \alpha)$ is as in (1.4.26).

Fix a function f in $S_0(X)$. Split $f = f^t + f_t$ as follows:

$$f^t(x) = \begin{cases} f(x) & \text{if } |f(x)| > f^*(\delta t^\gamma), \\ 0 & \text{if } |f(x)| \le f^*(\delta t^\gamma), \end{cases}$$

$$f_t(x) = \begin{cases} 0 & \text{if } |f(x)| > f^*(\delta t^\gamma), \\ f(x) & \text{if } |f(x)| \le f^*(\delta t^\gamma), \end{cases}$$

where δ is to be determined later and γ is the following nonzero real number:

$$\gamma = \frac{\frac{1}{q_0} - \frac{1}{q}}{\frac{1}{p_0} - \frac{1}{p}} = \frac{\frac{1}{q} - \frac{1}{q_1}}{\frac{1}{p} - \frac{1}{p_1}}\,.$$

Using Exercise 1.1.10 we write

$$d_{f^t}(v) = \begin{cases} d_f(v) & \text{when } v > f^*(\delta t^\gamma) \\ d_f(f^*(\delta t^\gamma)) & \text{when } v \le f^*(\delta t^\gamma) \end{cases}$$

$$d_{f_t}(v) = \begin{cases} 0 & \text{when } v \ge f^*(\delta t^\gamma) \\ d_f(v) - d_f(f^*(\delta t^\gamma)) & \text{when } v < f^*(\delta t^\gamma). \end{cases}$$

Observe the following facts

$$\begin{aligned} v \ge \delta t^\gamma \implies (f^t)^*(v) &\le \inf\big\{s \in (0, f^*(\delta t^\gamma)] :\ d_{f^t}(s) \le v\big\} \\ &= \inf\big\{s \in (0, f^*(\delta t^\gamma)] :\ d_f(f^*(\delta t^\gamma)) \le v\big\} \\ &= \inf(0, f^*(\delta t^\gamma)] \\ &= 0, \end{aligned}$$

$$\begin{aligned} v < \delta t^\gamma \implies (f^t)^*(v) &\le \inf\big\{s > f^*(\delta t^\gamma) :\ d_{f^t}(s) \le v\big\} \\ &= \inf\big\{s > f^*(\delta t^\gamma) :\ d_f(s) \le v\big\} \\ &= \inf\Big\{\big\{s > 0 :\ d_f(s) \le v\big\} \cap \big(f^*(\delta t^\gamma), \infty\big)\Big\} \\ &= f^*(v), \qquad\qquad \text{since } f^*(v) \ge f^*(\delta t^\gamma), \end{aligned}$$

$$\begin{aligned} v \geq \delta t^\gamma \implies (f_t)^*(v) &= \inf\{s>0:\ d_{f_t}(s) \leq v\} \\ &\leq \inf\{s>0:\ d_f(s) \leq v\} \qquad \text{since } d_{f_t} \leq d_f \\ &= f^*(v), \end{aligned}$$

$$\begin{aligned} v < \delta t^\gamma \implies (f_t)^*(v) &= \inf\{s>0:\ d_{f_t}(s) \leq v\} \\ &\leq f^*(\delta t^\gamma), \qquad \text{since } f^*(\delta t^\gamma) \in \{s>0:\ d_{f_t}(s) \leq v\}. \end{aligned}$$

We summarize these observations in a couple of inequalities:

$$(f^t)^*(s) \ \leq \begin{cases} f^*(s) & \text{if } 0 < s < \delta t^\gamma, \\ 0 & \text{if } s \geq \delta t^\gamma, \end{cases}$$

$$(f_t)^*(s) \ \leq \begin{cases} f^*(\delta t^\gamma) & \text{if } 0 < s < \delta t^\gamma, \\ f^*(s) & \text{if } s \geq \delta t^\gamma. \end{cases}$$

It follows from these inequalities that f^t lies in $L^{p_0,m}$ and f_t lies in $L^{p_1,m}$ for all $t>0$.

The quasi-linearity of the operator T and (1.4.9) imply

$$\begin{aligned} &\big\|T(f)\big\|_{L^{q,r}} \\ &= \big\|t^{\frac{1}{q}} T(f)^*(t)\big\|_{L^r(\frac{dt}{t})} \\ &\leq K\big\|t^{\frac{1}{q}} (|T(f_t)| + |T(f^t)|)^*(t)\big\|_{L^r(\frac{dt}{t})} \\ &\leq K\big\|t^{\frac{1}{q}} T(f_t)^*(\tfrac{t}{2}) + t^{\frac{1}{q}} T(f^t)^*(\tfrac{t}{2})\big\|_{L^r(\frac{dt}{t})} \\ &\leq K a_r \Big(\big\|t^{\frac{1}{q}} T(f_t)^*(\tfrac{t}{2})\big\|_{L^r(\frac{dt}{t})} + \big\|t^{\frac{1}{q}} T(f^t)^*(\tfrac{t}{2})\big\|_{L^r(\frac{dt}{t})}\Big) \\ &\leq K \max\{1, 2^{\frac{1}{r}-1}\}\Big(\big\|t^{\frac{1}{q}} T(f_t)^*(\tfrac{t}{2})\big\|_{L^r(\frac{dt}{t})} + \big\|t^{\frac{1}{q}} T(f^t)^*(\tfrac{t}{2})\big\|_{L^r(\frac{dt}{t})}\Big). \end{aligned} \tag{1.4.31}$$

It follows from (1.4.30) that

$$t^{\frac{1}{q_0}} T(f^t)^*(\tfrac{t}{2}) \ \leq 2^{\frac{1}{q_0}} \sup_{s>0} s^{\frac{1}{q_0}} T(f^t)^*(s) \leq 2^{\frac{1}{q_0}} M_0' \big\|f^t\big\|_{L^{p_0,m}}, \tag{1.4.32}$$

$$t^{\frac{1}{q_1}} T(f_t)^*(\tfrac{t}{2}) \ \leq 2^{\frac{1}{q_1}} \sup_{s>0} s^{\frac{1}{q_1}} T(f_t)^*(s) \leq 2^{\frac{1}{q_1}} M_1' \big\|f_t\big\|_{L^{p_1,m}}, \tag{1.4.33}$$

for all $t>0$. Now use (1.4.32), (1.4.33), and the facts that

$$t^{\frac{1}{q}} T(f^t)^*(\tfrac{t}{2}) = t^{\frac{1}{q}-\frac{1}{q_0}} t^{\frac{1}{q_0}} T(f^t)^*(\tfrac{t}{2}) \leq t^{\frac{1}{q}-\frac{1}{q_0}} 2^{\frac{1}{q_0}} M_0' \big\|f^t\big\|_{L^{p_0,m}}$$

$$t^{\frac{1}{q}} T(f^t)^*(\tfrac{t}{2}) = t^{\frac{1}{q}-\frac{1}{q_1}} t^{\frac{1}{q_1}} T(f^t)^*(\tfrac{t}{2}) \leq t^{\frac{1}{q}-\frac{1}{q_1}} 2^{\frac{1}{q_1}} M_1' \big\|f^t\big\|_{L^{p_1,m}},$$

to estimate (1.4.31) by

$$K\max\{1,2^{\frac{1}{r}-1}\}\left[2^{\frac{1}{q_0}}M_0'\left\|t^{\frac{1}{q}-\frac{1}{q_0}}\left\|f^t\right\|_{L^{p_0,m}}\right\|_{L^r(\frac{dt}{t})}+2^{\frac{1}{q_1}}M_1'\left\|t^{\frac{1}{q}-\frac{1}{q_1}}\left\|f_t\right\|_{L^{p_1,m}}\right\|_{L^r(\frac{dt}{t})}\right],$$

which is the same as

$$K\max\{1,2^{\frac{1}{r}-1}\}2^{\frac{1}{q_0}}M_0'\left\|t^{-\gamma(\frac{1}{p_0}-\frac{1}{p})}\left\|f^t\right\|_{L^{p_0,m}}\right\|_{L^r(\frac{dt}{t})} \tag{1.4.34}$$

$$+K\max\{1,2^{\frac{1}{r}-1}\}2^{\frac{1}{q_1}}M_1'\left\|t^{\gamma(\frac{1}{p}-\frac{1}{p_1})}\left\|f_t\right\|_{L^{p_1,m}}\right\|_{L^r(\frac{dt}{t})}. \tag{1.4.35}$$

Next, we change variables $u=\delta t^\gamma$ in the L^r quasi-norm in (1.4.34) to obtain

$$\begin{aligned}
&\left\|t^{-\gamma(\frac{1}{p_0}-\frac{1}{p})}\left\|f^t\right\|_{L^{p_0,m}}\right\|_{L^r(\frac{dt}{t})}\\
&\quad\le \frac{\delta^{\frac{1}{p_0}-\frac{1}{p}}}{|\gamma|^{\frac{1}{r}}}\left\|u^{-(\frac{1}{p_0}-\frac{1}{p})}\left(\int_0^u f^*(s)^m s^{\frac{m}{p_0}}\frac{ds}{s}\right)^{\frac{1}{m}}\right\|_{L^r(\frac{du}{u})}\\
&\quad\le \frac{\delta^{\frac{1}{p_0}-\frac{1}{p}}}{|\gamma|^{\frac{1}{r}}}\left[\frac{\frac{r}{m}}{r(\frac{1}{p_0}-\frac{1}{p})}\right]^{\frac{1}{m}}\left(\int_0^\infty (s^{\frac{1}{p_0}}f^*(s))^r s^{-r(\frac{1}{p_0}-\frac{1}{p})}\frac{ds}{s}\right)^{\frac{1}{r}}\\
&\quad= \frac{\delta^{\frac{1}{p_0}-\frac{1}{p}}}{m^{\frac{1}{m}}|\gamma|^{\frac{1}{r}}(\frac{1}{p_0}-\frac{1}{p})^{\frac{1}{m}}}\left\|f\right\|_{L^{p,r}},
\end{aligned}$$

where the last inequality is a consequence of Hardy's inequality:

$$\left(\int_0^\infty\left(\int_0^u g(s)\frac{ds}{s}\right)^p u^{-b}\frac{du}{u}\right)^{\frac{1}{p}}\le\frac{p}{b}\left(\int_0^\infty g(u)^p u^{-b}\frac{du}{u}\right)^{\frac{1}{p}} \tag{1.4.36}$$

with $g(s)=f^*(s)^m s^{m/p_0}\ge 0$, $p=r/m\ge 1$ and $b=r/p_0-r/p>0$. See Exercise 1.2.8 for the proof of (1.4.36).

Likewise, change variables $u=\delta t^\gamma$ in the L^r quasi-norm of (1.4.35) to obtain

$$\begin{aligned}
&\left\|t^{\gamma(\frac{1}{p}-\frac{1}{p_1})}\left\|f_t\right\|_{L^{p_1,m}}\right\|_{L^r(\frac{dt}{t})}\\
&\le \frac{\delta^{-(\frac{1}{p}-\frac{1}{p_1})}}{|\gamma|^{\frac{1}{r}}}\left\|u^{\frac{1}{p}-\frac{1}{p_1}}\left[\int_0^u f^*(u)^m s^{\frac{m}{p_1}}\frac{ds}{s}+\int_u^\infty f^*(s)^m s^{\frac{m}{p_1}}\frac{ds}{s}\right]^{\frac{1}{m}}\right\|_{L^r(\frac{du}{u})}\\
&= \frac{\delta^{-(\frac{1}{p}-\frac{1}{p_1})}}{|\gamma|^{\frac{1}{r}}}\left\|u^{\frac{m}{p}-\frac{m}{p_1}}\int_0^u f^*(u)^m s^{\frac{m}{p_1}}\frac{ds}{s}+\int_u^\infty f^*(s)^m s^{\frac{m}{p_1}}\frac{ds}{s}\right\|_{L^{r/m}(\frac{du}{u})}^{\frac{1}{m}}
\end{aligned}$$

$$\leq \frac{\delta^{-(\frac{1}{p}-\frac{1}{p_1})}}{|\gamma|^{\frac{1}{r}}} \Bigg\{ \Big\| u^{\frac{m}{p}-\frac{m}{p_1}} f^*(u)^m \int_0^u s^{\frac{m}{p_1}} \frac{ds}{s} \Big\|_{L^{r/m}(\frac{du}{u})} + \Big\| u^{\frac{m}{p}-\frac{m}{p_1}} \int_u^\infty f^*(s)^m s^{\frac{m}{p_1}} \frac{ds}{s} \Big\|_{L^{r/m}(\frac{du}{u})} \Bigg\}^{\frac{1}{m}}$$

$$\leq \frac{\delta^{-(\frac{1}{p}-\frac{1}{p_1})}}{|\gamma|^{\frac{1}{r}}} \Bigg\{ \frac{p_1}{m} \|f\|_{L^{p,r}}^m + \frac{\frac{r}{m}}{r(\frac{1}{p}-\frac{1}{p_1})} \bigg(\int_0^\infty \big(f^*(u)^m u^{\frac{m}{p_1}}\big)^{\frac{r}{m}} u^{\frac{r}{p}-\frac{r}{p_1}} \frac{du}{u} \bigg)^{\frac{m}{r}} \Bigg\}^{\frac{1}{m}}$$

$$= \frac{\delta^{-(\frac{1}{p}-\frac{1}{p_1})}}{m^{\frac{1}{m}}|\gamma|^{\frac{1}{r}}} \Bigg\{ \frac{\frac{p_1}{p}}{\frac{1}{p}-\frac{1}{p_1}} \Bigg\}^{\frac{1}{m}} \|f\|_{L^{p,r}},$$

where the last inequality above is Hardy's inequality:

$$\bigg(\int_0^\infty \Big(\int_u^\infty g(s) \frac{ds}{s} \Big)^p u^b \frac{du}{u} \bigg)^{\frac{1}{p}} \leq \frac{p}{b} \bigg(\int_0^\infty g(u)^p u^b \frac{du}{u} \bigg)^{\frac{1}{p}} \tag{1.4.37}$$

with $g(s) = f^*(s)^m s^{m/p_1} \geq 0$, $p = r/m \geq 1$ and $b = r/p - r/p_1 > 0$. See Exercise 1.2.8 for the proof of (1.4.37).

Combining these elements we deduce that given f in $S_0(X)$, we have that the expression in (1.4.34) plus the expression in (1.4.35) is at most

$$\frac{K \max\{1, 2^{\frac{1}{r}-1}\}}{m^{\frac{1}{m}}|\gamma|^{\frac{1}{r}}} \Bigg\{ \frac{2^{\frac{1}{q_0}} M_0' \delta^{\frac{1}{p_0}-\frac{1}{p}}}{(\frac{1}{p_0}-\frac{1}{p})^{\frac{1}{m}}} + \frac{2^{\frac{1}{q_1}} (\frac{p_1}{p})^{\frac{1}{m}} M_1' \delta^{-(\frac{1}{p}-\frac{1}{p_1})}}{(\frac{1}{p}-\frac{1}{p_1})^{\frac{1}{m}}} \Bigg\} \|f\|_{L^{p,r}}$$

We choose $\delta > 0$ such that the two terms in the curly brackets above are equal. We deduce that

$$\|T(f)\|_{L^{q,r}} \leq \frac{2K \max\{1, 2^{\frac{1}{r}-1}\}}{m^{\frac{1}{m}}|\gamma|^{\frac{1}{r}}} \Bigg\{ \frac{2^{\frac{1-\theta}{q_0}} (M_0')^{1-\theta}}{(\frac{1}{p_0}-\frac{1}{p})^{\frac{1-\theta}{m}}} \frac{2^{\frac{\theta}{q_1}} (\frac{p_1}{p})^{\frac{\theta}{m}} (M_1')^{\theta}}{(\frac{1}{p}-\frac{1}{p_1})^{\frac{\theta}{m}}} \Bigg\} \|f\|_{L^{p,r}}$$

where θ is as in (1.4.23), i.e.,

$$\theta = \frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}.$$

This proves (1.4.24) in the case $p_1, r < \infty$ with constant $C_*(p_0, q_0, p_1, q_1, K, r, \theta)$ equal to

$$\frac{2K \max\{1, 2^{\frac{1}{r}-1}\}}{m^{\frac{1}{m}}|\gamma|^{\frac{1}{r}}} \Bigg\{ \frac{2^{\frac{1-\theta}{q_0}} C(p_0, q_0, K, m)^{1-\theta} 2^{\frac{\theta}{q_1}} (\frac{p_1}{p})^{\frac{\theta}{m}} C(p_1, q_1, K, m)^{\theta}}{(\frac{1}{p_0}-\frac{1}{p})^{\frac{1-\theta}{m}} (\frac{1}{p}-\frac{1}{p_1})^{\frac{\theta}{m}}} \Bigg\},$$

where we recall that $m = \frac{1}{2}\min\big(q_0, q_1, \frac{\log 2}{\log 2K}, 2r\big)$ and $C(p_j, q_j, K, m)$ is as in Lemma 1.4.20.

We now turn to the remaining cases $p = \infty$ or $r = \infty$. The restriction $r < \infty$ can be removed since $C_*(p_0, q_0, p_1, q_1, K, r, \theta)$ has a finite limit as $r \to \infty$ and, moreover, $\|f\|_{L^{p,r}} = \|t^{1/p} f^*(t)\|_{L^r(dt/t)} \to \|t^{1/p} f^*(t)\|_{L^\infty(dt/t)} = \|f\|_{L^{p,\infty}}$ as $r \to \infty$ and likewise $\|T(f)\|_{L^{p,r}} \to \|T(f)\|_{L^{p,\infty}}$ as $r \to \infty$; see Exercise 1.1.3 (a). The restriction $p_1 < \infty$ can be removed as follows. Suppose that $p_1 = \infty$. Then, since $\theta \in (0,1)$ it follows that $p < \infty$ and we pick $p_2 > p$ and $p_2 < \infty$. It is easy to see T satisfies the restricted weak type (p_2, q_2) estimate

$$\sup_{\alpha>0} \alpha \nu(\{|T(\chi_A)| > \alpha\})^{\frac{1}{q_2}} \le M_0^{1-\varphi} M_1^{\varphi} \mu(A)^{\frac{1}{p_2}},$$

where

$$\frac{1-\varphi}{p_0} + \frac{\varphi}{\infty} = \frac{1}{p_2}, \qquad \frac{1-\varphi}{q_0} + \frac{\varphi}{q_1} = \frac{1}{q_2}. \tag{1.4.38}$$

Using the result obtained when $p_1 < \infty$ with p_2 in place of p_1 we obtain that

$$\big\|T(f)\big\|_{L^{q,r}} \le C_*(p_0, q_0, p_2, q_2, K, r, \rho) M_0^{1-\rho} (M_0^{1-\varphi} M_1^{\varphi})^{\rho} \big\|f\big\|_{L^{p,r}} \tag{1.4.39}$$

for all functions f in $S_0(X)$, where

$$\frac{1-\rho}{p_0} + \frac{\rho}{p_2} = \frac{1}{p}, \qquad \frac{1-\rho}{q_0} + \frac{\rho}{q_2} = \frac{1}{q}. \tag{1.4.40}$$

Combining (1.4.38) and (1.4.40) and using (1.4.23) we deduce that $\theta = \rho\varphi$ and hence (1.4.39) yields (1.4.24) in the case where $p_1 = \infty$. In this case we have

$$C_*(p_0, q_0, \infty, q_1, K, r, \theta) = C_*(p_0, q_0, \tfrac{p_0}{1-\varphi}, (\tfrac{1-\varphi}{q_0} + \tfrac{\varphi}{q_1})^{-1}, K, r, \tfrac{\theta}{\varphi}),$$

where φ is any number satisfying $1 > \varphi > 1 - \frac{p_0}{p}$.

Finally, we address the last assertion of the theorem which claims that when $p, r < \infty$ and $K = 1$, the linear (or sublinear with nonnegative values) operator T initially defined on finitely simple functions has a unique bounded extension from $L^{p,r}(X)$ to $L^{q,r}(Y)$, which also satisfies (1.4.24) (with the same constant). To obtain this conclusion, we will need to know that the space $S_0(X)$ is dense in $L^{p,r}(X)$ whenever $0 < p, r < \infty$. This is proved in Proposition 1.4.21 below. Assuming this proposition, we define the extension of T on $L^{p,r}(X)$ as follows:

Given f in $L^{p,r}(X)$ a sequence of functions f_j in $S_0(X)$ that converge to f in $L^{p,r}(X)$, notice that the linearity (or the sublinearity and the fact that $T(f) \ge 0$ for all f in $S_0(X)$) implies

$$|T(f_j) - T(f_k)| \le |T(f_j - f_k)|.$$

Using the boundedness of T from $L^{p,r}(X)$ to $L^{q,r}(Y)$ we obtain that the sequence $\{T(f_j)\}_j$ is Cauchy in $L^{q,r}(Y)$ and by the completeness of this space, it must converge to a limit which we call $\overline{T}(f)$. We observe that $\overline{T}(f)$ is independent of the choice of the sequence $\{f_j\}_j$ that converges to f in $L^{p,r}$. Moreover, one can show that $\overline{T}$ is linear (or sublinear with nonnegative values), $\overline{T}(f)$ coincides with $T(f)$

on $S_0(X)$ and $\overline{T}$ is bounded from $L^{p,r}(X)$ to $L^{q,r}(Y)$. Thus $\overline{T}$ is the unique bounded extension of T on the entire space $L^{p,r}(X)$. For details, see Exercise 1.4.17. □

Proposition 1.4.21. *For all* $0 < p, r < \infty$ *the space* $S_0(X)$ *is dense in* $L^{p,r}(X)$.

Proof. Let $f \in L^{p,r}(X)$ and assume first that $f \geq 0$. Using (1.4.5) and the fact that d_f is decreasing on $[0,\infty)$, we obtain for any $n \in \mathbf{Z}^+$,

$$\begin{aligned} \|f\|_{L^{p,r}(X)}^r &= p\int_0^\infty \left[d_f(s)^{\frac{1}{p}}s\right]^r \frac{ds}{s} \\ &\geq p\int_0^{2^{-n}} \left[d_f(2^{-n})\right]^{\frac{r}{p}} s^{r-1}\,ds \\ &= \frac{p2^{-nr}}{r}\left[d_f(2^{-n})\right]^{\frac{r}{p}}, \end{aligned}$$

which implies that $d_f(2^{-n}) < \infty$. Likewise, again in view of (1.4.5), we have

$$\|f\|_{L^{p,r}(X)}^r \geq p\int_0^{2^n} \left[d_f(s)\right]^{\frac{r}{p}} s^{r-1}\,ds = \frac{p2^{nr}}{r}\left[d_f(2^n)\right]^{\frac{r}{p}},$$

which implies that $\lim_{n\to\infty} d_f(2^n) = 0$. Thus, for any $n \in \mathbf{Z}^+$, there exists $k_n \in \mathbb{N}$ such that

$$d_f(2^{k_n}) = \mu\left(\left\{x \in X:\ f(x) > 2^{k_n}\right\}\right) < 2^{-n}.$$

Let $E_n = \left\{x \in X:\ 2^{-n} < f(x) \leq 2^{k_n}\right\}$ and note that $\mu\left(E_n\right) \leq d_f\left(2^{-n}\right) < \infty$ for each $n \in \mathbf{Z}^+$. We write $f\chi_{E_n}$ in binary expansion, that is, $f\chi_{E_n}(x) = \sum_{j=-k_n}^\infty d_j(x)2^{-j}$, where $d_j(x) = 0$ or 1. Let $B_j = \{x \in E_n : d_j(x) = 1\}$. Then, $\mu(B_j) \leq \mu(E_n)$ and $f\chi_{E_n}$ can be expressed as $f\chi_{E_n} = \sum_{j=-k_n}^\infty 2^{-j}\chi_{B_j}$.

Set $f_n = \sum_{j=-k_n}^n 2^{-j}\chi_{B_j}$. It is obvious that $f_n \in S_0^+(X)$ and $f_n \leq f\chi_{E_n} \leq f$. Observe that when $x \in E_n$, we have

$$f(x) - f_n(x) = \sum_{j=n+1}^\infty 2^{-j}\chi_{B_j} \leq 2^{-n},$$

and that when $x \notin E_n$, we have $f_n(x) = 0$ and $f(x) > 2^{k_n}$ or $f(x) \leq 2^{-n}$. It follows from these facts that

$$d_{f-f_n}(2^{-n}) = \mu\left(E_n \cap \{f - f_n > 2^{-n}\}\right) + \mu\left(E_n^c \cap \{f - f_n > 2^{-n}\}\right) < 2^{-n}.$$

Hence, for $2^{-n} \leq t < \infty$ one has

$$(f-f_n)^*(t) \leq (f-f_n)^*(2^{-n}) = \inf\left\{s > 0 : d_{f-f_n}(s) \leq 2^{-n}\right\} \leq 2^{-n}.$$

This implies that $\lim_{n\to\infty}(f-f_n)^*(t) = 0$ for all $t \in (0,\infty)$. By Proposition 1.4.5 (5), (6), we obtain for all $t \in (0,\infty)$

$$(f-f_n)^*(t) \leq f^*(t/2) + f_n^*(t/2) \leq 2f^*(t/2).$$

The Lebesgue dominated convergence theorem gives $\|f_n - f\|_{L^{p,r}(X)} \to 0$ as $n \to \infty$ which yields the required conclusion for nonnegative functions f in $L^{p,r}(X)$.

For a complex-valued function $f \in L^{p,r}(X)$, we write $f = f_1 - f_2 + i(f_3 - f_4)$, where f_j are nonnegative functions in $L^{p,r}(X)$. By the preceding conclusion, there exist sequences $\{f_n^j\}_{n\in\mathbf{Z}^+}$, $j = 1,2,3,4$, in $S_0^+(X)$ such that $f_n^j \to f_j$ in $L^{p,r}(X)$ as $n \to \infty$. Set $f_n = f_n^1 - f_n^2 + i(f_n^3 - f_n^4)$. Using the fact that $\|\cdot\|_{L^{p,r}(X)}$ is a quasi-norm we obtain

$$\|f - f_n\|_{L^{p,r}(X)} \le C(p,r) \sum_{j=1}^{4} \|f_j - f_n^j\|_{L^{p,r}(X)}$$

which tends to zero as $n \to \infty$. This completes the proof. □

Corollary 1.4.22. *Let T be as in the statement of Theorem 1.4.19 and let $0 < p_0 \neq p_1 \le \infty$ and $0 < q_0 \neq q_1 \le \infty$. If T is restricted weak type (p_0,q_0) and (p_1,q_1) with constants M_0 and M_1, respectively, and for some $0 < \theta < 1$ we have*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

and $p \le q$, then T satisfies the strong type estimate

$$\|T(f)\|_{L^q} \le C(p_0,q_0,p_1,q_1,\theta) M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \tag{1.4.41}$$

for all f in $S_0(X)$. Moreover, if T is linear (or sublinear with nonnegative values), then it has a unique bounded extension from $L^p(X,\mu)$ to $L^q(Y,\nu)$ that satisfies estimate (1.4.41) for all $f \in L^p(X)$ with the constant $C(p_0,q_0,p_1,q_1,\theta)$ replaced by $C(p_0,q_0,p_1,q_1,\theta)2^{2/p}\max(1,2^{1/p-1})^2$.

Proof. Since $\theta \in (0,1)$ we must have $p,q < \infty$. Take $r = q$ in Theorem 1.4.19 and note that $\|f\|_{L^{p,r}} \le \|f\|_{L^p}$ since $p \le q = r$; see Proposition 1.4.10. The last assertion follows using Exercise 1.4.17. □

We now give examples to indicate why the assumptions $p_0 \neq p_1$ and $q_0 \neq q_1$ cannot be dropped in Theorem 1.4.19.

Example 1.4.23. Let $X = Y = \mathbf{R}$ and

$$T(f)(x) = |x|^{-1/2} \int_0^1 f(t)\,dt\,.$$

Then $\alpha|\{x: \ |T(\chi_A)(x)| > \alpha\}|^{1/2} = 2^{1/2}|A \cap [0,1]|$ and thus T is of restricted weak types $(1,2)$ and $(3,2)$. But observe that T does not map $L^2 = L^{2,2}$ to $L^{q,2}$. Thus Theorem 1.4.19 fails if the assumption $q_0 \neq q_1$ is dropped. The dual operator

$$S(f)(x) = \chi_{[0,1]}(x) \int_{-\infty}^{+\infty} f(t)|t|^{-1/2}\,dt$$

satisfies $\alpha|\{x:\ |S(\chi_A)(x)|>\alpha\}|^{1/q}\le c|A|^{1/2}$ when $q=1$ or 3, and thus it furnishes an example of an operator of restricted weak types $(2,1)$ and $(2,3)$ that is not L^2 bounded. Thus Theorem 1.4.19 fails if the assumption $p_0\neq p_1$ is dropped.

As an application of Theorem 1.4.19, we give the following strengthening of Theorem 1.2.13.

We end this chapter with a corollary of the proof of Theorem 1.4.19.

Corollary 1.4.24. *Let $1\le r<\infty$, $1\le p_0\neq p_1<\infty$, and $0<q_0\neq q_1\le\infty$ and let (X,μ) and (Y,ν) be σ-finite measure spaces. Let T be a quasi-linear operator defined on $L^{p_0}(X)+L^{p_1}(X)$ and taking values in the set of measurable functions on Y. Assume that for some $M_0',M_1'<\infty$ the following estimates hold for $j=0,1$*

$$\big\|T(f)\big\|_{L^{q_j,\infty}(Y)}\le M_j'\big\|f\big\|_{L^{p_j}(X)}, \tag{1.4.42}$$

for all functions $f\in L^{p_j}(X)$. Fix $0<\theta<1$ and let

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\qquad \text{and}\qquad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}. \tag{1.4.43}$$

Then there exists a constant $C_*(p_0,q_0,p_1,q_1,K,r,\theta)<\infty$ such that for all functions f in $L^p(X)$ we have

$$\big\|T(f)\big\|_{L^{q,p}}\le C_*(p_0,q_0,p_1,q_1,K,r,\theta)(M_0')^{1-\theta}(M_1')^{\theta}\big\|f\big\|_{L^p}. \tag{1.4.44}$$

Proof. Since $L^p(X)$ is contained in the sum $L^{p_0}(X)+L^{p_1}(X)$, the operator T is well defined on $L^p(X)$. Hypothesis (1.4.42) implies that (1.4.30) holds for all $f\in L^{p_j,1}$. Repeat the proof of Theorem 1.4.19 starting from (1.4.30) fixing a function f in $L^p(X)$, $m=1$ and $r=p$. We obtain the required conclusion.

Theorem 1.4.25. ***(Young's inequality for weak type spaces)*** *Let G be a locally compact group with left Haar measure λ that satisfies (1.2.12) for all measurable subsets A of G. Let $1<p,q,r<\infty$ satisfy*

$$\frac{1}{q}+1=\frac{1}{p}+\frac{1}{r}. \tag{1.4.45}$$

Then there exists a constant $B_{p,q,r}>0$ such that for all f in $L^p(G)$ and g in $L^{r,\infty}(G)$ we have

$$\big\|f*g\big\|_{L^q(G)}\le B_{p,q,r}\big\|g\big\|_{L^{r,\infty}(G)}\big\|f\big\|_{L^p(G)}. \tag{1.4.46}$$

Proof. We fix $1<p,q<\infty$. Since p and q range in an open interval, we can find $p_0<p<p_1$, $q_0<q<q_1$, and $0<\theta<1$ such that (1.4.23) and (1.4.45) hold. Let $T(f)=f*g$, defined for all functions f on G. By Theorem 1.2.13, T extends to a bounded operator from L^{p_0} to $L^{q_0,\infty}$ and from L^{p_1} to $L^{q_1,\infty}$. It follows from the Corollary 1.4.24 that T extends to a bounded operator from $L^p(G)$ to $L^q(G)$. Notice that since G is locally compact, (G,λ) is a σ-finite measure space and for this reason, we were able to apply Corollary 1.4.24. □

Exercises

1.4.1. (a) Let g be a nonnegative integrable function on a measure space (X,μ) and let A be a measurable subset of X. Prove that

$$\int_A g\,d\mu \le \int_0^{\mu(A)} g^*(t)\,dt.$$

(b) (*G. H. Hardy and J. E. Littlewood*) For f and g measurable on a σ-finite measure space (X,μ), prove that

$$\int_X |f(x)g(x)|\,d\mu(x) \le \int_0^\infty f^*(t)g^*(t)\,dt.$$

Compare this result to the classical Hardy–Littlewood result asserting that for $a_j, b_j > 0$, the sum $\sum_j a_j b_j$ is greatest when both a_j and b_j are rearranged in decreasing order (for this see [148, p. 261]).

1.4.2. Let (X,μ) be a measure space. Prove that if $f \in L^{q_0,\infty}(X) \cap L^{q_1,\infty}(X)$ for some $0 < q_0 < q_1 \le \infty$, then $f \in L^{q,s}(X)$ for all $0 < s \le \infty$ and $q_0 < q < q_1$.

1.4.3. ([164]) Given $0 < p,q < \infty$, fix an $r = r(p,q) > 0$ such that $r \le 1$, $r \le q$ and $r < p$. Let (X,μ) be a measure space. For $t < \mu(X)$ define

$$f^{**}(t) = \sup_{\mu(E)\ge t} \left(\frac{1}{\mu(E)} \int_E |f|^r\,d\mu \right)^{1/r},$$

while for $t \ge \mu(X)$ (if $\mu(X) < \infty$) let

$$f^{**}(t) = \left(\frac{1}{t} \int_X |f|^r\,d\mu \right)^{1/r}.$$

Also define

$$|||\, f \,|||_{L^{p,q}} = \left(\int_0^\infty \left(t^{\frac{1}{p}} f^{**}(t)\right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

(The function f^{**} and the functional $f \to |||f|||_{L^{p,q}}$ depend on r.)
(a) Prove that the inequality $(((f+g)^{**})(t))^r \le (f^{**}(t))^r + (g^{**}(t))^r$ is valid for all $t \ge 0$. Since $r \le q$, conclude that the functional $f \to |||f|||_{L^{p,q}}^r$ is subadditive and hence it is a norm when $r = 1$ (this is possible only if $p > 1$).
(b) Show that for all f we have

$$\|f\|_{L^{p,q}} \le |||\, f \,|||_{L^{p,q}} \le \left(\frac{p}{p-r} \right)^{1/r} \|f\|_{L^{p,q}}.$$

(c) Conclude that $L^{p,q}(X)$ is metrizable and normable when $1 < p,q < \infty$.

1.4.4. Show that on a measure space (X,μ) the set of countable linear combinations of simple functions is dense in $L^{p,\infty}(X)$.
(b) Prove that finitely simple functions are not dense in $L^{p,\infty}(\mathbf{R})$ for any $0<p\le\infty$.
$\big[$*Hint:* Part (b): Show that the function $h(x)=x^{-1/p}\chi_{x>0}$ cannot be approximated in $L^{p,\infty}$ by a sequence of finitely simple functions. Given a finitely simple function s which is nonzero on a set A with $|A|>0$, show that $\|s-h\|_{L^{p,\infty}}\ge\sup_{0<\lambda<|A|^{1/p}}\lambda(\lambda^{-p}-|A|)^{1/p}=1.\big]$

1.4.5. Let (X,μ) be a nonatomic measure space.
(a) If $A_0\subsetneqq A_1\subsetneqq X$, $0<\mu(A_1)<\infty$, and $\mu(A_0)\le t\le\mu(A_1)$, show that there exists an $E_t\subsetneqq A_1$ with $\mu(E_t)=t$.
(b) Given a nonnegative continuous and decreasing function φ on $[0,\infty)$ such that $\varphi(t)=0$ whenever $t\ge\mu(X)$, prove that there exists a measurable function f on X with $f^*(t)=\varphi(t)$ for all $t>0$.
(c) Given $A\subsetneqq X$ with $0<\mu(A)<\infty$ and g an integrable function on X, show that there exists a subset $\widetilde{A}$ of X with $\mu(\widetilde{A})=\mu(A)$ such that

$$\int_{\widetilde{A}}|g|\,d\mu=\int_0^{\mu(A)}g^*(s)\,ds.$$

(d) If X is σ-finite, $f\in L^\infty(X)$, and $g\in L^1(X)$, prove that

$$\sup_{h:\,d_h=d_f}\left|\int_X h\,g\,d\mu\right|=\int_0^\infty f^*(s)g^*(s)\,ds,$$

where the supremum is taken over all functions h on X equidistributed with f.
$\big[$*Hint:* Part (a): Reduce matters to the situation in which $A_0=\emptyset$. Consider first the case that for all $A\subsetneqq X$ there exists a subset B of X satisfying $\frac{1}{10}\mu(A)\le\mu(B)\le\frac{9}{10}\mu(A)$. Then we can find subsets of A_1 of measure in any arbitrarily small interval, and by continuity the required conclusion follows. Next consider the case in which there is a subset A_1 of X such that every $B\subsetneqq A_1$ satisfies $\mu(B)<\frac{1}{10}\mu(A_1)$ or $\mu(B)>\frac{9}{10}\mu(A_1)$. Without loss of generality, normalize μ so that $\mu(A_1)=1$. Let $\mu_1=\sup\{\mu(C):\ C\subsetneqq A_1,\ \mu(C)<\frac{1}{10}\}$ and pick $B_1\subsetneqq A_1$ such that $\frac12\mu_1\le\mu(B_1)\le\mu_1$. Set $A_2=A_1\setminus B_1$ and define $\mu_2=\sup\{\mu(C):\ C\subsetneqq A_2,\ \mu(C)<\frac{1}{10}\}$. Continue in this way and define sets $A_1\supsetneqq A_2\supsetneqq A_3\supsetneqq\cdots$ and numbers $\frac{1}{10}\ge\mu_1\ge\mu_2\ge\mu_3\ge\cdots$. If $C\subsetneqq A_{n+1}$ with $\mu(C)<\frac{1}{10}$, then $C\cup B_n\subsetneqq A_n$ with $\mu(C\cup B_n)<\frac15<\frac{9}{10}$, and hence by assumption we must have $\mu(C\cup B_n)<\frac{1}{10}$. Conclude that $\mu_{n+1}\le\frac12\mu_n$ and that $\mu(A_n)\ge\frac45$ for all $n=1,2,\dots$. Then the set $\bigcap_{n=1}^\infty A_n$ must be an atom. Part (b): First show that when d is a simple right continuous decreasing function on $[0,\infty)$ there exists a measurable f on X such that $f^*=d$. For general continuous functions, use approximation. Part (c): Let $t=\mu(A)$ and define $A_1=\{x:\ |g(x)|>g^*(t)\}$ and $A_2=\{x:\ |g(x)|\ge g^*(t)\}$. Then $A_1\subsetneqq A_2$ and $\mu(A_1)\le t\le\mu(A_2)$. Pick $\widetilde{A}$ such that $A_1\subsetneqq\widetilde{A}\subsetneqq A_2$ and $\mu(\widetilde{A})=t=\mu(A)$ by part (a). Then $\int_{\widetilde{A}}g\,d\mu=\int_X g\chi_{\widetilde{A}}\,d\mu=\int_0^\infty(g\chi_{\widetilde{A}})^*\,ds=\int_0^{\mu(\widetilde{A})}g^*(s)\,ds$. Part (d): Reduce matters to functions $f,g\ge0$. Let

$f = \sum_{j=1}^{N} a_j \chi_{A_j}$ where $a_1 > a_2 > \cdots > a_N > 0$ and the A_j are pairwise disjoint. Write f as $\sum_{j=1}^{N} b_j \chi_{B_j}$, where $b_j = (a_j - a_{j+1})$ and $B_j = A_1 \cup \cdots \cup A_j$. Pick $\widetilde{B}_j$ as in part (c). Then $\widetilde{B}_1 \subsetneqq \cdots \subsetneqq \widetilde{B}_N$ and the function $f_1 = \sum_{j=1}^{N} b_j \chi_{\widetilde{B}_j}$ has the same distribution function as f. It follows from part (c) that $\int_X f_1 g \, d\mu = \int_0^\infty f^*(s) g^*(s) \, ds$. The case of a general $f \in L^\infty(X)$ follows by approximation by finitely simple functions.$\big]$

1.4.6. ([7], [297]) Let $K \geq 1$ and let $\| \cdot \|$ be a nonnegative functional on a vector space X that satisfies

$$\|x+y\| \leq K\big(\|x\| + \|y\|\big)$$

for all $x, y \in X$. For a fixed $\alpha \leq 1$ satisfying $(2K)^\alpha = 2$ show that

$$\|x_1 + \cdots + x_n\|^\alpha \leq 4\,(\|x_1\|^\alpha + \cdots + \|x_n\|^\alpha)$$

for all $n = 1, 2, \dots$ and all $x_1, x_2, \dots, x_n$ in X. This inequality is referred to as the *Aoki-Rolewicz theorem.*

$\big[$*Hint:* Quasi-linearity implies that $\|x_1 + \cdots + x_n\| \leq \max_{1 \leq j \leq n}[(2K)^j \|x_j\|]$ for all $x_1, \dots, x_n$ in X (use that $K \geq 1$). Define $H : X \to \mathbf{R}$ by setting $H(0) = 0$ and $H(x) = 2^{j/\alpha}$ if $2^{j-1} < \|x\|^\alpha \leq 2^j$. Then $\|x\| \leq H(x) \leq 2^{1/\alpha}\|x\|$ for all $x \in X$. Prove by induction that $\|x_1 + \cdots + x_n\|^\alpha \leq 2(H(x_1)^\alpha + \cdots + H(x_n)^\alpha)$. Suppose that this statement is true when $n = m$. To show its validity for $n = m+1$, without loss of generality assume that $\|x_1\| \geq \|x_2\| \geq \cdots \geq \|x_{m+1}\|$. Then $H(x_1) \geq H(x_2) \geq \cdots \geq H(x_{m+1})$. Assume that all the $H(x_j)$'s are distinct. Then since $H(x_j)^\alpha$ are distinct powers of 2, they must satisfy $H(x_j)^\alpha \leq 2^{-j+1} H(x_1)^\alpha$. Then

$$\begin{aligned}
\|x_1 + \cdots + x_{m+1}\|^\alpha &\leq \big[\max_{1 \leq j \leq m+1} (2K)^j \|x_j\|\big]^\alpha \\
&\leq \big[\max_{1 \leq j \leq m+1} (2K)^j H(x_j)\big]^\alpha \\
&\leq \big[\max_{1 \leq j \leq m+1} (2K)^j 2^{1/\alpha} 2^{-j/\alpha} H(x_1)\big]^\alpha \\
&= 2H(x_1)^\alpha \\
&\leq 2(H(x_1)^\alpha + \cdots + H(x_{m+1})^\alpha)\,.
\end{aligned}$$

We now consider the case that $H(x_j) = H(x_{j+1})$ for some $1 \leq j \leq m$. Then for some integer r we must have $2^{r-1} < \|x_{j+1}\|^\alpha \leq \|x_j\|^\alpha \leq 2^r$ and $H(x_j) = 2^{r/\alpha}$. Next note that

$$\|x_j + x_{j+1}\|^\alpha \leq K^\alpha(\|x_j\| + \|x_{j+1}\|)^\alpha \leq K^\alpha(2\,2^{r/\alpha})^\alpha = 2^{r+1}.$$

This implies

$$H(x_j + x_{j+1})^\alpha \leq 2^{r+1} = 2^r + 2^r = H(x_j)^\alpha + H(x_{j+1})^\alpha.$$

Now apply the inductive hypothesis to $x_1, \dots, x_{j-1}, x_j + x_{j+1}, x_{j+1}, \dots, x_m$ and use the previous inequality to obtain the required conclusion.$\big]$

1.4.7. (a) ([347]) Let (X, μ) and (Y, ν) be measure spaces. Let Z be a Banach space of complex-valued measurable functions on Y. Assume that Z is closed under absolute values and satisfies $\|f\|_Z = \| \, |f| \, \|_Z$. Suppose that T is a linear operator defined on the space of finitely simple functions on (X,μ) and taking values in Z. Suppose that for some constant $A > 0$ the following restricted weak type estimate

$$\big\|T(\chi_E)\big\|_Z \le A\mu(E)^{1/p}$$

holds for some $0 < p < \infty$ and for all E measurable subsets of X of finite measure. Show that for all finitely simply functions f on X we have

$$\big\|T(f)\big\|_Z \le p^{-1}A\big\|f\big\|_{L^{p,1}}.$$

Consequently T has a bounded extension from $L^{p,1}(X)$ to Z.
(b) ([172]) As an application of part (a) prove that for any U, V measurable subsets of $\mathbf{R}^n$ with $|U|,|V| < \infty$ and any f measurable on $U \times V$ we have

$$\left(\int_U \big\|f(u,\cdot)\big\|^2_{L^{2,1}(V)}\,du\right)^{\frac{1}{2}} \le \frac{1}{2}\,\big\|f\big\|_{L^{2,1}(U\times V)}.$$

[*Hint:* Part (a): Let $f = \sum_{j=1}^N a_j\chi_{E_j} \ge 0$, where $a_1 > a_2 > \cdots > a_N > 0$, $\mu(E_j) < \infty$ pairwise disjoint. Let $F_j = E_1 \cup \cdots \cup E_j$, $B_0 = 0$, and $B_j = \mu(F_j)$ for $j \ge 1$. Write $f = \sum_{j=1}^N (a_j - a_{j+1})\chi_{F_j}$, where $a_{N+1} = 0$. Then

$$\begin{aligned}
\big\|T(f)\big\|_Z &= \big\|\,|T(f)|\,\big\|_Z \\
&\le \sum_{j=1}^N (a_j - a_{j+1})\big\|T(\chi_{F_j})\big\|_Z \\
&\le A\sum_{j=1}^N (a_j - a_{j+1})(\mu(F_j))^{1/p} \\
&= A\sum_{j=0}^{N-1} a_{j+1}(B_{j+1}^{1/p} - B_j^{1/p}) \\
&= p^{-1}A\big\|f\big\|_{L^{p,1}},
\end{aligned}$$

where the penultimate equality follows by a summation by parts; see Appendix F.]

1.4.8. Let $0 < p,q,\alpha,\beta < \infty$. Also let $0 < q_1 < q_2 < \infty$.
(a) Show that the function $f_{\alpha,\beta}(t) = t^{-\alpha}(\log t^{-1})^{-\beta}\chi_{[0,e^{-\beta/\alpha})}(t)$ lies in $L^{p,q}(\mathbf{R})$ if and only if either $p < 1/\alpha$ or both $p = 1/\alpha$ and $q > 1/\beta$ hold. Conclude that the function $t \mapsto t^{-1/p}(\log t^{-1})^{-1/q_1}\chi_{[0,e^{-p/q_1})}(t)$ lies in $L^{p,q_2}(\mathbf{R})$ but not in $L^{p,q_1}(\mathbf{R})$.
(b) Find a necessary and sufficient condition in terms of p,α,β for the function $g_{\alpha,\beta}(t) = (1+t)^{-\alpha}(\log(2+t))^{-\beta}\chi_{[0,\infty)}$ to lie in $L^{p,q}(\mathbf{R})$.

(c) Let $\psi(t)$ be smooth decreasing function on $[0,\infty)$ and let $F(x)=\psi(|x|)$ for x in $\mathbf{R}^n$, where $|x|$ is the modulus of x. Show that $F^*(t)=f((t/v_n)^{1/n})$, where v_n is the volume of the unit ball. Use this formula to construct examples showing that $L^{p,q_1}(\mathbf{R}^n)\subsetneqq L^{p,q_2}(\mathbf{R}^n)$.
(d) On a general nonatomic measure space (X,μ) prove that there *does not* exist a constant $C(p,q_1,q_2)>0$ such that for all f in $L^{p,q_2}(X)$ the following is valid:

$$\big\|f\big\|_{L^{p,q_1}}\le C(p,q_1,q_2)\big\|f\big\|_{L^{p,q_2}}\,.$$

$\big[$*Hint:* Parts (a), (b): Use that $f_{\alpha,\beta}$ and $g_{\alpha,\beta}$ are equal to their decreasing rearrangements. Part (d): Use Exercise 1.4.5 (b) with $\varphi(t)=g_{1/p,1/q_1}(t)$.$\big]$

1.4.9. ([346]) Let $L^p(\omega)$ denote the space of all measurable functions f on $\mathbf{R}^n$ such that $\|f\|_{L^p(\omega)}^p=\int_{\mathbf{R}^n}|f(x)|^p\,\omega(x)\,dx<\infty$, where $0<\omega<\infty$ a.e. Let T be a sublinear operator that maps $L^{p_0}(\omega_0)$ to $L^{q_0,\infty}(\omega)$ and $L^{p_1}(\omega_1)$ to $L^{q_1,\infty}(\omega)$, where ω_0,ω_1,ω are positive functions and $1\le p_0<p_1<\infty$, $0<q_0,q_1<\infty$. Suppose that

$$\frac{1}{p_\theta}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\,,\qquad \frac{1}{q_\theta}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}\,.$$

Let $\Omega_\theta=\omega_0^{\frac{1-\theta}{p_0}p_\theta}\omega_1^{\frac{\theta}{p_1}p_\theta}$. Show that T maps $L^{p_\theta}\big(\Omega_\theta\big)\to L^{q_\theta,p_\theta}(\omega)$.
$\big[$*Hint:* Define $L(f)=(\omega_1/\omega_0)^{\frac{1}{p_1-p_0}}f$ and observe that for each $\theta\in[0,1]$, L maps $L^{p_\theta}\big(\Omega_\theta\big)\to L^{p_\theta}\big((\omega_0^{p_1}\omega_1^{-p_0})^{\frac{1}{p_1-p_0}}\big)$ isometrically. Then apply Corollary 1.4.24 to the sublinear operator $T\circ L^{-1}$.$\big]$

1.4.10. ([185], [349]) Let λ_n be a sequence of positive numbers with $\sum_n\lambda_n\le 1$ and $\sum_n\lambda_n\log(\frac{1}{\lambda_n})=K<\infty$. Suppose all sequences are indexed by a fixed countable set.
(a) Let f_n be a sequence of complex-valued functions in $L^{1,\infty}(X)$ with $\|f_n\|_{L^{1,\infty}}\le 1$ uniformly in n. Prove that $\sum_n\lambda_nf_n$ lies in $L^{1,\infty}(X)$ with norm at most $2(K+2)$. (This property is referred to as the *logconvexity* of $L^{1,\infty}$.)
(b) Let T_n be a sequence of sublinear operators that map $L^1(X)$ to $L^{1,\infty}(Y)$ with norms $\|T_n\|_{L^1\to L^{1,\infty}}\le B$ uniformly in n. Use part (a) to prove that $\sum_n\lambda_nT_n$ maps $L^1(X)$ to $L^{1,\infty}(Y)$ with norm at most $2B(K+2)$.
(c) Given $\delta>0$ pick $0<\varepsilon<\delta$ and use the simple estimate

$$\mu\big(\{\sum_{n=1}^{\infty}2^{-\delta n}f_n>\alpha\}\big)\le\sum_{n=1}^{\infty}\mu\big(\{2^{-\delta n}f_n>(2^\varepsilon-1)2^{-\varepsilon n}\alpha\}\big)$$

to obtain a simple proof of the statement in part (a) when $\lambda_n=2^{-\delta n}$, $n=1,2,\dots$.
$\big[$*Hint:* Part (a): For fixed $\alpha>0$, write $f_n=u_n+v_n+w_n$, where $u_n=f_n\chi_{|f_n|\le\frac{\alpha}{2}}$, $v_n=f_n\chi_{|f_n|>\frac{\alpha}{2\lambda_n}}$, and $w_n=f_n\chi_{\frac{\alpha}{2}<|f_n|\le\frac{\alpha}{2\lambda_n}}$. Let $u=\sum_n\lambda_nu_n$, $v=\sum_n\lambda_nv_n$, and $w=\sum_n\lambda_nw_n$. Clearly $|u|\le\frac{\alpha}{2}$. Also $\{v\neq 0\}\subseteqq\bigcup_n\{|f_n|>\frac{\alpha}{2\lambda_n}\}$; hence $\mu(\{v\neq 0\})\le\frac{2}{\alpha}$. Finally,

$$\begin{aligned}\int_X |w|\,d\mu &\le \sum_n \lambda_n \int_X |f_n|\chi_{\frac{\alpha}{2}<|f_n|\le\frac{\alpha}{2\lambda_n}}\,d\mu\\ &\le \sum_n \lambda_n\left[\int_{\alpha/2}^{\alpha/(2\lambda_n)} d_{f_n}(\beta)\,d\beta + \int_0^{\alpha/2} d_{f_n}(\alpha/2)\,d\beta\right]\\ &\le K+1\,.\end{aligned}$$

Using $\mu(\{|u+v+w|>\alpha\}) \le \mu(\{|u|>\alpha/2\})+\mu(\{|v|\neq 0\})+\mu(\{|w|>\alpha/2\})$, deduce the conclusion.]

1.4.11. Let $\{f_n\}_n$ be a sequence of measurable functions on a measure space (X,μ). Let $0<q,s\le\infty$.
(a) Suppose that $f_n \ge 0$ for all n. Show that

$$\big\|\liminf_{n\to\infty} f_n\big\|_{L^{q,s}} \le \liminf_{n\to\infty}\big\|f_n\big\|_{L^{q,s}}\,.$$

(b) Let $g_n \to g$ in $L^{q,s}$ as $n\to\infty$. Show that $\|g_n\|_{L^{q,s}} \to \|g\|_{L^{q,s}}$ as $n\to\infty$.

1.4.12. (a) Suppose that X is a quasi-Banach space and let X^* be its dual (which is always a Banach space). Prove that for all $T\in X^*$ we have

$$\big\|T\big\|_{X^*} = \sup_{\substack{x\in X\\ \|x\|_X\le 1}} |T(x)|\,.$$

(b) Now suppose that X is a Banach space. Use the Hahn–Banach theorem to prove that for every $x\in X$ we have

$$\|x\|_X = \sup_{\substack{T\in X^*\\ \|T\|_{X^*}\le 1}} |T(x)|\,.$$

Observe that this result may fail for quasi-Banach spaces. For example, if $X=L^{1,\infty}$, every linear functional on X^* vanishes on the set of simple functions.
(c) Let $1<p<\infty$, $X=L^{p,1}(Y)$, and $X^*=L^{p',\infty}(Y)$, where (Y,μ) is nonatomic σ-finite measure space. Conclude that

$$\begin{aligned}\big\|f\big\|_{L^{p,1}} &\approx \sup_{\|g\|_{L^{p',\infty}}\le 1}\left|\int_Y fg\,d\mu\right|,\\ \big\|f\big\|_{L^{p,\infty}} &\approx \sup_{\|g\|_{L^{p',1}}\le 1}\left|\int_Y fg\,d\mu\right|.\end{aligned}$$

1.4.13. Let $0<p,q<\infty$. Prove that any function in $L^{p,q}(X,\mu)$ can be written as

$$f=\sum_{n=-\infty}^{+\infty} c_n f_n,$$

where f_n is a function bounded by $2^{-n/p}$, supported on a set of measure 2^n, and the sequence $\{c_k\}_k$ lies in ℓ^q and satisfies

$$2^{-\frac{1}{p}}(\log 2)^{\frac{1}{q}}\big\|\{c_k\}_k\big\|_{\ell^q} \le \big\|f\big\|_{L^{p,q}} \le \big\|\{c_k\}_k\big\|_{\ell^q}\, 2^{\frac{1}{p}}(\log 2)^{\frac{1}{q}}.$$

[*Hint:* Let $c_n = 2^{n/p} f^*(2^n)$, $A_n = \{x : f^*(2^{n+1}) < |f(x)| \le f^*(2^n))\}$, and $f_n = c_n^{-1} f\chi_{A_n}$.]

1.4.14. (*T. Tao*) Let $0 < p < \infty$, $0 < \gamma < 1$, $A, B > 0$, and let f be a measurable function on a measure space (X,μ).
(a) Suppose that $\|f\|_{L^{p,\infty}} \le A$. Then for every measurable set E of finite measure there exists a measurable subset E' of E with $\mu(E') \ge \gamma\mu(E)$ such that f is integrable on E' and

$$\left|\int_{E'} f\,d\mu\right| \le (1-\gamma)^{-1/p} A\,\mu(E)^{1-\frac{1}{p}}.$$

(b) Suppose that (X,μ) is a σ-finite measure space and that f has the property that for any measurable subset E of X with $\mu(E) < \infty$ there is a measurable subset E' of E with $\mu(E') \ge \gamma\mu(E)$ such that f is integrable on E' and

$$\left|\int_{E'} f\,d\mu\right| \le B\,\mu(E)^{1-\frac{1}{p}}.$$

Then we have that $\|f\|_{L^{p,\infty}} \le B\,4^{1/p}\gamma^{-1}\sqrt{2}$.
(c) Conclude that if (X,μ) is a σ-finite measure space then

$$\big\|f\big\|_{L^{p,\infty}} \approx \sup_{\substack{E\subseteq X\\ 0<\mu(E)<\infty}} \ \inf_{\substack{E'\subseteq E\\ \mu(E')\ge\frac{1}{2}\mu(E)\\ f\in L^1(E')}} \mu(E)^{-1+\frac{1}{p}}\left|\int_{E'} f\,d\mu\right|.$$

[*Hint:* Part (a): Take $E' = E\setminus\{|f| > A(1-\gamma)^{-\frac{1}{p}}\mu(E)^{-\frac{1}{p}}\}$. Part (b): Write $X = \bigcup_{n=1}^\infty X_n$ with $\mu(X_n) < \infty$. Given $\alpha > 0$, note that the set $\{|f| > \alpha\}$ is contained in

$$\big\{\mathrm{Re}\, f > \tfrac{\alpha}{\sqrt{2}}\big\} \cup \big\{\mathrm{Im}\, f > \tfrac{\alpha}{\sqrt{2}}\big\} \cup \big\{\mathrm{Re}\, f < -\tfrac{\alpha}{\sqrt{2}}\big\} \cup \big\{\mathrm{Im}\, f < -\tfrac{\alpha}{\sqrt{2}}\big\}.$$

Let E_n be any of the preceding four sets intersected with X_n, let E_n' be a subset of it with measure at least $\gamma\mu(E_n)$ as in the hypothesis. Then $\left|\int_{E_n'} f\,d\mu\right| \ge \frac{\alpha}{\sqrt{2}}\gamma\mu(E_n)$, from which it follows that $\alpha\mu(E_n)^{1/p} \le B\sqrt{2}\,\gamma^{-1}$, and let $n\to\infty$.]

1.4.15. Let T be a linear operator defined on the set of finitely simple functions on a σ-finite measure space (X,μ) and taking values in the set of measurable functions on a σ-finite measure space (Y,ν) and T^t be a linear operator defined on the set of finitely simple functions on (Y,ν) and taking values in the set of measurable functions of (X,μ). Suppose that for all A subsets of X and B subsets of Y of finite measure we have

$$\int_B |T(\chi_A)|\,d\nu + \int_A |T^t(\chi_B)|\,d\mu < \infty$$

and that T and T^t are related via the "transpose identity"

$$\int_Y T(\chi_A)\,\chi_B\,d\nu = \int_X T^t(\chi_B)\,\chi_A\,d\mu = \Lambda(A,B)\,.$$

Assume that whenever $\mu(A_n)+\nu(B_n)\to 0$ as $n\to\infty$, we have $\Lambda(A_n,B_n)\to 0$. Suppose that T and T^t are *restricted weak type* $(1,1)$ operators, with constants C_1 and C_2, respectively. Show that, for all $1<p<\infty$, T is of *restricted weak type* (p,p). Precisely, show that there exists a constant K_p such that

$$\big\|T(\chi_A)\big\|_{L^p(Y)} \le K_p C_1^{\frac{1}{p}} C_2^{1-\frac{1}{p}} \mu(A)^{\frac{1}{p}}$$

for all measurable subsets A of X with $\mu(A)<\infty$.
$\big[$*Hint:* Suppose that $C_1\mu(F) > C_2\nu(E)$ and pick m so that $C_1\mu(F)\sim 2^m C_2\nu(E)$. Since T^t is restricted weak type $(1,1)$ there is an $F'\subseteqq F$ such that $\mu(F')\ge \frac12\mu(F)$ and $|\Lambda(F',E)|\le 2C_2\nu(E)$. Find by induction sets $F^{(j)}\subseteqq F\setminus(F'\cup\cdots\cup F^{(j-1)})$ such that $\mu(F^{(j)})\ge\frac12\mu(F\setminus(F'\cup\cdots\cup F^{(j-1)}))$ and $|\Lambda(F^{(j)},E)|\le 2C_2\nu(E)$, $j=1,2,\dots,m$. Stop when $F^{(m)}=F\setminus(F'\cup\cdots\cup F^{(m-1)})$ satisfies $C_1\mu(F^{(m)})\le C_2\nu(E)$. Since T is restricted weak type $(1,1)$ there is a subset E' of E such that $\nu(E')\ge\frac12\nu(E)$ and $|\Lambda(F^{(m)},E')|\le 2C_1\mu(F^{(m)})\le 2C_2\nu(E)$. Now write

$$\Lambda(F,E) = \sum_{j=1}^{m-1}\Lambda(F^{(j)},E) + \Lambda(F^{(m)},E') + \Lambda(F^{(m)},E\setminus E')$$

from which it follows that

$$|\Lambda(F,E)| \le 2C_2\nu(E)\left(1+\log_2\frac{C_1\mu(F)}{C_2\nu(E)}\right) + |\Lambda(F_1,E_1)|$$

where $F_1=F^{(m)}$ and $E_1=E\setminus E'$. Note that the first term in the sum above is at most $K_p'(C_1\mu(F))^{1/p}(C_2\nu(E))^{1/p'}$ and that the identical estimate holds if the roles of E and F are reversed. Also observe that $\mu(F_1)\le\frac12\mu(F)$ and $\nu(E_1)\le\frac12\nu(E)$. Continuing this process we find sets (F_n,E_n) with $\mu(F_{n+1})\le\frac12\mu(F_n)$ and $\nu(E_{n+1})\le\frac12\nu(E_n)$.Using $\Lambda(F_n,E_n)\to 0$ as $n\to\infty$ we deduce that $|\Lambda(F,E)|\le 2K_p'(C_1\mu(F))^{1/p}(C_2\nu(E))^{1/p'}$. Considering the sets $E_+=E\cap\{T(\chi_F)>0\}$ and $E_-=E\cap\{T(\chi_F)<0\}$, obtain that $\int_E\big|T(\chi_F)\big|\,d\nu\le 4K_p'\big(C_1\mu(F)\big)^{\frac1p}\big(C_2\nu(E)\big)^{\frac1{p'}}$ for all F and E measurable sets of finite measure. Exercise 1.1.12 (a) with $r=1$ yields that $\|T(\chi_F)\|_{L^{p,\infty}}\le 4K_pC_1^{1/p}C_2^{1/p'}\mu(F)^{1/p}$. $\big]$

1.4.16. ([35]) Let $0<p_0<p_1<\infty$ and $0<\alpha,\beta,A,B<\infty$. Suppose that a family of sublinear operators T_k is of restricted weak type (p_0,p_0) with constant $A\,2^{-k\alpha}$ and of restricted weak type (p_1,p_1) with constant $B\,2^{k\beta}$ for all $k\in\mathbf{Z}$. Show that there is a constant $C = C(\alpha,\beta,p_0,p_1)$ such that $\sum_{k\in\mathbf{Z}} T_k$ is of restricted weak type (p,p) with constant $CA^{1-\theta}B^{\theta}$, where $\theta = \alpha/(\alpha+\beta)$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

$\big[$*Hint:* Estimate $\mu(\{|T(\chi_E)| > \lambda\})$ by the sum $\sum_{k\ge k_0} \mu(\{|T_k(\chi_E)| > c\lambda 2^{\alpha'(k_0-k)}\})+$ $\sum_{k\le k_0} \mu(\{|T_k(\chi_E)| > c\lambda 2^{\beta'(k-k_0)}\})$, where c is a suitable constant and $0<\alpha'<\alpha$, $0<\beta'<\beta$. Apply the restricted weak type (p_0,p_0) hypothesis on each term of the first sum, the restricted weak type (p_1,p_1) hypothesis on each term of the second sum, and choose k_0 to optimize the resulting expression.$\big]$

1.4.17. Let (X,μ), (Y,ν) be measure spaces, $0<p,r,q,s\le\infty$ and $0<B<\infty$. Suppose that a sublinear operator T is defined on a dense subspace $\mathscr{D}$ of $L^{p,r}(X)$, takes values in the space of measurable functions of another measure space Y, and satisfies $T(f)\ge 0$ for all f in $\mathscr{D}$. Assume that

$$\big\|T(\varphi)\big\|_{L^{q,s}} \le B\big\|\varphi\big\|_{L^{p,r}}$$

for all φ in $\mathscr{D}$. Prove that T admits a unique sublinear extension $\overline{T}$ on $L^{p,r}(X)$ such that

$$\big\|\overline{T}(f)\big\|_{L^{q,s}} \le B\big\|f\big\|_{L^{p,r}}$$

for all $f\in L^{p,r}(X)$.

$\big[$*Hint:* Given $f\in L^{p,r}(X)$ find a sequence of functions φ_j in $\mathscr{D}$ such that $\varphi_j\to f$ in $L^{p,r}$. Use the inequality $|T(\varphi_j)-T(\varphi_k)|\le|T(\varphi_j-\varphi_k)|$, to obtain that the sequence $\{T(\varphi_j)\}_j$ is Cauchy in $L^{q,s}$ and thus it has a unique limit $\overline{T}(f)$ which is independent of the choice of sequence φ_j. Boundedness of $\overline{T}$ follows by density. To prove that $\overline{T}$ is sublinear use that convergence in $L^{q,s}$ implies convergence in measure and thus a subsequence of $T(\varphi_j)$ converges ν-a.e. to $\overline{T}(f)$. Also use Exercise 1.4.11.$\big]$

HISTORICAL NOTES

The modern theory of measure and integration was founded with the publication of Lebesgue's dissertation [214]; see also [215]. The theory of the Lebesgue integral reshaped the course of integration. The spaces $L^p([a,b])$, $1<p<\infty$, were first investigated by Riesz [290], who obtained many important properties of them. A rigorous treatise of harmonic analysis on general groups can be found in the book of Hewitt and Ross [152]. The best possible constant C_{pqr} in Young's inequality $\|f*g\|_{L^r(\mathbf{R}^n)}\le C_{pqr}\|f\|_{L^p(\mathbf{R}^n)}\|g\|_{L^q(\mathbf{R}^n)}$, $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$, $1<p,q,r<\infty$, was shown by Beckner [21] to be $C_{pqr}=(B_pB_qB_{r'})^n$, where $B_p^2=p^{1/p}(p')^{-1/p'}$.

Theorem 1.3.2 first appeared without proof in Marcinkiewicz's brief note [240]. After his death in World War II, this theorem seemed to have escaped attention until Zygmund reintroduced it in [387]. This reference presents the more difficult off-diagonal version of the theorem, derived by Zygmund. Stein and Weiss [347] strengthened Zygmund's theorem by assuming that the initial estimates are of restricted weak type whenever $1\le p_0,p_1,q_0,q_1\le\infty$. The extension of this result to the case $0<p_0,p_1,q_0,q_1<1$ in Theorem 1.4.19 is due to the author. The critical Lemma 1.4.20 was suggested by Kalton. Improvements of these results, in particular, the appearance of the space $S_0(X)$ and the presence of the factor $M_0^{1-\theta}M_1^{\theta}$ in (1.4.24) appeared in Liang, Liu, and Yang [224]. Equivalence of restricted weak type $(1,1)$ and weak type $(1,1)$ properties for certain maximal multipliers was obtained by Moon [257]. The following partial converse of Theorem 1.2.13 is due to Stepanov [351]: If a convolution operator maps $L^1(\mathbf{R}^n)$ to $L^{q,\infty}(\mathbf{R}^n)$ for some $1 < q < \infty$ then its kernel must be in $L^{q,\infty}$.

The extrapolation result of Exercise 1.3.7 is due to Yano [380]; see also Zygmund [389, pp. 119–120] and the related work of Carro [56], Soria [330], and Tao [356].

The original version of Theorem 1.3.4 was proved by Riesz [293] in the context of bilinear forms. This version is called the Riesz convexity theorem, since it says that the logarithm of the function $M(\alpha,\beta) = \inf_{x,y} \big|\sum_{j=1}^n \sum_{k=1}^m a_{jk}x_jy_k\big| \, \|x\|_{\ell^{1/\alpha}}^{-1} \|y\|_{\ell^{1/\beta}}^{-1}$ (where the infimum is taken over all sequences $\{x_j\}_{j=1}^n$ in $\ell^{1/\alpha}$ and $\{y_k\}_{k=1}^m$ in $\ell^{1/\beta}$) is a convex function of (α,β) in the triangle $0 \le \alpha,\beta \le 1$, $\alpha+\beta \ge 1$. Riesz's student Thorin [360] extended this triangle to the unit square $0 \le \alpha,\beta \le 1$ and generalized this theorem by replacing the maximum of a bilinear form with the maximum of the modulus of an entire function in many variables. After the end of World War II, Thorin published his thesis [361], building the subject and giving a variety of applications. The original proof of Thorin was rather long, but a few years later, Tamarkin and Zygmund [354] gave a very elegant short proof using the maximum modulus principle in a more efficient way. Today, this theorem is referred to as the Riesz–Thorin interpolation theorem.

Calderón [42] elaborated the complex-variables proof of the Riesz–Thorin theorem into a general method of interpolation between Banach spaces. The complex interpolation method can also be defined for pairs of quasi-Banach spaces, although certain complications arise in this setting; however, the Riesz–Thorin theorem is true for pairs of L^p spaces (with the "correct" geometric mean constant) for all $0 < p \le \infty$ and also for Lorentz spaces. In this setting, duality cannot be used, but a well-developed theory of analytic functions with values in quasi-Banach spaces is crucial. We refer to the articles of Kalton [186] and [187] for details. Complex interpolation for sublinear maps is also possible; see the article of Calderón and Zygmund [47]. Interpolation for analytic families of operators (Theorem 1.3.7) is due to Stein [331]. The critical Lemma 1.3.8 used in the proof was previously obtained by Hirschman [154].

The fact that nonatomic measure spaces contain subsets of all possible measures is classical. An extension of this result to countably additive vector measures with values in finite-dimensional Banach spaces was obtained by Lyapunov [236]; for a proof of this fact, see Diestel and Uhl [95, p. 264]. The Aoki–Rolewicz theorem (Exercise 1.4.6) was proved independently by Aoki [7] and Rolewicz [297]. For a proof of this fact and a variety of its uses in the context of quasi-Banach spaces we refer to the book of Kalton, Peck, and Roberts [188].

Decreasing rearrangements of functions were introduced by Hardy and Littlewood [146]; the authors attribute their motivation to understanding cricket averages. The $L^{p,q}$ spaces were introduced by Lorentz in [232] and in [233]. A general treatment of Lorentz spaces is given in the article of Hunt [164]. The normability of the spaces $L^{p,q}$ (which holds exactly when $1 < p \le \infty$ and $1 \le q \le \infty$) can be traced back to general principles obtained by Kolmogorov [199]. The introduction of the function f^{**}, which was used in Exercise 1.4.3, to explicitly define a norm on the normable spaces $L^{p,q}$ is due to Calderón [42]. These spaces appear as intermediate spaces in the general interpolation theory of Calderón [42] and in that of Lions and Peetre [225]. The latter was pointed out by Peetre [275]. For a systematic study of the duals of Lorentz spaces we refer to Cwikel [83] and Cwikel and Fefferman [84], [85]. An extension of the Marcinkiewicz interpolation theorem to Lorentz spaces was obtained by Hunt [163]. Carro, Raposo, and Soria [57] provide a comprehensive presentation of the theory of Lorentz spaces in the context of weighted inequalities. For further topics on interpolation one may consult the books of Bennett and Sharpley [24], Bergh and Löfström [25], Sadosky [309], Kislyakov and Kruglyak [194], and Chapter 5 in Stein and Weiss [348].