Chapter 5
Singular Integrals of Convolution Type

The topic of singular integrals is motivated by its intimate connection with some of the most important problems in Fourier analysis, such as that of the convergence of Fourier series. As we have seen, the L^p boundedness of the conjugate function on the circle is equivalent to the L^p convergence of Fourier series of L^p functions. And since the Hilbert transform on the real line provides an analogue of the conjugate function on the circle, it is deeply connected with the L^p convergence of Fourier integrals. It also appears in the theory of harmonic functions on the upper half space and has so many remarkable properties that deserve a careful investigation. The Hilbert transform is the prototype of all singular integrals and provides inspiration for subsequent development of the subject.

Historically, the theory of the Hilbert transform depended on techniques of complex analysis. With the development of the Calderón–Zygmund school, and the extension of one-dimensional theory to higher dimensions, real-variable methods slowly replaced complex analysis. The higher-dimensional framework proved to be flexible enough for generalizations and led to the introduction of singular integrals in other areas of mathematics. Singular integrals are nowadays intimately connected with partial differential equations, operator theory, several complex variables, and other fields. In this chapter we study singular integrals given by convolution with tempered distributions. We call such operators singular integrals of convolution type.

5.1 The Hilbert Transform and the Riesz Transforms

We begin the investigation of singular integrals with a careful study of the Hilbert transform which provides inspiration for the subsequent development of the theory.

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5.1.1 Definition and Basic Properties of the Hilbert Transform

There are several equivalent ways to introduce the Hilbert transform; in this exposition we first define it as a convolution operator with a certain principal value distribution, but we later discuss other equivalent definitions.

We begin by defining a distribution W_0 in $\mathscr{S}'(\mathbf{R})$ as follows:

$$\langle W_0, \varphi \rangle = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon \le |x| \le 1} \frac{\varphi(x)}{x}\, dx + \frac{1}{\pi} \int_{|x| \ge 1} \frac{\varphi(x)}{x}\, dx, \tag{5.1.1}$$

for φ in $\mathscr{S}(\mathbf{R})$. The function $1/x$ integrated over $[-1,-\varepsilon]\bigcup[\varepsilon,1]$ has mean value zero, and we may replace $\varphi(x)$ by $\varphi(x)-\varphi(0)$ in the first integral in (5.1.1). Since $(\varphi(x)-\varphi(0))x^{-1}$ is controlled by $\|\varphi'\|_{L^\infty}$, it follows that the limit in (5.1.1) exists. To see that W_0 is indeed in $\mathscr{S}'(\mathbf{R})$, we note that the estimate

$$\big|\langle W_0, \varphi \rangle\big| \le \frac{2}{\pi}\big\|\varphi'\big\|_{L^\infty} + \frac{2}{\pi} \sup_{x \in \mathbf{R}} |x\varphi(x)| \tag{5.1.2}$$

is valid. This says that $W_0 \in \mathscr{S}'(\mathbf{R})$.

Definition 5.1.1. The *truncated Hilbert transform* (at height ε) of a function f in $L^p(\mathbf{R})$, $1 \le p < \infty$, is defined by

$$H^{(\varepsilon)}(f)(x) = \frac{1}{\pi} \int_{|y| \ge \varepsilon} \frac{f(x-y)}{y}\, dy = \frac{1}{\pi} \int_{|x-y| \ge \varepsilon} \frac{f(y)}{x-y}\, dy. \tag{5.1.3}$$

The *Hilbert transform* of $\varphi \in \mathscr{S}(\mathbf{R})$ is defined by

$$H(\varphi)(x) = (W_0 * \varphi)(x) = \lim_{\varepsilon \to 0} H^{(\varepsilon)}(\varphi)(x). \tag{5.1.4}$$

Observe that $H^{(\varepsilon)}(f)$ is well defined for all $f \in L^p$, $1 \le p < \infty$. This follows from Hölder's inequality, since $1/x$ is integrable to the power p' on the set $|x| \ge \varepsilon$.

For Schwartz functions φ, the integral

$$\int_{-\infty}^{+\infty} \frac{\varphi(x-y)}{y}\, dy$$

may not converge absolutely for any real number x, but is defined as a limit of the absolutely convergent integrals

$$\int_{|y| \ge \varepsilon} \frac{\varphi(x-y)}{y}\, dy,$$

as $\varepsilon \to 0$. Such limits are called *principal value integrals* and are denoted by the letters p.v. Using this notation, the Hilbert transform of a Schwartz function φ is

$$H(\varphi)(x) = \frac{1}{\pi}\, \text{p.v.} \int_{-\infty}^{+\infty} \frac{\varphi(x-y)}{y}\, dy = \frac{1}{\pi}\, \text{p.v.} \int_{-\infty}^{+\infty} \frac{\varphi(y)}{x-y}\, dy. \tag{5.1.5}$$

Remark 5.1.2. We extend the definition of the Hilbert transform to a bigger class of functions. Suppose that f is an integrable function on $\mathbf{R}$ that satisfies a Hölder condition near every point x; that is, for any $x \in \mathbf{R}$ there are $C_x > 0$ and $\varepsilon_x > 0$ such that

$$|f(x) - f(y)| \le C_x |x-y|^{\varepsilon_x}$$

whenever $|y-x| < \delta_x$. Then we write

$$\begin{aligned} H^{(\varepsilon)}(f)(x) &= \frac{1}{\pi} \int\limits_{\varepsilon<|x-y|<\delta_x} \frac{f(y)}{x-y}\,dy + \frac{1}{\pi} \int\limits_{|x-y|\ge\delta_x} \frac{f(y)}{x-y}\,dy \\ &= \frac{1}{\pi} \int\limits_{\varepsilon<|x-y|<\delta_x} \frac{f(y)-f(x)}{x-y}\,dy + \frac{1}{\pi} \int\limits_{|x-y|\ge\delta_x} \frac{f(y)}{x-y}\,dy . \end{aligned}$$

Both integrals converge absolutely; hence the limit of $H^{(\varepsilon)}(f)(x)$ exists as $\varepsilon \to 0$.

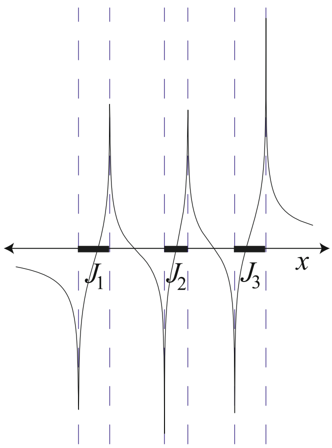

Fig. 5.1 The graph of the function $H(\chi_E)$ when E is a union of three disjoint intervals $J_1 \cup J_2 \cup J_3$.

Example 5.1.3. For the characteristic function $\chi_{[a,b]}$ of an interval $[a,b]$ we show that

$$H(\chi_{[a,b]})(x) = \frac{1}{\pi} \log \frac{|x-a|}{|x-b|} . \tag{5.1.6}$$

Let us verify this identity. Pick $\varepsilon < \min(|x-a|, |x-b|)$. To show (5.1.6) consider the three cases $0 < x-b$, $x-a < 0$, and $x-b < 0 < x-a$. In the first two cases, (5.1.6) follows immediately. In the third case we have

$$H(\chi_{[a,b]})(x) = \frac{1}{\pi} \lim_{\varepsilon\to 0} \left(\log \frac{|x-a|}{\varepsilon} + \log \frac{\varepsilon}{|x-b|} \right) , \tag{5.1.7}$$

which yields (5.1.6). Observe that the cancellation of ε in (5.1.7) reflects the fact that $1/x$ has integral zero on symmetric intervals $\varepsilon < |x| < c$. Note that $H(\chi_{[a,b]})(x)$ blows up logarithmically in x near the points a and b and decays like $|x|^{-1}$ as $x \to \infty$. See Figure 5.1.

Example 5.1.4. Let $\log^+ x = \log x$ when $x \ge 1$ and zero otherwise. Observe that the calculation in the previous example actually gives

$$H^{(\varepsilon)}(\chi_{[a,b]})(x) = \begin{cases} \dfrac{1}{\pi}\log^+ \dfrac{|x-a|}{\max(\varepsilon, |x-b|)} & \text{when } x > b, \\[2ex] -\dfrac{1}{\pi}\log^+ \dfrac{|x-b|}{\max(\varepsilon, |x-a|)} & \text{when } x < a, \\[2ex] \dfrac{1}{\pi}\log^+ \dfrac{|x-a|}{\varepsilon} - \dfrac{1}{\pi}\log^+ \dfrac{|x-b|}{\varepsilon} & \text{when } a < x < b. \end{cases}$$

We now give an alternative characterization of the Hilbert transform using the Fourier transform. To achieve this we need to compute the Fourier transform of the distribution W_0 defined in (5.1.1). Fix a Schwartz function φ on $\mathbf{R}$. Then

$$\begin{aligned} \left\langle \widehat{W_0}, \varphi \right\rangle &= \left\langle W_0, \widehat{\varphi} \right\rangle \qquad (5.1.8) \\ &= \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|\xi| \ge \varepsilon} \widehat{\varphi}(\xi)\, \frac{d\xi}{\xi} \\ &= \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\frac{1}{\varepsilon} \ge |\xi| \ge \varepsilon} \int_{\mathbf{R}} \varphi(x) e^{-2\pi i x \xi}\, dx\, \frac{d\xi}{\xi} \\ &= \lim_{\varepsilon \to 0} \int_{\mathbf{R}} \varphi(x) \left[\frac{1}{\pi} \int_{\frac{1}{\varepsilon} \ge |\xi| \ge \varepsilon} e^{-2\pi i x \xi}\, \frac{d\xi}{\xi} \right] dx \\ &= \lim_{\varepsilon \to 0} \int_{\mathbf{R}} \varphi(x) \left[\frac{-i}{\pi} \int_{\frac{1}{\varepsilon} \ge |\xi| \ge \varepsilon} \sin(2\pi x \xi)\, \frac{d\xi}{\xi} \right] dx \\ &= \lim_{\varepsilon \to 0} \int_{\mathbf{R}} \varphi(x) \left[\left(\frac{-i}{\pi} \operatorname{sgn} x \right) \int_{\frac{1}{2\pi\varepsilon} \ge |\xi| \ge \frac{\varepsilon}{2\pi}} \sin(|x|\xi)\, \frac{d\xi}{\xi} \right] dx. \qquad (5.1.9) \end{aligned}$$

Here we used the signum function

$$\operatorname{sgn} x = \begin{cases} +1 & \text{when } x > 0, \\ 0 & \text{when } x = 0, \\ -1 & \text{when } x < 0. \end{cases} \qquad (5.1.10)$$

Using the results (a) and (b) in Exercise 5.1.1 we obtain that the integrals inside the square brackets in (5.1.9) are uniformly bounded by 8 and converge to $2\frac{\pi}{2} = \pi$ as $\varepsilon \to 0$, whenever $x \ne 0$. These observations make possible the use of the Lebesgue dominated convergence theorem that allows the passage of the limit inside the integral in (5.1.9). We obtain that

$$\left\langle \widehat{W_0}, \varphi \right\rangle = \int_{\mathbf{R}} \varphi(x)(-i \operatorname{sgn}(x))\, dx. \qquad (5.1.11)$$

This implies that

$$\widehat{W_0}(\xi) = -i \operatorname{sgn} \xi \,. \tag{5.1.12}$$

In particular, identity (5.1.12) says that $\widehat{W_0}$ is a (bounded) function.

We now use identity (5.1.12) to write

$$H(f)(x) = \big(\widehat{f}(\xi)(-i \operatorname{sgn} \xi)\big)^{\vee}(x)\,. \tag{5.1.13}$$

This formula can be used to give an alternative definition of the Hilbert transform. An immediate consequence of (5.1.13) is that

$$\big\|H(f)\big\|_{L^2} = \big\|f\big\|_{L^2}\,, \tag{5.1.14}$$

that is, H is an isometry on $L^2(\mathbf{R})$. Moreover, H satisfies

$$H^2 = HH = -I\,, \tag{5.1.15}$$

where I is the identity operator. Equation (5.1.15) is a simple consequence of the fact that $(-i \operatorname{sgn} \xi)^2 = -1$. The adjoint operator H^* of H is uniquely defined via the identity

$$\big\langle f \,|\, H(g) \big\rangle = \int_{\mathbf{R}} f\,\overline{H(g)}\,dx = \int_{\mathbf{R}} H^*(f)\,\overline{g}\,dx = \big\langle H^*(f) \,|\, g \big\rangle\,,$$

and we can easily obtain that H^* has multiplier $\overline{-i \operatorname{sgn} \xi} = i \operatorname{sgn} \xi$. We conclude that $H^* = -H$. Likewise, we obtain $H^t = -H$.

5.1.2 Connections with Analytic Functions

We now investigate connections of the Hilbert transform with the Poisson kernel. Recall the definition of the Poisson kernel P_y given in Example 1.2.17. Then for a real-valued function f in $L^p(\mathbf{R})$, $1 \le p < \infty$, we have

$$(P_y * f)(x) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{(x-t)^2 + y^2}\,dt\,, \tag{5.1.16}$$

and the integral in (5.1.16) converges absolutely by Hölder's inequality, since the function $t \mapsto ((x-t)^2 + y^2)^{-1}$ is in $L^{p'}(\mathbf{R})$ whenever $y > 0$.

Let $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and imaginary parts of a complex number z. Observe that

$$(P_y * f)(x) = \operatorname{Re}\left(\frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{x - t + iy}\,dt\right) = \operatorname{Re}\left(\frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{z - t}\,dt\right),$$

where $z = x + iy$. The function

$$F_f(z) = \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{z-t}\, dt$$

defined on

$$\mathbf{R}^2_+ = \{z = x+iy :\ y > 0\}$$

is analytic, since its $\partial/\partial\bar{z}$ derivative is zero. The real part of $F_f(x+iy)$ is $(P_y * f)(x)$. The imaginary part of $F_f(x+iy)$ is

$$\operatorname{Im}\left(\frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{f(t)}{x-t+iy}\, dt\right) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(t)(x-t)}{(x-t)^2+y^2}\, dt = (f * Q_y)(x)\,,$$

where Q_y is called the *conjugate Poisson kernel* and is given by

$$Q_y(x) = \frac{1}{\pi}\frac{x}{x^2+y^2}\,. \tag{5.1.17}$$

The function $u_f + iv_f$ is analytic and thus $u_f(x+iy) = (f * P_y)(x)$ and $v_f(x+iy) = (f * Q_y)(x)$ are *conjugate harmonic functions*. Since the family P_y, $y > 0$, is an approximate identity, it follows from Theorem 1.2.19 that $P_y * f \to f$ in $L^p(\mathbf{R})$ as $y \to 0$. The following question therefore arises: What is the limit of $f * Q_y$ as $y \to 0$? The next result addresses this question.

Theorem 5.1.5. *Let $1 \le p < \infty$. For any $f \in L^p(\mathbf{R})$ we have*

$$f * Q_\varepsilon - H^{(\varepsilon)}(f) \to 0 \tag{5.1.18}$$

in L^p and almost everywhere as $\varepsilon \to 0$. Moreover, for φ in $\mathscr{S}(\mathbf{R})$ we have

$$F_\varphi(x+iy) = \frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{\varphi(t)}{x+iy-t}\, dt \to \varphi(x) + iH(\varphi)(x) \tag{5.1.19}$$

as $y \to 0+$ for all $x \in \mathbf{R}$.

Proof. We see that

$$(Q_\varepsilon * f)(x) - \frac{1}{\pi}\int_{|t|\ge\varepsilon} \frac{f(x-t)}{t}\, dt = \frac{1}{\pi}(f * \psi_\varepsilon)(x),$$

where $\psi_\varepsilon(x) = \varepsilon^{-1}\psi(\varepsilon^{-1}x)$ and

$$\psi(t) = \begin{cases} \frac{t}{t^2+1} - \frac{1}{t} & \text{when } |t| \ge 1, \\ \frac{t}{t^2+1} & \text{when } |t| < 1. \end{cases} \tag{5.1.20}$$

Note that ψ is integrable over the line and has integral zero. Furthermore, the integrable function

$$\Psi(t) = \begin{cases} \frac{1}{t^2+1} & \text{when } |t| \geq 1, \\ 1 & \text{when } |t| < 1, \end{cases} \tag{5.1.21}$$

is a radially decreasing majorant of ψ, i.e., it is even, decreasing on $[0,\infty)$, and satisfies $|\psi| \leq \Psi$. It follows from Theorem 1.2.21 (with $a=0$) that $f * \psi_\varepsilon \to 0$ in L^p. Also Corollary 2.1.19 (with $a=0$) implies that $f * \psi_\varepsilon \to 0$ almost everywhere as $\varepsilon \to 0$.

Assertion (5.1.19) is a consequence of (5.1.18), the discussion preceding Theorem 5.1.5, and the observation that $H^{(\varepsilon)}(\varphi)$ converges to $H(\varphi)$ pointwise everywhere as $\varepsilon \to 0$. $\square$

Remark 5.1.6. We will show later that for $f \in L^p(\mathbf{R})$, $1 \leq p < \infty$, the expressions $H^{(\varepsilon)}(f)$ converge a.e. (and also in L^p when $p > 1$) to a function $\widetilde{H}(f)$. This will be a consequence of Theorem 5.1.12 (or Corollary 5.3.6 when $p=1$), combined with Theorem 2.1.14 and the observation that for Schwartz functions φ, $H^{(\varepsilon)}(\varphi)$ converge to $H(\varphi)$ as $\varepsilon \to 0$. The linear operator $\widetilde{H}$ defined in this way extends the Hilbert transform H initially defined on Schwartz functions and will still be denoted by H. Thus for $f \in L^p(\mathbf{R})$, $1 \leq p < \infty$, one has

$$\lim_{\varepsilon \to 0} f * Q_\varepsilon = H(f) \quad \text{a.e.}$$

This convergence is also valid in L^p in view of the preceding observations and Theorem 5.1.5.

5.1.3 L^p Boundedness of the Hilbert Transform

As a consequence of the result in Exercise 5.1.4 and of the fact that

$$x \leq \tfrac{1}{2}(e^x - e^{-x}), \qquad x \geq 0,$$

we obtain that

$$\big|\{x : \, |H(\chi_E)(x)| > \alpha\}\big| \leq \frac{2}{\pi}\frac{|E|}{\alpha}, \qquad \alpha > 0, \tag{5.1.22}$$

for all subsets E of the real line of finite measure. Theorem 1.4.19 with $p_0 = q_0 = 1$ and $p_1 = q_1 = 2$ now implies that H is bounded on L^p for $1 < p < 2$. Duality gives that $H^* = -H$ is bounded on L^p for $2 < p < \infty$ and hence so is H.

We give another proof of the boundedness of the Hilbert transform H on $L^p(\mathbf{R})$, which has the advantage that it gives the best possible constant in the resulting norm inequality when p is a power of 2.

Theorem 5.1.7. *For all $1 < p < \infty$, there exists a positive constant C_p such that*

$$\big\|H(f)\big\|_{L^p} \le C_p \big\|f\big\|_{L^p}$$

for all f in $\mathscr{S}(\mathbf{R})$. Moreover, the constant C_p satisfies $C_p \le 2p$ for $2 \le p < \infty$ and $C_p \le 2p/(p-1)$ for $1 < p \le 2$. Therefore, the Hilbert transform H admits an extension to a bounded operator on $L^p(\mathbf{R})$ when $1 < p < \infty$.

Proof. The proof we give is based on the interesting identity

$$H(f)^2 = f^2 + 2H(fH(f)), \tag{5.1.23}$$

which is valid whenever f is a real-valued Schwartz function. We prove (5.1.23) in two different ways. First we consider the analytic function

$$F_f(z) = \frac{i}{\pi}\int_{\mathbf{R}} \frac{f(t)}{z-t}\,dt$$

defined on the upper half space. We compute its square. Fix $z \in \mathbf{C}$ with $\operatorname{Re} z > 0$ and f a real-valued Schwartz function. Then for $\varepsilon > 0$ we have

$$\begin{aligned}
F_f(z)^2 &= \Big(\frac{i}{\pi}\Big)^2 \int_{\mathbf{R}}\int_{\mathbf{R}} \frac{f(t)f(s)}{(z-t)(z-s)}\,dt\,ds \\
&= \Big(\frac{i}{\pi}\Big)^2 \int_{\mathbf{R}} \int\limits_{|t-s|>\varepsilon} \frac{f(t)f(s)}{t-s}\Big(\frac{1}{z-t} - \frac{1}{z-s}\Big)\,dt\,ds - \frac{1}{\pi^2}\iint\limits_{|t-s|\le\varepsilon} \frac{f(t)f(s)\,dt\,ds}{(z-t)(z-s)} \\
&= \Big(\frac{i}{\pi}\Big)^2 \int_{\mathbf{R}} f(t) \int\limits_{|t-s|>\varepsilon} \frac{f(s)}{t-s}\,ds\,\frac{dt}{z-t} - \Big(\frac{i}{\pi}\Big)^2 \int_{\mathbf{R}} f(s) \int\limits_{|t-s|>\varepsilon} \frac{f(t)}{t-s}\,dt\,\frac{ds}{z-s} \\
&\qquad\qquad - \frac{1}{\pi^2}\iint\limits_{|t-s|\le\varepsilon} \frac{f(t)f(s)\,dt\,ds}{(z-t)(z-s)}\,.
\end{aligned}$$

Letting $\varepsilon \to 0$ and passing the limit inside the integral by the Lebesgue dominated convergence theorem, we deduce

$$F_f(z)^2 = i\,\frac{i}{\pi}\int_{\mathbf{R}} \frac{2f(t)H(f)(t)}{z-t}\,dt\,. \tag{5.1.24}$$

We now let $\operatorname{Im} z \to 0+$ in (5.1.24) and use (5.1.19) in Theorem 5.1.5. We obtain

$$f^2 - H(f)^2 + i2fH(f) = \big(f + iH(f)\big)^2 = i\Big(2fH(f) + iH\big(2fH(f)\big)\Big),$$

and equating the real parts we deduce (5.1.23).

To give an alternative proof of (5.1.23) we take Fourier transforms. Let

$$m(\xi) = -i\operatorname{sgn}\xi$$

be the symbol of the Hilbert transform. We have

$$\begin{aligned}
&\widehat{f}^2(\xi)+2[H(fH(f))]^{\widehat{}}(\xi) \\
&\qquad= (\widehat{f}*\widehat{f})(\xi)+2m(\xi)(\widehat{f}*\widehat{H(f)})(\xi) \\
&\qquad= \int_{\mathbf{R}}\widehat{f}(\eta)\widehat{f}(\xi-\eta)\,d\eta+2m(\xi)\int_{\mathbf{R}}\widehat{f}(\eta)\widehat{f}(\xi-\eta)m(\eta)\,d\eta \qquad (5.1.25)\\
&\qquad= \int_{\mathbf{R}}\widehat{f}(\eta)\widehat{f}(\xi-\eta)\,d\eta+2m(\xi)\int_{\mathbf{R}}\widehat{f}(\eta)\widehat{f}(\xi-\eta)m(\xi-\eta)\,d\eta. \qquad (5.1.26)
\end{aligned}$$

Averaging (5.1.25) and (5.1.26) we obtain

$$\widehat{f}^2(\xi)+2[H(fH(f))]^{\widehat{}}(\xi)=\int_{\mathbf{R}}\widehat{f}(\eta)\widehat{f}(\xi-\eta)\big[1+m(\xi)\big(m(\eta)+m(\xi-\eta)\big)\big]\,d\eta\,.$$

But the last displayed expression is equal to

$$\int_{\mathbf{R}}\widehat{f}(\eta)\widehat{f}(\xi-\eta)m(\eta)m(\xi-\eta)\,d\eta=(\widehat{H(f)}*\widehat{H(f)})(\xi)$$

in view of the identity

$$m(\eta)m(\xi-\eta)=1+m(\xi)m(\eta)+m(\xi)m(\xi-\eta),$$

which is valid for all $(\xi,\eta)\in\mathbf{R}^2\setminus\{(0,0)\}$ for the function $m(\xi)=-i\operatorname{sgn}\xi$.

Having established (5.1.23), we can easily obtain L^p bounds for H when $p=2^k$ is a power of 2. We already know that H is bounded on L^p with norm one when $p=2^k$ and $k=1$. Suppose that H is bounded on L^p with bound c_p for $p=2^k$ for some $k\in\mathbf{Z}^+$. Then for a nonzero real-valued function f in $\mathscr{C}_0^\infty$ we have

$$\begin{aligned}
\big\|H(f)\big\|_{L^{2p}}=\big\|H(f)^2\big\|_{L^p}^{\frac12} &\le \big(\big\|f^2\big\|_{L^p}+\big\|2H(fH(f))\big\|_{L^p}\big)^{\frac12}\\
&\le \big(\big\|f\big\|_{L^{2p}}^2+2c_p\big\|fH(f)\big\|_{L^p}\big)^{\frac12}\\
&\le \big(\big\|f\big\|_{L^{2p}}^2+2c_p\big\|f\big\|_{L^{2p}}\big\|H(f)\big\|_{L^{2p}}\big)^{\frac12}.
\end{aligned}$$

Since $\|H(f)\|_{L^{2p}}<\infty$, we obtain that

$$\left(\frac{\big\|H(f)\big\|_{L^{2p}}}{\big\|f\big\|_{L^{2p}}}\right)^2-2c_p\frac{\big\|H(f)\big\|_{L^{2p}}}{\big\|f\big\|_{L^{2p}}}-1\le 0.$$

If follows that

$$\frac{\big\|H(f)\big\|_{L^{2p}}}{\big\|f\big\|_{L^{2p}}}\le c_p+\sqrt{c_p^2+1}\,,$$

and from this we conclude that H is bounded on L^{2p} with bound

$$c_{2p}\le c_p+\sqrt{c_p^2+1}\,. \qquad (5.1.27)$$

This completes the induction. We have proved that H maps L^p to L^p when $p = 2^k$, $k = 1,2,\dots$. Interpolation now gives that H maps L^p to L^p for all $p \ge 2$. Since $H^* = -H$, duality gives that H is also bounded on L^p for $1 < p \le 2$.

The previous proof of the boundedness of the Hilbert transform provides us with some useful information about the norm of this operator on $L^p(\mathbf{R})$. Let us begin with the identity

$$\cot\frac{x}{2} = \cot x + \sqrt{1+\cot^2 x},$$

valid for $0 < x < \frac{\pi}{2}$. If $c_p \le \cot\frac{\pi}{2p}$, then (5.1.27) gives that

$$c_{2p} \le c_p + \sqrt{c_p^2+1} \le \cot\frac{\pi}{2p} + \sqrt{1+\cot^2\frac{\pi}{2p}} = \cot\frac{\pi}{2\cdot 2p},$$

and since $1 = \cot\frac{\pi}{4} = \cot\frac{\pi}{2\cdot 2}$, we obtain by induction that the numbers $\cot\frac{\pi}{2p}$ are indeed bounds for the norm of H on L^p when $p = 2^k$, $k = 1,2,\dots$. Duality now gives that the numbers $\cot\frac{\pi}{2p'} = \tan\frac{\pi}{2p}$ are bounds for the norm of H on L^p when $p = \frac{2^k}{2^k-1}$, $k = 1,2,\dots$. These bounds allow us to derive good estimates for the norm $\|H\|_{L^p\to L^p}$ as $p \to 1$ and $p \to \infty$. Indeed, since $\cot\frac{\pi}{2p} \le p$ when $p \ge 2$, the Riesz–Thorin interpolation theorem gives that $\|H\|_{L^p\to L^p} \le 2p$ for $2 \le p < \infty$ and by duality $\|H\|_{L^p\to L^p} \le \frac{2p}{p-1}$ for $1 < p \le 2$. This completes the proof which is worth comparing with that of Theorem 4.1.7. □

Remark 5.1.8. The numbers $\cot\frac{\pi}{2p}$ for $2 \le p < \infty$ and $\tan\frac{\pi}{2p}$ for $1 < p \le 2$ are indeed equal to the norms of the Hilbert transform H on $L^p(\mathbf{R})$. This requires a more delicate argument; see Exercise 5.1.12.

Remark 5.1.9. We may wonder what happens when $p = 1$ or $p = \infty$. The Hilbert transform of $\chi_{[a,b]}$ computed in Example 5.1.3 is easily seen to be unbounded and not integrable, since it behaves like $1/|x|$ as $x \to \infty$. This behavior near infinity suggests that the Hilbert transform may map L^1 to $L^{1,\infty}$. This is indeed the case, but this will not be shown until Section 5.3.

We now introduce the maximal Hilbert transform.

Definition 5.1.10. The *maximal Hilbert transform* is the operator

$$H^{(*)}(f)(x) = \sup_{\varepsilon>0}\left|H^{(\varepsilon)}(f)(x)\right| \tag{5.1.28}$$

defined for all f in L^p, $1 \le p < \infty$. For such f, $H^{(\varepsilon)}(f)$ is well defined as a convergent integral by Hölder's inequality. Hence $H^{(*)}(f)$ makes sense for $f \in L^p(\mathbf{R})$, although for some values of x, $H^{(*)}(f)(x)$ may be infinite.

Example 5.1.11. Using the result of Example 5.1.4, we obtain that

$$H^{(*)}(\chi_{[a,b]})(x) = \frac{1}{\pi}\left|\log\frac{|x-a|}{|x-b|}\right|. \tag{5.1.29}$$

We see that in general, $H^{(*)}(f)(x) \neq |H(f)(x)|$ by taking f to be the characteristic function of the union of two disjoint closed intervals.

The definition of H gives that $H^{(\varepsilon)}(f)$ converges pointwise to $H(f)$ whenever f lies in $\mathscr{C}_0^\infty(\mathbf{R})$. If we have the estimate $\|H^{(*)}(f)\|_{L^p} \le C_p\|f\|_{L^p}$ for $f \in L^p(\mathbf{R})$, Theorem 2.1.14 yields that $H^{(\varepsilon)}(f)$ converges to $H(f)$ a.e. as $\varepsilon \to 0$ for any $f \in L^p$. This almost everywhere limit provides a way to describe $H(f)$ for general $f \in L^p(\mathbf{R})$. Note that Theorem 5.1.7 implies only that H has a (unique) bounded extension on L^p, but it does not provide a way to describe $H(f)$ when f is a general L^p function.

The next theorem is a simple consequence of these ideas.

Theorem 5.1.12. *There exists a constant C such that for all $1 < p < \infty$ we have*

$$\big\|H^{(*)}(f)\big\|_{L^p} \le C\max\big(p,(p-1)^{-2}\big)\big\|f\big\|_{L^p}. \tag{5.1.30}$$

Moreover, for all f in $L^p(\mathbf{R})$, $H^{(\varepsilon)}(f)$ converges to $H(f)$ a.e. and in L^p.

Proof. Another proof of this theorem is given in Theorem 4.2.4 in [131] in which the asserted bound is improved.

Recall the kernels P_ε and Q_ε defined in (5.1.16) and (5.1.17). Fix $1 < p < \infty$ and suppose momentarily that

$$f * Q_\varepsilon = H(f) * P_\varepsilon\,, \qquad \varepsilon > 0\,, \tag{5.1.31}$$

holds whenever f is an L^p function. Then we have

$$H^{(\varepsilon)}(f) = H^{(\varepsilon)}(f) - f * Q_\varepsilon + H(f) * P_\varepsilon\,. \tag{5.1.32}$$

Using the identity

$$H^{(\varepsilon)}(f)(x) - (f * Q_\varepsilon)(x) = -\frac{1}{\pi}\int_{\mathbf{R}} f(x-t)\,\psi_\varepsilon(t)\,dt\,, \tag{5.1.33}$$

where ψ is as in (5.1.20), and applying Corollary 2.1.12, we obtain the estimate

$$\sup_{\varepsilon>0}|H^{(\varepsilon)}(f)(x) - (f * Q_\varepsilon)(x)| \le \frac{1}{\pi}\big\|\Psi\big\|_{L^1}M(f)(x)\,, \tag{5.1.34}$$

where Ψ is as in (5.1.21) and M is the Hardy–Littlewood maximal function. In view of (5.1.32) and (5.1.34), we obtain for $f \in L^p(\mathbf{R}^n)$ that

$$|H^{(*)}(f)(x)| \le \big\|\Psi\big\|_{L^1}M(f)(x) + M(H(f))(x)\,. \tag{5.1.35}$$

It follows immediately from (5.1.35) that $H^{(*)}$ is L^p bounded with norm at most $C\max\big(p,(p-1)^{-2}\big)$.

We now turn to the proof of (5.1.31). It suffices to prove (5.1.31) for Schwartz functions since, given $f \in L^p$ there is a sequence $\phi_j \in \mathscr{S}$ such that $\|f - \phi_j\|_{L^p} \to 0$ as $j \to \infty$ and P_ε, Q_ε lie in $L^{p'}$. Taking Fourier transforms, we see that (5.1.31) is a consequence of the identity

$$\big((-i\,\mathrm{sgn}\,\xi)e^{-2\pi|\xi|}\big)^\vee(x) = \frac{1}{\pi}\frac{x}{x^2+1}\,. \tag{5.1.36}$$

To prove (5.1.36) we write

$$\begin{aligned}
\big((-i\,\mathrm{sgn}\,\xi)e^{-2\pi|\xi|}\big)^\vee(x) &= \int_{-\infty}^{+\infty} e^{-2\pi|\xi|}(-i\,\mathrm{sgn}\,\xi)e^{2\pi i x\xi}\,d\xi \\
&= 2\int_0^\infty e^{-2\pi\xi}\sin(2\pi x\xi)\,d\xi \\
&= \frac{1}{\pi}\int_0^\infty e^{-\xi}\sin(x\xi)\,d\xi \qquad (5.1.37) \\
&= \frac{1}{\pi}\int_0^\infty (e^{-\xi})''\sin(x\xi)\,d\xi \\
&= -\frac{x}{\pi}\int_0^\infty (e^{-\xi})'\cos(x\xi)\,d\xi \\
&= -\frac{x}{\pi}\bigg[-1 + x\int_0^\infty e^{-\xi}\sin(x\xi)\,d\xi\bigg] \qquad (5.1.38)
\end{aligned}$$

and we equate (5.1.38) and (5.1.37).

The statement in the theorem about the almost everywhere convergence of $H^{(\varepsilon)}(f)$ to $H(f)$ is a consequence of (5.1.30), of the fact that the alleged convergence holds for Schwartz functions, and of Theorem 2.1.14. Finally, the L^p convergence follows from the almost everywhere convergence and the Lebesgue dominated convergence theorem in view of the validity of (5.1.35). □

5.1.4 The Riesz Transforms

We now study an n-dimensional analogue of the Hilbert transform. It turns out that there exist n operators in $\mathbf{R}^n$, called the Riesz transforms, with properties analogous to those of the Hilbert transform on $\mathbf{R}$.

To define the Riesz transforms, we first introduce tempered distributions W_j on $\mathbf{R}^n$, for $1 \le j \le n$, as follows. For $\varphi \in \mathscr{S}(\mathbf{R}^n)$, let

$$\langle W_j, \varphi\rangle = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\lim_{\varepsilon\to 0}\int_{|y|\ge\varepsilon}\frac{y_j}{|y|^{n+1}}\varphi(y)\,dy.$$

One should check that indeed $W_j \in \mathscr{S}'(\mathbf{R}^n)$. Observe that the normalization of W_j is similar to that of the Poisson kernel.

Definition 5.1.13. For $1 \le j \le n$, the jth *Riesz transform* of f is given by convolution with the distribution W_j, that is,

$$R_j(f)(x) = (f * W_j)(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \,\text{p.v.} \int_{\mathbf{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\,dy\,, \tag{5.1.39}$$

for all $f \in \mathscr{S}(\mathbf{R}^n)$. Definition 5.1.13 makes sense for any integrable function f that has the property that for all x there exist $C_x > 0$, $\varepsilon_x > 0$, and $\delta_x > 0$ such that for y satisfying $|y-x| < \delta_x$ we have $|f(x) - f(y)| \le C_x |x-y|^{\varepsilon_x}$. The principal value integral in (5.1.39) is as in Definition 5.1.1.

We now give a characterization of R_j using the Fourier transform. For this we need to compute the Fourier transform of W_j.

Proposition 5.1.14. *The jth Riesz transform R_j is given on the Fourier transform side by multiplication by the function $-i\xi_j/|\xi|$. That is, for any f in $\mathscr{S}(\mathbf{R}^n)$ we have*

$$R_j(f)(x) = \Big(-\frac{i\xi_j}{|\xi|} \widehat{f}(\xi) \Big)^{\vee}(x)\,. \tag{5.1.40}$$

Proof. The proof is essentially a reprise of the corresponding proof for the Hilbert transform, but it involves a few technical difficulties. Fix a Schwartz function φ on $\mathbf{R}^n$. Then for $1 \le j \le n$ we have

$$\begin{aligned}
\big\langle \widehat{W_j}, \varphi \big\rangle &= \big\langle W_j, \widehat{\varphi} \big\rangle \qquad\qquad (5.1.41)\\
&= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \lim_{\varepsilon \to 0} \int_{|\xi| \ge \varepsilon} \widehat{\varphi}(\xi) \frac{\xi_j}{|\xi|^{n+1}}\,d\xi \\
&= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \lim_{\varepsilon \to 0} \int_{\frac{1}{\varepsilon} \ge |\xi| \ge \varepsilon} \int_{\mathbf{R}^n} \varphi(x) e^{-2\pi i x\cdot\xi}\,dx\, \frac{\xi_j}{|\xi|^{n+1}}\,d\xi \\
&= \lim_{\varepsilon \to 0} \int_{\mathbf{R}^n} \varphi(x) \left[\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\frac{1}{\varepsilon} \ge |\xi| \ge \varepsilon} e^{-2\pi i x\cdot\xi} \frac{\xi_j}{|\xi|^{n+1}}\,d\xi \right] dx \\
&= \lim_{\varepsilon \to 0} \int_{\mathbf{R}^n} \varphi(x) \left[\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbf{S}^{n-1}} \int_{\varepsilon \le r \le \frac{1}{\varepsilon}} e^{-2\pi i r x\cdot\theta} \frac{r}{r^{n+1}} r^{n-1}\,dr\,\theta_j\,d\theta \right] dx \\
&= \int_{\mathbf{R}^n} \varphi(x) \left[-i \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbf{S}^{n-1}} \int_0^\infty \sin(2\pi r x\cdot\theta) \frac{dr}{r}\, \theta_j\,d\theta \right] dx \\
&= \int_{\mathbf{R}^n} \varphi(x) \left[-i \frac{\pi}{2} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbf{S}^{n-1}} \operatorname{sgn}(x\cdot\theta)\, \theta_j\,d\theta \right] dx \\
&= \int_{\mathbf{R}^n} -i\varphi(x) \frac{x_j}{|x|}\,dx\,,
\end{aligned}$$

where in the penultimate equality we used the identity $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$, for which we refer to Exercise 5.1.1, while in the last equality we used the identity

$$-i\frac{\pi}{2}\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\int_{\mathbf{S}^{n-1}} \operatorname{sgn}(x\cdot\theta)\theta_j\,d\theta = -i\frac{x_j}{|x|}\,, \tag{5.1.42}$$

which needs to be established. The passage of the limit inside the integral in the previous calculation is a consequence of the Lebesgue dominated convergence theorem, which is justified from the fact that

$$\left|\int_\varepsilon^{1/\varepsilon} \frac{\sin(2\pi r\theta)}{r}\,dr\right| \le 4 \tag{5.1.43}$$

for all $\varepsilon > 0$. For a proof of (5.1.43) we again refer to Exercise 5.1.1. $\square$

It remains to establish (5.1.42). Let us recall that $O(n)$ is the set of all orthogonal $n\times n$ matrices with real entries. An invertible matrix A is called orthogonal if its transpose A^t is equal to its inverse A^{-1}, that is, $AA^t = A^tA = I$.

Lemma 5.1.15. *The following identity is valid for all $\xi \in \mathbf{R}^n\setminus\{0\}$:*

$$\int_{\mathbf{S}^{n-1}} \operatorname{sgn}(\xi\cdot\theta)\,\theta_j\,d\theta = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})}\frac{\xi_j}{|\xi|}\,. \tag{5.1.44}$$

Therefore (5.1.42) *holds.*

Proof. We begin with the identity

$$\int_{\mathbf{S}^{n-1}} \operatorname{sgn}(\theta_k)\,\theta_j\,d\theta = \begin{cases} 0 & \text{if } k\neq j,\\[2ex] \displaystyle\int_{\mathbf{S}^{n-1}}|\theta_j|\,d\theta & \text{if } k=j, \end{cases} \tag{5.1.45}$$

which can be proved by noting that for $k\neq j$, $\operatorname{sgn}(\theta_k)$ has a constant sign on the hemispheres $\theta_k>0$ and $\theta_k<0$, on either of which the function $\theta\mapsto\theta_j$ has integral zero.

It suffices to prove (5.1.44) for a unit vector ξ. Given $\xi\in\mathbf{S}^{n-1}$, pick an orthogonal $n\times n$ matrix $A=(a_{kl})_{k,l}$ such that $Ae_j=\xi$. Then the jth column of the matrix A is the vector $(\xi_1,\xi_2,\dots,\xi_n)^t$. We have

$$\begin{aligned}
\int_{\mathbf{S}^{n-1}} \operatorname{sgn}(\xi\cdot\theta)\,\theta_j\,d\theta &= \int_{\mathbf{S}^{n-1}} \operatorname{sgn}(Ae_j\cdot\theta)\,\theta_j\,d\theta\\
&= \int_{\mathbf{S}^{n-1}} \operatorname{sgn}(e_j\cdot A^t\theta)\,(AA^t\theta)_j\,d\theta\\
&= \int_{\mathbf{S}^{n-1}} \operatorname{sgn}(e_j\cdot\theta)\,(A\theta)_j\,d\theta\\
&= \int_{\mathbf{S}^{n-1}} \operatorname{sgn}(\theta_j)\,(a_{j1}\theta_1+\cdots+\xi_j\theta_j+\cdots+a_{jn}\theta_n)\,d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \xi_j \int_{\mathbf{S}^{n-1}} \operatorname{sgn}(\theta_j)\, \theta_j \, d\theta + \sum_{1 \le m \ne j \le n} 0 \\
&= \frac{\xi_j}{|\xi|} \int_{\mathbf{S}^{n-1}} |\theta_j| \, d\theta .
\end{aligned}
$$

Next, for all $j \in \{1, 2, \dots, n\}$, we compute the value of the integral

$$
\int_{\mathbf{S}^{n-1}} |\theta_j| \, d\theta = \int_{\mathbf{S}^{n-1}} |\theta_1| \, d\theta ,
$$

which is obviously independent of j by symmetry. In view of the result of Appendix D.2, we write

$$
\begin{aligned}
\int_{\mathbf{S}^{n-1}} |\theta_1| \, d\theta &= \int_{-1}^{1} |s| \int_{\sqrt{1-s^2}\,\mathbf{S}^{n-2}} d\varphi \frac{ds}{(1-s^2)^{\frac{1}{2}}} \\
&= \omega_{n-2} \int_{-1}^{1} |s| (1-s^2)^{\frac{n-3}{2}} \, ds \\
&= \omega_{n-2} \int_{0}^{1} u^{\frac{n-3}{2}} \, du \\
&= \frac{2\omega_{n-2}}{n-1} \\
&= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2}) \frac{n-1}{2}} \\
&= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})} ,
\end{aligned}
$$

having used the expression for ω_{n-2} in Appendix A.3. This proves (5.1.44). The proof of the lemma and hence that of Proposition 5.1.14 is complete. $\square$

Proposition 5.1.16. *The Riesz transforms satisfy*

$$
-I = \sum_{j=1}^{n} R_j^2 , \qquad \text{on } L^2(\mathbf{R}^n), \tag{5.1.46}
$$

where I is the identity operator.

Proof. Use the Fourier transform and the identity $\sum_{j=1}^{n} (-i\xi_j/|\xi|)^2 = -1$ to obtain that $\sum_{j=1}^{n} R_j^2(f) = -f$ for any f in $L^2(\mathbf{R}^n)$. $\square$

We can express the mixed derivatives of Schwartz function in terms of its Laplacian using the Riesz transforms.

Proposition 5.1.17. *For φ in $\mathscr{S}(\mathbf{R}^n)$ and $1 \le j,k \le n$ we have*

$$\partial_j\partial_k\varphi(x) = -R_jR_k\Delta\varphi(x) \tag{5.1.47}$$

for all $x \in \mathbf{R}^n$.

Proof. We verify the claimed identity by taking Fourier transforms. We have

$$\begin{aligned}\big(\partial_j\partial_k\varphi\big)^\frown(\xi) &= (2\pi i\xi_j)(2\pi i\xi_k)\widehat{\varphi}(\xi)\\ &= -\Big(-\frac{i\xi_j}{|\xi|}\Big)\Big(-\frac{i\xi_k}{|\xi|}\Big)(-4\pi^2|\xi|^2)\widehat{\varphi}(\xi)\\ &= -\big(R_jR_k\Delta\varphi\big)^\frown(\xi)\end{aligned}$$

and taking the inverse Fourier transform, identity (5.1.47) follows. □

Next we discuss a use of the Riesz transforms to partial differential equations.

Example 5.1.18. Suppose that f is a given function in $L^2(\mathbf{R}^n)$ and that u is a tempered distribution on $\mathbf{R}^n$ that solves *Laplace's equation*

$$\Delta u = f\,. \tag{5.1.48}$$

We express all second-order derivatives of u in terms of the Riesz transforms of f.

To solve equation (5.1.48) we first show that the tempered distribution

$$\big(\partial_j\partial_k u + R_jR_k(f)\big)^\frown$$

is supported at $\{0\}$. In view of Proposition 2.4.1, this implies that

$$\partial_j\partial_k u = -R_jR_k(f) + P$$

where P is a polynomial of n variables (that depends on j and k) and provides a way to express the mixed partials of u in terms of the Riesz transforms of f.

To verify that $\big(\partial_j\partial_k u + R_jR_k(f)\big)^\frown$ is supported at $\{0\}$, we fix a Schwartz function ψ whose support does not contain the origin. Then ψ vanishes in a neighborhood of zero and we can pick $\mathscr{C}^\infty$ function η which vanishes in a smaller neighborhood of zero and is equal to 1 on the support of ψ. We define

$$\zeta(\xi) = -\eta(\xi)\Big(-\frac{i\xi_j}{|\xi|}\Big)\Big(-\frac{i\xi_k}{|\xi|}\Big)$$

and we notice that ζ is a bounded $\mathscr{C}^\infty$ function and so are all of its derivatives; also

$$\eta(\xi)(2\pi i\xi_j)(2\pi i\xi_k) = \zeta(\xi)(-4\pi^2|\xi|^2)\,.$$

Taking the Fourier transform of both sides of (5.1.48) we obtain

$$(-4\pi^2|\xi|^2)\,\widehat{u}(\xi) = \widehat{f}(\xi)$$

and multiplying by ζ (which is allowed since all derivatives of ζ lie in $L^\infty \cap \mathscr{C}^\infty$)

$$\zeta(\xi)(-4\pi^2|\xi|^2)\,\widehat{u} = \zeta(\xi)\,\widehat{\Delta u} = \zeta(\xi)\widehat{f}(\xi)\,.$$

It follows that for all $1 \le j,k \le n$ we have

$$\begin{aligned}
\big\langle (\partial_j\partial_k u)\widehat{\ }, \psi\big\rangle &= \big\langle (2\pi i\xi_j)(2\pi i\xi_k)\widehat{u}, \psi\big\rangle \\
&= \big\langle (2\pi i\xi_j)(2\pi i\xi_k)\widehat{u}, \eta\,\psi\big\rangle \\
&= \big\langle \eta(\xi)(2\pi i\xi_j)(2\pi i\xi_k)\widehat{u}, \psi\big\rangle \\
&= \big\langle \zeta(\xi)(-4\pi^2|\xi|^2)\widehat{u}, \psi\big\rangle \\
&= \big\langle \zeta(\xi)\widehat{f}(\xi), \psi\big\rangle \\
&= \big\langle -\eta(\xi)\big(-\tfrac{i\xi_j}{|\xi|}\big)\big(-\tfrac{i\xi_k}{|\xi|}\big)\widehat{f}(\xi), \psi\big\rangle \\
&= \big\langle -\eta(\xi)(R_jR_k(f))\widehat{\ }(\xi), \psi\big\rangle \\
&= -\big\langle (R_jR_k(f))\widehat{\ }, \eta\,\psi\big\rangle \\
&= -\big\langle (R_jR_k(f))\widehat{\ }, \psi\big\rangle
\end{aligned}$$

and since this holds for all Schwartz functions ψ whose support does not contain the origin, it follows that $\big(\partial_j\partial_k u + R_jR_k(f)\big)\widehat{\ }$ is supported at $\{0\}$.

Exercises

5.1.1. (a) Show that for all $0 < a < b < \infty$ we have

$$\left| \int_a^b \frac{\sin x}{x}\,dx \right| \le 4\,.$$

(b) For $a > 0$ define

$$I(a) = \int_0^\infty \frac{\sin x}{x} e^{-ax}\,dx$$

and show that $I(a)$ is continuous at zero. Differentiate in a and look at the behavior of $I(a)$ as $a \to \infty$ to obtain the identity

$$I(a) = \frac{\pi}{2} - \arctan(a)\,.$$

Deduce that $I(0) = \frac{\pi}{2}$ and also derive the following identity used in (5.1.10):

$$\int_{-\infty}^{+\infty} \frac{\sin(bx)}{x}\,dx = \pi\,\mathrm{sgn}\,(b)\,.$$

(c) Argue as in part (b) to prove for $a \ge 0$ the identity

$$\int_0^\infty \frac{1-\cos x}{x^2} e^{-ax}\,dx = \frac{\pi}{2} - \arctan(a) + a\log\frac{a}{\sqrt{1+a^2}}\,.$$

$\big[$*Hint:* Part (a): Consider the cases $b \le 1$, $a \le 1 \le b$, $1 \le a$. When $a \ge 1$, integrate by parts.$\big]$

5.1.2. (a) Let φ be a compactly supported $\mathscr{C}^{m+1}$ function on $\mathbf{R}$ for some m in $\mathbf{Z}^+ \bigcup\{0\}$. Prove that if $\varphi^{(m)}$ is the mth derivative of φ, then

$$|H(\varphi^{(m)})(x)| \le C_{m,\varphi}\,(1+|x|)^{-m-1}$$

for some $C_{m,\varphi} > 0$.
(b) Let φ be a compactly supported $\mathscr{C}^{m+1}$ function on $\mathbf{R}^n$ for some $m \in \mathbf{Z}^+$. Show that

$$|R_j(\partial^\alpha \varphi)(x)| \le C_{n,m,\varphi}\,(1+|x|)^{-n-m}$$

for some $C_{n,m,\varphi} > 0$ and all multi-indices α with $|\alpha| = m$.
(c) Let I be an interval on the line and assume that a function h is equal to 1 on the left half of I, is equal to -1 on the right half of I, and vanishes outside I. Prove that for $x \notin 2I$ we have

$$|H(h)(x)| \le 4|I|^2|x-\text{center}(I)|^{-2}\,.$$

$\big[$*Hint:* Use that when $|t| \le \frac{1}{2}$ we have $\log(1+t) = t + R_1(t)$, where $|R_1(t)| \le 2|t|^2$.$\big]$

5.1.3. (a) Using identity (5.1.13) one may define $H(f)$ as an element of $\mathscr{S}'(\mathbf{R})$ for bounded functions f on the line whose Fourier transform vanishes in a neighborhood of the origin. Using this interpretation, prove that

$$\begin{aligned} H(e^{ix}) &= -ie^{ix}\,,\\ H(\cos x) &= \sin x\,,\\ H(\sin x) &= -\cos x\,,\\ H(\sin(\pi x)/\pi x) &= (1-\cos(\pi x))/\pi x\,. \end{aligned}$$

(b) Show that the operators given by convolution with the smooth function $\sin(t)/t$ and the distribution p.v. $\cos(t)/t$ are bounded on $L^p(\mathbf{R})$ whenever $1 < p < \infty$.
$\big[$*Hint:* Use that the Fourier transform of the distribution e^{ix} is $\delta_{1/2\pi}$. $\big]$

5.1.4. ([347]) Show that the distribution function of the Hilbert transform of the characteristic function of a measurable subset E of the real line of finite measure is

$$d_{H(\chi_E)}(\alpha) = \frac{4|E|}{e^{\pi\alpha} - e^{-\pi\alpha}}\,, \qquad \alpha > 0\,.$$

$\big[$*Hint:* First take $E = \bigcup_{j=1}^N (a_j, b_j)$, where $b_j < a_{j+1}$. Show that the equation $H(\chi_E)(x) = \alpha$ has exactly one root ρ_j in each open interval (a_j, b_j) for $1 \le j \le N$ and exactly one root r_j in each interval (b_j, a_{j+1}) for $1 \le j \le N$, ($a_{N+1} = \infty$). Then

$|\{x \in \mathbf{R} : \ H(\chi_E)(x) > \alpha\}| = \sum_{j=1}^{N} r_j - \sum_{j=1}^{N} \rho_j$, and this can be expressed in terms of $\sum_{j=1}^{N} a_j$ and $\sum_{j=1}^{N} b_j$. Argue similarly for the set $\{x \in \mathbf{R} : \ H(\chi_E)(x) < -\alpha\}$. For a general measurable set E, find sets E_n such that each E_n is a finite union of intervals and that $\chi_{E_n} \to \chi_E$ in L^2. Then $H(\chi_{E_n}) \to H(\chi_E)$ in measure; thus $H(\chi_{E_{n_k}}) \to H(\chi_E)$ a.e. for some subsequence n_k. The Lebesgue dominated convergence theorem gives $d_{H(\chi_{E_{n_k}})} \to d_{H(\chi_E)}$. See Figure 5.1.$\big]$

5.1.5. Let $1 \le p < \infty$ and let T be a linear operator defined on the space of Schwartz functions that commutes with dilations, i.e., $T(\delta^\lambda f) = \delta^\lambda T(f)$ for all $f \in \mathscr{S}(\mathbf{R}^n)$ and all $\lambda > 0$. (Here $\delta^\lambda(f)(x) = f(\lambda x)$.) Suppose that there exists a constant $C > 0$ such that for all $f \in \mathscr{S}(\mathbf{R}^n)$ with L^p norm one we have

$$|\{x : \ |T(f)(x)| > 1\}| \le C.$$

Prove that T admits a bounded extension from $L^p(\mathbf{R}^n)$ to $L^{p,\infty}(\mathbf{R}^n)$ with norm at most $C^{1/p}$.
$\big[$*Hint:* Try functions of the form $\lambda^{-n/p} f(\lambda^{-1}x)/\|f\|_{L^p}$ with $\lambda > 0$.$\big]$

5.1.6. Let φ be in $\mathscr{S}(\mathbf{R})$. Prove that

$$\lim_{N\to\infty} \text{p.v.} \int_{\mathbf{R}} \frac{e^{2\pi i N x}}{x} \varphi(x)\,dx = \varphi(0)\pi i,$$
$$\lim_{N\to-\infty} \text{p.v.} \int_{\mathbf{R}} \frac{e^{2\pi i N x}}{x} \varphi(x)\,dx = -\varphi(0)\pi i.$$

5.1.7. Let T_α, $\alpha \in \mathbf{R}$, be the operator given by convolution with the distribution whose Fourier transform is the function

$$u_\alpha(\xi) = e^{-\pi i \alpha \operatorname{sgn} \xi}.$$

(a) Show that the T_α's are isometries on $L^2(\mathbf{R})$ that satisfy

$$(T_\alpha)^{-1} = T_{2-\alpha}.$$

(b) Express T_α in terms of the identity operator and the Hilbert transform.

5.1.8. Let $Q_y^{(j)}$ be the jth *conjugate Poisson kernel* of P_y defined by

$$Q_y^{(j)}(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{x_j}{(|x|^2 + y^2)^{\frac{n+1}{2}}}.$$

Prove that

$$(Q_y^{(j)})^\wedge(\xi) = -i \frac{\xi_j}{|\xi|} e^{-2\pi y |\xi|}.$$

Conclude that $R_j(P_y) = Q_y^{(j)}$ and that for f in $L^2(\mathbf{R}^n)$ we have $R_j(f) * P_y = f * Q_y^{(j)}$. These results are analogous to the statements $\widehat{Q_y}(\xi) = -i \operatorname{sgn}(\xi) \widehat{P_y}(\xi)$, $H(P_y) = Q_y$, and $H(f) * P_y = f * Q_y$.

5.1.9. Fix $n \ge 2$. Let $f_0, f_1, \dots, f_n$ be in $L^2(\mathbf{R}^n)$ and, for $0 \le j \le n$, let $u_j(x,x_0) = (P_{x_0} * f_j)(x)$ be the Poisson integrals of f_j where $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ and $x_0 > 0$. Show that a necessary and sufficient condition for

$$f_j = R_j(f_0), \qquad j = 1, \dots, n,$$

is that the following system of *generalized Cauchy-Riemann equations* holds:

$$\sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j}(x,x_0) = 0,$$
$$\frac{\partial u_j}{\partial x_k}(x,x_0) = \frac{\partial u_k}{\partial x_j}(x,x_0), \quad 0 \le j \ne k \le n.$$

5.1.10. Prove the distributional identity

$$\partial_j |x|^{-n+1} = (1-n)\text{p.v.}\, \frac{x_j}{|x|^{n+1}}.$$

Then take Fourier transforms of both sides and use Theorem 2.4.6 to obtain another proof of Proposition 5.1.14.

5.1.11. (a) Prove that if T is a bounded linear operator on $L^2(\mathbf{R})$ that commutes with translations and dilations and anticommutes with the reflection $f(x) \mapsto \widetilde{f}(x) = f(-x)$, then T is a constant multiple of the Hilbert transform.
(b) Prove that if T is a bounded linear operator on $L^2(\mathbf{R})$ that commutes with translations and dilations and vanishes when applied to functions whose Fourier transform is supported in $[0, \infty)$, then T is a constant multiple of the operator $f \mapsto \big(\widehat{f}\chi_{(-\infty,0]}\big)^{\vee}$.

5.1.12. ([282]) Fix $1 < p \le 2$.
(a) Show that the function $G(x,y) = \text{Re}\,(|x| + iy)^p$ is subharmonic on $\mathbf{R}^2$.
(b) Let $u(x,y), v(x,y)$ be real-valued functions on $\mathbf{R}^2$ such that $u + iv$ is a holomorphic function of $x + iy$. Prove that $G(u,v)$ is a subharmonic function on $\mathbf{R}^2$.
(c) Prove that there is a constant B_p such that for all a and b reals we have

$$|b|^p \le \Big(\tan \frac{\pi}{2p}\Big)^p |a|^p - B_p \text{Re}\,(|a| + ib)^p.$$

(d) Prove that for f in $\mathscr{C}_0^\infty(\mathbf{R})$ we have

$$\int_{\mathbf{R}} \text{Re}\,(|f(x)| + iH(f)(x))^p \, dx \ge 0.$$

(e) Combine the results in parts (d) and (c) with $a = f(x)$, $b = H(f)(x)$ to obtain that

$$\big\|H\big\|_{L^p \to L^p} \le \tan \frac{\pi}{2p}.$$

(f) To deduce that this constant is sharp, take $\pi/2p' < \gamma < \pi/2p$ and let $f_\gamma(x) = (x+1)^{-1}|x+1|^{2\gamma/\pi}|x-1|^{-2\gamma/\pi}\cos\gamma$. Then

$$H(f_\gamma)(x) = \begin{cases} \frac{1}{x+1}\frac{|x+1|}{|x-1|}^{2\gamma/\pi}\sin\gamma & \text{when } |x|>1, \\ -\frac{1}{x+1}\frac{|x+1|}{|x-1|}^{2\gamma/\pi}\sin\gamma & \text{when } |x|<1. \end{cases}$$

$\big[$*Hint:* Part (d): Let C_R be the circle of radius R centered at $(0,R)$ in $\mathbf{R}^2$. Use that the integral of the subharmonic function $G((P_y * f)(x), Q_y * f)(x))$ over C_R is at least $2\pi R \,\mathrm{Re}\,(|(P_R * f)(0)| + i(Q_R * f)(0))^p$ and let $R\to\infty$. Part (f): The formula for $H(f_\gamma)$ is best derived by considering the restriction of the analytic function

$$F(z) = (z+1)^{-1}\left(\frac{iz+i}{z-1}\right)^{2\gamma/\pi}$$

on the real line.$\big]$

5.2 Homogeneous Singular Integrals and the Method of Rotations

So far we have introduced the Hilbert and the Riesz transforms and we have derived the L^p boundedness of the former. The boundedness properties of the Riesz transforms on L^p spaces are consequences of the results discussed in this section.

5.2.1 Homogeneous Singular and Maximal Singular Integrals

We introduce singular integral operators on $\mathbf{R}^n$ that appropriately generalize the Riesz transforms on $\mathbf{R}^n$. Here is the setup. We fix Ω to be an integrable function of the unit sphere $\mathbf{S}^{n-1}$ with mean value zero. Observe that the kernel

$$K_\Omega(x) = \frac{\Omega(x/|x|)}{|x|^n}, \qquad x \neq 0, \tag{5.2.1}$$

is homogeneous of degree $-n$ just like the functions $x_j/|x|^{n+1}$. Since K_Ω is not in $L^1(\mathbf{R}^n)$, convolution with K_Ω cannot be defined as an operation on Schwartz functions on $\mathbf{R}^n$. For this reason we introduce a distribution W_Ω in $\mathscr{S}'(\mathbf{R}^n)$ by setting

$$\big\langle W_\Omega, \varphi\big\rangle = \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} K_\Omega(x)\varphi(x)\,dx = \lim_{\varepsilon\to 0}\int_{\varepsilon\leq|x|\leq\varepsilon^{-1}} K_\Omega(x)\varphi(x)\,dx \tag{5.2.2}$$

for $\varphi \in \mathscr{S}(\mathbf{R}^n)$. Using the fact that Ω has mean value zero, we can easily see that W_Ω is a well defined tempered distribution on $\mathbf{R}^n$. Indeed, since K_Ω has integral zero over all annuli centered at the origin, we have

$$\begin{aligned}
\big|\langle W_\Omega, \varphi\rangle\big| &= \left|\lim_{\varepsilon\to 0}\int_{\varepsilon\le|x|\le 1}\frac{\Omega(x/|x|)}{|x|^n}(\varphi(x)-\varphi(0))\,dx+\int_{|x|\ge 1}\frac{\Omega(x/|x|)}{|x|^n}\varphi(x)\,dx\right| \\
&\le \big\|\nabla\varphi\big\|_{L^\infty}\int_{|x|\le 1}\frac{|\Omega(x/|x|)|}{|x|^{n-1}}\,dx+\sup_{y\in\mathbf{R}^n}|y|\,|\varphi(y)|\int_{|x|\ge 1}\frac{|\Omega(x/|x|)|}{|x|^{n+1}}\,dx \\
&\le C_1\big\|\nabla\varphi\big\|_{L^\infty}\big\|\Omega\big\|_{L^1}+C_2\sum_{|\alpha|\le 1}\big\|\varphi(x)x^\alpha\big\|_{L^\infty}\big\|\Omega\big\|_{L^1},
\end{aligned}$$

for suitable C_1 and C_2, where we used (2.2.2) in the last estimate. Note that the distribution W_Ω coincides with the function K_Ω on $\mathbf{R}^n\setminus\{0\}$.

The Hilbert transform and the Riesz transforms are examples of these general operators T_Ω. For instance, the function $\Omega(\theta)=\frac{\theta}{\pi|\theta|}=\frac{1}{\pi}\operatorname{sgn}\theta$ defined on the unit sphere $\mathbf{S}^0=\{-1,1\}\subseteqq\mathbf{R}$ gives rise to the Hilbert transform, while the function

$$\Omega(\theta)=\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\frac{\theta_j}{|\theta|}$$

defined on $\mathbf{S}^{n-1}\subseteqq\mathbf{R}^n$ gives rise to the jth Riesz transform.

Definition 5.2.1. Let Ω be integrable on the sphere $\mathbf{S}^{n-1}$ with mean value zero. For $0<\varepsilon<N$ and $f\in\bigcup_{1\le p<\infty}L^p(\mathbf{R}^n)$ we define the *truncated singular integral*

$$T_\Omega^{(\varepsilon,N)}(f)(x)=\int_{\varepsilon\le|y|\le N}f(x-y)\,\frac{\Omega(y/|y|)}{|y|^n}\,dy. \tag{5.2.3}$$

Note that for $f\in L^p(\mathbf{R}^n)$ we have

$$\big\|T_\Omega^{(\varepsilon,N)}(f)\big\|_{L^p}\le\big\|\Omega\big\|_{L^1}\log(N/\varepsilon)\,\big\|f\big\|_{L^p(\mathbf{R}^n)},$$

which implies that (5.2.3) is finite a.e. and therefore well defined. We denote by T_Ω the singular integral operator whose kernel is the distribution W_Ω, that is,

$$T_\Omega(f)(x)=(f*W_\Omega)(x)=\lim_{\substack{\varepsilon\to 0\\ N\to\infty}}T_\Omega^{(\varepsilon,N)}(f)(x),$$

defined for $f\in\mathscr{S}(\mathbf{R}^n)$. The associated *maximal singular integral* is defined by

$$T_\Omega^{(**)}(f)=\sup_{0<N<\infty}\sup_{0<\varepsilon<N}\big|T_\Omega^{(\varepsilon,N)}(f)\big|. \tag{5.2.4}$$

We note that if Ω is bounded, there is no need to use the upper truncations in the definition of $T_\Omega^{(\varepsilon,N)}$ given in (5.2.3). In this case the maximal singular integrals could be defined as

$$T_\Omega^{(*)}(f) = \sup_{\varepsilon>0} \big|T_\Omega^{(\varepsilon)}(f)\big|, \tag{5.2.5}$$

where for $f \in \bigcup_{1\le p<\infty} L^p(\mathbf{R})$, $\varepsilon > 0$, and $x \in \mathbf{R}^n$, $T_\Omega^{(\varepsilon)}(f)(x)$ is defined in terms of the absolutely convergent integral

$$T_\Omega^{(\varepsilon)}(f)(x) = \int_{|y|\ge\varepsilon} f(x-y)\frac{\Omega(y/|y|)}{|y|^n}\,dy.$$

To examine the relationship between $T_\Omega^{(*)}$ and $T_\Omega^{(**)}$ for $\Omega \in L^\infty(\mathbf{S}^{n-1})$, notice that

$$\left|\int_{\varepsilon\le|y|\le N} f(x-y)\frac{\Omega(y/|y|)}{|y|^n}\,dy\right| \le \sup_{0<N<\infty} \big|T_\Omega^{(\varepsilon,N)}(f)(x)\big|. \tag{5.2.6}$$

Then for $f \in L^p(\mathbf{R}^n)$, $1 \le p < \infty$, we let $N \to \infty$ on the left in (5.2.6) and we note that the limit exists in view of the absolute convergence of the integral. Then we take the supremum over $\varepsilon > 0$ to deduce that $T_\Omega^{(*)}$ is pointwise bounded by $T_\Omega^{(**)}$. Since $T_\Omega^{(\varepsilon,N)} = T_\Omega^{(\varepsilon)} - T_\Omega^{(N)}$, it also follows that $T_\Omega^{(**)} \le 2T_\Omega^{(*)}$; thus $T_\Omega^{(*)}$ and $T_\Omega^{(**)}$ are pointwise comparable when Ω lies in $L^\infty(\mathbf{S}^{n-1})$. This is the case with the Hilbert transform, that is, $H^{(**)}$ is comparable to $H^{(*)}$; likewise with the Riesz transforms.

A certain class of multipliers can be realized as singular integral operators of the kind discussed. Recall from Proposition 2.4.7 that if m is homogeneous of degree 0 and infinitely differentiable on the sphere, then $m^\vee$ is given by

$$m^\vee = c\,\delta_0 + W_\Omega,$$

for some complex constant c and some smooth Ω on $\mathbf{S}^{n-1}$ with mean value zero. Therefore, all convolution operators whose multipliers are homogeneous of degree zero smooth functions on $\mathbf{S}^{n-1}$ can be realized as a constant multiple of the identity plus an operator of the form T_Ω.

Example 5.2.2. Let $P(\xi) = \sum_{|\alpha|=k} b_\alpha \xi^\alpha$ be a homogeneous polynomial of degree k in $\mathbf{R}^n$ that vanishes only at the origin. Let α be a multi-index of order k. Then the function

$$m(\xi) = \frac{\xi^\alpha}{P(\xi)} \tag{5.2.7}$$

is infinitely differentiable on the sphere and homogeneous of degree zero. The operator given by multiplication on the Fourier transform by $m(\xi)$ is a constant multiple of the identity plus an operator given by convolution with a distribution of the form W_Ω for some Ω in $\mathscr{C}^\infty(\mathbf{S}^{n-1})$ with mean value zero. In this section we establish the L^p boundedness of such operators when Ω has appropriate smoothness on the sphere. This, in particular, implies that $m(\xi)$ defined by (5.2.7) lies in the space $\mathscr{M}_p(\mathbf{R}^n)$, defined in Section 2.5, for $1 < p < \infty$.

5.2.2 L^2 Boundedness of Homogeneous Singular Integrals

Next we would like to compute the Fourier transform of W_Ω. This provides information as to whether the operator given by convolution with K_Ω is L^2 bounded. We have the following result.

Proposition 5.2.3. *Let $n \ge 2$ and $\Omega \in L^1(\mathbf{S}^{n-1})$ have mean value zero. Then the Fourier transform of W_Ω is a (finite a.e.) function given by the formula*

$$\widehat{W_\Omega}(\xi) = \int_{\mathbf{S}^{n-1}} \Omega(\theta)\left(\log\frac{1}{|\xi\cdot\theta|} - \frac{i\pi}{2}\operatorname{sgn}(\xi\cdot\theta)\right)d\theta\,. \tag{5.2.8}$$

Remark 5.2.4. We need to show that the function of ξ on the right in (5.2.8) is well defined and finite for almost all ξ in $\mathbf{R}^n$. Write $\xi = |\xi|\xi'$ where $\xi' \in \mathbf{S}^{n-1}$ and notice that

$$\log\frac{1}{|\xi\cdot\theta|} = \log\frac{1}{|\xi|} + \log\frac{1}{|\xi'\cdot\theta|}\,.$$

Since Ω has mean value zero, the term $\log\frac{1}{|\xi|}$ multiplied by $\Omega(\theta)$ vanishes when integrated over the sphere.

We need to show that

$$\int_{\mathbf{S}^{n-1}} |\Omega(\theta)|\log\frac{1}{|\xi'\cdot\theta|}\,d\theta < \infty \tag{5.2.9}$$

for almost all $\xi' \in \mathbf{S}^{n-1}$. Integrate (5.2.9) over $\xi' \in \mathbf{S}^{n-1}$ and apply Fubini's theorem to obtain

$$\begin{aligned}
&\int_{\mathbf{S}^{n-1}} |\Omega(\theta)|\int_{\mathbf{S}^{n-1}} \log\frac{1}{|\xi'\cdot\theta|}\,d\xi'\,d\theta \\
&\qquad = \int_{\mathbf{S}^{n-1}} |\Omega(\theta)|\int_{\mathbf{S}^{n-1}} \log\frac{1}{|\xi_1|}\,d\xi\,d\theta \\
&\qquad = \omega_{n-2}\int_{\mathbf{S}^{n-1}} |\Omega(\theta)|\int_{-1}^{+1}\left(\log\frac{1}{|s|}\right)(1-s^2)^{\frac{n-3}{2}}\,ds\,d\theta \\
&\qquad = C_n\big\|\Omega\big\|_{L^1(\mathbf{S}^{n-1})} < \infty,
\end{aligned}$$

since we are assuming that $n \ge 2$. (The second-to-last identity follows from the identity in Appendix D.2.) We conclude that (5.2.9) holds for almost all $\xi' \in \mathbf{S}^{n-1}$.

Since the function of ξ on the right in (5.2.8) is homogeneous of degree zero, it follows that it is a locally integrable function on $\mathbf{R}^n$.

Before we return to the proof of Proposition 5.2.3, we discuss the following lemma:

Lemma 5.2.5. *Let a be a nonzero real number. Then for $0 < \varepsilon < N < \infty$ we have*

$$\lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_\varepsilon^N \frac{\cos(ra)-\cos(r)}{r}\,dr \;=\; \log\frac{1}{|a|}\,, \tag{5.2.10}$$

$$\left|\int_{\varepsilon}^{N}\frac{\cos(ra)-\cos(r)}{r}\,dr\right| \le 2\left|\log\frac{1}{|a|}\right| \qquad \textit{for all } N>\varepsilon>0, \tag{5.2.11}$$

$$\lim_{\substack{\varepsilon\to 0\\ N\to\infty}}\int_{\varepsilon}^{N}\frac{e^{-ira}-\cos(r)}{r}\,dr = \log\frac{1}{|a|}-i\frac{\pi}{2}\operatorname{sgn} a, \tag{5.2.12}$$

$$\left|\int_{\varepsilon}^{N}\frac{e^{-ira}-\cos(r)}{r}\,dr\right| \le 2\left|\log\frac{1}{|a|}\right|+4 \qquad \textit{for all } N>\varepsilon>0. \tag{5.2.13}$$

Proof. We first prove (5.2.10) and (5.2.11). By the fundamental theorem of calculus we write

$$\begin{aligned}
\int_{\varepsilon}^{N}\frac{\cos(ra)-\cos(r)}{r}\,dr &= \int_{\varepsilon}^{N}\frac{\cos(r|a|)-\cos(r)}{r}\,dr\\
&= -\int_{\varepsilon}^{N}\int_{1}^{|a|}\sin(tr)\,dt\,dr\\
&= -\int_{1}^{|a|}\int_{\varepsilon}^{N}\sin(tr)\,dr\,dt\\
&= -\int_{1}^{|a|}\frac{\cos(\varepsilon t)}{t}\,dt+\int_{N}^{N|a|}\frac{\cos(t)}{t}\,dt,
\end{aligned}$$

and from this expression, we clearly obtain (5.2.11). But the first integral of the same expression converges to $-\log|a|$ as $\varepsilon\to 0$ while the second integral converges to zero as $N\to\infty$ by an integration by parts. This proves (5.2.10).

To prove (5.2.12) and (5.2.13) we need to know that the expressions

$$\left|\int_{\varepsilon}^{N}\frac{\sin(ra)}{r}\,dr\right| = \left|\int_{\varepsilon|a|}^{N|a|}\frac{\sin(r)}{r}\,dr\right| \tag{5.2.14}$$

tend to $\frac{\pi}{2}$ as $\varepsilon\to 0$ and $N\to\infty$ and are bounded by 4. Both statements follow from Exercise 5.1.1. □

Let us now prove Proposition 5.2.3.

Proof. Let us set $\xi'=\xi/|\xi|$. We have the following:

$$\begin{aligned}
\left\langle \widehat{W_\Omega},\varphi\right\rangle &= \left\langle W_\Omega,\widehat{\varphi}\right\rangle\\
&= \lim_{\varepsilon\to 0}\int_{|x|\ge\varepsilon}\frac{\Omega(x/|x|)}{|x|^n}\,\widehat{\varphi}(x)\,dx\\
&= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}}\int_{\varepsilon\le|x|\le N}\frac{\Omega(x/|x|)}{|x|^n}\,\widehat{\varphi}(x)\,dx\\
&= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}}\int_{\mathbf{R}^n}\varphi(\xi)\int_{\varepsilon\le|x|\le N}\frac{\Omega(x/|x|)}{|x|^n}e^{-2\pi i x\cdot\xi}\,dx\,d\xi\\
&= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}}\int_{\mathbf{R}^n}\varphi(\xi)\int_{\mathbf{S}^{n-1}}\Omega(\theta)\int_{\varepsilon\le r\le N}e^{-2\pi i r\theta\cdot\xi}\,\frac{dr}{r}\,d\theta\,d\xi
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{\mathbf{R}^n} \varphi(\xi) \int_{\mathbf{S}^{n-1}} \Omega(\theta) \int_{\varepsilon\le r\le N} \left(e^{-2\pi r|\xi| i\theta\cdot\xi'} - \cos(2\pi r|\xi|)\right) \frac{dr}{r}\, d\theta\, d\xi \\
&= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{\mathbf{R}^n} \varphi(\xi) \int_{\mathbf{S}^{n-1}} \Omega(\theta) \int_{\frac{\varepsilon}{2\pi|\xi|}\le r\le \frac{N}{2\pi|\xi|}} \frac{e^{-ir\theta\cdot\xi'} - \cos(r)}{r}\, dr\, d\theta\, d\xi \\
&= \int_{\mathbf{R}^n} \varphi(\xi) \int_{\mathbf{S}^{n-1}} \Omega(\theta) \left(\log\frac{1}{|\xi'\cdot\theta|} - \frac{i\pi}{2}\operatorname{sgn}(\xi\cdot\theta)\right) d\theta\, d\xi,
\end{aligned}
$$

where we used the Lebesgue dominated convergence theorem to pass the limit inside, Lemma 5.2.5, and Remark 5.2.4. We were able to subtract $\cos(2\pi r|\xi|)$ from the r integral in the previous calculation, since Ω has mean value zero over the sphere. Also, the use of the dominated convergence theorem is justified from the fact that the function

$$(\theta,\xi) \mapsto |\Omega(\theta)|\,|\varphi(\xi)|\left(\log\frac{1}{|\xi'\cdot\theta|} + 4\right)$$

lies in $L^1(\mathbf{S}^{n-1}\times\mathbf{R}^n)$. Moreover, all the interchanges of integrals are well justified by Fubini's theorem. □

Corollary 5.2.6. *Let $\Omega \in L^1(\mathbf{S}^{n-1})$ have mean value zero. Then for almost all ξ' in $\mathbf{S}^{n-1}$ the integral*

$$\int_{\mathbf{S}^{n-1}} \Omega(\theta) \log\frac{1}{|\xi'\cdot\theta|}\, d\theta \tag{5.2.15}$$

converges absolutely. Moreover, the associated operator T_Ω maps $L^2(\mathbf{R}^n)$ to itself if and only if

$$\operatorname*{ess.sup}_{\xi'\in\mathbf{S}^{n-1}} \left| \int_{\mathbf{S}^{n-1}} \Omega(\theta) \log\frac{1}{|\xi'\cdot\theta|}\, d\theta \right| < \infty. \tag{5.2.16}$$

Proof. To obtain the absolute convergence of the integral in (5.2.15) we integrate over $\xi' \in \mathbf{S}^{n-1}$ and we apply Fubini's theorem. The assertion concerning the boundedness of T_Ω on L^2 is an immediate consequence of Proposition 5.2.3 and Theorem 2.5.10. □

There exist functions Ω in $L^1(\mathbf{S}^{n-1})$ with mean value zero such that the expressions in (5.2.16) are equal to infinity; consequently, not all such Ω give rise to bounded operators on $L^2(\mathbf{R}^n)$. Observe, however, that for Ω odd i.e., $\Omega(-\theta) = -\Omega(\theta)$ for all $\theta \in \mathbf{S}^{n-1}$, (5.2.16) trivially holds, since $\log\frac{1}{|\xi\cdot\theta|}$ is even and its product against an odd function must have integral zero over $\mathbf{S}^{n-1}$. We conclude that singular integrals T_Ω with odd Ω are always L^2 bounded.

5.2.3 The Method of Rotations

Having settled the issue of L^2 boundedness for singular integrals of the form T_Ω with Ω odd, we turn our attention to their L^p boundedness. A simple procedure called the method of rotations plays a crucial role in the study of operators T_Ω when Ω is an odd function. This method is based on the use of the directional Hilbert transforms. Fix a unit vector θ in $\mathbf{R}^n$. For a Schwartz function f on $\mathbf{R}^n$ let

$$\mathscr{H}_\theta(f)(x) = \frac{1}{\pi}\,\mathrm{p.v.}\int_{-\infty}^{+\infty} f(x-t\theta)\frac{dt}{t}. \tag{5.2.17}$$

We call $\mathscr{H}_\theta(f)$ the *directional Hilbert transform* of f in the direction θ. For functions $f\in\mathscr{S}(\mathbf{R}^n)$ the integral in (5.2.17) is well defined, since it converges rapidly at infinity and by subtracting the constant $f(x)$, it also converges near zero.

Likewise, we define the *directional maximal Hilbert transforms*. For a function f in $\bigcup_{1\le p<\infty} L^p(\mathbf{R}^n)$ and $0<\varepsilon<N<\infty$ we let

$$\begin{aligned}
\mathscr{H}_\theta^{(\varepsilon,N)}(f)(x) &= \frac{1}{\pi}\int_{\varepsilon\le|t|\le N} f(x-t\theta)\frac{dt}{t},\\
\mathscr{H}_\theta^{(**)}(f)(x) &= \sup_{0<\varepsilon<N<\infty}\left|\mathscr{H}_\theta^{(\varepsilon,N)}(f)(x)\right|.
\end{aligned}$$

We observe that for any fixed $0<\varepsilon<N<\infty$ and $f\in L^p(\mathbf{R}^n)$, $\mathscr{H}_\theta^{(\varepsilon,N)}(f)$ is well defined almost everywhere. Indeed, by Minkowski's integral inequality we obtain

$$\big\|\mathscr{H}_\theta^{(\varepsilon,N)}(f)\big\|_{L^p(\mathbf{R}^n)} \le \frac{2}{\pi}\|f\|_{L^p(\mathbf{R}^n)}\log\frac{N}{\varepsilon}<\infty,$$

which implies that $\mathscr{H}_\theta^{(\varepsilon,N)}(f)(x)$ is finite for almost all $x\in\mathbf{R}^n$. Thus $\mathscr{H}_\theta^{(**)}(f)$ is well defined for f in $\bigcup_{1\le p<\infty} L^p(\mathbf{R}^n)$.

Theorem 5.2.7. *If Ω is odd and integrable over $\mathbf{S}^{n-1}$, then T_Ω and $T_\Omega^{(**)}$ are L^p bounded for all $1<p<\infty$. More precisely, T_Ω initially defined on Schwartz functions has a bounded extension on $L^p(\mathbf{R}^n)$ (which is also denoted by T_Ω).*

Proof. Let e_j be the usual unit vectors in $\mathbf{S}^{n-1}$. The operator $\mathscr{H}_{e_1}$ is obtained by applying the Hilbert transform in the first variable followed by the identity operator in the remaining variables. Clearly, $\mathscr{H}_{e_1}$ is bounded on $L^p(\mathbf{R}^n)$ with norm equal to that of the Hilbert transform on $L^p(\mathbf{R})$. Next observe that the following identity is valid for all matrices $A\in O(n)$:

$$\mathscr{H}_{A(e_1)}(f)(x) = \mathscr{H}_{e_1}(f\circ A)(A^{-1}x). \tag{5.2.18}$$

This implies that the L^p boundedness of $\mathscr{H}_\theta$ can be reduced to that of $\mathscr{H}_{e_1}$. We conclude that $\mathscr{H}_\theta$ is L^p bounded for $1<p<\infty$ with norm bounded by the norm of the Hilbert transform on $L^p(\mathbf{R})$ for every $\theta\in\mathbf{S}^{n-1}$.

Identity (5.2.18) is also valid for $\mathscr{H}_\theta^{(\varepsilon,N)}$ and $\mathscr{H}_\theta^{(**)}$. Consequently, $\mathscr{H}_\theta^{(**)}$ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ with norm at most that of $H^{(**)}$ on $L^p(\mathbf{R})$.

Next we realize a general singular integral T_Ω with Ω odd as an average of the directional Hilbert transforms $\mathscr{H}_\theta$. We start with f in $\bigcup_{1\le p<\infty} L^p(\mathbf{R}^n)$ and the following identities:

$$\begin{aligned}\int_{\varepsilon\le|y|\le N} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\,dy &= +\int_{\mathbf{S}^{n-1}} \Omega(\theta) \int_{r=\varepsilon}^{N} f(x-r\theta)\,\frac{dr}{r}\,d\theta \\ &= -\int_{\mathbf{S}^{n-1}} \Omega(\theta) \int_{r=\varepsilon}^{N} f(x+r\theta)\,\frac{dr}{r}\,d\theta,\end{aligned}$$

where the first follows by switching to polar coordinates and the second one is a consequence of the first one and the fact that Ω is odd via the change variables $\theta \mapsto -\theta$. Averaging the two identities, we obtain

$$\begin{aligned}&\int_{\varepsilon\le|y|\le N} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\,dy \\ &\qquad = \frac{1}{2}\int_{\mathbf{S}^{n-1}} \Omega(\theta) \int_{r=\varepsilon}^{N} \frac{f(x-r\theta)-f(x+r\theta)}{r}\,dr\,d\theta \\ &\qquad = \frac{\pi}{2}\int_{\mathbf{S}^{n-1}} \Omega(\theta)\,\mathscr{H}_\theta^{(\varepsilon,N)}(f)(x)\,d\theta.\end{aligned} \tag{5.2.19}$$

It follows from the identity in (5.2.19) that

$$\int_{\varepsilon\le|y|\le N} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\,dy = \frac{\pi}{2}\int_{\mathbf{S}^{n-1}} \Omega(\theta)\,\mathscr{H}_\theta^{(\varepsilon,N)}(f)(x)\,d\theta\,, \tag{5.2.20}$$

from which we conclude that

$$T_\Omega^{(**)}(f)(x) \le \frac{\pi}{2}\int_{\mathbf{S}^{n-1}} |\Omega(\theta)|\,\mathscr{H}_\theta^{(**)}(f)(x)\,d\theta\,. \tag{5.2.21}$$

Using the Lebesgue dominated convergence theorem, we see that for f in $\mathscr{S}(\mathbf{R}^n)$, we can pass the limits as $\varepsilon \to 0$ and $N \to \infty$ inside the integral in (5.2.20), concluding that

$$T_\Omega(f)(x) = \frac{\pi}{2}\int_{\mathbf{S}^{n-1}} \Omega(\theta)\,\mathscr{H}_\theta(f)(x)\,d\theta\,, \tag{5.2.22}$$

for $f \in \mathscr{S}(\mathbf{R}^n)$. The L^p boundedness of T_Ω and $T_\Omega^{(**)}$ for Ω odd are then trivial consequences of (5.2.22) and (5.2.21) via Minkowski's integral inequality. □

Corollary 5.2.8. *The Riesz transforms R_j and the maximal Riesz transforms $R_j^{(*)}$ are bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

Proof. The assertion follows from the fact that the Riesz transforms have odd kernels. Since the kernel of R_j decays like $|x|^{-n}$ near infinity, it follows that $R_j^{(*)}(f)$ is well defined for $f \in L^p(\mathbf{R}^n)$. Since $R_j^{(*)}$ is pointwise bounded by $2R_j^{(**)}$, the conclusion follows from Theorem 5.2.7. □

Remark 5.2.9. It follows from the proof of Theorem 5.2.7 and from Theorems 5.1.7 and 5.1.12 that whenever Ω is an odd function on $\mathbf{S}^{n-1}$, we have

$$\big\|T_\Omega\big\|_{L^p\to L^p} \le \big\|\Omega\big\|_{L^1}\begin{cases} a\,p & \text{when } p\ge 2,\\ a\,(p-1)^{-1} & \text{when } 1<p\le 2,\end{cases}$$
$$\big\|T_\Omega^{(**)}\big\|_{L^p\to L^p} \le \big\|\Omega\big\|_{L^1}\begin{cases} a\,p & \text{when } p\ge 2,\\ a\,(p-1)^{-1} & \text{when } 1<p\le 2,\end{cases}$$

for some $a>0$ independent of p and the dimension.

5.2.4 Singular Integrals with Even Kernels

Since a general integrable function Ω on $\mathbf{S}^{n-1}$ with mean value zero can be written as a sum of an odd and an even function, it suffices to study singular integral operators T_Ω with even kernels. For the rest of this section, fix an integrable even function Ω on $\mathbf{S}^{n-1}$ with mean value zero. The following idea is fundamental in the study of such singular integrals. Proposition 5.1.16 implies that

$$T_\Omega = -\sum_{j=1}^n R_jR_jT_\Omega\,. \tag{5.2.23}$$

If R_jT_Ω were another singular integral operator of the form T_{Ω_j} for some odd Ω_j, then the boundedness of T_Ω would follow from that of T_{Ω_j} via the identity (5.2.23) and Theorem 5.2.7. It turns out that R_jT_Ω does have an odd kernel, but it may not be integrable on $\mathbf{S}^{n-1}$ unless Ω itself possesses an additional amount of integrability. The amount of extra integrability needed is logarithmic, more precisely of this sort:

$$c_\Omega = \int_{\mathbf{S}^{n-1}} |\Omega(\theta)|\log^+|\Omega(\theta)|\,d\theta<\infty. \tag{5.2.24}$$

Observe that

$$\big\|\Omega\big\|_{L^1}\le c_\Omega + e\omega_{n-1}\le C_n\,(c_\Omega+1)\,,$$

which says that the norm $\|\Omega\|_{L^1}$ is always controlled by a dimensional constant multiple of $c_\Omega+1$. The following theorem is the main result of this section.

Theorem 5.2.10. *Let $n\ge 2$ and let Ω be an even integrable function on $\mathbf{S}^{n-1}$ with mean value zero that satisfies (5.2.24). Then the corresponding singular integral T_Ω is bounded on $L^p(\mathbf{R}^n)$, $1<p<\infty$, with norm at most a dimensional constant multiple of the quantity* $\max\big((p-1)^{-2},p^2\big)(c_\Omega+1)$.

If the operator T_Ω in Theorem 5.2.10 is weak type $(1,1)$, then the estimate on the L^p operator norm of T_Ω can be improved to $\|T_\Omega\|_{L^p\to L^p}\le C_n(p-1)^{-1}$ as $p\to 1$. This is indeed the case; see the historical comments at the end of this chapter.

Proof. Let W_Ω be the distributional kernel of T_Ω. We have that W_Ω coincides with the function $\Omega(x/|x|)|x|^{-n}$ on $\mathbf{R}^n\setminus\{0\}$. Using Proposition 5.2.3 and the fact that Ω is an even function, we obtain the formula

$$\widehat{W_\Omega}(\xi)=\int_{\mathbf{S}^{n-1}}\Omega(\theta)\log\frac{1}{|\xi\cdot\theta|}\,d\theta\,, \tag{5.2.25}$$

which implies that $\widehat{W_\Omega}$ is itself an even function. Now, using Exercise 5.2.3 and condition (5.2.24), we conclude that $\widehat{W_\Omega}$ is a bounded function. Therefore, T_Ω is L^2 bounded. To obtain the L^p boundedness of T_Ω, we use the idea mentioned earlier involving the Riesz transforms. In view of (5.1.46), we have that

$$T_\Omega=-\sum_{j=1}^{n}R_jT_j, \tag{5.2.26}$$

where $T_j=R_jT_\Omega$. Equality (5.2.26) makes sense as an operator identity on $L^2(\mathbf{R}^n)$, since T_Ω and each R_j are well defined and bounded on $L^2(\mathbf{R}^n)$.

The kernel of the operator T_j is the inverse Fourier transform of the distribution $-i\frac{\xi_j}{|\xi|}\widehat{W_\Omega}(\xi)$, which we denote by K_j. At this point we know only that K_j is a tempered distribution whose Fourier transform is the function $-i\frac{\xi_j}{|\xi|}\widehat{W_\Omega}(\xi)$. Our first goal is to show that K_j coincides with an integrable function on an annulus. To prove this assertion we write

$$W_\Omega=W_\Omega^0+W_\Omega^1+W_\Omega^\infty\,,$$

where W_Ω^0 is a distribution and $W_\Omega^1,W_\Omega^\infty$ are functions defined by

$$\begin{aligned}\left\langle W_\Omega^0,\varphi\right\rangle &=\lim_{\varepsilon\to0}\int_{\varepsilon<|x|\le\frac12}\frac{\Omega(x/|x|)}{|x|^n}\varphi(x)\,dx\,,\\ W_\Omega^1(x)&=\frac{\Omega(x/|x|)}{|x|^n}\chi_{\frac12\le|x|\le2}\,,\\ W_\Omega^\infty(x)&=\frac{\Omega(x/|x|)}{|x|^n}\chi_{2<|x|}\,.\end{aligned}$$

We now fix a $j\in\{1,2,\dots,n\}$ and we write

$$K_j=K_j^0+K_j^1+K_j^\infty\,,$$

where

$$\begin{aligned}K_j^0&=\big(-i\tfrac{\xi_j}{|\xi|}\widehat{W_\Omega^0}(\xi)\big)^\vee\,,\\ K_j^1&=\big(-i\tfrac{\xi_j}{|\xi|}\widehat{W_\Omega^1}(\xi)\big)^\vee\,,\\ K_j^\infty&=\big(-i\tfrac{\xi_j}{|\xi|}\widehat{W_\Omega^\infty}(\xi)\big)^\vee\,.\end{aligned}$$

Notice that K_j^0 is well defined via Theorem 2.3.21.

Define the annulus

$$A = \{x \in \mathbf{R}^n : \ 2/3 < |x| < 3/2\}.$$

For a smooth function ϕ supported in the annulus $2/3 < |x| < 3/2$ we have

$$\begin{aligned}
\langle K_j^0, \phi \rangle &= \big\langle \big(-i\tfrac{\xi_j}{|\xi|}\widehat{W_\Omega^0}(\xi)\big)^\vee, \phi \big\rangle \\
&= \big\langle -i\tfrac{\xi_j}{|\xi|}\widehat{W_\Omega^0}(\xi), \phi^\vee(\xi) \big\rangle \\
&= \big\langle \widehat{W_\Omega^0}(\xi), -i\tfrac{\xi_j}{|\xi|}\phi^\vee(\xi) \big\rangle \\
&= \big\langle W_\Omega^0, \big(-i\tfrac{\xi_j}{|\xi|}\phi^\vee(\xi)\big)^\wedge \big\rangle \\
&= -\big\langle W_\Omega^0, \widetilde{R_j(\phi)} \big\rangle \\
&= -\lim_{\varepsilon\to 0} \int\limits_{\varepsilon<|y|<1/2} \frac{\Omega(y/|y|)}{|y|^n} R_j(\phi)(-y)\, dy \qquad (\Omega \text{ is even}) \\
&= -\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \lim_{\varepsilon\to 0} \int\limits_{\varepsilon<|y|<1/2} \frac{\Omega(y/|y|)}{|y|^n} \int\limits_{R^n} \frac{y_j - x_j}{|y-x|^{n+1}} \phi(x)\, dx\, dy,
\end{aligned}$$

where the action of the distribution W_Ω^0 on $R_j(\phi)$ is justified by fact that $R_j(\phi)(y)$ is smooth on the support of W_Ω^0; note $|x-y| \geq 1/6$. Moreover, $\langle \widehat{W_\Omega^0}(\xi), -i\frac{\xi_j}{|\xi|}\phi^\vee(\xi)\rangle$ should be interpreted as a convergent integral.

It follows that for $x \in A$, the absolute value of the convolution of W_Ω^0 with the kernel of the Riesz transform R_j is

$$\left| \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \lim_{\varepsilon\to 0} \int_{\varepsilon<|y|<\frac{1}{2}} \frac{x_j - y_j}{|x-y|^{n+1}} \frac{\Omega(y/|y|)}{|y|^n}\, dy \right| \tag{5.2.27}$$

$$= \left| \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{|y|<\frac{1}{2}} \left(\frac{x_j - y_j}{|x-y|^{n+1}} - \frac{x_j}{|x|^{n+1}} \right) \frac{\Omega(y/|y|)}{|y|^n}\, dy \right| \tag{5.2.28}$$

$$\begin{aligned}
&\leq \int_{|y|\leq\frac{1}{2}} C_n |y| \frac{|\Omega(y/|y|)|}{|y|^n}\, dy \\
&= C_n' \big\|\Omega\big\|_{L^1},
\end{aligned}$$

where we used the fact that $\Omega(y/|y|)|y|^{-n}$ has integral zero over annuli of the form $\varepsilon < |y| < \frac{1}{2}$, the mean value theorem applied to the function $x_j|x|^{-(n+1)}$, and the fact that $|x-y| \geq 1/6$ for x in the annulus A. We conclude that on A, K_j^0 coincides with the bounded function inside the absolute value in (5.2.27).

Likewise, for $x \in A$ we have

$$\begin{aligned} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \bigg| \int_{|y|>2} \frac{x_j - y_j}{|x-y|^{n+1}} \frac{\Omega(y/|y|)}{|y|^n} \, dy \bigg| & \\ \le \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{|y|>2} \frac{1}{|x-y|^n} \frac{|\Omega(y/|y|)|}{|y|^n} \, dy & \\ \le \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{|y|>2} \frac{4^n}{|y|^{2n}} |\Omega(y/|y|)| \, dy & \\ = C \big\|\Omega\big\|_{L^1}, & \end{aligned} \tag{5.2.29}$$

from which it follows that on the annulus A, K_j^∞ coincides with the bounded function inside the absolute value in (5.2.29) or in (5.2.28).

Now observe that condition (5.2.24) gives that the function W_Ω^1 satisfies

$$\begin{aligned} \int_{|x|\le 2} |W_\Omega^1(x)| \log^+ |W_\Omega^1(x)| \, dx & \\ \le \int_{1/2}^2 \int_{\mathbf{S}^{n-1}} \frac{|\Omega(\theta)|}{r^n} \log^+ [2^n |\Omega(\theta)|] \, d\theta \, r^{n-1} \frac{dr}{r} & \\ \le (\log 4) \left[n(\log 2) \big\|\Omega\big\|_{L^1} + c_\Omega \right] < \infty . & \end{aligned}$$

Since the Riesz transform R_j is countably subadditive and maps L^p to L^p with norm at most $4(p-1)^{-1}$ for $1 < p < 2$, it follows from Exercise 1.3.7 that $K_j^1 = R_j(W_\Omega^1)$ is integrable over the ball $|x| \le 3/2$ and moreover, it satisfies

$$\int_A |K_j^1(x)| \, dx \le C_n \left[\int_{|x|\le 2} |W_\Omega^1(x)| \log^+ |W_\Omega^1(x)| \, dx + 1 \right] \le C_n' (c_\Omega + 1) .$$

Furthermore, since $\widehat{K_j}$ is homogeneous of degree zero, K_j is a homogeneous distribution of degree $-n$ (Exercise 2.3.9). This means that for all test functions φ and all $\lambda > 0$ we have

$$\left\langle K_j, \delta^\lambda(\varphi) \right\rangle = \left\langle K_j, \varphi \right\rangle , \tag{5.2.30}$$

where $\delta^\lambda(\varphi)(x) = \varphi(\lambda x)$. But for $\varphi \in \mathscr{C}_0^\infty$ supported in the annulus $3/4 < |x| < 4/3$ and for λ in $(8/9, 9/8)$ we have that $\delta^{\lambda^{-1}}(\varphi)$ is supported in A and thus we can express (5.2.30) as convergent integrals as follows:

$$\int_{\mathbf{R}^n} K_j(x) \varphi(x) \, dx = \int_{\mathbf{R}^n} K_j(x) \varphi(\lambda^{-1} x) \, dx = \int_{\mathbf{R}^n} \lambda^n K_j(\lambda x) \varphi(x) \, dx . \tag{5.2.31}$$

From this it would be ideal to be able to directly obtain that $K_j(x) = \lambda^n K_j(\lambda x)$ for all $8/9 < |x| < 9/8$ and $8/9 < \lambda < 9/8$, in particular when $\lambda = |x|^{-1}$. But unfortunately, we can only deduce that for every $\lambda \in (8/9, 9/8)$, $K_j(x) = \lambda^n K_j(\lambda x)$ holds for all x in the annulus except a set of measure zero that depends on λ. To be able to define the restriction of K_j on $\mathbf{S}^{n-1}$, we employ a more delicate argument.

For any J subinterval of $[8/9, 9/8]$ we obtain from (5.2.31) that

$$\int_{\mathbf{R}^n} K_j(x)\varphi(x)\,dx = \int_{\mathbf{R}^n} \fint_J \lambda^n K_j(\lambda x)\,d\lambda\,\varphi(x)\,dx,$$

where integral with the slashed integral denotes the average of a function over the set J. Since φ was an arbitrary $\mathscr{C}_0^\infty$ function supported in the annulus $3/4 < |x| < 4/3$, it follows that for every J subinterval of $[8/9, 9/8]$, there is a null subset E_J of the annulus $A' = \{x:\ 27/32 < |x| < 32/27\}$ such that

$$K_j(x) = \fint_J \lambda^n K_j(\lambda x)\,d\lambda \tag{5.2.32}$$

for all $x \in A' \setminus E_J$.

Let $J_0 = [\sqrt{8/9}, \sqrt{9/8}]$. We claim that there is a set of null subset E of A' such that for all $x \in A' \setminus E$ we have

$$\fint_{J_0} \lambda^n K_j(\lambda x)\,d\lambda = \fint_{rJ_0} \lambda^n K_j(\lambda x)\,d\lambda \tag{5.2.33}$$

for every r in J_0. Indeed, let E be the union of E_{rJ_0} over all r in $J_0 \cap \mathbf{Q}$. Then in view of (5.2.32), identity (5.2.33) holds for $x \in A' \setminus E$ and $J_0 \cap \mathbf{Q}$. But for a fixed x in $A' \setminus E$, the function of r on the right hand side of (5.2.33) is constant on the rationals and is also continuous (in r), hence it must be constant for all $r \in J_0$. Thus the claim follows since both sides of (5.2.33) are equal to (5.2.32).

Writing $x = \delta\theta$, where $27/32 < \delta < 32/27$ and $\theta \in \mathbf{S}^{n-1}$, it follows by Fubini's theorem that there is a $\delta \in (27/32, 32/27)$ (in fact almost all δ have this property) such that

$$\fint_{J_0} \lambda^n K_j(\lambda\delta\theta)\,d\lambda = \fint_{rJ_0} \lambda^n K_j(\lambda\delta\theta)\,d\lambda \tag{5.2.34}$$

for almost all $\theta \in \mathbf{S}^{n-1}$ and all $r \in J_0$. We fix such a δ, which we denote δ_0.

We now define a function Ω_j on $\mathbf{S}^{n-1}$ by setting

$$\Omega_j(\theta) = \fint_{J_0} \delta_0^n \lambda^n K_j(\lambda\delta_0\theta)\,d\lambda = \fint_{rJ_0} \delta_0^n \lambda^n K_j(\lambda\delta_0\theta)\,d\lambda$$

for all $r \in J_0$. The function Ω_j is defined almost everywhere and is integrable over $\mathbf{S}^{n-1}$, since K_j is integrable over the annulus A.

Let $e_1 = (1, 0, \dots, 0)$. Let Ψ be a $\mathscr{C}_0^\infty(\mathbf{R}^n)$ nonzero, nonnegative, radial, and supported in the annulus $32/(27\sqrt{2}) < |x| < 27\sqrt{2}/32$ around $\mathbf{S}^{n-1}$. We start with

$$\Omega_j(\theta) = \fint_{r^{-1}J_0} \delta_0^n \lambda^n K_j(\lambda\delta_0\theta)\,d\lambda = \fint_{J_0} \delta_0^n r^n \lambda^n K_j(r\lambda\delta_0\theta)\,d\lambda,$$

which holds for all $r \in J_0$, we multiply by $\Psi(re_1)$, and we integrate over $\mathbf{S}^{n-1}$ and over $(0, \infty)$ with respect to the measure dr/r. We obtain

$$
\begin{aligned}
\int_0^\infty \Psi(re_1)\frac{dr}{r}\int_{\mathbf{S}^{n-1}}\Omega_j(\theta)\,d\theta &= \fint_{J_0}\int_0^\infty\int_{\mathbf{S}^{n-1}}\delta_0^n\lambda^n K_j(\lambda\delta_0 r\theta)\Psi(re_1)r^n d\theta\frac{dr}{r}d\lambda \\
&= \fint_{J_0}\int_{\mathbf{R}^n}\delta_0^n\lambda^n K_j(\lambda\delta_0 x)\,\Psi(x)dxd\lambda \\
&= \fint_{J_0}\int_{\mathbf{R}^n}K_j(x)\Psi((\lambda\delta_0)^{-1}x)\,dxd\lambda \\
&= \fint_{J_0}\big\langle K_j,\Psi\big\rangle\,d\lambda\,, \\
&= \big\langle K_j,\Psi\big\rangle
\end{aligned}
$$

in view of the homogeneity of K_j. But also, for some constant c'_Ψ we have

$$
\big\langle K_j,\Psi\big\rangle = \big\langle\widehat{K_j},\widehat{\Psi}\,\big\rangle = \int_{\mathbf{R}^n}\frac{-i\xi_j}{|\xi|}\widehat{W_\Omega}(\xi)\widehat{\Psi}(\xi)\,d\xi = c'_\Psi\int_{\mathbf{S}^{n-1}}\frac{-i\theta_j}{|\theta|}\widehat{W_\Omega}(\theta)\,d\theta = 0,
$$

since by (5.2.25), $\frac{-i\xi_j}{|\xi|}\widehat{W_\Omega}(\xi)$ is an odd function. We conclude that Ω_j has mean value zero over $\mathbf{S}^{n-1}$.

Thus $\Omega \in L^1(\mathbf{S}^{n-1})$ has mean value zero and the distribution W_{Ω_j} is well defined. We claim that

$$
K_j = W_{\Omega_j}\,. \tag{5.2.35}
$$

To establish (5.2.35), we show first that $\big\langle K_j,\varphi\big\rangle = \big\langle W_{\Omega_j},\varphi\big\rangle$ whenever φ is supported in the annulus $8/9<|x|<9/8$. Using (5.2.32) we have

$$
\begin{aligned}
\int_{\mathbf{R}^n}K_j(x)\varphi(x)\,dx &= \int_{\mathbf{R}^n}\fint_{J_0}K_j(\delta_0\lambda x)\delta_0^n\lambda^n d\lambda\,\varphi(x)\,dx \\
&= \int_0^\infty\int_{\mathbf{S}^{n-1}}\fint_{J_0}K_j(\delta_0\lambda r\theta)\delta_0^n\lambda^n r^n\,d\lambda\,\varphi(r\theta)\,d\theta\frac{dr}{r} \\
&= \int_0^\infty\int_{\mathbf{S}^{n-1}}\fint_{rJ_0}K_j(\delta_0\lambda'\theta)\delta_0^n(\lambda')^n d\lambda'\,\varphi(r\theta)\,d\theta\frac{dr}{r} \\
&= \int_0^\infty\int_{\mathbf{S}^{n-1}}\Omega_j(\theta)\varphi(r\theta)\,d\theta\frac{dr}{r} \\
&= \big\langle W_{\Omega_j},\varphi\big\rangle,
\end{aligned}
$$

having used (5.2.34) in the second to last equality.

Given a general $\mathscr{C}_0^\infty$ function φ whose support is contained in an annulus of the form $M^{-1}<|x|<M$, for some $M>0$, via a smooth partition of unity, we write φ as a finite sum of smooth functions φ_k whose supports are contained in annuli of the form $8s/9<|x|<9s/8$ for some $s>0$. These annuli can be brought inside the annulus $8/9<|x|<9/8$ by a dilation. Since both K_j and W_{Ω_j} are homogeneous distributions of degree $-n$ and agree on the annulus $8/9<|x|<9/8$ they must agree on annuli $8s/9<|x|<9s/8$. Consequently, $\langle K_j,\varphi\rangle = \langle W_{\Omega_j},\varphi\rangle$ for all $\varphi\in\mathscr{C}_0^\infty(\mathbf{R}^n\setminus\{0\})$. Therefore, $K_j - W_{\Omega_j}$ is supported at the origin, and since it is homogeneous of degree $-n$, it must be equal to $b\delta_0$, a constant multiple of the Dirac mass. But $\widehat{K_j}$ is an odd function and hence K_j is also odd. It follows that W_{Ω_j} is an odd function on $\mathbf{R}^n\setminus\{0\}$, which implies that Ω_j is an odd function. We say that $u\in\mathscr{S}'(\mathbf{R}^n)$ is odd if $\widetilde{u}=-u$, where $\widetilde{u}$ is defined by $\langle\widetilde{u},\psi\rangle=\langle u,\widetilde{\psi}\rangle$ for all $\psi\in\mathscr{S}(\mathbf{R}^n)$ and $\widetilde{\psi}(x)=\psi(-x)$. We have that $K_j-W_{\Omega_j}$ is an odd distribution, and thus $b\delta_0$ must be an odd distribution. But if $b\delta_0$ is odd, then $b=0$. We conclude that for each j there exists an odd integrable function Ω_j on $\mathbf{S}^{n-1}$ with $\|\Omega_j\|_{L^1}$ controlled by a constant multiple of $c_\Omega+1$ such that (5.2.35) holds.

Then we use (5.2.26) and (5.2.35) to write

$$T_\Omega=-\sum_{j=1}^n R_jT_{\Omega_j}\,,$$

and appealing to the boundedness of each T_{Ω_j} (Theorem 5.2.7) and to that of the Riesz transforms, we obtain the required L^p boundedness for T_Ω. □

We note that Theorem 5.2.10 holds for all $\Omega\in L^1(\mathbf{S}^{n-1})$ that satisfy (5.2.24), not necessarily even Ω. Simply write $\Omega=\Omega_e+\Omega_o$, where Ω_e is even and Ω_o is odd, and check that condition (5.2.24) holds for Ω_e.

5.2.5 Maximal Singular Integrals with Even Kernels

We have the corresponding theorem for maximal singular integrals.

Theorem 5.2.11. *Let Ω be an even integrable function on $\mathbf{S}^{n-1}$ with mean value zero that satisfies (5.2.24). Then the corresponding maximal singular integral $T_\Omega^{(**)}$, defined in (5.2.4), is bounded on $L^p(\mathbf{R}^n)$ for $1<p<\infty$ with norm at most a dimensional constant multiple of* $\max(p^2,(p-1)^{-2})(c_\Omega+1)$.

Proof. For $f\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, x define the maximal function of f in the direction θ by setting

$$M_\theta(f)(x)=\sup_{a>0}\frac{1}{2a}\int_{|r|\le a}|f(x-r\theta)|\,dr\,. \tag{5.2.36}$$

In view of Exercise 5.2.5 we have that M_θ is bounded on $L^p(\mathbf{R}^n)$ with norm at most $3\,p\,(p-1)^{-1}$.

Fix Φ a smooth radial function such that $\Phi(x)=0$ for $|x|\le 1/4$, $\Phi(x)=1$ for $|x|\ge 3/4$, and $0\le\Phi(x)\le 1$ for all x in $\mathbf{R}^n$. For $f\in L^p(\mathbf{R}^n)$ and $0<\varepsilon<N<\infty$ we introduce the smoothly truncated singular integral

$$\widetilde{T}_\Omega^{(\varepsilon,N)}(f)(x)=\int_{\mathbf{R}^n}\frac{\Omega\big(\frac{y}{|y|}\big)}{|y|^n}\Big(\Phi\big(\tfrac{y}{\varepsilon}\big)-\Phi\big(\tfrac{y}{N}\big)\Big)f(x-y)\,dy$$

and the corresponding maximal singular integral operator

$$\widetilde{T}_\Omega^{(**)}(f)=\sup_{0<N<\infty}\sup_{0<\varepsilon<N}|\widetilde{T}_\Omega^{(\varepsilon,N)}(f)|\,. \tag{5.2.37}$$

Computing the supremum in (5.2.37), we first consider the case where $N>4\varepsilon$.

For f in $L^p(\mathbf{R}^n)$ (for some $1 < p < \infty$), we have

$$\begin{aligned}
&\big|\widetilde{T}_\Omega^{(\varepsilon,N)}(f)(x) - T_\Omega^{(\varepsilon,N)}(f)(x)\big| \\
&\quad = \Bigg| \int\limits_{\frac{\varepsilon}{4}\le|y|\le\varepsilon} \frac{\Omega\big(\frac{y}{|y|}\big)}{|y|^n}\Phi\big(\tfrac{y}{\varepsilon}\big) f(x-y)\,dy - \int\limits_{\frac{N}{4}\le|y|\le N} \frac{\Omega\big(\frac{y}{|y|}\big)}{|y|^n}\Phi\big(\tfrac{y}{N}\big) f(x-y)\,dy\Bigg| \\
&\quad \le \Bigg[\int\limits_{\frac{\varepsilon}{4}\le|y|\le\varepsilon} \frac{|\Omega\big(\frac{y}{|y|}\big)|}{|y|^n}|f(x-y)|\,dy + \int\limits_{\frac{N}{4}\le|y|\le N} \frac{|\Omega\big(\frac{y}{|y|}\big)|}{|y|^n}|f(x-y)|\,dy\Bigg] \\
&\quad \le \int\limits_{\mathbf{S}^{n-1}} |\Omega(\theta)| \Bigg[\frac{4}{\varepsilon}\int_{\frac{\varepsilon}{4}}^{\varepsilon} |f(x-r\theta)|\,dr + \frac{4}{N}\int_{\frac{N}{4}}^{N} |f(x-r\theta)|\,dr\Bigg]\,d\theta \\
&\quad \le 16\int_{\mathbf{S}^{n-1}} |\Omega(\theta)| M_\theta(f)(x)\,d\theta\,.
\end{aligned}$$

Now if $N \le 4\varepsilon$, then the function $\Phi\big(\frac{y}{\varepsilon}\big) - \Phi\big(\frac{y}{N}\big) - \chi_{\varepsilon\le|y|\le N}$ is bounded by 3 and is supported in the annulus $\frac{\varepsilon}{4} \le |y| \le 4\varepsilon$. In this case we obtain

$$\begin{aligned}
\big|\widetilde{T}_\Omega^{(\varepsilon,N)}(f)(x) - T_\Omega^{(\varepsilon,N)}(f)(x)\big| &\le 3\int\limits_{\mathbf{S}^{n-1}} |\Omega(\theta)| \int_{\frac{\varepsilon}{4}}^{4\varepsilon} |f(x-r\theta)|\,\frac{dr}{r}\,d\theta \\
&\le 96\int_{\mathbf{S}^{n-1}} |\Omega(\theta)| M_\theta(f)(x)\,d\theta\,.
\end{aligned}$$

We deduce from these estimates that

$$\sup_{0<\varepsilon<N<\infty} \big|\widetilde{T}_\Omega^{(\varepsilon,N)}(f)(x) - T_\Omega^{(\varepsilon,N)}(f)(x)\big| \le 96\int_{\mathbf{S}^{n-1}} |\Omega(\theta)| M_\theta(f)(x)\,d\theta\,.$$

Using the result of Exercise 5.2.5 we conclude that

$$\big\|\widetilde{T}_\Omega^{(**)}(f) - T_\Omega^{(**)}(f)\big\|_{L^p} \le 600\,\big\|\Omega\big\|_{L^1} \max(p,(p-1)^{-1})\,\big\|f\big\|_{L^p}\,.$$

This implies that it suffices to obtain the required L^p bound for the smoothly truncated maximal singular integral operator $\widetilde{T}_\Omega^{(**)}$.

Let K_j, Ω_j, and T_j be as in the previous theorem, and let F_j be the Riesz transform of the function $\Omega(x/|x|)\Phi(x)|x|^{-n}$. Let $f \in L^p(\mathbf{R}^n)$. A calculation yields the identity

$$\begin{aligned}
\widetilde{T}_\Omega^{(\varepsilon,N)}(f)(x) &= \int_{\mathbf{R}^n} \Bigg[\frac{1}{\varepsilon^n}\frac{\Omega(\frac{y}{\varepsilon}/|\frac{y}{\varepsilon}|)}{|\frac{y}{\varepsilon}|^n}\Phi(\tfrac{y}{\varepsilon}) - \frac{1}{N^n}\frac{\Omega(\frac{y}{N}/|\frac{y}{N}|)}{|\frac{y}{N}|^n}\Phi(\tfrac{y}{N})\Bigg] f(x-y)\,dy \\
&= -\Big(\sum_{j=1}^n \Big[\tfrac{1}{\varepsilon^n}F_j\big(\tfrac{\cdot}{\varepsilon}\big) - \tfrac{1}{N^n}F_j\big(\tfrac{\cdot}{N}\big)\Big] * R_j(f)\Big)(x)\,,
\end{aligned}$$

where in the last step we used Proposition 5.1.16. Therefore we may write

$$
\begin{aligned}
-\widetilde{T}_{\Omega}^{(\varepsilon,N)}(f)(x) &= \sum_{j=1}^{n} \int_{\mathbf{R}^n} \Big[\tfrac{1}{\varepsilon^n} F_j\left(\tfrac{x-y}{\varepsilon}\right) - \tfrac{1}{N^n} F_j\left(\tfrac{x-y}{N}\right)\Big] R_j(f)(y)\,dy \\
&= A_1^{(\varepsilon,N)}(f)(x) + A_2^{(\varepsilon,N)}(f)(x) + A_3^{(\varepsilon,N)}(f)(x),
\end{aligned}
\tag{5.2.38}
$$

where

$$
\begin{aligned}
A_1^{(\varepsilon,N)}(f)(x) &= \sum_{j=1}^{n} \frac{1}{\varepsilon^n} \int_{|x-y|\le\varepsilon} F_j\left(\tfrac{x-y}{\varepsilon}\right) R_j(f)(y)\,dy \\
&\qquad\qquad - \sum_{j=1}^{n} \frac{1}{N^n} \int_{|x-y|\le N} F_j\left(\tfrac{x-y}{N}\right) R_j(f)(y)\,dy, \\
A_2^{(\varepsilon,N)}(f)(x) &= \sum_{j=1}^{n} \int_{\mathbf{R}^n} \Big[\tfrac{1}{\varepsilon^n} \chi_{|x-y|>\varepsilon} \left\{F_j\left(\tfrac{x-y}{\varepsilon}\right) - K_j\left(\tfrac{x-y}{\varepsilon}\right)\right\} \\
&\qquad\qquad - \tfrac{1}{N^n} \chi_{|x-y|>N} \left\{F_j\left(\tfrac{x-y}{N}\right) - K_j\left(\tfrac{x-y}{N}\right)\right\}\Big] R_j(f)(y)\,dy, \\
A_3^{(\varepsilon,N)}(f)(x) &= \sum_{j=1}^{n} \int_{\mathbf{R}^n} \Big[\tfrac{1}{\varepsilon^n} \chi_{|x-y|>\varepsilon} K_j\left(\tfrac{x-y}{\varepsilon}\right) - \tfrac{1}{N^n} \chi_{|x-y|>N} K_j\left(\tfrac{x-y}{N}\right)\Big] R_j(f)(y)\,dy.
\end{aligned}
$$

It follows from the definitions of F_j and K_j that

$$
\begin{aligned}
F_j(z) - K_j(z) &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \lim_{\varepsilon\to 0} \int_{\varepsilon\le|y|} \frac{\Omega(y/|y|)}{|y|^n} \big(\Phi(y) - 1\big) \frac{z_j - y_j}{|z-y|^{n+1}}\,dy \\
&= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \int_{|y|\le\frac{3}{4}} \frac{\Omega(y/|y|)}{|y|^n} \big(\Phi(y) - 1\big) \left\{\frac{z_j - y_j}{|z-y|^{n+1}} - \frac{z_j}{|z|^{n+1}}\right\} dy
\end{aligned}
$$

whenever $|z| \ge 1$. But using the mean value theorem, the last expression is easily seen to be bounded by

$$
C_n \int_{|y|\le\frac{3}{4}} \frac{\Omega(y/|y|)}{|y|^n} \frac{|y|}{|z|^{n+1}}\,dy = C_n' \big\|\Omega\big\|_{L^1} |z|^{-(n+1)},
$$

whenever $|z| \ge 1$. Using this estimate, we obtain that the jth term in $A_2^{(\varepsilon,N)}(f)(x)$ is bounded by

$$
C_n \frac{\|\Omega\|_{L^1}}{\varepsilon^n} \int_{|x-y|>\varepsilon} \frac{|R_j(f)(y)|\,dy}{(|x-y|/\varepsilon)^{n+1}} \le C_n \frac{2\|\Omega\|_{L^1}}{2^{-n}\varepsilon^n} \int_{\mathbf{R}^n} \frac{|R_j(f)(y)|\,dy}{\left(1 + \frac{|x-y|}{\varepsilon}\right)^{n+1}}.
$$

It follows that for functions f in L^p we have

$$
\sup_{0<\varepsilon<N<\infty} |A_2^{(\varepsilon,N)}(f)| \le C_n \big\|\Omega\big\|_{L^1} M(R_j(f)),
$$

in view of Theorem 2.1.10. (M here is the Hardy–Littlewood maximal operator.) By Theorem 2.1.6, M maps $L^p(\mathbf{R}^n)$ to itself with norm bounded by a dimensional constant multiple of $\max(1,(p-1)^{-1})$. Since by Remark 5.2.9 the norm $\|R_j\|_{L^p\to L^p}$ is controlled by a dimensional constant multiple of $\max(p,(p-1)^{-1})$, it follows that

$$\Big\|\sup_{0<\varepsilon<N<\infty}|A_2^{(\varepsilon,N)}(f)|\Big\|_{L^p}\le C_n\big\|\Omega\big\|_{L^1}\max(p,(p-1)^{-1})\big\|f\big\|_{L^p}. \tag{5.2.39}$$

Next, recall that in the proof of Theorem 5.2.10 we showed that

$$K_j(x)=\frac{\Omega_j(x/|x|)}{|x|^n},$$

where Ω_j are integrable functions on $\mathbf{S}^{n-1}$ that satisfy

$$\big\|\Omega_j\big\|_{L^1}\le C_n(c_\Omega+1). \tag{5.2.40}$$

Consequently, for functions f in $L^p(\mathbf{R}^n)$ we have

$$\sup_{0<\varepsilon<N<\infty}|A_3^{(\varepsilon,N)}(f)|\le 2\sum_{j=1}^n T_{\Omega_j}^{(**)}(R_j(f)),$$

and by Remark 5.2.9 this last expression has L^p norm at most a dimensional constant multiple of $\big\|\Omega_j\big\|_{L^1}\max(p,(p-1)^{-1})\big\|R_j(f)\big\|_{L^p}$. It follows that

$$\Big\|\sup_{0<\varepsilon<N<\infty}|A_3^{(\varepsilon,N)}(f)|\Big\|_{L^p}\le C_n\max(p^2,(p-1)^{-2})(c_\Omega+1)\,\big\|f\big\|_{L^p}. \tag{5.2.41}$$

Finally, we turn our attention to the term $A_1^{(\varepsilon,N)}(f)$. To prove the required estimate, we first show that there exist nonnegative homogeneous of degree zero functions G_j on $\mathbf{R}^n$ that satisfy

$$|F_j(x)|\le G_j(x) \qquad \text{when } |x|\le 1 \tag{5.2.42}$$

and

$$\int_{\mathbf{S}^{n-1}}|G_j(\theta)|\,d\theta\le C_n(c_\Omega+1). \tag{5.2.43}$$

To prove (5.2.42), first note that if $|x|\le 1/8$, then

$$\begin{aligned}|F_j(x)| &= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\left|\int_{\mathbf{R}^n}\frac{\Omega(y/|y|)}{|y|^n}\Phi(y)\frac{x_j-y_j}{|x-y|^{n+1}}dy\right|\\ &\le C_n\int_{|y|\ge\frac14}\frac{|\Omega(y/|y|)|}{|y|^{2n}}\,dy\\ &\le C_n'\big\|\Omega\big\|_{L^1}.\end{aligned}$$

We now fix an x satisfying $1/8 \le |x| \le 1$ and we write

$$\begin{aligned}
|F_j(x)| &\le \Phi(x)|K_j(x)| + |F_j(x) - \Phi(x)K_j(x)| \\
&\le |K_j(x)| + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\left|\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} \frac{x_j - y_j}{|x-y|^{n+1}}\big(\Phi(y) - \Phi(x)\big)\frac{\Omega(y/|y|)}{|y|^n}dy\right| \\
&= |K_j(x)| + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\big(P_1(x) + P_2(x) + P_3(x)\big),
\end{aligned}$$

where

$$\begin{aligned}
P_1(x) &= \left|\int_{|y|\le\frac{1}{16}}\left(\frac{x_j - y_j}{|x-y|^{n+1}} - \frac{x_j}{|x|^{n+1}}\right)\big(\Phi(y) - \Phi(x)\big)\frac{\Omega(y/|y|)}{|y|^n}\,dy\right|, \\
P_2(x) &= \left|\int_{\frac{1}{16}\le|y|\le 2} \frac{x_j - y_j}{|x-y|^{n+1}}\big(\Phi(y) - \Phi(x)\big)\frac{\Omega(y/|y|)}{|y|^n}\,dy\right|, \\
P_3(x) &= \left|\int_{|y|\ge 2} \frac{x_j - y_j}{|x-y|^{n+1}}\big(\Phi(y) - \Phi(x)\big)\frac{\Omega(y/|y|)}{|y|^n}\,dy\right|.
\end{aligned}$$

But since $1/8 \le |x| \le 1$, we see that

$$P_1(x) \le C_n \int_{|y|\le\frac{1}{16}} \frac{|y|}{|x|^{n+1}}\frac{|\Omega(y/|y|)|}{|y|^n}\,dy \le C_n'\big\|\Omega\big\|_{L^1}$$

and that

$$P_3(x) \le C_n \int_{|y|\ge 2} \frac{|\Omega(y/|y|)|}{|y|^{2n}}\,dy \le C_n'\big\|\Omega\big\|_{L^1}.$$

For $P_2(x)$ we use the estimate $|\Phi(y) - \Phi(x)| \le C|x-y|$ to obtain

$$\begin{aligned}
P_2(x) &\le \int_{\frac{1}{16}\le|y|\le 2} \frac{C}{|x-y|^{n-1}}\frac{|\Omega(y/|y|)|}{|y|^n}\,dy \\
&\le 4C\int_{\frac{1}{16}\le|y|\le 2} \frac{|\Omega(y/|y|)|}{|x-y|^{n-1}|y|^{n-\frac{1}{2}}}\,dy \\
&\le 4C\int_{\mathbf{R}^n} \frac{|\Omega(y/|y|)|}{|x-y|^{n-1}|y|^{n-\frac{1}{2}}}\,dy.
\end{aligned}$$

Recall that $K_j(x) = \Omega_j(x/|x|)|x|^{-n}$. We now set

$$G_j(x) = C_n\left(\big\|\Omega\big\|_{L^1} + \left|\Omega_j\left(\frac{x}{|x|}\right)\right| + |x|^{n-\frac{3}{2}}\int_{\mathbf{R}^n}\frac{|\Omega(y/|y|)|\,dy}{|x-y|^{n-1}|y|^{n-\frac{1}{2}}}\right) \tag{5.2.44}$$

and we observe that G_j is a homogeneous of degree zero function, it satisfies (5.2.42), and it is integrable over the annulus $\frac{1}{2} \le |x| \le 2$. To verify the last assertion, we split up the double integral

$$I = \int_{\frac{1}{2}\le|x|\le 2}\int_{\mathbf{R}^n} \frac{|\Omega(y/|y|)|\,dy}{|x-y|^{n-1}|y|^{n-\frac{1}{2}}}\,dx$$

into the pieces $1/4 \le |y| \le 4$, $|y| > 4$, and $|y| < 1/4$. The part of I where $1/4 \le |y| \le 4$ is pointwise bounded by a constant multiple of

$$\int_{\frac{1}{4}\le|y|\le 4} \left|\Omega\Big(\frac{y}{|y|}\Big)\right| \int_{\frac{1}{2}\le|x|\le 2} \frac{dx}{|y-x|^{n-1}}\,dy \le \int_{\frac{1}{4}\le|y|\le 4} \left|\Omega\Big(\frac{y}{|y|}\Big)\right| \int_{|x-y|\le 6} \frac{dx}{|y-x|^{n-1}}\,dy\,,$$

which is pointwise controlled by a constant multiple of $\|\Omega\|_{L^1}$. In the part of I where $|y| > 4$ we use that $|x-y|^{-n+1} \le (|y|/2)^{-n+1}$ to obtain rapid decay in y and hence a bound by a constant multiple of $\|\Omega\|_{L^1}$. Finally, in the part of I where $|y| < 1/4$ we use that $|x-y|^{-n+1} \le (1/4)^{-n+1}$, and then we also obtain a similar bound. It follows from (5.2.44) and (5.2.40) that

$$\int_{\frac{1}{2}\le|x|\le 2} |G_j(x)|\,dx \le C_n\big(\|\Omega\|_{L^1} + \|\Omega_j\|_{L^1} + \|\Omega\|_{L^1}\big) \le C_n(c_\Omega + 1).$$

Since G_j is homogeneous of degree zero, we deduce (5.2.43).

To complete the proof, we argue as follows:

$$\begin{aligned}
&\sup_{0<\varepsilon<N<\infty} |A_1^{(\varepsilon,N)}(f)(x)| \\
&\qquad \le 2\sup_{\varepsilon>0}\sum_{j=1}^{n} \frac{1}{\varepsilon^n}\int_{|z|\le\varepsilon} |F_j(z)|\,|R_j(f)(x-z)|\,dz \\
&\qquad \le 2\sup_{\varepsilon>0}\sum_{j=1}^{n} \frac{1}{\varepsilon^n}\int_{r=0}^{\varepsilon}\int_{\mathbf{S}^{n-1}} |F_j(r\theta)|\,|R_j(f)(x-r\theta)|\,r^{n-1}\,d\theta\,dr \\
&\qquad \le 2\sum_{j=1}^{n}\int_{\mathbf{S}^{n-1}} |G_j(\theta)| \left\{\sup_{\varepsilon>0}\frac{1}{\varepsilon^n}\int_{r=0}^{\varepsilon} |R_j(f)(x-r\theta)|\,r^{n-1}\,dr\right\} d\theta \\
&\qquad \le 4\sum_{j=1}^{n}\int_{\mathbf{S}^{n-1}} |G_j(\theta)| M_\theta(R_j(f))(x)\,d\theta\,.
\end{aligned}$$

Using (5.2.43) together with the L^p boundedness of the Riesz transforms and of M_θ we obtain

$$\Big\|\sup_{0<\varepsilon<N<\infty} |A_1^{(\varepsilon,N)}(f)|\Big\|_{L^p} \le C_n \max(p,(p-1)^{-2})(c_\Omega+1)\big\|f\big\|_{L^p}\,. \tag{5.2.45}$$

Combining (5.2.45), (5.2.39), and (5.2.41), we obtain the required conclusion. □

The following corollary is a consequence of Theorem 5.2.11.

Corollary 5.2.12. *Let Ω be as in Theorem 5.2.11. Then for $1 < p < \infty$ and f in $L^p(\mathbf{R}^n)$ the functions $T_\Omega^{(\varepsilon,N)}(f)$ converge to $T_\Omega(f)$ in L^p and almost everywhere as $\varepsilon \to 0$ and $N \to \infty$.*

Proof. The a.e. convergence is a consequence of Theorem 2.1.14. The L^p convergence is a consequence of the Lebesgue dominated convergence theorem since for $f \in L^p(\mathbf{R}^n)$ we have that $|T_\Omega^{(\varepsilon,N)}(f)| \le T_\Omega^{(**)}(f)$ and $T_\Omega^{(**)}(f)$ is in $L^p(\mathbf{R}^n)$. $\square$

Exercises

5.2.1. Show that the directional Hilbert transform $\mathscr{H}_\theta$ is given by convolution with the distribution w_θ in $\mathscr{S}'(\mathbf{R}^n)$ defined by

$$\langle w_\theta, \varphi \rangle = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{+\infty} \frac{\varphi(t\theta)}{t}\, dt.$$

Compute the Fourier transform of w_θ and prove that $\mathscr{H}_\theta$ maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$.

5.2.2. Extend the definitions of W_Ω and T_Ω to $\Omega = d\mu$ a finite signed Borel measure on $\mathbf{S}^{n-1}$ with mean value zero. Compute the Fourier transform of $W_{d\mu}$ and find a necessary and sufficient condition on measures $d\mu$ so that $T_{d\mu}$ is L^2 bounded. Notice that the directional Hilbert transform $\mathscr{H}_\theta$ is a special case of such an operator $T_{d\mu}$.

5.2.3. Use the inequality $AB \le A\log A + e^B$ for $A \ge 1$ and $B > 0$ to prove that if Ω satisfies (5.2.24) then it must satisfy (5.2.16). Conclude that if $|\Omega|\log^+|\Omega|$ is in $L^1(\mathbf{S}^{n-1})$, then T_Ω is L^2 bounded.
$\big[$*Hint:* Use that $\int_{\mathbf{S}^{n-1}} |\xi\cdot\theta|^{-\alpha}\, d\theta$ converges when $\alpha < 1$. See Appendix D.3.$\big]$

5.2.4. Let Ω be a nonzero integrable function on $\mathbf{S}^{n-1}$ with mean value zero. Let $f \ge 0$ be nonzero and integrable over $\mathbf{R}^n$. Prove that $T_\Omega(f)$ is not in $L^1(\mathbf{R}^n)$.
$\big[$*Hint:* Show that $\widehat{T_\Omega(f)}$ cannot be continuous at zero.$\big]$

5.2.5. Let $\theta \in \mathbf{S}^{n-1}$. Use an identity similar to (5.2.18) to show that the maximal operators

$$\sup_{a>0} \frac{1}{a}\int_0^a |f(x-r\theta)|\, dr, \qquad \sup_{a>0} \frac{1}{2a}\int_{-a}^{+a} |f(x-r\theta)|\, dr$$

are $L^p(\mathbf{R}^n)$ bounded for $1 < p < \infty$ with norm at most $3\,p\,(p-1)^{-1}$.

5.2.6. For $\Omega \in L^1(\mathbf{S}^{n-1})$ and f locally integrable on $\mathbf{R}^n$, define

$$M_\Omega(f)(x) = \sup_{R>0} \frac{1}{v_n R^n} \int_{|y|\le R} |\Omega(y/|y|)|\,|f(x-y)|\, dy.$$

Apply the method of rotations to prove that M_Ω maps $L^p(\mathbf{R}^n)$ to itself for $1 < p < \infty$.

5.2.7. Let $\Omega(x,\theta)$ be a function on $\mathbf{R}^n \times \mathbf{S}^{n-1}$ satisfying
(a) $\Omega(x,-\theta) = -\Omega(x,\theta)$ for all x and θ.
(b) $\sup_x |\Omega(x,\theta)|$ is in $L^1(\mathbf{S}^{n-1})$.

Use the method of rotations to prove that

$$T_\Omega(f)(x) = \text{p.v.} \int_{\mathbf{R}^n} \frac{\Omega(x, y/|y|)}{|y|^n} f(x-y)\,dy$$

is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.

5.2.8. Let $\Omega \in L^1(\mathbf{S}^{n-1})$ have mean value zero. Prove that if T_Ω maps $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, then $p = q$.
$\big[$*Hint:* Use dilations.$\big]$

5.2.9. Prove that for all $1 < p < \infty$ there exists a constant $A_p > 0$ such that for every complex-valued $\mathscr{C}^2(\mathbf{R}^2)$ function f with compact support we have the bound

$$\big\|\partial_{x_1} f\big\|_{L^p} + \big\|\partial_{x_2} f\big\|_{L^p} \le A_p \big\|\partial_{x_1} f + i\partial_{x_2} f\big\|_{L^p}.$$

5.2.10. (a) Let $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ be the usual Laplacian on $\mathbf{R}^n$. Prove that for all $1 < p < \infty$ there exists a constant $A_p > 0$ such that for all $\mathscr{C}^2$ functions f with compact support we have the bound

$$\big\|\partial_{x_j}\partial_{x_k} f\big\|_{L^p} \le A_p \big\|\Delta f\big\|_{L^p}.$$

(b) Let $\Delta^m = \overbrace{\Delta \circ \cdots \circ \Delta}^{m \text{ times}}$. Show that for any $1 < p < \infty$ there exists a $C_p > 0$ such that for all f of class $\mathscr{C}^{2m}$ with compact support and all differential monomials ∂_x^α of order $|\alpha| = 2m$ we have

$$\big\|\partial_x^\alpha f\big\|_{L^p} \le C_p \big\|\Delta^m f\big\|_{L^p}.$$

5.2.11. Use the same idea as in Lemma 5.2.5 to show that if f is continuous on $[0,\infty)$, differentiable in $(0,\infty)$, and satisfies

$$\lim_{N\to\infty} \int_N^{Na} \frac{f(u)}{u}\,du = 0$$

for all $a > 0$, then

$$\lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_\varepsilon^N \frac{f(at) - f(t)}{t}\,dt = f(0)\log\frac{1}{a}.$$

5.2.12. Let Ω_o be an odd integrable function on $\mathbf{S}^{n-1}$ and Ω_e an even function on $\mathbf{S}^{n-1}$ that satisfies (5.2.24). Let f be a function supported in a ball B in $\mathbf{R}^n$. Prove that
(a) If $|f|\log^+|f|$ is integrable over a ball B, then $T_{\Omega_o}(f)$ and $T_{\Omega_o}^{(**)}(f)$ are integrable over B.
(b) If $|f|(\log^+|f|)^2$ is integrable over a ball B, then $T_{\Omega_e}(f)$ and $T_{\Omega_e}^{(**)}(f)$ are integrable over B.
$\big[$*Hint:* Use Exercise 1.3.7.$\big]$

5.2.13. ([324]) Let Ω be integrable on $\mathbf{S}^{n-1}$ with mean value zero. Use Jensen's inequality to show that for some $C > 0$ and every radial function $f \in L^2(\mathbf{R}^n)$ we have

$$\big\|T_\Omega(f)\big\|_{L^2} \le C\big\|f\big\|_{L^2}.$$

This inequality subsumes that T_Ω is well defined on radial $L^2(\mathbf{R}^n)$ functions.

5.3 The Calderón–Zygmund Decomposition and Singular Integrals

The behavior of singular integral operators on $L^1(\mathbf{R}^n)$ is a more subtle issue than that on L^p for $1 < p < \infty$. It turns out that singular integrals are not bounded from L^1 to L^1. See Example 5.1.3 and also Exercise 5.2.4. In this section we see that singular integrals map L^1 into the larger space $L^{1,\infty}$. This result strengthens their L^p boundedness.

5.3.1 The Calderón–Zygmund Decomposition

To make some advances in the theory of singular integrals, we need to introduce the Calderón–Zygmund decomposition. This is a powerful stopping-time construction that has many other interesting applications. We have already encountered an example of a stopping-time argument in Section 2.1.

Recall that a dyadic cube in $\mathbf{R}^n$ is the set

$$[2^k m_1, 2^k(m_1+1)) \times \cdots \times [2^k m_n, 2^k(m_n+1)),$$

where $k, m_1, \dots, m_n \in \mathbf{Z}$. Two dyadic cubes are either disjoint or related by inclusion.

Theorem 5.3.1. *Let $f \in L^1(\mathbf{R}^n)$ and $\alpha > 0$. Then there exist functions g and b on $\mathbf{R}^n$ such that*

(1) $f = g + b$.

(2) $\|g\|_{L^1} \le \|f\|_{L^1}$ *and* $\|g\|_{L^\infty} \le 2^n \alpha$.

(3) $b = \sum_j b_j$, *where each* b_j *is supported in a dyadic cube* Q_j. *Furthermore, the cubes* Q_k *and* Q_j *are disjoint when* $j \neq k$.

(4) $\displaystyle\int_{Q_j} b_j(x)\,dx = 0.$

(5) $\|b_j\|_{L^1} \le 2^{n+1}\alpha |Q_j|$.

(6) $\sum_j |Q_j| \le \alpha^{-1}\|f\|_{L^1}$.

Remark 5.3.2. This decomposition is called the *Calderón–Zygmund decomposition* of f at height α. The function g is called the *good function* of the decomposition, since it is both integrable and bounded; hence the letter g. The function b is called the *bad function*, since it contains the singular part of f (hence the letter b), but it is carefully chosen to have mean value zero. It follows from (1) and (2) that the bad function b is integrable and satisfies

$$\|b\|_{L^1} \le \|f\|_{L^1} + \|g\|_{L^1} \le 2\|f\|_{L^1}.$$

By (2) the good function is integrable and bounded; hence it lies in all the L^p spaces for $1 \le p \le \infty$. More specifically, we have the following estimate:

$$\|g\|_{L^p} \le \|g\|_{L^1}^{\frac{1}{p}} \|g\|_{L^\infty}^{1-\frac{1}{p}} \le \|f\|_{L^1}^{\frac{1}{p}} (2^n\alpha)^{1-\frac{1}{p}} = 2^{\frac{n}{p'}} \alpha^{\frac{1}{p'}} \|f\|_{L^1}^{\frac{1}{p}}. \tag{5.3.1}$$

Proof. Decompose $\mathbf{R}^n$ into a mesh of disjoint dyadic cubes of the same size such that

$$|Q| \ge \frac{1}{\alpha}\|f\|_{L^1}$$

for every cube Q in the mesh. Call these cubes of zero generation. Subdivide each cube of zero generation into 2^n congruent cubes by bisecting each of its sides. We now have a new mesh of dyadic cubes, which we call of generation one. Select a cube Q of generation one if

$$\frac{1}{|Q|}\int_Q |f(x)|\,dx > \alpha. \tag{5.3.2}$$

Let $S^{(1)}$ be the set of all selected cubes of generation one. Now subdivide each nonselected cube of generation one into 2^n congruent subcubes by bisecting each side and call these cubes of generation two. Then select all cubes Q of generation two if (5.3.2) holds. Let $S^{(2)}$ be the set of all selected cubes of generation two. Repeat this procedure indefinitely.

The set of all selected cubes $\bigcup_{m=1}^\infty S^{(m)}$ is countable and is exactly the set of the cubes Q_j proclaimed in the proposition. Note that in some instances this set may be empty, in which case $b = 0$ and $g = f$. Let us observe that the selected cubes are disjoint, for otherwise some Q_k would be a proper subset of some Q_j, which is impossible since the selected cube Q_j was never subdivided. Now define

$$b_j = \left(f - \frac{1}{|Q_j|}\int_{Q_j} f\,dx \right)\chi_{Q_j},$$

$b = \sum_j b_j$, and $g = f - b$.

For a selected cube Q_j there exists a unique nonselected cube Q' with twice its side length that contains Q_j. Let us call this cube the parent of Q_j. Since the parent Q' of Q_j was not selected, we have $|Q'|^{-1}\int_{Q'} |f|\,dx \le \alpha$. Then

$$\frac{1}{|Q_j|}\int_{Q_j}|f(x)|\,dx \le \frac{1}{|Q_j|}\int_{Q'}|f(x)|\,dx = \frac{2^n}{|Q'|}\int_{Q'}|f(x)|\,dx \le 2^n\alpha.$$

Consequently,

$$\int_{Q_j}|b_j|\,dx \le \int_{Q_j}|f|\,dx + |Q_j|\left|\frac{1}{|Q_j|}\int_{Q_j} f\,dx\right| \le 2\int_{Q_j}|f|\,dx \le 2^{n+1}\alpha|Q_j|,$$

which proves (5). To prove (6), simply observe that

$$\sum_j |Q_j| \le \frac{1}{\alpha}\sum_j \int_{Q_j}|f|\,dx = \frac{1}{\alpha}\int_{\bigcup_j Q_j}|f|\,dx \le \frac{1}{\alpha}\|f\|_{L^1}.$$

Next we need to obtain the estimates concerning g. We obviously have

$$g = \begin{cases} f & \text{on } \mathbf{R}^n \setminus \bigcup_j Q_j, \\ \frac{1}{|Q_j|}\int_{Q_j} f\,dx & \text{on } Q_j. \end{cases} \tag{5.3.3}$$

On the cube Q_j, g is equal to the constant $|Q_j|^{-1}\int_{Q_j} f\,dx$, and this is bounded by $2^n\alpha$. It suffices to show that g is bounded outside the union of the Q_j's. Indeed, for each $x \in \mathbf{R}^n \setminus \bigcup_j Q_j$ and for each $k = 0,1,2,\dots$ there exists a unique nonselected dyadic cube $Q_x^{(k)}$ of generation k that contains x. Then for each $k \ge 0$, we have

$$\left|\frac{1}{|Q_x^{(k)}|}\int_{Q_x^{(k)}} f(y)\,dy\right| \le \frac{1}{|Q_x^{(k)}|}\int_{Q_x^{(k)}}|f(y)|\,dy \le \alpha.$$

The intersection of the closures of the cubes $Q_x^{(k)}$ is the singleton $\{x\}$. Using Corollary 2.1.16, we deduce that for almost all $x \in \mathbf{R}^n \setminus \bigcup_j Q_j$ we have

$$f(x) = \lim_{k\to\infty}\frac{1}{|Q_x^{(k)}|}\int_{Q_x^{(k)}} f(y)\,dy.$$

Since these averages are at most α, we conclude that $|f| \le \alpha$ a.e. on $\mathbf{R}^n \setminus \bigcup_j Q_j$, hence $|g| \le \alpha$ a.e. on this set. Finally, it follows from (5.3.3) that $\|g\|_{L^1} \le \|f\|_{L^1}$. This finishes the proof of the theorem. $\square$

We now apply the Calderón–Zygmund decomposition to obtain weak type $(1,1)$ bounds for a wide class of singular integral operators that includes the operators T_Ω we studied in the previous section.

5.3.2 General Singular Integrals

The kernels of the general singular integrals we will study are tempered distributions that coincide with functions away from the origin. The setup as follows. Let K be a measurable function defined on $\mathbf{R}^n \setminus \{0\}$ that is integrable on compact subsets of $\mathbf{R}^n \setminus \{0\}$ and satisfies the size condition

$$\sup_{R>0} \int_{R \le |x| \le 2R} |K(x)|\,dx = A_1 < \infty. \tag{5.3.4}$$

This condition is less restrictive than the standard size estimate

$$\sup_{x \in \mathbf{R}^n} |x|^n |K(x)| < \infty, \tag{5.3.5}$$

but it is strong enough to capture size properties of kernels $K(x) = \Omega(x/|x|)/|x|^n$, where $\Omega \in L^1(\mathbf{S}^{n-1})$. We also note that condition (5.3.4) is equivalent to

$$\sup_{R>0} \frac{1}{R} \int_{|x| \le R} |K(x)|\,|x|\,dx < \infty. \tag{5.3.6}$$

See Exercise 5.3.1.

The size condition (5.3.4) is sufficient to make the restriction of $K(x)$ on $|x| > \delta$ a tempered distribution (for any $\delta > 0$). Indeed, for $\varphi \in \mathscr{S}(\mathbf{R}^n)$ we have

$$\begin{aligned} \int_{|x| \ge 1} |K(x)\varphi(x)|\,dx &\le \sum_{m=0}^{\infty} \int_{2^{m+1} \ge |x| \ge 2^m} \frac{|K(x)|(1+|x|)^N |\varphi(x)|}{(1+2^m)^N}\,dx \\ &\le \sum_{m=0}^{\infty} \frac{A_1}{(1+2^m)^N} \sup_{x \in \mathbf{R}^n} (1+|x|)^N |\varphi(x)|, \end{aligned}$$

and this expression is bounded by a constant times a finite sum of Schwartz seminorms of φ.

We are interested in tempered distributions W on $\mathbf{R}^n$ that extend the function K defined on $\mathbf{R}^n \setminus \{0\}$ and have the form

$$\langle W, \varphi \rangle = \lim_{j \to \infty} \int_{|x| \ge \delta_j} K(x)\varphi(x)\,dx, \qquad \varphi \in \mathscr{S}(\mathbf{R}^n), \tag{5.3.7}$$

for some sequence $\delta_j \downarrow 0$ as $j \to \infty$. It is not hard to see that there exists a tempered distribution W satisfying (5.3.7) for all $\varphi \in \mathscr{S}(\mathbf{R}^n)$ if and only if

$$\lim_{j \to \infty} \int_{1 \ge |x| \ge \delta_j} K(x)\,dx = L \tag{5.3.8}$$

exists. See Exercise 5.3.2. If such a distribution W exists it may not be unique, since it depends on the choice of the sequence δ_j. Two different sequences tending to zero may give two different tempered distributions W of the form (5.3.7), both coinciding with the function K on $\mathbf{R}^n \setminus \{0\}$. See Example 5.4.2 and Remark 5.4.3.

If condition (5.3.8) is satisfied, we can define

$$\langle W, \varphi \rangle = \lim_{j\to\infty} \int_{j \geq |x| \geq \delta_j} K(x)\varphi(x)\,dx \tag{5.3.9}$$

and the limit exists as $j \to \infty$ for all $\varphi \in \mathscr{S}(\mathbf{R}^n)$ and is equal to

$$\langle W, \varphi \rangle = \int_{|x|\leq 1} K(x)(\varphi(x) - \varphi(0))\,dx + \varphi(0)L + \int_{|x|\geq 1} K(x)\varphi(x)\,dx\,.$$

Moreover, the previous calculations show that W is an element of $\mathscr{S}'(\mathbf{R}^n)$.

Next we assume that the given function K on $\mathbf{R}^n \setminus \{0\}$ satisfies a certain smoothness condition. There are three kinds of smoothness conditions that we encounter: first, the *gradient condition*

$$|\nabla K(x)| \leq A_2 |x|^{-n-1}, \qquad x \neq 0; \tag{5.3.10}$$

next, the weaker *Lipschitz condition*,

$$|K(x-y) - K(x)| \leq A_2 \frac{|y|^\delta}{|x|^{n+\delta}}\,, \qquad \text{whenever } |x| \geq 2|y|; \tag{5.3.11}$$

and finally the even weaker smoothness condition

$$\sup_{y\neq 0} \int_{|x|\geq 2|y|} |K(x-y) - K(x)|\,dx = A_2\,, \tag{5.3.12}$$

for some $A_2 < \infty$. One should verify that (5.3.12) is a weaker condition than (5.3.11), which in turn is weaker than (5.3.10). Condition (5.3.12) is often referred to as *Hörmander's condition*.

5.3.3 L^r Boundedness Implies Weak Type $(1,1)$ Boundedness

This next theorem provides the most classical application of the Calderón–Zygmund decomposition.

Theorem 5.3.3. *Let K be a function on $\mathbf{R}^n \setminus \{0\}$ that satisfies (5.3.4)[1] and (5.3.12) for some $A_1, A_2 < \infty$. Let W be an element of $\mathscr{S}'(\mathbf{R}^n)$ related to K as in (5.3.7). Suppose that the operator T given by convolution with W has a bounded extension that maps $L^r(\mathbf{R}^n)$ to itself with norm B for some $1 < r \leq \infty$. Then T has an extension that maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$ with norm*

[1] this condition could be replaced by the assumption that K is integrable over any compact set that does not contain the origin.

$$\big\|T\big\|_{L^1\to L^{1,\infty}} \le C_n\,(A_2+B), \tag{5.3.13}$$

and T also extends to a bounded operator from $L^p(\mathbf{R}^n)$ to itself for $1<p<\infty$ with norm

$$\big\|T\big\|_{L^p\to L^p} \le C_n'\max\big(p,(p-1)^{-1}\big)\,(A_2+B), \tag{5.3.14}$$

where C_n, C_n' are constants that depend on the dimension but not on r or p.

Proof. We discuss the case $r<\infty$ and we refer to Exercise 5.3.7 for the case $r=\infty$. Let $\alpha>0$ be given. We fix a step function f given as a finite linear combination of characteristic functions of disjoint dyadic intervals. The class of such functions is dense in all the L^p spaces. Once (5.3.13) is obtained for such functions, a density argument gives that T admits an extension on L^1 that also satisfies (5.3.13). Therefore it suffices to prove (5.3.13) for such a function f.

Apply the Calderón–Zygmund decomposition to f at height $\gamma\alpha$, where γ is a positive constant to be chosen later. That is, write the function f as the sum

$$f = g+b = g+\sum_j b_j,$$

where conditions (1)–(6) of Theorem 5.3.1 are satisfied with the constant α replaced by $\gamma\alpha$. Since f is a finite linear combination of characteristic functions of disjoint dyadic cubes, there are only finitely many cubes Q_j that appear in the Calderón–Zygmund decomposition to f. Each b_j is supported in a dyadic cube Q_j with center y_j and the Q_j's are pairwise disjoint. We denote by $\ell(Q)$ the side length of a cube Q. Let Q_j^* be the unique cube with sides parallel to the axes having the same center as Q_j and having side length $\ell(Q_j^*)=2\sqrt{n}\,\ell(Q_j)$. Because of the form of f, each b_j is a bounded function supported in $\overline{Q_j}$, hence it is in L^r, thus each $T(b_j)$ is a well-defined L^r function. We observe that for all j and all $x\notin Q_j^*$ we have

$$T(b_j)(x) = \lim_{k\to\infty}\int_{k\ge|x-y|\ge\delta_k} K(x-y)b_j(y)\,dy = \int_{Q_j} K(x-y)b_j(y)\,dy,$$

where the last integral converges absolutely. This is a consequence of the Lebesgue dominated convergence theorem, based on the facts that b_j is bounded, that K is integrable over any compact annulus that does not contain the origin (cf. (5.3.4)), and that $x-Q_j$ is contained in such a compact annulus, since $x\notin Q_j^*$.

Next we use the cancellation of b_j in the following way:

$$\begin{aligned}
&\int_{(\cup_i Q_i^*)^c}\sum_j\big|T(b_j)(x)\big|\,dx\\
&\qquad=\int_{(\bigcup_i Q_i^*)^c}\sum_j\bigg|\int_{Q_j} b_j(y)\big(K(x-y)-K(x-y_j)\big)\,dy\bigg|\,dx\\
&\qquad\le\sum_j\int_{(Q_j^*)^c}\int_{Q_j}|b_j(y)||K(x-y)-K(x-y_j)|\,dy\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \sum_j \int_{Q_j} |b_j(y)| \int_{(Q_j^*)^c} |K(x-y) - K(x-y_j)|\, dx\, dy \\
&= \sum_j \int_{Q_j} |b_j(y)| \int_{-y_j+(Q_j^*)^c} |K(x-(y-y_j)) - K(x)|\, dx\, dy \\
&\le \sum_j \int_{Q_j} |b_j(y)| \int_{|x|\ge 2|y-y_j|} |K(x-(y-y_j)) - K(x)|\, dx\, dy \\
&\le A_2 \sum_j \big\|b_j\big\|_{L^1} \\
&\le A_2 2^{n+1} \big\|f\big\|_{L^1} < \infty,
\end{aligned}
$$

where we used (5.3.12) since if $x \in -y_j + (Q_j^*)^c$ then $|x| \ge \frac{1}{2}\ell(Q_j^*) = \sqrt{n}\,\ell(Q_j)$ and since $y - y_j \in -y_j + Q_j$ we have $|y-y_j| \le \frac{\sqrt{n}}{2}\,\ell(Q_j)$, thus $|x| \ge 2|y-y_j|$. See Figure 5.2.

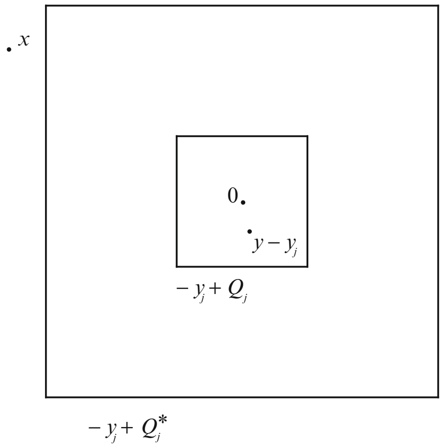

Fig. 5.2 The cubes $-y_j + Q_j$ and $-y_j + Q_j^*$.

Thus we proved that

$$
\int_{(\bigcup_i Q_i^*)^c} \sum_j |T(b_j)(x)|\, dx \le 2^{n+1} A_2 \big\|f\big\|_{L^1},
$$

an inequality we use below. We have

$$
\begin{aligned}
&\big|\big\{x \in \mathbf{R}^n : |T(f)(x)| > \alpha\big\}\big| \\
&\qquad \le \Big|\Big\{x \in \mathbf{R}^n :\ |T(g)(x)| > \frac{\alpha}{2}\Big\}\Big| + \Big|\Big\{x \in \mathbf{R}^n :\ |T(b)(x)| > \frac{\alpha}{2}\Big\}\Big| \\
&\qquad \le \frac{2^r}{\alpha^r}\big\|T(g)\big\|_{L^r}^r + \Big|\bigcup_i Q_i^*\Big| + \Big|\Big\{x \notin \bigcup_i Q_i^* :\ |T\big(\sum_j b_j\big)(x)| > \frac{\alpha}{2}\Big\}\Big|
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2^r}{\alpha^r}\|T(g)\|_{L^r}^r + \Big|\bigcup_i Q_i^*\Big| + \Big|\Big\{x \notin \bigcup_i Q_i^* : \,|\sum_j T(b_j)(x)| > \frac{\alpha}{2}\Big\}\Big| \\
&\le \frac{2^r}{\alpha^r}B^r\|g\|_{L^r}^r + \sum_i |Q_i^*| + \frac{2}{\alpha}\int_{(\bigcup_i Q_i^*)^c}\sum_j |T(b_j)(x)|\,dx \\
&\le \frac{2^r}{\alpha^r}2^{\frac{nr}{r'}}B^r(\gamma\alpha)^{\frac{r}{r'}}\|f\|_{L^1} + (2\sqrt{n})^n\frac{\|f\|_{L^1}}{\gamma\alpha} + \frac{2}{\alpha}2^{n+1}A_2\|f\|_{L^1} \\
&\le \left(\frac{(2^{n+1}B\gamma)^r}{2^n\gamma} + \frac{(2\sqrt{n})^n}{\gamma} + 2^{n+2}A_2\right)\frac{\|f\|_{L^1}}{\alpha}\,.
\end{aligned}
$$

Choosing $\gamma = 2^{-(n+1)}B^{-1}$, we deduce the weak type $(1,1)$ estimate (5.3.13) for T with $C_n = 2 + 2^{n+1}(2\sqrt{n})^n + 2^{n+2}$.

It follows from Exercise 1.3.2 that T has an extension that maps L^p to L^p with bound at most $C_n'(A_2+B)(p-1)^{-1/p}$ when $1 < p < r$. The adjoint operator T^* of T, defined by

$$
\langle T(f)\,|\,g\rangle = \langle f\,|\,T^*(g)\rangle\,,
$$

has a kernel that coincides with the function $K^*(x) = \overline{K(-x)}$ on $\mathbf{R}^n \setminus \{0\}$. We notice that since K satisfies (5.3.12), then so does K^* and with the same bound. Therefore, T^*, which maps $L^{r'}$ to $L^{r'}$, has a kernel that satisfies (5.3.12). By the preceding argument, T^* maps $L^{p'}$ to $L^{p'}$ with bound at most $C_n'(A_2+B)(p-1)^{-1/p}$ whenever $1 < p' < r'$. By duality this yields that T maps $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ with bound at most $C_n'(A_2+B)(p-1)^{1-1/p}$ whenever $r' < p < \infty$. Using interpolation we obtain that T maps L^p to itself with norm at most $C_n'(A_2+B)\max((p-1)^{-1/p},(p-1)^{1-1/p})$ for p in the interval (r,r'), which is nonempty only if $r<2$. Then (5.3.14) holds since $\max((p-1)^{-1/p},(p-1)^{1-1/p}) \le \max((p-1)^{-1},p)$. $\square$

5.3.4 Discussion on Maximal Singular Integrals

In this subsection we introduce maximal singular integrals and we derive their boundedness under certain smoothness conditions on the kernels, assuming boundedness of the associated linear operator.

Suppose that K is a kernel on $\mathbf{R}^n \setminus \{0\}$ that satisfies the size condition

$$
|K(x)| \le A_1|x|^{-n} \tag{5.3.15}
$$

for $x \neq 0$. Then for any $\varepsilon > 0$ the function $K^{(\varepsilon)}(x) = K(x)\chi_{|x|\ge\varepsilon}$ lies in $L^{p'}(\mathbf{R}^n)$ (with norm bounded by $c_{p,n}A_1\varepsilon^{-n/p}$) for all $1 \le p < \infty$. Consequently, by Hölder's inequality, the integral

$$
(f * K^{(\varepsilon)})(x) = \int_{|y|\ge\varepsilon} f(x-y)K(y)\,dy
$$

converges absolutely for all $x \in \mathbf{R}^n$ and all $f \in L^p(\mathbf{R}^n)$, when $1 \le p < \infty$.

Let $f \in \bigcup_{1\le p<\infty} L^p(\mathbf{R}^n)$. We define the truncated singular integrals $T^{(\varepsilon)}(f)$ associated with the kernel K by setting

$$T^{(\varepsilon)}(f) = f * K^{(\varepsilon)};$$

we also define the *maximal truncated singular integral operator* associated with K by setting

$$T^{(*)}(f) = \sup_{\varepsilon>0} |(f * K^{(\varepsilon)})| = \sup_{\varepsilon>0} |T^{(\varepsilon)}(f)|.$$

This operator is well defined, but possibly infinite, for certain points in $\mathbf{R}^n$.

We now consider the situation in which the kernel K satisfies an integrability condition over concentric annuli centered at the origin, a condition that is certainly a weaker condition than (5.3.15). Precisely, suppose that K is a measurable function on $\mathbf{R}^n \setminus \{0\}$, that is integrable on compact subsets of $\mathbf{R}^n \setminus \{0\}$, for which there is a constant $A_1 < \infty$ such that

$$\sup_{R>0} \int_{R\le|x|\le 2R} |K(x)|\,dx \le A_1 < \infty. \tag{5.3.16}$$

Such kernels may not be integrable to the power $p' > 1$ over the region $|x| \ge \varepsilon$. For this reason, it is not possible to define $T^{(\varepsilon)}$ as an absolutely convergent integral. To overcome this difficulty, we consider double truncations. We define the doubly truncated kernel $K^{(\varepsilon,N)}$ by setting

$$K^{(\varepsilon,N)}(x) = K(x)\chi_{\varepsilon\le|x|\le N}(x). \tag{5.3.17}$$

A repeated application of (5.3.16) yields that

$$\int |K^{(\varepsilon,N)}(x)|\,dx \le A_1\Big(\Big[\log_2 \frac{N}{\varepsilon}\Big] + 1\Big),$$

which implies that $K^{(\varepsilon,N)}$ is integrable over concentric annuli centered at the origin. Next, we define the doubly truncated singular integrals $T^{(\varepsilon,N)}$ by setting

$$T^{(\varepsilon,N)}(f) = f * K^{(\varepsilon,N)},$$

and we observe that these operators are well defined when f in L^p, for $1 \le p \le \infty$. Indeed, Theorem 1.2.10 yields that

$$\big\|T^{(\varepsilon,N)}(f)\big\|_{L^p} \le \big\|f\big\|_{L^p} \int |K^{(\varepsilon,N)}(x)|\,dx < \infty$$

for functions f in L^p, $1 \le p \le \infty$. Consequently, for almost every $x \in \mathbf{R}^n$ we have

$$|T^{(\varepsilon,N)}(f)(x)| < \infty.$$

For functions in $\bigcup_{1\le p\le\infty} L^p(\mathbf{R}^n)$ we define the *doubly truncated maximal singular integral operator* $T^{(**)}$ associated with K by setting

$$T^{(**)}(f) = \sup_{0<\varepsilon<N<\infty} |T^{(\varepsilon,N)}(f)|. \tag{5.3.18}$$

For such functions and for almost all $x \in \mathbf{R}^n$, $T^{(**)}(f)(x)$ is well defined, but potentially infinite.

One observation is that under condition (5.3.16), one can also define $T^{(*)}(g)$ for general integrable functions g with compact support. In this case, say that the ball $B(0,R)$ contains the support of g. Let $x \in B(0,M)$ and $N = M+R$. Then $|T^{(\varepsilon)}(g)(x)| \le |g| * |K^{(\varepsilon,N)}|(x)$, which is finite a.e. as the convolution of two L^1 functions; consequently, the integral defining $T^{(\varepsilon)}(g)(x)$ converges absolutely for all $x \in B(0,R)$. Since $R > 0$ is arbitrary, $T^{(\varepsilon)}(g)(x)$ is defined and finite for almost all $x \in \mathbf{R}^n$.

Obviously $T^{(*)}$ and $T^{(**)}$ are related. If K satisfies condition (5.3.15), then

$$\left| \int_{\varepsilon\le|y|} f(x-y)K(y)\,dy \right| \le \sup_{N>0} \left| \int_{\varepsilon\le|y|\le N} f(x-y)K(y)\,dy \right|,$$

which implies that

$$T^{(*)}(f) \le T^{(**)}(f)$$

for all $f \in \bigcup_{1\le p<\infty} L^p$. Also, $T^{(\varepsilon,N)}(f) = T^{(\varepsilon)}(f) - T^{(N)}(f)$; hence

$$T^{(**)}(f) \le 2T^{(*)}(f).$$

Therefore, for kernels satisfying (5.3.15), $T^{(**)}$ and $T^{(*)}$ are comparable and the boudnedness properties of $T^{(**)}$ and $T^{(*)}$ are equivalent

Theorem 5.3.4. ***(Cotlar's inequality)*** *Let $0 < A_1, A_2, A_3 < \infty$ and suppose that K is defined on $\mathbf{R}^n \setminus \{0\}$ and satisfies the size condition,*

$$|K(x)| \le A_1 |x|^{-n}, \qquad x \ne 0, \tag{5.3.19}$$

the smoothness condition

$$|K(x-y) - K(x)| \le A_2 |y|^{\delta}\, |x|^{-n-\delta}, \tag{5.3.20}$$

whenever $|x| \ge 2|y| > 0$, and the cancellation condition

$$\sup_{0<r<R<\infty} \left| \int_{r<|x|<R} K(x)\,dx \right| \le A_3. \tag{5.3.21}$$

Let W in $\mathscr{S}'(\mathbf{R}^n)$ be related to K via (5.3.7) and let T be the operator given by convolution with W. Then there is a constant $C_{n,\delta}$ such that the following inequality is valid:

$$T^{(*)}(f) \le M(T(f)) + C_{n,\delta}\,(A_1 + A_2 + A_3)\,M(f), \tag{5.3.22}$$

for all $f \in \mathscr{S}(\mathbf{R}^n)$*, where* M *is the Hardy–Littlewood maximal operator. Consequently,* $T^{(*)}$ *is bounded on* $L^p(\mathbf{R}^n)$ *when* $1 < p < \infty$*.*

Proof. Let φ be a radially decreasing smooth function with integral 1 supported in the ball $B(0,1/2)$. For a function g and $\varepsilon > 0$ we use the notation $g_\varepsilon(x) = \varepsilon^{-n} g(\varepsilon^{-1}x)$. For a distribution W we define W_ε analogously, i.e. as the unique distribution with the property $\langle W_\varepsilon, \psi\rangle = \varepsilon^{-n}\langle W, \psi_{\varepsilon^{-1}}\rangle$. We begin by observing that $K_{\varepsilon^{-1}}(x) = \varepsilon^n K(\varepsilon x)$ satisfies (5.3.19), (5.3.20), and (5.3.21) uniformly in $\varepsilon > 0$.

Set, as before, $K^{(\varepsilon)}(x) = K(x)\chi_{|x|\geq\varepsilon}$. Fix $f \in \mathscr{S}(\mathbf{R}^n)$ for some $1 < p < \infty$. Obviously we have

$$f * K^{(\varepsilon)} = f * \big((K_{\varepsilon^{-1}})^{(1)}\big)_\varepsilon = f * W * \varphi_\varepsilon + f * \big((K_{\varepsilon^{-1}})^{(1)} - W_{\varepsilon^{-1}} * \varphi\big)_\varepsilon . \tag{5.3.23}$$

Next we prove the following estimate for all $\varepsilon > 0$:

$$\Big|\big((K_{\varepsilon^{-1}})^{(1)} - W_{\varepsilon^{-1}} * \varphi\big)(x)\Big| \leq C(A_1+A_2+A_3)(1+|x|)^{-n-\delta} \tag{5.3.24}$$

for all $x \in \mathbf{R}^n$. Indeed, for $|x| \geq 1$ we express the left-hand side in (5.3.24) as

$$\bigg|\int_{\mathbf{R}^n} \big(K_{\varepsilon^{-1}}(x) - K_{\varepsilon^{-1}}(x-y)\big)\varphi(y)\,dy\bigg| .$$

Since φ is supported in $|y| \leq 1/2$, we have $|x| \geq 2|y|$, and condition (5.3.20) yields that the expression on the left-hand side of (5.3.24) is bounded by

$$\frac{A_2}{|x|^{n+\delta}} \int_{\mathbf{R}^n} |y|^\delta |\varphi(y)|\,dy \leq c\,\frac{A_2}{(1+|x|)^{n+\delta}} ,$$

which proves (5.3.24) in the case $|x| \geq 1$. When $|x| < 1$, the left-hand side of (5.3.24) equals

$$(W_{\varepsilon^{-1}} * \varphi)(x) = \lim_{\delta_j\to 0}\int_{|x-y|\geq\delta_j} K_{\varepsilon^{-1}}(x-y)\varphi(y)\,dy \tag{5.3.25}$$

for some sequence $\delta_j \downarrow 0$; see the discussion in Section 5.3.2. The expression in (5.3.25) is equal to

$$I_1 + I_2 + I_3 ,$$

where

$$\begin{aligned}
I_1 &= \int_{|x-y|>\frac18} K_{\varepsilon^{-1}}(x-y)\varphi(y)\,dy ,\\
I_2 &= \int_{|x-y|\leq\frac18} K_{\varepsilon^{-1}}(x-y)\big(\varphi(y)-\varphi(x)\big)\,dy ,\\
I_3 &= \varphi(x)\lim_{\delta_j\to 0}\int_{\frac18\geq|x-y|\geq\delta_j} K_{\varepsilon^{-1}}(x-y)\,dy .
\end{aligned}$$

In I_1 we have $1/8 \leq |x-y| \leq 1+1/2 = 3/2$; hence I_1 is bounded by a multiple of A_1. Since $|\varphi(x)-\varphi(y)| \leq c|x-y|$, the same is valid for I_2. Finally, I_3 is bounded by a multiple of A_3. Combining these facts yields the proof of (5.3.24) in the case $|x| < 1$ as well.

Use Corollary 2.1.12 to deduce that

$$\sup_{\varepsilon>0}\left|f*\big((K_{\varepsilon^{-1}})^{(1)}-K_{\varepsilon^{-1}}*\varphi\big)_{\varepsilon}\right|\le c\,(A_1+A_2+A_3)M(f)\,.$$

Finally, take the supremum over $\varepsilon>0$ in (5.3.23) and use (5.3.24) and Corollary 2.1.12 one more time to deduce the estimate

$$T^{(*)}(f)\le M(f*W)+C\,(A_1+A_2+A_3)M(f)\,,$$

where C depends on n and δ; this concludes the proof of (5.3.22) for all functions $f\in\mathscr{S}(\mathbf{R}^n)$. Thus $T^{(*)}$ is bounded on L^p, $1<p<\infty$, when restricted to Schwartz functions.

Now given a general function g in $L^p(\mathbf{R}^n)$ we find a sequence h_j in $\mathscr{S}(\mathbf{R}^n)$ such that $\|h_j-g\|_{L^p}\to 0$ as $j\to\infty$. Then we have the pointwise estimate

$$|T^{(\varepsilon)}(g)|\le|T^{(\varepsilon)}(g-h_j)|+|T^{(\varepsilon)}(h_j)|\le c_{p,n}A_1\varepsilon_0^{-\frac{n}{p}}\|g-h_j\|_{L^p}+|T^{(*)}(h_j)|$$

for all $\varepsilon\ge\varepsilon_0$. Taking the supremum over $\varepsilon\ge\varepsilon_0$ and then L^p norm over the ball $B(0,R)$, we obtain

$$\Big\|\sup_{\varepsilon\ge\varepsilon_0}|T^{(\varepsilon)}(g)|\Big\|_{L^p(B(0,R))}\le c'_{p,n}A_1\varepsilon_0^{-\frac{n}{p}}R^{\frac{n}{p}}\|g-h_j\|_{L^p}+C'\,(A_1+A_2+A_3)\|h_j\|_{L^p}\,.$$

Now we let $j\to\infty$ first, and then $R\to\infty$ and $\varepsilon_0\to 0$ to deduce the boundedness of $T^{(*)}$ on $L^p(\mathbf{R}^n)$ via the Lebesgue monotone convergence theorem. $\square$

5.3.5 Boundedness for Maximal Singular Integrals Implies Weak Type $(1,1)$ Boundedness

We now state and prove a result analogous to that in Theorem 5.3.3 for maximal singular integrals.

Theorem 5.3.5. *Let $K(x)$ be function on $\mathbf{R}^n\setminus\{0\}$ satisfying (5.3.4) with constant $A_1<\infty$ and Hörmander's condition (5.3.12) with constant $A_2<\infty$. Suppose that the operator $T^{(**)}$ as defined in (5.3.18) maps $L^2(\mathbf{R}^n)$ to itself with norm B. Then $T^{(**)}$ maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$ with norm*

$$\big\|T^{(**)}\big\|_{L^1\to L^{1,\infty}}\le C_n(A_1+A_2+B),$$

where C_n is some dimensional constant.

Proof. The proof of this theorem is only a little more involved than the proof of Theorem 5.3.3. We fix an $L^1(\mathbf{R}^n)$ function f. We apply the Calderón–Zygmund decomposition of f at height $\gamma\alpha$ for some $\gamma,\alpha>0$. We then write $f=g+b$, where

$b = \sum_j b_j$ and each b_j is supported in some cube Q_j. We define Q_j^* as the cube with the same center as Q_j and with sides parallel to the sides of Q_j having length $\ell(Q_j^*) = 5\sqrt{n}\,\ell(Q_j)$. This is only a minor change compared with the definition of Q_j in Theorem 5.3.3. The main change in the proof is in the treatment of the term

$$\Big|\Big\{x \in \Big(\bigcup_j Q_j^*\Big)^c : \ |T^{(**)}(b)(x)| > \frac{\alpha}{2}\Big\}\Big|. \tag{5.3.26}$$

We show that for all $\gamma \le (2^{n+5}A_1)^{-1}$ we have

$$\Big|\Big\{x \in \Big(\bigcup_j Q_j^*\Big)^c : \ |T^{(**)}(b)(x)| > \frac{\alpha}{2}\Big\}\Big| \le 2^{n+8} A_2 \frac{\|f\|_{L^1}}{\alpha}. \tag{5.3.27}$$

Let us conclude the proof of the theorem assuming for the moment the validity of (5.3.27). As in the proof of Theorem 5.3.3, we can show that

$$\Big|\Big\{x \in \mathbf{R}^n : \ |T^{(**)}(g)(x)| > \frac{\alpha}{2}\Big\}\Big| + \Big|\bigcup_j Q_j^*\Big| \le \Big(2^{n+2}B^2\gamma + \frac{(5\sqrt{n})^n}{\gamma}\Big)\frac{\|f\|_{L^1}}{\alpha}.$$

Combining this estimate with (5.3.27) and choosing

$$\gamma = (2^{n+5}(A_1 + A_2 + B))^{-1},$$

we obtain the required estimate

$$\big|\big\{x \in \mathbf{R}^n : \ |T^{(**)}(f)(x)| > \alpha\big\}\big| \le C_n(A_1 + A_2 + B)\frac{\|f\|_{L^1}}{\alpha}$$

with $C_n = 2^{-3} + (5\sqrt{n})^n 2^{n+5} + 2^{n+8}$.

It remains to prove (5.3.27). This estimate will be a consequence of the fact that for $x \in \big(\bigcup_j Q_j^*\big)^c$ we have the key inequality

$$T^{(**)}(b)(x) \le 4E_1(x) + 2^{n+2}\alpha\gamma E_2(x) + 2^{n+3}\alpha\gamma A_1, \tag{5.3.28}$$

where

$$\begin{aligned} E_1(x) &= \sum_j \int_{Q_j} |K(x-y) - K(x-y_j)|\,|b_j(y)|\,dy, \\ E_2(x) &= \sum_j \int_{Q_j} |K(x-y) - K(x-y_j)|\,dy, \end{aligned}$$

and y_j is the center of Q_j.

If we had (5.3.28), then we could easily derive (5.3.27). Indeed, fix a γ satisfying $\gamma \le (2^{n+5}A_1)^{-1}$. Then we have $2^{n+3}\alpha\gamma A_1 < \frac{\alpha}{3}$, and using (5.3.28), we obtain

$$
\begin{aligned}
&\Big|\Big\{x\in\Big(\bigcup_j Q_j^*\Big)^c:\ |T^{(**)}(b)(x)|>\frac{\alpha}{2}\Big\}\Big| \\
&\qquad\le\Big|\Big\{x\in\Big(\bigcup_j Q_j^*\Big)^c:\ 4E_1(x)>\frac{\alpha}{12}\Big\}\Big| \\
&\qquad\qquad\qquad+\Big|\Big\{x\in\Big(\bigcup_j Q_j^*\Big)^c:\ 2^{n+2}\alpha\gamma E_2(x)>\frac{\alpha}{12}\Big\}\Big| \\
&\qquad\le\frac{48}{\alpha}\int_{(\bigcup_j Q_j^*)^c}E_1(x)\,dx+2^{n+6}\gamma\int_{(\bigcup_j Q_j^*)^c}E_2(x)\,dx,
\end{aligned}
\tag{5.3.29}
$$

since $\frac{\alpha}{2}=\frac{\alpha}{3}+\frac{\alpha}{12}+\frac{\alpha}{12}$. We have

$$
\begin{aligned}
&\int_{(\bigcup_j Q_j^*)^c}E_1(x)\,dx \\
&\qquad\le\sum_j\int_{Q_j}|b_j(y)|\int_{(Q_j^*)^c}|K(x-y)-K(x-y_j)|\,dx\,dy \\
&\qquad\le\sum_j\int_{Q_j}|b_j(y)|\int_{|x-y_j|\ge 2|y-y_j|}|K(x-y)-K(x-y_j)|\,dx\,dy \\
&\qquad\le A_2\sum_j\int_{Q_j}|b_j(y)|\,dy=A_2\sum_j\big\|b_j\big\|_{L^1}\le A_2 2^{n+1}\big\|f\big\|_{L^1},
\end{aligned}
\tag{5.3.30}
$$

where we used the fact that if $x\in(Q_j^*)^c$, then $|x-y_j|\ge\frac{1}{2}\ell(Q_j^*)=\frac{5}{2}\sqrt{n}\,\ell(Q_j)$. But since $|y-y_j|\le\frac{\sqrt{n}}{2}\ell(Q_j)$, this implies that $|x-y_j|\ge 2|y-y_j|$. Here we used the fact that the diameter of a cube is equal to $\sqrt{n}$ times its side length. Likewise, we obtain that

$$
\int_{(\bigcup_j Q_j^*)^c}E_2(x)\,dx\le A_2\sum_j|Q_j|\le A_2\frac{\big\|f\big\|_{L^1}}{\alpha\gamma}. \tag{5.3.31}
$$

Combining (5.3.30) and (5.3.31) with (5.3.29) yields (5.3.27).

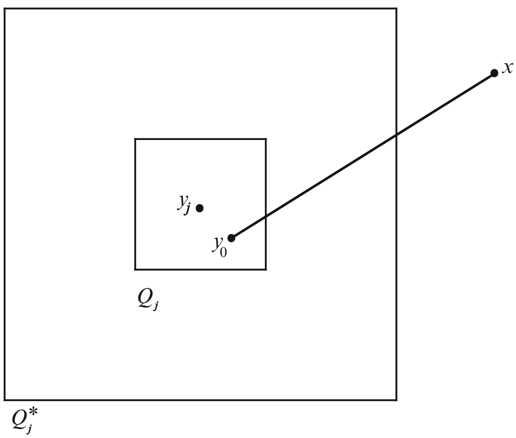

Fig. 5.3 The cubes Q_j and Q_j^*.

Therefore, the main task of the proof is to prove (5.3.28). Since $b = \sum_j b_j$, to estimate $T^{(**)}(b)$, it suffices to estimate each $|T^{(\varepsilon,N)}(b_j)|$ uniformly in ε and N. To achieve this we use the estimate

$$|T^{(\varepsilon,N)}(b_j)| \leq |T^{(\varepsilon)}(b_j)| + |T^{(N)}(b_j)|, \tag{5.3.32}$$

noting that the truncated singular integrals $T^{(\varepsilon)}(b_j)$ are well defined. Indeed, say x lies in a compact set K_0. Pick M such that $K_0 - Q_j$ is contained in a ball $B(0,M)$. Then

$$|T^{(\varepsilon)}(b_j)(x)| \leq |b_j| * |K^{(\varepsilon,M)}|(x),$$

which is finite a.e. as the convolution of two L^1 functions; thus the integral defining $T^{(\varepsilon)}(b_j)(x)$ converges absolutely and the expression $T^{(\varepsilon)}(b_j)(x)$ is well defined for almost all x.

We work with $T^{(\varepsilon)}$ and we note that $T^{(N)}$ can be treated similarly. Fix $x \notin \bigcup_j Q_j^*$ and $\varepsilon > 0$ and define

$$\begin{aligned} J_1(x,\varepsilon) &= \{j : \ \forall y \in Q_j \text{ we have } |x-y| < \varepsilon\}, \\ J_2(x,\varepsilon) &= \{j : \ \forall y \in Q_j \text{ we have } |x-y| > \varepsilon\}, \\ J_3(x,\varepsilon) &= \{j : \ \exists y \in Q_j \text{ we have } |x-y| = \varepsilon\}. \end{aligned}$$

Note that

$$T^{(\varepsilon)}(b_j)(x) = 0$$

whenever $x \notin \bigcup_j Q_j^*$ and $j \in J_1(x,\varepsilon)$. Also note that

$$K^{(\varepsilon)}(x-y) = K(x-y)$$

whenever $x \notin \bigcup_j Q_j^*$, $j \in J_2(x,\varepsilon)$ and $y \in Q_j$. Therefore,

$$\sup_{\varepsilon>0} |T^{(\varepsilon)}(b)(x)| \leq \sup_{\varepsilon>0} \Big| \sum_{j\in J_2(x,\varepsilon)} T(b_j)(x) \Big| + \sup_{\varepsilon>0} \Big| \sum_{j\in J_3(x,\varepsilon)} T(b_j \chi_{|x-\cdot|\geq\varepsilon})(x) \Big|,$$

but since

$$\sup_{\varepsilon>0} \Big| \sum_{j\in J_2(x,\varepsilon)} T(b_j)(x) \Big| \leq \sum_j |T(b_j)(x)| \leq E_1(x), \tag{5.3.33}$$

it suffices to estimate the term

$$\sup_{\varepsilon>0} \Big| \sum_{j\in J_3(x,\varepsilon)} T(b_j \chi_{|x-\cdot|\geq\varepsilon})(x) \Big|.$$

We now make some geometric observations; see Figure 5.3. Fix $\varepsilon > 0$ and a cube Q_j with $j \in J_3(x,\varepsilon)$; recall that x lies in $(\bigcup_j Q_j^*)^c$. Then we have

$$\varepsilon \geq \frac{1}{2}\big(\ell(Q_j^*) - \ell(Q_j)\big) = \frac{1}{2}(5\sqrt{n}-1)\ell(Q_j) \geq 2\sqrt{n}\,\ell(Q_j). \tag{5.3.34}$$

Since $j \in J_3(x,\varepsilon)$, there exists a $y_0 \in Q_j$ with

$$|x-y_0| = \varepsilon\,.$$

Using (5.3.34), we obtain that for any $y \in Q_j$ we have

$$\frac{\varepsilon}{2} \le \varepsilon - \sqrt{n}\,\ell(Q_j) \le |x-y_0| - |y-y_0| \le |x-y|\,,$$
$$|x-y| \le |x-y_0| + |y-y_0| \le \varepsilon + \sqrt{n}\,\ell(Q_j) \le \frac{3\varepsilon}{2}\,.$$

Therefore, we have proved that

$$\bigcup_{j\in J_3(x,\varepsilon)} Q_j \subseteqq B(x,\tfrac{3\varepsilon}{2}) \setminus B(x,\tfrac{\varepsilon}{2})\,.$$

Letting

$$c_j(\varepsilon) = \frac{1}{|Q_j|}\int_{Q_j} b_j(y)\chi_{|x-y|\ge\varepsilon}(y)\,dy\,,$$

we note that in view of property (5) of the Calderón–Zygmund decomposition (Theorem 5.3.1), the estimate

$$|c_j(\varepsilon)| \le 2^{n+1}\alpha\gamma$$

holds. Then

$$\begin{aligned}
&\sup_{\varepsilon>0}\Big|\sum_{j\in J_3(x,\varepsilon)}\int_{Q_j} K(x-y)b_j(y)\chi_{|x-y|\ge\varepsilon}(y)\,dy\Big| \\
&\quad\le \sup_{\varepsilon>0}\Big|\sum_{j\in J_3(x,\varepsilon)}\int_{Q_j} K(x-y)\big(b_j(y)\chi_{|x-y|\ge\varepsilon}(y) - c_j(\varepsilon)\big)\,dy\Big| \\
&\qquad\qquad + \sup_{\varepsilon>0}\Big|\sum_{j\in J_3(x,\varepsilon)} c_j(\varepsilon)\int_{Q_j} K(x-y)\,dy\Big| \\
&\quad\le \sup_{\varepsilon>0}\Big|\sum_{j\in J_3(x,\varepsilon)}\int_{Q_j}\big(K(x-y)-K(x-y_j)\big)\big(b_j(y)\chi_{|x-y|\ge\varepsilon}(y) - c_j(\varepsilon)\big)\,dy\Big| \\
&\qquad\qquad + 2^{n+1}\alpha\gamma\sup_{\varepsilon>0}\int_{B(x,\frac{3\varepsilon}{2})\setminus B(x,\frac{\varepsilon}{2})} |K(x-y)|\,dy \\
&\quad\le \sum_j \int_{Q_j}\big|K(x-y)-K(x-y_j)\big|\big(|b_j(y)| + 2^{n+1}\alpha\gamma\big)\,dy \\
&\qquad\qquad + 2^{n+1}\alpha\gamma\sup_{\varepsilon>0}\int_{\frac{\varepsilon}{2}\le|x-y|\le\frac{3\varepsilon}{2}} |K(x-y)|\,dy \\
&\quad\le E_1(x) + 2^{n+1}\alpha\gamma E_2(x) + 2^{n+1}\alpha\gamma(2A_1)\,.
\end{aligned}$$

The last estimate, together with (5.3.33), with (5.3.32), and with the analogous estimate for $\sup_{N>0}|T^{(N)}(b_j)(x)|$ (which is similarly obtained), yields (5.3.28). $\square$

The value of the previous theorem lies in the following: Since we know that for some sequences $\varepsilon_j \downarrow 0$, $N_j \uparrow \infty$ the pointwise limit $T^{(\varepsilon_j,N_j)}(f)$ exists a.e. for all f in a dense subclass of L^1, then Theorem 5.3.5 allows us to deduce that $T^{(\varepsilon_j,N_j)}(f)$ exists a.e. for all f in $L^1(\mathbf{R}^n)$.

If the singular integrals have kernels of the form $\Omega(x/|x|)|x|^{-n}$ with Ω in L^∞, such as the Hilbert transform and the Riesz transforms, then the upper truncations are not needed for K in (5.3.17). In this case

$$T_\Omega^{(\varepsilon)}(f)(x) = \int_{|y|\ge\varepsilon} f(x-y)\frac{\Omega(y/|y|)}{|y|^n}\,dy$$

is well defined for $f \in \bigcup_{1\le p<\infty} L^p(\mathbf{R}^n)$ by Hölder's inequality and is equal to

$$\lim_{N\to\infty}\int_{\varepsilon\le|y|\le N} f(x-y)\frac{\Omega(y/|y|)}{|y|^n}\,dy\,.$$

Corollary 5.3.6. *The maximal Hilbert transform $H^{(*)}$ and the maximal Riesz transforms $R_j^{(*)}$ are weak type $(1,1)$. Secondly, $\lim_{\varepsilon\to 0} H^{(\varepsilon)}(f)$ and $\lim_{\varepsilon\to 0} R_j^{(\varepsilon)}(g)$ exist a.e. for all $f \in L^1(\mathbf{R})$ and $g \in L^1(\mathbf{R}^n)$, as $\varepsilon \to 0$.*

Proof. Since the kernels $1/x$ on $\mathbf{R}$ and $x_j/|x|^n$ on $\mathbf{R}^n$ satisfy (5.3.10), the first statement in the corollary is an immediate consequence of Theorem 5.3.5. The second statement follows from Theorem 2.1.14 and Corollary 5.2.8, since these limits exist for Schwartz functions. □

Corollary 5.3.7. *Under the hypotheses of Theorem 5.3.5, $T^{(**)}$ maps $L^p(\mathbf{R}^n)$ to itself for $1 < p < 2$ with norm*

$$\big\|T^{(**)}\big\|_{L^p\to L^p} \le \frac{C_n(A_1+A_2+B)}{p-1}\,,$$

where C_n is some dimensional constant.

Exercises

5.3.1. Let A_1 be defined in (5.3.4). Prove that

$$\frac{1}{2}A_1 \le \sup_{R>0}\frac{1}{R}\int_{|x|\le R}|K(x)|\,|x|\,dx \le 2A_1\,;$$

thus the expressions in (5.3.6) and (5.3.4) are equivalent.

5.3.2. Suppose that K is a locally integrable function on $\mathbf{R}^n\setminus\{0\}$ that satisfies (5.3.4). Suppose that $\delta_j \downarrow 0$. Prove that the principal value operation

$$\langle W,\varphi\rangle = \lim_{j\to\infty}\int_{\delta_j\le|x|\le 1} K(x)\varphi(x)\,dx$$

defines a distribution in $\mathscr{S}'(\mathbf{R}^n)$ if and only if the following limit exists:

$$\lim_{j\to\infty}\int_{\delta_j\le|x|\le 1} K(x)\,dx\,.$$

5.3.3. Suppose that a function K on $\mathbf{R}^n\setminus\{0\}$ satisfies condition (5.3.4) with constant A_1 and condition (5.3.12) with constant A_2.
(a) Show that the functions $K(x)\chi_{|x|\ge\varepsilon}$ also satisfy condition (5.3.12) uniformly in $\varepsilon>0$ with constant A_1+A_2.
(b) Obtain the same conclusion for the upper truncations $K(x)\chi_{|x|\le N}$.
(c) Deduce a similar conclusion for the double truncations $K^{(\varepsilon,N)}(x)=K(x)\chi_{\varepsilon\le|x|\le N}$.

5.3.4. Modify the proof of Theorem 5.3.5 to prove that if $T^{(**)}$ maps L^r to $L^{r,\infty}$ for some $1<r<\infty$, and K satisfies condition (5.3.12), then $T^{(**)}$ maps L^1 to $L^{1,\infty}$.

5.3.5. Assume that T is a linear operator acting on measurable functions on $\mathbf{R}^n$ such that whenever a function f is supported in a cube Q, then $T(f)$ is supported in a fixed multiple of Q.
(a) Suppose that T maps L^p to itself for some $1<p<\infty$ with norm B. Prove that T extends to a bounded operator from L^1 to $L^{1,\infty}$ with norm a constant multiple of B.
(b) Suppose that T maps L^p to L^q for some $1<q<p<\infty$ with norm B. Prove that T extends to a bounded operator from L^1 to $L^{s,\infty}$ with norm a multiple of B, where

$$\frac{1}{p'}+\frac{1}{q}=\frac{1}{s}\,.$$

5.3.6. (a) Let $1<q<\infty$. Show that the good function g in Theorem 5.3.1 lies in the Lorentz space $L^{q,1}$ and that $\|g\|_{L^{q,1}}\le C_{n,q}\alpha^{1/q'}\|f\|_{L^1}^{1/q}$ for some constant $C_{n,q}$.
(b) Let $1<r<\infty$. Obtain a generalization of Theorem 5.3.3 in which the assumption that T maps L^r to L^r is replaced by that T maps $L^{r,1}$ to $L^{r,\infty}$ with norm B.
(c) Let $1<r<\infty$. Obtain a further generalization of Theorem 5.3.3 in which the assumption that T maps L^r to L^r is replaced by that it is of restricted weak type (r,r), i.e., it satisfies

$$|\{x:\ |T(\chi_E)(x)|>\alpha\}|\le B^r\,\frac{|E|}{\alpha^r}$$

for all subsets E of $\mathbf{R}^n$ with finite measure.

5.3.7. Let K satisfy (5.3.12) for some $A_2>0$, let $W\in\mathscr{S}'(\mathbf{R}^n)$ be an extension of K on $\mathbf{R}^n$ as in (5.3.7), and let T be the operator given by convolution with W. Obtain the case $r=\infty$ in Theorem 5.3.3. Precisely, prove that if T maps $L^\infty(\mathbf{R}^n)$ to itself with constant B, then T has an extension on L^1+L^∞ that satisfies

$$\big\|T\big\|_{L^1\to L^{1,\infty}}\le C_n'\,(A_2+B),$$

and for $1 < p < \infty$ it satisfies

$$\|T\|_{L^p \to L^p} \le C_n \frac{1}{(p-1)^{1/p}} (A_2 + B),$$

where C_n, C_n' are constants that depend only on the dimension.
[*Hint:* Apply the Calderón–Zygmund decomposition $f = g + b$ at height $\alpha\gamma$, where $\gamma = (2^{n+1}B)^{-1}$. Since $|g| \le 2^n \alpha\gamma$, observe that

$$|\{x :\ |T(f)(x)| > \alpha\}| \le |\{x :\ |T(b)(x)| > \alpha/2\}|.$$

For the interpolation use the result of Exercise 1.3.2.]

5.3.8. (*Calderón–Zygmund decomposition on* L^q) Fix a function $f \in L^q(\mathbf{R}^n)$ for some $1 \le q < \infty$ and let $\alpha > 0$. Then there exist functions g and b on $\mathbf{R}^n$ such that

(1) $f = g + b$.

(2) $\|g\|_{L^q} \le \|f\|_{L^q}$ and $\|g\|_{L^\infty} \le 2^{\frac{n}{q}} \alpha$.

(3) $b = \sum_j b_j$, where each b_j is supported in a cube Q_j. Furthermore, the cubes Q_k and Q_j have disjoint interiors when $j \ne k$.

(4) $\|b_j\|_{L^q}^q \le 2^{n+q}\alpha^q |Q_j|$.

(5) $\int_{Q_j} b_j(x)\,dx = 0$.

(6) $\sum_j |Q_j| \le \alpha^{-q}\|f\|_{L^q}^q$.

(7) $\|b\|_{L^q} \le 2^{\frac{n+q}{q}} \|f\|_{L^q}$ and $\|b\|_{L^1} \le 2\alpha^{1-q}\|f\|_{L^q}^q$.

[*Hint:* Imitate the basic idea of the proof of Theorem 5.3.1, but select a cube Q if $\big(\frac{1}{|Q|}\int_Q |f(x)|^q\,dx\big)^{1/q} > \alpha$. Define g and b as in the proof of Theorem 5.3.1.]

5.3.9. Let $f \in L^1(\mathbf{R}^n)$. Then for any $\alpha > 0$, prove that there exist disjoint cubes Q_j in $\mathbf{R}^n$ such that the set $E_\alpha = \{x \in \mathbf{R}^n :\ M_c(f)(x) > \alpha\}$ is contained in $\bigcup_j 3Q_j$ and $\frac{\alpha}{4^n} < \frac{1}{|Q_j|}\int_{Q_j} |f(t)|\,dt \le \frac{\alpha}{2^n}$.
[*Hint:* For given $\alpha > 0$, select all maximal dyadic cubes $Q_j(\alpha)$ such that the average of f over them is bigger than α. Given $x \in E_\alpha$, pick a cube R that contains x such that the average of $|f|$ over R is bigger than α and find a dyadic cube Q such that $2^{-n}|Q| < |R| \le |Q|$ and that $\int_{R\cap Q} |f|\,dx > 2^{-n}\alpha|R|$. Conclude that Q is contained in some $Q_k(4^{-n}\alpha)$ and thus R is contained in $3Q_k(4^{-n}\alpha)$. The collection of all $Q_j = Q_j(4^{-n}\alpha)$ is the required one.]

5.3.10. Let $K(x)$ be a function on $\mathbf{R}^n \setminus \{0\}$ that satisfies $|K(x)| \le A|x|^{-n}$. Let $\eta(x)$ be a smooth function equal to 1 when $|x| \ge 2$ and vanishing when $|x| \le 1$. For $f \in L^p$, $1 \le p < \infty$, define truncated singular integral operators

$$T^{(\varepsilon)}(f)(x) = \int_{|y|\geq\varepsilon} K(y)f(x-y)\,dy,$$
$$T_\eta^{(\varepsilon)}(f)(x) = \int_{\mathbf{R}^n} \eta(y/\varepsilon)K(y)f(x-y)\,dy.$$

Show that the truncated maximal singular integral $T^{(*)}(f) = \sup_{\varepsilon>0}|T^{(\varepsilon)}(f)|$ is L^p bounded for $1<p<\infty$ if and only if the smoothly truncated maximal singular integral $T_\eta^{(*)}(f) = \sup_{\varepsilon>0}|T_\eta^{(\varepsilon)}(f)|$ is L^p bounded. Formulate an analogous statement for $p=1$.

5.3.11. (M. Mastyło) Let $1\leq p<\infty$. Suppose that T_ε are linear operators defined on $L^p(\mathbf{R}^n)$ such that for all $f\in L^p(\mathbf{R}^n)$ we have $|T_\varepsilon(f)|\leq A\,\varepsilon^{-a}\|f\|_{L^p}$ for some $0<a,A<\infty$. Also suppose that there is a constant $C<\infty$ such that the maximal operator $T_*(f) = \sup_{\varepsilon>0}|T_\varepsilon(f)|$ satisfies $\big\|T_*(h)\|_{L^p}\leq C\|h\|_{L^p}$ for all $h\in\mathscr{S}(\mathbf{R}^n)$. Prove that the same inequality is valid for all $f\in L^p(\mathbf{R}^n)$.
$\big[$*Hint:* For a fixed $\delta>0$ define $S_\delta(f)=\sup_{\varepsilon>\delta}|T_\varepsilon(f)|$, which is a subadditive functional on $L^p(\mathbf{R}^n)$. For a fixed $f_0\in L^p(\mathbf{R}^n)$ define a linear space $X_0=\{\lambda f_0:\ \lambda\in\mathbf{C}\}$ and a linear functional T_0 on X_0 by setting $T_0(\lambda f_0)=\lambda S_\delta(f_0)$. By the Hahn–Banach theorem there is an extension $\widetilde{T_0}$ of T_0 that satisfies $|\widetilde{T_0}(f)|\leq S_\delta(f)$ for all $f\in L^p(\mathbf{R}^n)$. Since S_δ is L^p is bounded on Schwartz functions with norm at most C, then so is $\widetilde{T_0}$. But $\widetilde{T_0}$ is linear and by density it is bounded on $L^p(\mathbf{R}^n)$ with norm at most C; consequently, $\|S_\delta(f_0)\|_{L^p}=\|T_0(f_0)\|_{L^p}=\|\widetilde{T_0}(f_0)\|_{L^p}\leq C\|f_0\|_{L^p}$. The required conclusion for T_* follows by Fatou's lemma.$\big]$

5.4 Sufficient Conditions for L^p Boundedness

We have used the Calderón–Zygmund decomposition to prove weak type $(1,1)$ boundedness for singular integral and maximal singular integral operators, assuming that these operators are already L^2 bounded. It is therefore natural to ask for sufficient conditions that imply L^2 boundedness for such operators. Precisely, what are sufficient conditions on functions K on $\mathbf{R}^n\setminus\{0\}$ so that the corresponding singular and maximal singular integral operators associated with K are L^2 bounded? We saw in Section 5.2 that if K has the special form $K(x)=\Omega(x/|x|)/|x|^n$ for some $\Omega\in L^1(\mathbf{S}^{n-1})$ with mean value zero, then condition (5.2.16) is necessary and sufficient for the L^2 boundedness of T, while the L^2 boundedness of $T^{(*)}$ requires the stronger smoothness condition (5.2.24).

For the general K considered in this section (for which the corresponding operator does not necessarily commute with dilations), we only give some sufficient conditions for L^2 boundedness of T and $T^{(**)}$.

Throughout this section K denotes a locally integrable function on $\mathbf{R}^n\setminus\{0\}$ that satisfies the "size" condition

$$\sup_{R>0}\int_{R\leq|x|\leq 2R}|K(x)|\,dx = A_1<\infty, \tag{5.4.1}$$

the "smoothness" condition

$$\sup_{y\neq 0}\int_{|x|\geq 2|y|}|K(x-y)-K(x)|\,dx = A_2 < \infty, \tag{5.4.2}$$

and the "cancellation" condition

$$\sup_{0<R_1<R_2<\infty}\left|\int\limits_{R_1<|x|<R_2} K(x)\,dx\right| = A_3 < \infty, \tag{5.4.3}$$

for some $A_1, A_2, A_3 > 0$. As mentioned earlier, condition (5.4.2) is often referred to as Hörmander's condition. In this section we show that these three conditions give rise to convolution operators that are bounded on L^p.

5.4.1 Sufficient Conditions for L^p Boundedness of Singular Integrals

We first note that under conditions (5.4.1), (5.4.2), and (5.4.3), there exists a tempered distribution W of the form (5.3.7) that coincides with K on $\mathbf{R}^n\setminus\{0\}$. Indeed, condition (5.4.3) implies that there exists a sequence $\delta_j\downarrow 0$ such that

$$\lim_{j\to\infty}\int_{\delta_j<|x|\leq 1} K(x)\,dx = L$$

exists. Using (5.3.8), we conclude that there exists such a tempered distribution W. Note that we must have $|L|\leq A_3$.

We observe that the difference of two distributions W and W' that coincide with K on $\mathbf{R}^n\setminus\{0\}$ must be supported at the origin.

Theorem 5.4.1. *Assume that K satisfies (5.4.1), (5.4.2), and (5.4.3), and let W be a tempered distribution of the form (5.3.7) that coincides with K on $\mathbf{R}^n\setminus\{0\}$. Then we have*

$$\sup_{0<\varepsilon<N<\infty}\sup_{\xi\in\mathbf{R}^n}|(K\chi_{\varepsilon<|\cdot|<N})^\wedge(\xi)| \leq 15(A_1+A_2+A_3) \tag{5.4.4}$$

and consequently

$$\sup_{\xi\in\mathbf{R}^n}|\widehat{W}(\xi)|\leq 15(A_1+A_2+A_3)\,. \tag{5.4.5}$$

Thus the operator given by convolution with W maps $L^2(\mathbf{R}^n)$ to itself with norm at most $15(A_1+A_2+A_3)$. Consequently, it also maps $L^1(\mathbf{R}^n)$ to $L^{1,\infty}(\mathbf{R}^n)$ with bound at most a dimensional constant multiple of $A_1+A_2+A_3$ and $L^p(\mathbf{R}^n)$ to itself with bound at most $C_n\max(p,(p-1)^{-1})(A_1+A_2+A_3)$, when $1<p<\infty$, where C_n is a dimensional constant.

Proof. Let us set $K^{(\varepsilon,N)}(x) = K(x)\chi_{\varepsilon<|x|<N}$. Estimate (5.4.4) implies that for all f in $\mathscr{S}(\mathbf{R}^n)$ we have

$$\big\|f * K^{(\delta_j,j)}\big\|_{L^2} \le 15(A_1+A_2+A_3)\big\|f\big\|_{L^2}$$

uniformly in j. Using this, (5.3.9), and Fatou's lemma, we obtain that

$$\big\|f * W\big\|_{L^2} \le 15(A_1+A_2+A_3)\big\|f\big\|_{L^2},$$

for all $f \in \mathscr{S}(\mathbf{R}^n)$, and this is equivalent to (5.4.5).
Case 1: Suppose that $\varepsilon < |\xi|^{-1} < N$. Then we write

$$\widehat{K^{(\varepsilon,N)}}(\xi) = I_1(\xi) + I_2(\xi),$$

where

$$\begin{aligned} I_1(\xi) &= \int_{\varepsilon<|x|<|\xi|^{-1}} K(x)e^{-2\pi i x\cdot\xi}\,dx,\\ I_2(\xi) &= \int_{|\xi|^{-1}<|x|<N} K(x)e^{-2\pi i x\cdot\xi}\,dx. \end{aligned}$$

We now have

$$I_1(\xi) = \int_{\varepsilon<|x|<|\xi|^{-1}} K(x)\,dx + \int_{\varepsilon<|x|<|\xi|^{-1}} K(x)\,(e^{-2\pi i x\cdot\xi}-1)\,dx. \tag{5.4.6}$$

It follows that

$$|I_1(\xi)| \le A_3 + 2\pi|\xi|\int_{|x|<|\xi|^{-1}} |x|\,|K(x)|\,dx \le A_3 + 2\pi(2A_1)$$

uniformly in ε. Let us now examine $I_2(\xi)$. Let $z = \frac{\xi}{2|\xi|^2}$ so that $e^{2\pi i z\cdot\xi} = -1$ and $2|z| = |\xi|^{-1}$. By changing variables $x = x'-z$, rewrite I_2 as

$$I_2(\xi) = -\int_{|\xi|^{-1}<|x'-z|<N} K(x'-z)\,e^{-2\pi i x'\cdot\xi}\,dx';$$

hence averaging gives

$$I_2(\xi) = \frac{1}{2}\int_{|\xi|^{-1}<|x|<N} K(x)e^{-2\pi i x\cdot\xi}\,dx - \frac{1}{2}\int_{|\xi|^{-1}<|x-z|<N} K(x-z)\,e^{-2\pi i x\cdot\xi}\,dx.$$

Now use that

$$\int_A F\,dx - \int_B G\,dx = \int_B (F-G)\,dx + \int_{A\setminus B} F\,dx - \int_{B\setminus A} F\,dx \tag{5.4.7}$$

to write $I_2(\xi) = J_1(\xi) + J_2(\xi) + J_3(\xi) + J_4(\xi) + J_5(\xi)$, where

$$J_1(\xi) = +\frac{1}{2} \int\limits_{|\xi|^{-1}<|x-z|<N} \big(K(x) - K(x-z)\big) e^{-2\pi i x\cdot\xi}\, dx\,, \tag{5.4.8}$$

$$J_2(\xi) = +\frac{1}{2} \int\limits_{\substack{|\xi|^{-1}<|x|<N \\ |x-z|\le|\xi|^{-1}}} K(x)\, e^{-2\pi i x\cdot\xi}\, dx\,, \tag{5.4.9}$$

$$J_3(\xi) = +\frac{1}{2} \int\limits_{\substack{|\xi|^{-1}<|x|<N \\ |x-z|\ge N}} K(x)\, e^{-2\pi i x\cdot\xi}\, dx\,, \tag{5.4.10}$$

$$J_4(\xi) = -\frac{1}{2} \int\limits_{\substack{|\xi|^{-1}<|x-z|<N \\ |x|\le|\xi|^{-1}}} K(x)\, e^{-2\pi i x\cdot\xi}\, dx\,, \tag{5.4.11}$$

$$J_5(\xi) = -\frac{1}{2} \int\limits_{\substack{|\xi|^{-1}<|x-z|<N \\ |x|\ge N}} K(x)\, e^{-2\pi i x\cdot\xi}\, dx\,. \tag{5.4.12}$$

Since $2|z| = |\xi|^{-1}$, $J_1(\xi)$ is bounded in absolute value by $\frac{1}{2}A_2$, in view of (5.4.2).

Next observe that $|\xi|^{-1} \le |x| \le \frac{3}{2}|\xi|^{-1}$ in (5.4.9), while $\frac{1}{2}|\xi|^{-1} \le |x| \le |\xi|^{-1}$ in (5.4.11); hence both J_2 and J_4 are bounded by $\frac{1}{2}A_1$. Finally, we have $\frac{1}{2}N < |x| < N$ in (5.4.10) (since $|x| > N - \frac{1}{2}|\xi|^{-1}$), and similarly we have $N \le |x| < \frac{3}{2}N$ in (5.4.12). Thus both J_3 and J_5 are bounded above by $\frac{1}{2}A_1$.

Case 2: If $\varepsilon < N \le |\xi|^{-1}$, then we write

$$\int_{\varepsilon<|x|<N} K(x) e^{-2\pi i x\cdot\xi}\, dx = \int_{\varepsilon<|x|<N} K(x)\, dx + \int_{\varepsilon<|x|<N} K(x)(e^{-2\pi i x\cdot\xi} - 1)\, dx$$

which is bounded in absolute value by

$$A_3 + 2\pi|\xi| \int_{|x|\le|\xi|^{-1}} |K(x)|\,|x|\, dx \le A_3 + 4\pi A_1.$$

Notice that if $\xi = 0$, only the first term appears which is bounded by A_3.

Case 2: If $|\xi|^{-1} \le \varepsilon < N$, we write

$$\int_{\varepsilon<|x|<N} K(x) e^{-2\pi i x\cdot\xi} dx = \int_{|\xi|^{-1}<|x|<N} K(x) e^{-2\pi i x\cdot\xi} dx - \int_{|\xi|^{-1}<|x|<\varepsilon} K(x) e^{-2\pi i x\cdot\xi} dx,$$

and both of the terms on the right are similar to $I_2(\xi)$ which was shown to be bounded by $2A_1 + \frac{1}{2}A_2$.

In all cases (5.4.4) holds. $\square$

5.4.2 An Example

We now give an example of a distribution that satisfies conditions (5.4.1), (5.4.2), and (5.4.3).

Example 5.4.2. Let τ be a nonzero real number and let $K(x) = \frac{1}{|x|^{n+i\tau}}$ defined for $x \in \mathbf{R}^n \setminus \{0\}$. For a sequence $\delta_k \downarrow 0$ and φ a Schwartz function on $\mathbf{R}^n$, define

$$\big\langle W, \varphi \big\rangle = \lim_{k\to\infty} \int_{\delta_k \le |x|} \varphi(x) \, \frac{dx}{|x|^{n+i\tau}}, \tag{5.4.13}$$

whenever the limit exists. We claim that for some choices of sequences δ_k, W is a well defined tempered distribution on $\mathbf{R}^n$. Take, for example, $\delta_k = e^{-2\pi k/\tau}$. For this sequence δ_k, observe that

$$\int_{\delta_k \le |x| \le 1} \frac{1}{|x|^{n+i\tau}} \, dx = \omega_{n-1} \frac{1-(e^{-2\pi k/\tau})^{-i\tau}}{-i\tau} = 0\,,$$

and thus

$$\big\langle W, \varphi \big\rangle = \int_{|x|\le 1} (\varphi(x) - \varphi(0)) \, \frac{dx}{|x|^{n+i\tau}} + \int_{|x|\ge 1} \varphi(x) \frac{dx}{|x|^{n+i\tau}}\,, \tag{5.4.14}$$

which implies that $W \in \mathscr{S}'(\mathbf{R}^n)$, since

$$\big|\big\langle W, \varphi \big\rangle\big| \le C\big[\big\|\nabla\varphi\big\|_{L^\infty} + \big\|\,|x|\,\varphi(x)\big\|_{L^\infty}\big]\,.$$

If φ is supported in $\mathbf{R}^n \setminus \{0\}$, then

$$\big\langle W, \varphi \big\rangle = \int_{\mathbf{R}^n} K(x)\varphi(x)\, dx\,.$$

Therefore W coincides with the function K away from the origin. Moreover, (5.4.1) and (5.4.2) are clearly satisfied for K, while (5.4.3) is also satisfied, since

$$\left|\int_{R_1<|x|<R_2} \frac{1}{|x|^{n+i\tau}} \, dx\right| = \omega_{n-1} \left|\frac{R_1^{-i\tau} - R_2^{-i\tau}}{-i\tau}\right| \le \frac{2\omega_{n-1}}{|\tau|}\,.$$

Remark 5.4.3. It is important to emphasize that the limit in (5.4.13) *may not* exist for all sequences $\delta_k \to 0$. For example, the limit in (5.4.13) does not exist if $\delta_k = e^{-\pi k/\tau}$. Moreover, for a different choice of a sequence δ_k for which the limit in (5.4.13) exists (for example, $\delta_k = e^{-\pi(2k+1)/\tau}$), we obtain a different distribution W_1 that coincides with the function $K(x) = |x|^{-n-i\tau}$ on $\mathbf{R}^n \setminus \{0\}$.

We discuss a point of caution. We can directly check that the distributions W defined by (5.4.13) are not homogeneous distributions of degree $-n-i\tau$. In fact, the only homogeneous distribution of degree $-n-i\tau$ that coincides with the function $|x|^{-n-i\tau}$ away from zero is a multiple of the distribution $u_{-n-i\tau}$, where u_z is defined in (2.4.7). Let us investigate the relationship between $u_{-n-i\tau}$ and W defined in (5.4.14). Recall that (2.4.8) gives

$$\begin{aligned}\left\langle u_{-n-i\tau},\varphi\right\rangle = & \int_{|x|\geq 1}\varphi(x)\frac{\pi^{-i\frac{\tau}{2}}}{\Gamma(-i\frac{\tau}{2})}|x|^{-n-i\tau}\,dx \\ & +\int_{|x|\leq 1}(\varphi(x)-\varphi(0))\frac{\pi^{-i\frac{\tau}{2}}}{\Gamma(-i\frac{\tau}{2})}|x|^{-n-i\tau}\,dx+\frac{\omega_{n-1}\pi^{-i\frac{\tau}{2}}}{-i\tau\Gamma(-i\frac{\tau}{2})}\varphi(0)\,.\end{aligned}$$

Using (5.4.14), we conclude that $u_{-n-i\tau}-c_1W=c_2\delta_0$ for suitable nonzero constants c_1 and c_2. Since the Dirac mass at the origin is not a homogeneous distribution of degree $-n-i\tau$, it follows that neither is W.

Since

$$\widehat{u_{-n-i\tau}}=u_{i\tau}=c_3|\xi|^{i\tau}\,,$$

the identity $u_{-n-i\tau}-c_1W=c_2\delta_0$ can be used to obtain a formula for the Fourier transform of W and thus produce a different proof that convolution with W is a bounded operator on $L^2(\mathbf{R}^n)$.

5.4.3 Necessity of the Cancellation Condition

Although conditions (5.4.1), (5.4.2), and (5.4.3) are sufficient for L^2 boundedness, they might not necessary. However, we show that (5.4.3) is a necessary condition, given (5.4.1).

Proposition 5.4.4. *Suppose that K is a function on $\mathbf{R}^n\setminus\{0\}$ that satisfies (5.4.1). Let W be a tempered distribution on $\mathbf{R}^n$ related to K as in (5.3.7). If the operator T given by convolution with W maps $L^2(\mathbf{R}^n)$ to itself (equivalently if $\widehat{W}$ is an L^∞ function), then the function K must satisfy (5.4.3).*

Proof. Pick a radial $\mathscr{C}^\infty$ function φ supported in the ball $|x|\leq 2$ with $0\leq\varphi\leq 1$, and $\varphi(x)=1$ when $|x|\leq 1$. For $R>0$ let $\varphi^R(x)=\varphi(x/R)$. Fourier inversion for distributions gives the second equality,

$$(W*\varphi^R)(0)=\left\langle W,\varphi^R\right\rangle=\left\langle \widehat{W},\widehat{\varphi^R}\right\rangle=\int_{\mathbf{R}^n}\widehat{W}(\xi)R^n\widehat{\varphi}(R\xi)\,d\xi\,,$$

and the preceding identity implies that

$$|(W*\varphi^R)(0)|\leq\big\|\widehat{W}\big\|_{L^\infty}\big\|\widehat{\varphi}\big\|_{L^1}=\big\|T\big\|_{L^2\to L^2}\big\|\widehat{\varphi}\big\|_{L^1}$$

uniformly in $R>0$. Fix $0<R_1<R_2<\infty$. If $R_2\leq 2R_1$, we have

$$\left|\int\limits_{R_1<|x|<R_2}K(x)\,dx\right|\leq\int\limits_{R_1<|x|<2R_1}|K(x)|\,dx\leq A_1\,,$$

which implies the required conclusion. We may therefore assume that $2R_1 < R_2$. Since the part of the integral in (5.4.3) over the set $R_1 < |x| < 2R_1$ is controlled by A_1, it suffices to control the integral of $K(x)$ over the set $2R_1 < |x| < R_2$. Since the function $\varphi^{R_2} - \varphi^{R_1}$ is supported away from the origin, the action of the distribution W on it can be written as integration against the function K. We have

$$\int_{\mathbf{R}^n} K(x)(\varphi^{R_2}(x) - \varphi^{R_1}(x))\,dx$$
$$= \int\limits_{2R_1<|x|<R_2} K(x)\,dx + \int\limits_{R_1<|x|<2R_1} K(x)(1-\varphi^{R_1}(x))\,dx + \int\limits_{R_2<|x|<2R_2} K(x)\varphi^{R_2}(x)\,dx.$$

The sum of the last two integrals is bounded by $3A_1$ (since $0 \le \varphi \le 1$), while the first integral is equal to

$$(W * \varphi^{R_2})(0) - (W * \varphi^{R_1})(0)$$

and is therefore bounded by $2\|T\|_{L^2\to L^2}\|\widehat{\varphi}\|_{L^1}$. We conclude that the function K must satisfy (5.4.3) with constant

$$A_3 \le 3A_1 + 2\|\widehat{\varphi}\|_{L^1}\|T\|_{L^2\to L^2} \le c\big(A_1 + \|T\|_{L^2\to L^2}\big)\,.$$

This establishes the assertion. $\square$

5.4.4 Sufficient Conditions for L^p Boundedness of Maximal Singular Integrals

We now discuss the analogous result to Theorem 5.4.1 for the maximal singular integral operator $T^{(**)}$.

Theorem 5.4.5. *Suppose that K satisfies (5.4.1), (5.4.2), and (5.4.3) and let $T^{(**)}$ be as in* (5.3.18). *Then $T^{(**)}$ is bounded on $L^p(\mathbf{R}^n)$, $1 < p < \infty$, with norm*

$$\big\|T^{(**)}\big\|_{L^p\to L^p} \le C_n \max(p,(p-1)^{-1})(A_1 + A_2 + A_3),$$

where C_n is a dimensional constant.

Proof. We first define an operator T associated with K that satisfies (5.4.1), (5.4.2), and (5.4.3). Because of condition (5.4.3), there exists a sequence $\delta_j \downarrow 0$ such that

$$\lim_{j\to\infty}\int_{\delta_j<|x|\le 1} K(x)\,dx$$

exists. Therefore, for $\varphi \in \mathscr{S}(\mathbf{R}^n)$ we can define a tempered distribution

$$\big\langle W,\varphi\big\rangle = \lim_{j\to\infty}\int_{\delta_j\le|x|\le j} K(x)\varphi(x)\,dx$$

and an operator T given by $T(f) = f * W$ for $f \in \mathscr{S}(\mathbf{R}^n)$. In view of Theorems 5.4.1 and 5.3.3, T admits an L^p bounded extension ($1 < p < \infty$) with

$$\big\|T\big\|_{L^p \to L^p} \le c_n \max(p, (p-1)^{-1})(A_1 + A_2 + A_3) \tag{5.4.15}$$

and is weak type $(1,1)$. This extension is still denoted by T.

Fix $1 < p < \infty$ and $f \in L^p(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ with compact support. We have

$$\begin{aligned}
&T^{(\varepsilon,N)}(f)(x) \\
&\quad = \int_{\varepsilon \le |x-y| < N} K(x-y) f(y)\, dy = T^{(\varepsilon)}(f)(x) - T^{(N)}(f)(x) \\
&\quad = \int_{\varepsilon \le |x-y|} K(x-y) f(y)\, dy - \int_{N \le |x-y|} K(x-y) f(y)\, dy \\
&\quad = \int_{\varepsilon \le |x-y|} (K(x-y) - K(z_1 - y)) f(y)\, dy + \int_{\varepsilon \le |x-y|} K(z_1 - y) f(y)\, dy \\
&\qquad\qquad - \int_{N \le |x-y|} (K(x-y) - K(z_2 - y)) f(y)\, dy - \int_{N \le |x-y|} K(z_2 - y) f(y)\, dy \\
&\quad = \int_{\varepsilon \le |x-y|} (K(x-y) - K(z_1 - y)) f(y)\, dy + T(f)(z_1) - T(f\chi_{|x-\cdot|<\varepsilon})(z_1) \\
&\qquad\quad - \int_{N \le |x-y|} (K(x-y) - K(z_2 - y)) f(y)\, dy - T(f)(z_2) + T(f\chi_{|x-\cdot|<N})(z_2),
\end{aligned}$$

where z_1 and z_2 are arbitrary points in $\mathbf{R}^n$ that satisfy $|z_1 - x| \le \frac{\varepsilon}{2}$ and $|z_2 - x| \le \frac{N}{2}$. We used that f has compact support in order to be able to write $T^{(\varepsilon)}(f)(x)$ and $T^{(N)}(f)(x)$ as convergent integrals for almost every x.

At this point we take absolute values, average over $|z_1 - x| \le \frac{\varepsilon}{2}$ and $|z_2 - x| \le \frac{N}{2}$, and we apply Hölder's inequality in two terms. We obtain the estimate

$$\begin{aligned}
&|T^{(\varepsilon,N)}(f)(x)| \\
&\quad \le \frac{1}{v_n}\left(\frac{2}{\varepsilon}\right)^n \int_{|z_1 - x| \le \frac{\varepsilon}{2}} \int_{|x-y| \ge \varepsilon} |K(x-y) - K(z_1 - y)|\, |f(y)|\, dy\, dz_1 \\
&\qquad + \frac{1}{v_n}\left(\frac{2}{\varepsilon}\right)^n \int_{|z_1 - x| \le \frac{\varepsilon}{2}} |T(f)(z_1)|\, dz_1 \\
&\qquad + \left(\frac{1}{v_n}\left(\frac{2}{\varepsilon}\right)^n \int_{|z_1 - x| \le \frac{\varepsilon}{2}} |T(f\chi_{|x-\cdot|<\varepsilon})(z_1)|^p\, dz_1\right)^{\frac{1}{p}} \\
&\qquad + \frac{1}{v_n}\left(\frac{2}{N}\right)^n \int_{|z_2 - x| \le \frac{N}{2}} \int_{|x-y| \ge N} |K(x-y) - K(z_2 - y)|\, |f(y)|\, dy\, dz_2 \\
&\qquad + \frac{1}{v_n}\left(\frac{2}{N}\right)^n \int_{|z_2 - x| \le \frac{N}{2}} |T(f)(z_2)|\, dz_2 \\
&\qquad + \left(\frac{1}{v_n}\left(\frac{2}{N}\right)^n \int_{|z_2 - x| \le \frac{N}{2}} |T(f\chi_{|x-\cdot|<N})(z_2)|^p\, dz_2\right)^{\frac{1}{p}},
\end{aligned}$$

where v_n is the volume of the unit ball in $\mathbf{R}^n$. Applying condition (5.4.2) and estimate (5.4.15), we obtain for f in $L^p(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$ with compact support that

$$\begin{aligned}
&|T^{(\varepsilon,N)}(f)(x)| \\
&\quad\le \frac{1}{v_n}\left(\frac{2}{\varepsilon}\right)^n \int_{|z_1-x|\le\frac{\varepsilon}{2}} |T(f)(z_1)|\,dz_1 + \frac{1}{v_n}\left(\frac{2}{N}\right)^n \int_{|z_2-x|\le\frac{N}{2}} |T(f)(z_2)|\,dz_2 \\
&\qquad + c_n\Big(\sum_{j=1}^3 A_j\Big)\max(p,(p-1)^{-1})\left(\frac{1}{v_n}\left(\frac{2}{\varepsilon}\right)^n \int_{|z_1-x|\le\varepsilon} |f(z_1)|^p\,dz_1\right)^{\frac{1}{p}} \\
&\qquad + c_n\Big(\sum_{j=1}^3 A_j\Big)\max(p,(p-1)^{-1})\left(\frac{1}{v_n}\left(\frac{2}{N}\right)^n \int_{|z_2-x|\le N} |f(z_2)|^p\,dz_2\right)^{\frac{1}{p}} \\
&\qquad + 2A_2\big\|f\big\|_{L^\infty}.
\end{aligned}$$

We now use density to remove the compact support condition on f and obtain the last displayed estimate for all functions f in $L^p(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$. Taking the supremum over all $0<\varepsilon<N$ and over all $N>0$, we deduce that for all f in $L^p(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$ we have the estimate

$$T^{(**)}(f)(x) \le 2A_2\big\|f\big\|_{L^\infty} + S_p(f)(x), \tag{5.4.16}$$

where S_p is the sublinear operator defined by

$$S_p(f)(x) = 2M(T(f))(x) + 3^{n+1}c_n\Big(\sum_{j=1}^3 A_j\Big)\max(p,(p-1)^{-1})(M(|f|^p)(x))^{\frac{1}{p}},$$

and M is the Hardy–Littlewood maximal operator.

Recalling that M maps L^1 to $L^{1,\infty}$ with bound at most 3^n and also L^p to $L^{p,\infty}$ with bound at most $2\cdot 3^{n/p}$ for $1<p<\infty$ (Exercise 2.1.4), we conclude that S_p maps $L^p(\mathbf{R}^n)$ to $L^{p,\infty}(\mathbf{R}^n)$ with norm at most

$$\big\|S_p\big\|_{L^p\to L^{p,\infty}} \le \widetilde{c}_n(A_1+A_2+A_3)\max(p,(p-1)^{-1}), \tag{5.4.17}$$

where $\widetilde{c}_n$ is another dimensional constant.

Now write $f = f_\alpha + f^\alpha$, where

$$f_\alpha = f\chi_{|f|\le\alpha/(16A_2)} \qquad \text{and} \qquad f^\alpha = f\chi_{|f|>\alpha/(16A_2)}.$$

The function f_α is in $L^\infty\cap L^p$ and f^α is in $L^1\cap L^p$. Moreover, we see that

$$\big\|f^\alpha\big\|_{L^1} \le (16A_2/\alpha)^{p-1}\big\|f\big\|_{L^p}^p. \tag{5.4.18}$$

Apply the Calderón–Zygmund decomposition (Theorem 5.3.1) to the function f^α at height $\alpha\gamma$ to write $f^\alpha = g^\alpha + b^\alpha$, where g^α is the good function and b^α is the bad function of this decomposition. Using (5.3.1), we obtain

$$\big\|g^\alpha\big\|_{L^p} \le 2^{n/p'}(\alpha\gamma)^{1/p'}\big\|f^\alpha\big\|_{L^1}^{1/p} \le 2^{(n+4)/p'}(A_2\gamma)^{1/p'}\big\|f\big\|_{L^p}. \tag{5.4.19}$$

We now use (5.4.16) to get

$$|\{x\in\mathbf{R}^n :\ T^{(**)}(f)(x) > \alpha\}| \le b_1 + b_2 + b_3\,, \tag{5.4.20}$$

where

$$\begin{aligned} b_1 &= |\{x\in\mathbf{R}^n :\ 2A_2\big\|f_\alpha\big\|_{L^\infty} + S_p(f_\alpha)(x) > \alpha/4\}|\,,\\ b_2 &= |\{x\in\mathbf{R}^n :\ 2A_2\big\|g^\alpha\big\|_{L^\infty} + S_p(g^\alpha)(x) > \alpha/4\}|\,,\\ b_3 &= |\{x\in\mathbf{R}^n :\ T^{(**)}(b^\alpha)(x) > \alpha/2\}|\,. \end{aligned}$$

Observe that $2A_2\|f_\alpha\|_{L^\infty} \le \alpha/8$. Selecting $\gamma = 2^{-n-5}(A_1+A_2)^{-1}$ and using property (2) in Theorem 5.3.1, we obtain

$$2A_2\big\|g^\alpha\big\|_{L^\infty} \le A_2 2^{n+1}\alpha\gamma \le \alpha 2^{-4} < \frac{\alpha}{8}$$

and therefore

$$\begin{aligned} b_1 &\le |\{x\in\mathbf{R}^n :\ S_p(f_\alpha)(x) > \alpha/8\}|\,,\\ b_2 &\le |\{x\in\mathbf{R}^n :\ S_p(g^\alpha)(x) > \alpha/8\}|\,. \end{aligned} \tag{5.4.21}$$

Since $\gamma \le (2^{n+5}A_1)^{-1}$, it follows from (5.3.27) that

$$b_3 \le \Big|\bigcup_j Q_j^*\Big| + 2^{n+8}A_2\frac{\big\|f^\alpha\big\|_{L^1}}{\alpha} \le \left(\frac{(5\sqrt{n})^n}{\gamma} + 2^{n+8}A_2\right)\frac{\big\|f^\alpha\big\|_{L^1}}{\alpha}\,,$$

and using (5.4.18), we obtain

$$b_3 \le C_n(A_1+A_2)^p\alpha^{-p}\big\|f\big\|_{L^p}^p.$$

Using Chebyshev's inequality in (5.4.21) and (5.4.17), we finally obtain that

$$b_1 + b_2 \le (8/\alpha)^p\,(\widetilde{c}_n)^p(A_1{+}A_2{+}A_3)^p\max(p,(p-1)^{-1})^p\big(\big\|f\big\|_{L^p}^p + \big\|g^\alpha\big\|_{L^p}^p\big).$$

Combining the estimates for b_1, b_2, and b_3 and using (5.4.19), we deduce

$$\big\|T^{(**)}(f)\big\|_{L^{p,\infty}} \le C_n(A_1+A_2+A_3)\max(p,(p-1)^{-1})\big\|f\big\|_{L^p(\mathbf{R}^n)}. \tag{5.4.22}$$

Finally, we need to obtain a similar estimate to (5.4.22), in which the weak L^p norm on the left is replaced by the L^p norm. This is a consequence of Theorem 1.3.2 via interpolation between the estimates $L^{\frac{p+1}{2}} \to L^{\frac{p+1}{2},\infty}$ and $L^{2p} \to L^{2p,\infty}$ for $2 < p < \infty$ and between the estimates $L^{2p} \to L^{2p,\infty}$ and $L^1 \to L^{1,\infty}$ for $1 < p < 2$. The latter estimate follows from Theorem 5.3.5. See also Corollary 5.3.7. □

Exercises

5.4.1. Let T be a convolution operator that is L^2 bounded. Suppose that a given function $f_0 \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ has vanishing integral and that $T(f_0)$ is integrable. Prove that $T(f_0)$ also has vanishing integral.

5.4.2. Let K satisfy (5.4.1), (5.4.2), and (5.4.3) and let $W \in \mathscr{S}'$ be an extension of K on $\mathbf{R}^n$. Let f be a compactly supported $\mathscr{C}^1$ function on $\mathbf{R}^n$ with mean value zero. Prove that the function $f * W$ is in $L^1(\mathbf{R}^n)$.

5.4.3. Suppose K is a function on $\mathbf{R}^n \setminus \{0\}$ that satisfies (5.4.1), (5.4.2), and (5.4.3). Let $K^{(\varepsilon,N)}(x) = K(x)\chi_{\varepsilon<|x|<N}$ for $0 < \varepsilon < N < \infty$ and let $T^{(\varepsilon,N)}$ be the operator given by convolution with $K^{(\varepsilon,N)}$. Let $1 < p < \infty$ and $f \in L^p(\mathbf{R}^n)$. Prove that for some sequence $\varepsilon_j \downarrow 0$, $T^{(\varepsilon_j,N)}(f)$ converges almost everywhere as $j \to \infty$ and $N \to \infty$.
[*Hint:* Use Theorems 5.4.5 and 2.1.14.]

5.4.4. (a) Prove that for all $x, y \in \mathbf{R}^n$ that satisfy $0 \neq x \neq y$ we have

$$\left| \frac{x-y}{|x-y|} - \frac{x}{|x|} \right| \leq 2\,\frac{|y|}{|x|}.$$

(b) Let Ω be an integrable function with mean value zero on the sphere $\mathbf{S}^{n-1}$. Suppose that Ω satisfies a *Lipschitz (Hölder) condition* of order $0 < \alpha < 1$ on $\mathbf{S}^{n-1}$. This means that

$$|\Omega(\theta_1) - \Omega(\theta_2)| \leq B_0 |\theta_1 - \theta_2|^{\alpha}$$

for all $\theta_1, \theta_2 \in \mathbf{S}^{n-1}$. Prove that $K(x) = \Omega(x/|x|)/|x|^n$ satisfies Hörmander's condition (5.4.3) with constant at most a multiple of $B_0 + \|\Omega\|_{L^\infty}$.

5.4.5. Let Ω be an L^1 function on $\mathbf{S}^{n-1}$ with mean value zero.
(a) Let $\omega_\infty(t) = \sup\{|\Omega(\theta_1) - \Omega(\theta_2)| : \ \theta_1, \theta_2 \in \mathbf{S}^{n-1},\ |\theta_1 - \theta_2| \leq t\}$ and suppose that the following *Dini condition* holds:

$$\int_0^1 \omega_\infty(t)\,\frac{dt}{t} < \infty.$$

Prove that the function $K(x) = \Omega(x/|x|)|x|^{-n}$ satisfies Hörmander's condition.
(b) (*A. Calderón and A. Zygmund*) For $A \in O(n)$, let

$$\|A\| = \sup\{|\theta - A(\theta)| : \ \theta \in \mathbf{S}^{n-1}\}.$$

Suppose that Ω satisfies the more general Dini-type condition

$$\int_0^1 \omega_1(t)\frac{dt}{t} < \infty,$$

where

$$\omega_1(t) = \sup_{\substack{A\in O(n)\\ \|A\|\le t}} \int_{\mathbf{S}^{n-1}} |\Omega(A(\theta)) - \Omega(\theta)|\, d\theta\,.$$

Prove the same conclusion as in part (a).
$\big[$*Hint:* Part (b): Use the result in part (a) of Exercise 5.4.4 and switch to polar coordinates.$\big]$

5.5 Vector-Valued Inequalities

Certain nonlinear expressions that appear in Fourier analysis, such as maximal functions and square functions, can be viewed as linear quantities taking values in some Banach space. This point of view provides the motivation for a systematic study of Banach-valued operators. Let us illustrate this line of thinking via an example. Let T be a linear operator acting on L^p of some measure space (X,μ) and taking values in the set of measurable functions of another measure space (Y,ν). The seemingly nonlinear inequality

$$\Big\|\Big(\sum_j |T(f_j)|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \le C_p \Big\|\Big(\sum_j |f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \tag{5.5.1}$$

can be transformed to a linear one with only a slight change of view. Let us denote by $L^p(X,\ell^2)$ the Banach space of all sequences $\{f_j\}_j$ of measurable functions on X that satisfy

$$\big\|\{f_j\}_j\big\|_{L^p(X,\ell^2)} = \bigg(\int_X \Big(\sum_j |f_j|^2\Big)^{\frac{p}{2}} d\mu\bigg)^{\frac{1}{p}} < \infty. \tag{5.5.2}$$

Define a linear operator acting on such sequences by setting

$$\vec{T}(\{f_j\}_j) = \{T(f_j)\}_j. \tag{5.5.3}$$

Then (5.5.1) is equivalent to the inequality

$$\big\|\vec{T}(\{f_j\}_j)\big\|_{L^p(Y,\ell^2)} \le C_p \big\|\{f_j\}_j\big\|_{L^p(X,\ell^2)}, \tag{5.5.4}$$

in which $\vec{T}$ is thought of as a linear operator acting on the L^p space of ℓ^2-valued functions on X. This is the basic idea of vector-valued inequalities. A nonlinear inequality such as (5.5.1) can be viewed as a linear norm estimate for an operator acting and taking values in suitable Banach spaces.

5.5.1 ℓ^2-Valued Extensions of Linear Operators

The following result is classical and fundamental in the subject of vector-valued inequalities.

Theorem 5.5.1. *Let $0 < p,q < \infty$ and let (X,μ) and (Y,ν) be two σ-finite measure spaces. The following are valid:*
(a) Suppose that T is a bounded linear operator from $L^p(X)$ to $L^q(Y)$ with norm N. Then T has an ℓ^2-valued extension, that is, for all complex-valued functions f_j in $L^p(X)$ we have

$$\Big\|\Big(\sum_j |T(f_j)|^2\Big)^{\frac{1}{2}}\Big\|_{L^q(Y)} \le C_{p,q} N \Big\|\Big(\sum_j |f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(X)} \tag{5.5.5}$$

for some constant $C_{p,q}$ that satisfies $C_{p,q} = 1$ if $p \le q$. Moreover, if T maps real-valued L^p functions to real-valued L^q functions with norm N_{real}, then (5.5.5) holds for real-valued functions f_j with N_{real} in place of N.
(b) Suppose that T is a bounded linear operator from $L^p(X)$ to $L^{q,\infty}(Y)$ with norm M. Then T has an ℓ^2-valued extension, that is,

$$\Big\|\Big(\sum_j |T(f_j)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{q,\infty}(Y)} \le D_{p,q} M \Big\|\Big(\sum_j |f_j|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(X)} \tag{5.5.6}$$

for some constant $D_{p,q}$ that depends only on p and q. Moreover, if T maps real-valued L^p functions to real-valued $L^{q,\infty}$ functions with norm M_{real}, then (5.5.5) holds for real-valued functions f_j with M_{real} in place of M.

To prove this theorem, we need the following identities.

Lemma 5.5.2. *For any $0 < r < \infty$, define constants*

$$A_r = \left(\frac{\Gamma(\frac{r+1}{2})}{\pi^{\frac{r+1}{2}}}\right)^{\frac{1}{r}} \qquad \text{and} \qquad B_r = \left(\frac{\Gamma(\frac{r}{2}+1)}{\pi^{\frac{r}{2}}}\right)^{\frac{1}{r}}. \tag{5.5.7}$$

Then for any $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbf{R}$ we have

$$\left(\int_{\mathbf{R}^n} |\lambda_1 x_1 + \cdots + \lambda_n x_n|^r e^{-\pi|x|^2} dx\right)^{\frac{1}{r}} = A_r\,(\lambda_1^2 + \cdots + \lambda_n^2)^{\frac{1}{2}}\,, \tag{5.5.8}$$

where $dx = dx_1 \cdots dx_n$. Also for all $w_1, w_2, \dots, w_n \in \mathbf{C}$ we have

$$\left(\int_{\mathbf{C}^n} |w_1 z_1 + \cdots + w_n z_n|^r e^{-\pi|z|^2} dz\right)^{\frac{1}{r}} = B_r(|w_1|^2 + \cdots + |w_n|^2)^{\frac{1}{2}}\,, \tag{5.5.9}$$

where $dz = dz_1 \cdots dz_n = dx_1 dy_1 \cdots dx_n dy_n$ if $z_j = x_j + iy_j$.

Proof. Dividing both sides of (5.5.8) by $(\lambda_1^2+\cdots+\lambda_n^2)^{\frac{1}{2}}$, we reduce things to the situation in which $\lambda_1^2+\cdots+\lambda_n^2=1$. Let $e_1=(1,0,\dots,0)^t$ be the standard basis column unit vector on $\mathbf{R}^n$ and find an orthogonal $n\times n$ matrix $A\in O(n)$ (orthogonal means a real matrix satisfying $A^t=A^{-1}$) such that $A^{-1}e_1=(\lambda_1,\dots,\lambda_n)^t$. Then the first coordinate of Ax is

$$(Ax)_1 = Ax\cdot e_1 = x\cdot A^t e_1 = x\cdot A^{-1}e_1 = \lambda_1 x_1+\cdots+\lambda_n x_n\,.$$

Now change variables $y=Ax$ in the integral in (5.5.8) and use the fact that $|Ax|=|x|$ to obtain

$$\begin{aligned}
\left(\int_{\mathbf{R}^n}|\lambda_1x_1+\cdots+\lambda_nx_n|^r e^{-\pi|x|^2}dx\right)^{\frac{1}{r}} &= \left(\int_{\mathbf{R}^n}|y_1|^r e^{-\pi|y|^2}dy\right)^{\frac{1}{r}}\\
&= \left(2\int_0^\infty t^r e^{-\pi t^2}dt\right)^{\frac{1}{r}}\\
&= \left(\int_0^\infty s^{\frac{r-1}{2}} e^{-\pi s}ds\right)^{\frac{1}{r}}\\
&= \left(\frac{\Gamma\big(\frac{r+1}{2}\big)}{\pi^{\frac{r+1}{2}}}\right)^{\frac{1}{r}}\\
&= A_r\,,
\end{aligned}$$

which proves (5.5.8). We used the fact that $\int_{\mathbf{R}} e^{-\pi|x|^2}dx=1$.

The proof of (5.5.9) is almost identical. We normalize by assuming that

$$|w_1|^2+\cdots+|w_n|^2=1\,,$$

and we let ε_1 be the column vector of $\mathbf{C}^n$ having 1 in the first entry and zero elsewhere. We find a unitary $n\times n$ matrix $\mathscr{A}$ such that $\mathscr{A}^{-1}\varepsilon_1=(\overline{w_1},\dots,\overline{w_n})^t$. Unitary means $\mathscr{A}^{-1}=\mathscr{A}^*$, where $\mathscr{A}^*$ is the conjugate transpose matrix of $\mathscr{A}$, i.e., the matrix whose entries are the complex conjugates of $\mathscr{A}^t$ and that satisfies $u\cdot\overline{\mathscr{A}v}=\mathscr{A}^*u\cdot\overline{v}$ for all $u,v\in\mathbf{C}^n$. Then $(\mathscr{A}z)_1=w_1z_1+\cdots+w_nz_n$ and also $|\mathscr{A}z|=|z|$; therefore, changing variables $\zeta=\mathscr{A}z$ in the integral in (5.5.9), we can rewrite that integral as

$$\begin{aligned}
\left(\int_{\mathbf{C}^n}|\zeta_1|^r e^{-\pi|\zeta|^2}d\zeta\right)^{\frac{1}{r}} &= \left(\int_{\mathbf{C}}|\zeta_1|^r e^{-\pi|\zeta_1|^2}d\zeta_1\right)^{\frac{1}{r}}\\
&= \left(2\pi\int_0^\infty t^r e^{-\pi t^2}t\,dt\right)^{\frac{1}{r}}\\
&= \left(\pi\int_0^\infty s^{\frac{r}{2}} e^{-\pi s}ds\right)^{\frac{1}{r}}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\Gamma(\frac{r}{2}+1)}{\pi^{\frac{r}{2}}} \right)^{\frac{1}{r}} \\
&= B_r \,,
\end{aligned}
$$

where the second equality follows by polar coordinates and the third by the change of variables $s = t^2$. □

We now continue with the proof of Theorem 5.5.1.

Proof. The proof is based on conclusion (5.5.9) of Lemma 5.5.2.

Part (a): Assume first that $q < p$ and let B_r be as in (5.5.7). We may assume that the sequence $\{f_j\}_j$ is indexed by $\mathbf{Z}^+$. Use successively identity (5.5.9), the boundedness of T, Hölder's inequality with exponents p/q and $(p/q)'$ with respect to the measure $e^{-\pi|z|^2}dz$, and identity (5.5.9) again to deduce for $n \in \mathbf{Z}^+$

$$
\begin{aligned}
\Big\| \Big(\sum_{j=1}^n |T(f_j)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(Y)}^q &= (B_q)^{-q} \int_Y \int_{\mathbf{C}^n} |z_1 T(f_1) + \dots + z_n T(f_n)|^q e^{-\pi|z|^2} dz\, d\nu \\
&= (B_q)^{-q} \int_{\mathbf{C}^n} \int_Y |T(z_1 f_1 + \dots + z_n f_n)|^q \, d\nu \, e^{-\pi|z|^2} dz \\
&\le (B_q)^{-q} N^q \int_{\mathbf{C}^n} \Big(\int_X |z_1 f_1 + \dots + z_n f_n|^p \, d\mu \Big)^{\frac{q}{p}} e^{-\pi|z|^2} dz \\
&\le (B_q)^{-q} N^q \Big(\int_{\mathbf{C}^n} \int_X |z_1 f_1 + \dots + z_n f_n|^p \, d\mu \, e^{-\pi|z|^2} dz \Big)^{\frac{q}{p}} \\
&= (B_q)^{-q} N^q \Big(B_p^p \int_X \Big(\sum_{j=1}^n |f_j|^2 \Big)^{\frac{p}{2}} d\mu \Big)^{\frac{q}{p}} \\
&= (B_p B_q^{-1})^q N^q \Big\| \Big(\sum_{j=1}^n |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(X)}^q .
\end{aligned}
$$

Now, letting $n \to \infty$ in the previous inequality, we obtain the required conclusion with $C_{p,q} = B_p B_q^{-1}$ when $q < p$.

We now turn to the case $q \ge p$. Using similar reasoning, we obtain

$$
\begin{aligned}
\Big\| \Big(\sum_{j=1}^n |T(f_j)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(Y)}^q &= (B_q)^{-q} \int_Y \int_{\mathbf{C}^n} |z_1 T(f_1) + \dots + z_n T(f_n)|^q e^{-\pi|z|^2} dz\, d\nu \\
&= (B_q)^{-q} \int_{\mathbf{C}^n} \int_Y |T(z_1 f_1 + \dots + z_n f_n)|^q \, d\nu \, e^{-\pi|z|^2} dz \\
&\le (N B_q^{-1})^q \int_{\mathbf{C}^n} \Big(\int_X |z_1 f_1 + \dots + z_n f_n|^p \, d\mu \Big)^{\frac{q}{p}} e^{-\pi|z|^2} dz \\
&= (N B_q^{-1})^q \Big\| \int_X |z_1 f_1 + \dots + z_n f_n|^p \, d\mu \Big\|_{L^{\frac{q}{p}}(\mathbf{C}^n, e^{-\pi|z|^2} dz)}^{q/p} \\
&\le (N B_q^{-1})^q \Big\{ \int_X \Big\| |z_1 f_1 + \dots + z_n f_n|^p \Big\|_{L^{\frac{q}{p}}(\mathbf{C}^n, e^{-\pi|z|^2} dz)} d\mu \Big\}^{\frac{q}{p}}
\end{aligned}
$$

$$= (NB_q^{-1})^q \Big\{ \int_X \Big(\int_{\mathbf{C}^n} |z_1 f_1 + \cdots + z_n f_n|^q e^{-\pi |z|^2} dz \Big)^{\frac{p}{q}} d\mu \Big\}^{\frac{q}{p}}$$

$$= (NB_q^{-1})^q \Big\{ \int_X (B_q)^p \Big(\sum_{j=1}^n |f_j|^2 \Big)^{\frac{p}{2}} d\mu \Big\}^{\frac{q}{p}}$$

$$= N^q \Big\| \Big(\sum_{j=1}^n |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(X)}^q .$$

Note that we made use of Minkowski's integral inequality (Exercise 1.1.6) in the last inequality. Letting $n \to \infty$ proves the required conclusion with $C_{p,q} = 1$ if $q \ge p$.

If T happens to map real-valued functions to real-valued functions, then we adapt the preceding argument by taking f_j to be real-valued functions, we replace B_p by A_p, B_q by A_q, N by N_{real} and we use identity (5.5.8) instead of (5.5.9).

Part (b): Inequality (5.5.6) will be a consequence of (5.5.5) and of the following result of Exercise 1.1.12 which holds when Y is σ-finite:

$$\|g\|_{L^{q,\infty}} \le \sup_{0<\nu(E)<\infty} \nu(E)^{\frac{1}{q}-\frac{1}{r}} \Big(\int_E |g|^r \, d\nu \Big)^{\frac{1}{r}} \le \Big(\frac{q}{q-r} \Big)^{\frac{1}{r}} \|g\|_{L^{q,\infty}}, \tag{5.5.10}$$

where $0 < r < q$ and the supremum is taken over all subsets E of Y of finite measure. Using (5.5.10), we obtain

$$\Big\| \Big(\sum_j |T(f_j)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{q,\infty}(Y)}$$

$$\le \sup_{0<\nu(E)<\infty} \nu(E)^{\frac{1}{q}-\frac{1}{r}} \Big(\int_E \Big(\sum_j |T(f_j)|^2 \Big)^{\frac{r}{2}} d\nu \Big)^{\frac{1}{r}}$$

$$= \sup_{0<\nu(E)<\infty} \nu(E)^{\frac{1}{q}-\frac{1}{r}} \Big(\int_Y \Big(\sum_j |\chi_E T(f_j)|^2 \Big)^{\frac{r}{2}} d\nu \Big)^{\frac{1}{r}}$$

$$\le \sup_{0<\nu(E)<\infty} \nu(E)^{\frac{1}{q}-\frac{1}{r}} \big\| T_E \big\|_{L^p \to L^r} C_{p,r} \Big(\int_X \Big(\sum_j |f_j|^2 \Big)^{\frac{p}{2}} d\mu \Big)^{\frac{1}{p}}, \tag{5.5.11}$$

where T_E is defined by $T_E(f) = \chi_E T(f)$. Since for any function f in $L^p(X)$ we have

$$\nu(E)^{\frac{1}{q}-\frac{1}{r}} \big\| T_E(f) \big\|_{L^r} \le \Big(\frac{q}{q-r} \Big)^{\frac{1}{r}} \big\| T(f) \big\|_{L^{q,\infty}} \le \Big(\frac{q}{q-r} \Big)^{\frac{1}{r}} M \big\| f \big\|_{L^p},$$

it follows that for any measurable set E of finite measure the estimate

$$\nu(E)^{\frac{1}{q}-\frac{1}{r}} \big\| T_E \big\|_{L^p \to L^r} \le \Big(\frac{q}{q-r} \Big)^{\frac{1}{r}} M \tag{5.5.12}$$

is valid. Inserting (5.5.12) in (5.5.11), we obtain (5.5.6) with $D_{p,q} = C_{p,r} \big(\frac{q}{q-r} \big)^{\frac{1}{r}}$ and $0 < r < q$. Recall that $C_{p,r} = 1$ if $r \ge p$ and $C_{p,r} = B_p B_r^{-1}$ if $r < p$.

If T happens to map real-valued functions to real-valued functions, then we take f_j to be real-valued and we replace M by M_{real}, B_r by A_r, and B_q by A_q in the preceding argument. $\square$

5.5.2 Applications and ℓ^r-Valued Extensions of Linear Operators

Here is an application of Theorem 5.5.1:

Example 5.5.3. On the real line consider the intervals $I_j = [b_j, \infty)$ for $j \in \mathbf{Z}$. Let T_j be the operator given by multiplication on the Fourier transform by the characteristic function of I_j. Then we have the following two inequalities:

$$\Big\| \Big(\sum_{j \in \mathbf{Z}} |T_j(f_j)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbf{R})} \le C_p \Big\| \Big(\sum_{j \in \mathbf{Z}} |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbf{R})}, \tag{5.5.13}$$

$$\Big\| \Big(\sum_{j \in \mathbf{Z}} |T_j(f_j)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{1,\infty}(\mathbf{R})} \le C \Big\| \Big(\sum_{j \in \mathbf{Z}} |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^1(\mathbf{R})}, \tag{5.5.14}$$

for $1 < p < \infty$. To prove these, first observe that the operator $T = \frac{1}{2}(I + iH)$ is given on the Fourier transform by multiplication by the characteristic function of the half-axis $[0, \infty)$ [precisely, the Fourier multiplier of T is equal to 1 on the set $(0, \infty)$ and $1/2$ at the origin; this function is almost everywhere equal to the characteristic function of the half-axis $[0, \infty)$]. Moreover, each T_j is given by

$$T_j(f)(x) = e^{2\pi i b_j x} T(e^{-2\pi i b_j (\cdot)} f)(x)$$

and thus with $g_j(x) = e^{-2\pi i b_j x} f(x)$, (5.5.13) and (5.5.14) can be written respectively as

$$\Big\| \Big(\sum_{j \in \mathbf{Z}} |T(g_j)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbf{R})} \le C_p \Big\| \Big(\sum_{j \in \mathbf{Z}} |g_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbf{R})},$$

$$\Big\| \Big(\sum_{j \in \mathbf{Z}} |T(g_j)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{1,\infty}(\mathbf{R})} \le C \Big\| \Big(\sum_{j \in \mathbf{Z}} |g_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^1(\mathbf{R})}.$$

Theorem 5.5.1 gives that both of the previous estimates are valid by in view of the boundedness of $T = \frac{1}{2}(I + iH)$ from L^p to L^p and from $L^1 \to L^{1,\infty}$. For a slight generalization and an extension to higher dimensions, see Exercise 5.6.1.

We have now seen that bounded operators from L^p to L^q (or to $L^{q,\infty}$) always admit ℓ^2-valued extensions. It is natural to ask whether they also admit ℓ^r-valued extensions for some $r \neq 2$. For some values of r we may answer this question. Here is a straightforward corollary of Theorem 5.5.1.

Corollary 5.5.4. *Let (X,μ) and (Y,ν) be σ-finite measure spaces. Suppose that T is a linear bounded operator from $L^p(X)$ to $L^p(Y)$ with norm A for some $1<p<\infty$. Let r be a number between p and 2. Then we have*

$$\Big\|\Big(\sum_j |T(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^p(Y)} \le A\Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p(X)}. \tag{5.5.15}$$

Proof. Using Exercise 5.5.2 we interpolate between the trivial bound $L^p(X,\ell^p)\to L^p(Y,\ell^p)$ and the bound $L^p(X,\ell^2)\to L^p(Y,\ell^2)$, which follows from Theorem 5.5.1. We obtain the bound $L^p(X,\ell^r)\to L^p(Y,\ell^r)$ since r lies between p and 2. □

Example 5.5.5. The result of Corollary 5.5.4 may fail if r does not lie in the interval with endpoints p and 2. Let us take, for example, $1<p<2$ and consider an $r<p$. Take $X=Y=\mathbf{R}$ and define a linear operator T by setting

$$T(f)(x)=\widehat{f}(x)\chi_{[0,1]}(x).$$

Then T is L^p bounded, since $\|T(f)\|_{L^p}\le\|T(f)\|_{L^{p'}}\le\|f\|_{L^p}$. Now take $f_j=\chi_{[j-1,j]}$ for $j=1,\dots,N$. A simple calculation gives

$$\Big(\sum_{j=1}^N |T(f_j)(x)|^r\Big)^{\frac{1}{r}} = N^{\frac{1}{r}}\left|\frac{e^{-2\pi ix}-1}{-2\pi ix}\chi_{[0,1]}(x)\right|,$$

while

$$\Big(\sum_{j=1}^N |f_j|^r\Big)^{\frac{1}{r}} = \chi_{[0,N]}.$$

It follows that $N^{1/r}\le CN^{1/p}$ for all $N>1$, and hence (5.5.15) cannot hold if $p>r$.

We have now seen that ℓ^r-valued extensions for $r\neq 2$ may fail in general. But do they fail for some specific operators of interest in Fourier analysis? For instance, is the inequality

$$\Big\|\Big(\sum_{j\in\mathbf{Z}} |H(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^p} \le C_{p,r}\Big\|\Big(\sum_{j\in\mathbf{Z}} |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p} \tag{5.5.16}$$

true for the Hilbert transform H whenever $1<p,r<\infty$? The answer to this question is affirmative. Inequality (5.5.16) is indeed valid and was first proved using complex function theory. In the next section we plan to study inequalities such as (5.5.16) for general singular integrals using the Calderón–Zygmund theory of the previous section applied to the context of Banach-valued functions.

5.5.3 General Banach-Valued Extensions

Let $\mathscr{B}$ be a Banach space over the field of complex numbers with norm $\|\ \|_{\mathscr{B}}$, and let $\mathscr{B}^*$ be its dual (with norm $\|\ \|_{\mathscr{B}^*}$). A function F defined on a measure space (X,μ)

and taking values in $\mathscr{B}$ is called $\mathscr{B}$-measurable if there exists a measurable subset X_0 of X such that $\mu(X \setminus X_0) = 0$, $F[X_0]$ is contained in some separable subspace $\mathscr{B}_0$ of $\mathscr{B}$, and for every $u^* \in \mathscr{B}^*$ the complex-valued map

$$x \mapsto \langle u^*, F(x) \rangle$$

is measurable. A consequence of this definition is that the positive function $x \mapsto \|F(x)\|_{\mathscr{B}}$ on X is measurable; to see this, use the relevant result in [382, p. 131].

For $0 < p \le \infty$ we denote by $L^p(X)$ the space $L^p(X, \mathbf{C})$. Let $L^p(X) \otimes \mathscr{B}$ be the set of all finite linear combinations of elements of $\mathscr{B}$ with coefficients in $L^p(X)$, that is, elements of the form

$$F = f_1 u_1 + \cdots + f_m u_m, \tag{5.5.17}$$

where $f_j \in L^p(X)$, $u_j \in \mathscr{B}$, and $m \in \mathbf{Z}^+$. We define $L^p(X, \mathscr{B})$ to be the space of all $\mathscr{B}$-measurable functions F on X satisfying

$$\left(\int_X \|F(x)\|_{\mathscr{B}}^p \, d\mu(x) \right)^{\frac{1}{p}} < \infty, \tag{5.5.18}$$

with the obvious modification when $p = \infty$. Similarly define $L^{p,\infty}(X, \mathscr{B})$ as the space of all $\mathscr{B}$-measurable functions F on X satisfying

$$\Big\| \|F(\cdot)\|_{\mathscr{B}} \Big\|_{L^{p,\infty}(X)} < \infty. \tag{5.5.19}$$

Then $L^p(X, \mathscr{B})$ (respectively, $L^{p,\infty}(X, \mathscr{B})$) is called the L^p (respectively, $L^{p,\infty}$) space of functions on X with values in $\mathscr{B}$. The quantity in (5.5.18) (respectively, in (5.5.19)) is the *norm* of F in $L^p(X, \mathscr{B})$ (respectively, in $L^{p,\infty}(X, \mathscr{B})$).

Proposition 5.5.6. *Let $\mathscr{B}$ be a Banach space and (X, μ) a σ-finite measure space.*
(a) The set $\{\sum_{j=1}^m \chi_{E_j} u_j : u_j \in \mathscr{B}, E_j \subseteq X$ are pairwise disjoint and $\mu(E_j) < \infty\}$ *is dense in $L^p(X, \mathscr{B})$ whenever $0 < p < \infty$.*
(b) The set $\{\sum_{j=0}^\infty \chi_{E_j} u_j : u_j \in \mathscr{B}, E_j \subseteq X$ are pairwise disjoint and $X = \cup_{j=0}^\infty E_j\}$ *is dense in $L^\infty(X, \mathscr{B})$.*
(c) The space $\mathscr{C}_0^\infty \otimes \mathscr{B}$ of functions of the form $\sum_{j=1}^m \varphi_j u_j$, where $u_j \in \mathscr{B}$, φ_j are in $\mathscr{C}_0^\infty(\mathbf{R}^n)$, is dense in $L^p(\mathbf{R}^n, \mathscr{B})$ for $1 \le p < \infty$.

Proof. If $F \in L^p(X, \mathscr{B})$ for $0 < p \le \infty$, then F is $\mathscr{B}$-measurable; thus there exists $X_0 \subseteq X$ satisfying $\mu(X \setminus X_0) = 0$ and $F[X_0] \subseteq \mathscr{B}_0$, where $\mathscr{B}_0$ is some separable subspace of $\mathscr{B}$. Choose a countable dense sequence $\{u_j\}_{j=1}^\infty$ of $\mathscr{B}_0$.

(a) First assume that $p < \infty$. Since X is σ-finite, for any $\varepsilon > 0$, there exists a measurable subset X_1 of X_0 with $\mu(X_1) < \infty$ such that

$$\int_{X \setminus X_1} \|F(x)\|_{\mathscr{B}}^p \, d\mu < \frac{\varepsilon^p}{3}.$$

Setting

$$\widetilde{B}(u_j, \varepsilon) = \left\{ u \in \mathscr{B}_0 : \|u - u_j\|_{\mathscr{B}} < \varepsilon (3\mu(X_1))^{-\frac{1}{p}} \right\},$$

we have $\mathscr{B}_0 \subsetneqq \bigcup_{j=1}^{\infty} \widetilde{B}(u_j,\varepsilon)$. Let $A_1 = \widetilde{B}(u_1,\varepsilon)$ and $A_j = \widetilde{B}(u_j,\varepsilon) \setminus (\bigcup_{i=1}^{j-1} \widetilde{B}(u_i,\varepsilon))$ for $j \geq 2$. It is easily seen that $\{A_j\}_{j=1}^{\infty}$ are pairwise disjoint and $\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} \widetilde{B}(u_j,\varepsilon)$. Set $E_j = F^{-1}[A_j] \cap X_1$. Then $X_1 = \bigcup_{j=1}^{\infty} E_j$ and $\{E_j\}_{j=1}^{\infty}$ are pairwise disjoint. Since $\mu(X_1) = \sum_{j=1}^{\infty} \mu(E_j) < \infty$, it follows that $\mu(E_j) < \infty$ and also that for some $m \in \mathbf{Z}^+$,

$$\int_{\bigcup_{j=m+1}^{\infty} E_j} \|F(x)\|_{\mathscr{B}}^p \, d\mu < \frac{\varepsilon^p}{3}. \tag{5.5.20}$$

Moreover, one can easily verify that $\sum_{j=1}^{m} \chi_{E_j} u_j$ is $\mathscr{B}$-measurable. Notice that $\|F(x) - u_j\|_{\mathscr{B}} < \varepsilon(3\mu(X_1))^{-1/p}$ for any $x \in E_j$ and $j \in \{1,\dots,m\}$. This fact combined with (5.5.20) and the mutual disjointness of $\{E_j\}_{j=1}^{m}$ yields that

$$\begin{aligned}
\int_X \Big\| F(x) - \sum_{j=1}^{m} \chi_{E_j}(x) u_j \Big\|_{\mathscr{B}}^p d\mu &= \int_{X \setminus X_1} \|F(x)\|_{\mathscr{B}}^p \, d\mu + \int_{\bigcup_{j=m+1}^{\infty} E_j} \|F(x)\|_{\mathscr{B}}^p \, d\mu \\
&\quad + \int_{\bigcup_{j=1}^{m} E_j} \Big\| \sum_{j=1}^{m} \chi_{E_j}(x)[F(x) - u_j] \Big\|_{\mathscr{B}}^p d\mu \\
&< \frac{\varepsilon^p}{3} + \frac{\varepsilon^p}{3} + \frac{\varepsilon^p}{3} = \varepsilon^p.
\end{aligned}$$

(b) Now consider the case $p = \infty$. Obviously we have $\mathscr{B}_0 \subsetneqq \bigcup_{j=1}^{\infty} B(u_j,\varepsilon)$, where $B(u_j,\varepsilon) = \{u \in \mathscr{B}_0 : \|u - u_j\|_{\mathscr{B}} < \varepsilon\}$. Let $A_1 = B(u_1,\varepsilon)$ and for $j \geq 2$ define sets $A_j = B(u_j,\varepsilon) \setminus (\bigcup_{i=1}^{j-1} B(u_i,\varepsilon))$. Let $E_j = F^{-1}[A_j]$ for $j \geq 1$ and $E_0 = X \setminus (\bigcup_{j=1}^{\infty} E_j)$. Then $\mu(E_0) = 0$. As in the proof of the case $p < \infty$, we have that $\{E_j\}_{j=0}^{\infty}$ are pairwise disjoint and $X_0 \subsetneqq \bigcup_{j=0}^{\infty} E_j$. Pick $u_0 = 0$. Notice that $\sum_{j=0}^{\infty} \chi_{E_j} u_j$ is $\mathscr{B}$-measurable. Since $\|F(x) - u_j\|_{\mathscr{B}} < \varepsilon$ for any $x \in E_j$ and $j \geq 0$, we have

$$\Big\| F - \sum_{j=0}^{\infty} \chi_{E_j} u_j \Big\|_{L^\infty(X,\mathscr{B})} = \Big\| \sum_{j=0}^{\infty} \chi_{E_j}(F - u_j) \Big\|_{L^\infty(X,\mathscr{B})} < \varepsilon,$$

which completes the proof in the case $p = \infty$.

(c) For the last assertion, we fix a smooth function with supported in the unit ball of $\mathbf{R}^n$ with integral one. Let $\varphi_\delta(x) = \delta^{-n}\varphi(x/\delta)$ for $x \in \mathbf{R}^n$ and $\delta > 0$. Given a function $\sum_{j=1}^{m} \chi_{E_j} u_j$ as in part (a) approximating a given f in $L^p \otimes \mathscr{B}$, we consider the function $\sum_{j=1}^{m} (\chi_{E_j} * \varphi_\delta) u_j$ which lies in $\mathscr{C}_0^\infty \otimes \mathscr{B}$. Since $\|\chi_{E_j} * \varphi_\delta - \chi_{E_j}\|_{L^p} \to 0$ as $\delta \to 0$ when $1 \leq p < \infty$, $\sum_{j=1}^{m} (\chi_{E_j} * \varphi_\delta) u_j$ tends to $\sum_{j=1}^{m} \chi_{E_j} u_j$ in $L^p \otimes \mathscr{B}$ as $\delta \to 0$, and the conclusion follows. $\square$

Let (X,μ) be a measure space. If F is an element of $L^1(X) \otimes \mathscr{B}$ given as in (5.5.17), we define its integral (which is an element of $\mathscr{B}$) by setting

$$\int_X F(x) \, d\mu(x) = \sum_{j=1}^{m} \left(\int_X f_j(x) \, d\mu(x) \right) u_j.$$

Observe that for every $F\in L^1(X)\otimes\mathscr{B}$ we have

$$\begin{aligned}\Big\|\int_X F(x)\,d\mu(x)\Big\|_{\mathscr{B}} &= \sup_{\|u^*\|_{\mathscr{B}^*}\le 1}\Big|\Big\langle u^*,\sum_{j=1}^m\Big(\int_X f_j\,d\mu\Big)u_j\Big\rangle\Big|\\ &= \sup_{\|u^*\|_{\mathscr{B}^*}\le 1}\Big|\int_X\big\langle u^*,\sum_{j=1}^m f_ju_j\big\rangle\,d\mu\Big|\\ &\le \int_X\sup_{\|u^*\|_{\mathscr{B}^*}\le 1}\big|\big\langle u^*,\sum_{j=1}^m f_ju_j\big\rangle\big|\,d\mu\\ &= \big\|F\big\|_{L^1(X,\mathscr{B})}\,.\end{aligned}$$

Thus the linear operator

$$F\mapsto I_F=\int_X F(x)\,d\mu(x)$$

is bounded from $L^1(X)\otimes\mathscr{B}$ into $\mathscr{B}$. Since every element of $L^1(X,\mathscr{B})$ is a (norm) limit (Proposition 5.5.6 (c)) of a sequence of elements in $L^1(X)\otimes\mathscr{B}$, by continuity, the operator $F\mapsto I_F$ has a unique extension on $L^1(X,\mathscr{B})$ that we call the *Bochner integral* of F and denote by

$$\int_X F(x)\,d\mu(x)\,.$$

$L^1(X,\mathscr{B})$ is called the space of all Bochner integrable functions from X to $\mathscr{B}$. Since the Bochner integral is an extension of I_F, for each $F\in L^1(X,\mathscr{B})$ we have

$$\Big\|\int_X F(x)\,dx\Big\|_{\mathscr{B}}\le\int_X\|F(x)\|_{\mathscr{B}}\,dx\,.$$

Consequently, measurable functions F with $\int_X\|F(x)\|_{\mathscr{B}}\,dx<\infty$ are Bochner integrable over X. It is not difficult to show that the Bochner integral of F is the only element of $\mathscr{B}$ that satisfies

$$\Big\langle u^*,\int_X F(x)\,d\mu(x)\Big\rangle=\int_X\big\langle u^*,F(x)\big\rangle\,d\mu(x) \tag{5.5.21}$$

for all $u^*\in\mathscr{B}^*$. The next result concerns duality in this context when $X=\mathbf{R}^n$.

Proposition 5.5.7. *Let $\mathscr{B}$ be a Banach space and $1\le p\le\infty$.*
(a) For any $F\in L^p(\mathbf{R}^n,\mathscr{B})$ we have

$$\|F\|_{L^p(\mathbf{R}^n,\mathscr{B})}=\sup_{\|G\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}^*)}\le 1}\Big|\int_{\mathbf{R}^n}\langle G(x),F(x)\rangle\,dx\Big|\,.$$

Consequently, $L^p(\mathbf{R}^n,\mathscr{B})$ isometrically embeds in $(L^{p'}(\mathbf{R}^n,\mathscr{B}^*))^*$.
(b) for any $G\in L^{p'}(\mathbf{R}^n,\mathscr{B}^*)$ one has

$$\|G\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}^*)}=\sup_{\|F\|_{L^p(\mathbf{R}^n,\mathscr{B})}\le 1}\Big|\int_{\mathbf{R}^n}\langle G(x),F(x)\rangle\,dx\Big|$$

and thus $L^{p'}(\mathbf{R}^n,\mathscr{B}^*)$ isometrically embeds in $(L^p(\mathbf{R}^n,\mathscr{B}))^*$.

Proof. Hölder's inequality yields that the right-hand side of (a) is controlled by its left-hand side. It remains to establish the reverse inequality.

For $F \in L^p(\mathbf{R}^n, \mathscr{B})$ and $\varepsilon > 0$, by Proposition 5.5.6, there is $F_\varepsilon(x) = \sum_{j=1}^m \chi_{E_j}(x) u_j$ with $m \in \mathbf{Z}^+$ or $m = \infty$ (when $p = \infty$) such that $\|F_\varepsilon - F\|_{L^p(\mathbf{R}^n,\mathscr{B})} < \varepsilon/2$, where $\{E_j\}_{j=1}^m$ are pairwise disjoint subsets of $\mathbf{R}^n$ and $u_j \in \mathscr{B}$. Since $F_\varepsilon \in L^p(\mathbf{R}^n, \mathscr{B})$, we choose a nonnegative function h satisfying $\|h\|_{L^{p'}(\mathbf{R}^n)} \le 1$ such that

$$\big\|F_\varepsilon\big\|_{L^p(\mathbf{R}^n,\mathscr{B})} = \bigg(\int_{\mathbf{R}^n} \big\|F_\varepsilon(x)\big\|_{\mathscr{B}}^p \, dx\bigg)^{\frac{1}{p}} < \int_{\mathbf{R}^n} h(x) \big\|F_\varepsilon(x)\big\|_{\mathscr{B}} \, dx + \frac{\varepsilon}{4}. \tag{5.5.22}$$

When $1 \le p < \infty$, we can further choose $h \in L^{p'}(\mathbf{R}^n)$ to be a function with bounded support, which ensures that it is integrable. For given $u_j \in \mathscr{B}$, there exists $u_j^* \in \mathscr{B}^*$ satisfying $\|u_j^*\|_{\mathscr{B}^*} = 1$ and

$$\|u_j\|_{\mathscr{B}} < \langle u_j^*, u_j\rangle + \frac{\varepsilon}{4(\|h\|_{L^1(\mathbf{R}^n)} + 1)}. \tag{5.5.23}$$

Set $G(x) = \sum_{j=1}^m h(x)\chi_{E_j}(x) u_j^*$. Clearly G is $\mathscr{B}^*$-measurable and $\|G\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}^*)} \le 1$. It follows from (5.5.22) and (5.5.23) that

$$\begin{aligned}
\int_{\mathbf{R}^n} \langle G(x), F_\varepsilon(x)\rangle \, dx &= \int_{\mathbf{R}^n} h(x) \sum_{j=1}^m \chi_{E_j}(x) \langle u_j^*, u_j\rangle \, dx \\
&\ge \int_{\mathbf{R}^n} h(x) \sum_{j=1}^m \bigg(\|u_j\|_{\mathscr{B}} - \frac{\varepsilon}{4(\|h\|_{L^1(\mathbf{R}^n)} + 1)} \bigg) \chi_{E_j}(x) \, dx \\
&\ge \big\|F_\varepsilon\big\|_{L^p(\mathbf{R}^n,\mathscr{B})} - \frac{\varepsilon}{4} - \frac{\varepsilon}{4}.
\end{aligned}$$

Using Hölder's inequality we have

$$\bigg| \int_{\mathbf{R}^n} \langle G(x), F_\varepsilon(x) - F(x)\rangle \, dx \bigg| \le \|G\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}^*)} \|F - F_\varepsilon\|_{L^p(\mathbf{R}^n,\mathscr{B})} < \frac{\varepsilon}{2},$$

hence we obtain

$$\big\|F_\varepsilon\big\|_{L^p(\mathbf{R}^n,\mathscr{B})} \le \sup_{\|G\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}^*)} \le 1} \bigg| \int_{\mathbf{R}^n} \langle G(x), F(x)\rangle \, dx \bigg| + \varepsilon.$$

Letting $\varepsilon \to 0$ yields the desired inequality in part (a).

(b) The duality statement $|\langle G(x), F(x)\rangle| \le \|G(x)\|_{\mathscr{B}^*} \|F(x)\|_{\mathscr{B}}$ together with Hölder's inequality imply the $\ge$ inequality in part (b). We prove the $\le$ inequality via an argument symmetric with that in case (a). For completeness we include the details. For a given G in $L^{p'}(\mathbf{R}^n, \mathscr{B})$ and $\varepsilon > 0$, by Proposition 5.5.6, there is $G_\varepsilon(x) = \sum_{j=1}^m \chi_{E_j}(x) u_j^*$ with $m \in \mathbf{Z}^+$ or $m = \infty$ (when $p = 1$) such that $\|G_\varepsilon - G\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}^*)} < \varepsilon/2$, where $\{E_j\}_{j=1}^m$ are pairwise disjoint subsets of $\mathbf{R}^n$ and

$u_j^* \in \mathscr{B}^*$. Since G_ε lies in $L^{p'}(\mathbf{R}^n, \mathscr{B}^*)$, we choose a nonnegative function h satisfying $\|h\|_{L^p(\mathbf{R}^n)} \le 1$ such that

$$\big\|G_\varepsilon\big\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}^*)} = \left(\int_{\mathbf{R}^n} \|G_\varepsilon(x)\|_{\mathscr{B}^*}^{p'}\,dx\right)^{\frac{1}{p'}} < \int_{\mathbf{R}^n} h(x)\big\|G_\varepsilon(x)\big\|_{\mathscr{B}^*}\,dx + \frac{\varepsilon}{4}. \quad (5.5.24)$$

When $1 < p \le \infty$, we can further choose $h \in L^p(\mathbf{R}^n)$ to be a function with bounded support, which ensures that it is integrable. For each $u_j^* \in \mathscr{B}$, there exists $u_j \in \mathscr{B}$ satisfying $\|u_j\|_{\mathscr{B}} = 1$ and

$$\|u_j^*\|_{\mathscr{B}^*} < \langle u_j^*, u_j\rangle + \frac{\varepsilon}{4(\|h\|_{L^1(\mathbf{R}^n)}+1)}. \quad (5.5.25)$$

Set $F(x) = \sum_{j=1}^m h(x)\chi_{E_j}(x)u_j$. Clearly F is $\mathscr{B}$-measurable and $\|F\|_{L^p(\mathbf{R}^n,\mathscr{B})} \le 1$. It follows from (5.5.24) and (5.5.25) that

$$\begin{aligned}\int_{\mathbf{R}^n} \langle G_\varepsilon(x), F(x)\rangle\,dx &= \int_{\mathbf{R}^n} h(x)\sum_{j=1}^m \chi_{E_j}(x)\langle u_j^*, u_j\rangle\,dx \\ &\ge \int_{\mathbf{R}^n} h(x)\sum_{j=1}^m \left(\|u_j^*\|_{\mathscr{B}^*} - \frac{\varepsilon}{4(\|h\|_{L^1(\mathbf{R}^n)}+1)}\right)\chi_{E_j}(x)\,dx \\ &\ge \big\|G_\varepsilon\big\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}^*)} - \frac{\varepsilon}{2}.\end{aligned}$$

Hence, for any $\varepsilon > 0$, we have

$$\big\|G_\varepsilon\big\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}^*)} \le \sup_{\|F\|_{L^p(\mathbf{R}^n,\mathscr{B})}\le 1} \left|\int_{\mathbf{R}^n} \langle G(x), F(x)\rangle\,dx\right| + \varepsilon$$

which implies the reverse inequality in part (b) by letting $\varepsilon \to 0$. □

Definition 5.5.8. Let X, Y be measure spaces. Let T be a linear operator that maps $L^p(X)$ to $L^q(Y)$ (respectively, $L^p(X)$ to $L^{q,\infty}(Y)$) for some $0 < p, q \le \infty$. We define another operator $\vec{T}$ acting on $L^p(X)\otimes\mathscr{B}$ by setting

$$\vec{T}\Big(\sum_{j=1}^m f_j u_j\Big) = \sum_{j=1}^m T(f_j)\,u_j.$$

If $\vec{T}$ happens to have a bounded extension from $L^p(X,\mathscr{B})$ to $L^q(Y,\mathscr{B})$ (respectively from $L^p(X,\mathscr{B})$ to $L^{q,\infty}(Y,\mathscr{B})$), then we say that T has a bounded *$\mathscr{B}$-valued extension*. In this case we also denote by $\vec{T}$ the $\mathscr{B}$-valued extension of T.

Example 5.5.9. Let $\mathscr{B} = \ell^r$ for some $1 \le r < \infty$. Then a measurable function $F : X \to \mathscr{B}$ is just a sequence $\{f_j\}_j$ of measurable functions $f_j : X \to \mathbf{C}$. The space $L^p(X, \ell^r)$ consists of all measurable complex-valued sequences $\{f_j\}_j$ on X that satisfy

$$\big\|\{f_j\}_j\big\|_{L^p(X,\ell^r)} = \Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p(X)} < \infty.$$

The space $L^p(X)\otimes\ell^r$ is the set of all finite sums

$$\sum_{j=1}^{m}(a_{j1},a_{j2},a_{j3},\dots)\,g_j\,,$$

where $g_j\in L^p(X)$ and $(a_{j1},a_{j2},a_{j3},\dots)\in\ell^r$, $j=1,\dots,m$. This is certainly a subspace of $L^p(X,\ell^r)$. Now given $(f_1,f_2,\dots)\in L^p(X,\ell^r)$, let $F_m=e_1f_1+\dots+e_mf_m$, where e_j is the infinite sequence with zeros everywhere except at the jth entry, where it has 1. Then $F_m\in L^p(X)\otimes\ell^r$ and approximates f in the norm of $L^p(X,\ell^r)$. This shows the density of $L^p(X)\otimes\ell^r$ in $L^p(X,\ell^r)$.

If T is a linear operator bounded from $L^p(X)$ to $L^q(Y)$, then $\vec{T}$ is defined by

$$\vec{T}(\{f_j\}_j)=\{T(f_j)\}_j.$$

According to Definition 5.5.8, T has a bounded ℓ^r-extension if and only if the inequality

$$\Big\|\Big(\sum_j|T(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^q}\le C\Big\|\Big(\sum_j|f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p}$$

is valid.

A linear operator T acting on measurable functions is called *positive* if it satisfies $f\ge 0 \implies T(f)\ge 0$. It is straightforward to verify that positive operators satisfy

$$\begin{aligned}
&f\le g \implies T(f)\le T(g)\,,\\
&|T(f)|\le T(|f|)\,,\\
&\sup_j|T(f_j)|\le T\big(\sup_j|f_j|\big)\,,
\end{aligned}\tag{5.5.26}$$

for all f,g,f_j measurable functions. We have the following result regarding vector-valued extensions of positive operators:

Proposition 5.5.10. *Let $0<p,q\le\infty$ and (X,μ), (Y,ν) be two measure spaces. Let T be a positive linear operator mapping $L^p(X)$ to $L^q(Y)$ (respectively, to $L^{q,\infty}(Y)$) with norm A. Let $\mathscr{B}$ be a Banach space. Then T has a $\mathscr{B}$-valued extension $\vec{T}$ that maps $L^p(X,\mathscr{B})$ to $L^q(Y,\mathscr{B})$ (respectively, to $L^{q,\infty}(Y,\mathscr{B})$) with the same norm.*

Proof. Let us first understand this theorem when $\mathscr{B}=\ell^r$ for $1\le r\le\infty$. The two endpoint cases $r=1$ and $r=\infty$ can be checked easily using the properties in (5.5.26). For instance, for $r=1$ we have

$$\Big\|\sum_j|T(f_j)|\Big\|_{L^q}\le\Big\|\sum_j T(|f_j|)\Big\|_{L^q}=\Big\|T\Big(\sum_j|f_j|\Big)\Big\|_{L^q}\le A\Big\|\sum_j|f_j|\Big\|_{L^p}\,,$$

while for $r=\infty$ we have

$$\Big\|\sup_j|T(f_j)|\Big\|_{L^q}\le\Big\|T(\sup_j|f_j|)\Big\|_{L^q}\le A\,\big\|\sup_j|f_j|\big\|_{L^p}\,.$$

The required inequality for $1 < r < \infty$,

$$\Big\| \Big(\sum_j |T(f_j)|^r \Big)^{\frac{1}{r}} \Big\|_{L^q} \le A \Big\| \Big(\sum_j |f_j|^r \Big)^{\frac{1}{r}} \Big\|_{L^p},$$

follows from the Riesz–Thorin interpolation theorem (see Exercise 5.5.2).

The result for a general Banach space $\mathscr{B}$ can be proved using the following inequality:

$$\big\| \vec{T}(F)(x) \big\|_{\mathscr{B}} \le T\big(\|F\|_{\mathscr{B}} \big)(x), \qquad x \in X, \tag{5.5.27}$$

by simply taking L^q norms. To prove (5.5.27), let us take $F = \sum_{j=1}^n f_j u_j$. Then

$$\begin{aligned} \big\| \vec{T}(F)(x) \big\|_{\mathscr{B}} = \Big\| \sum_{j=1}^n T(f_j)(x) u_j \Big\|_{\mathscr{B}} &= \sup_{\|u^*\|_{\mathscr{B}^*} \le 1} \Big| \Big\langle u^*, \sum_{j=1}^n T(f_j)(x) u_j \Big\rangle \Big| \\ &= \sup_{\|u^*\|_{\mathscr{B}^*} \le 1} \Big| T\Big(\sum_{j=1}^n f_j \langle u^*, u_j \rangle \Big)(x) \Big| \\ &\le T\Big(\sup_{\|u^*\|_{\mathscr{B}^*} \le 1} \Big| \Big\langle u^*, \sum_{j=1}^n f_j u_j \Big\rangle \Big| \Big)(x) \\ &= T\Big(\Big\| \sum_{j=1}^n f_j u_j \Big\|_{\mathscr{B}} \Big)(x) \\ &= T\big(\|F\|_{\mathscr{B}} \big)(x), \end{aligned}$$

where the inequality makes use of the fact that T is a positive operator. □

Exercises

5.5.1. ([207]) Let (X,μ) and (Y,ν) be σ-finite measure spaces and suppose that $0 < p_1, p_1 \le \infty$, $1 \le q_0, q_1 \le \infty$, and that $p_0 > p_1$. For $0 < \theta < 1$ define p, q by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be Banach spaces and let $\vec{T}$ be a linear operator that maps $L^{p_0}(X,\mathscr{B}_1)$ to $L^{q_0}(Y,\mathscr{B}_2)$ with norm A_0 and $L^{p_1}(X,\mathscr{B}_1)$ to $L^{q_1}(Y,\mathscr{B}_2)$ with norm A_1. Show that $\vec{T}$ has an extension that maps $L^p(X,\mathscr{B}_1)$ to $L^q(Y,\mathscr{B}_2)$ with norm at most $9A_0^{1-\theta}A_1^{\theta}$, by following the steps below:
(a) Let $i \in \{0,1\}$. If F_j, $j = 1,\dots,m$ are in $L^{p_i}(\mathscr{B}_1)$ with disjoint supports, using $\max_{j\in\{1,\dots,m\}} \|\vec{T}(F_j)(y)\|_{\mathscr{B}_2} \le \frac{1}{2^m} \sum_{\varepsilon_k=\pm 1} \big\| \sum_{k=1}^m \varepsilon_k \vec{T}(F_k)(y) \big\|_{\mathscr{B}_2}$ for $y \in Y$ show that

$$\Big\| \max_{j=1,\dots,m} \|\vec{T}(F_j)\|_{\mathscr{B}_2} \Big\|_{L^{q_i}} \le A_i \Big\| \sum_{j=1}^m F_j \Big\|_{L^{p_i}(X,\mathscr{B}_1)}.$$

(b) Assume that F lies in a dense subspace of $\mathscr{B}_1$, it satisfies $\|\|F\|_{\mathscr{B}_1}\|_{L^p} = 1$, and it takes only finitely many values. For a $\lambda > 1$ pick a large integer N such that $\lambda^{-N} < \|F(x)\|_{\mathscr{B}_1} \le \lambda^N$ for all $x \in X$ such that $\|F(x)\|_{\mathscr{B}_1} \neq 0$ and define $F_j = F\chi_{\Omega_j}$, where $\Omega_j = \{x : 2^j < \|F\|_{\mathscr{B}_1} \le 2^{j+1}\}$. Let $a = \frac{p}{p_1} - \frac{p}{p_0}$. Prove the inequalities

$$\Big\|\sum_j \lambda^{-ja\theta} F_j\Big\|_{L^{p_0}(X,\mathscr{B}_1)}^{p_0} \le \lambda^{a\theta p_0} \qquad \text{and} \qquad \Big\|\sum_j \lambda^{ja\theta} F_j\Big\|_{L^{p_1}(X,\mathscr{B}_1)}^{p_1} \le 1\,.$$

(c) Define $g_0(y) = \max_j \lambda^{-ja\theta} \|\vec{T}(F_j)(y)\|_{\mathscr{B}_2}$, $g_1(y) = \max_j \lambda^{ja(1-\theta)} \|\vec{T}(F_j)(y)\|_{\mathscr{B}_2}$ for $y \in Y$ and show that

$$\|g_0\|_{L^{q_0}(Y)} \le A_0\,\lambda^{a\theta} \qquad \text{and} \qquad \|g_1\|_{L^{q_1}(Y)} \le A_1\,.$$

(d) Prove that for all $y \in Y$ we have

$$\|\vec{T}(F)(y)\|_{\mathscr{B}_2} \le \sum_j \|\vec{T}(F_j)(y)\|_{\mathscr{B}_2} \le g_0(y)^{1-\theta} g_1(y)^{\theta}\bigg(2 + \frac{1}{\lambda^{a\theta}-1} + \frac{1}{\lambda^{a(1-\theta)}-1}\bigg)$$

and conclude that $\|\vec{T}(F)\|_{L^p(Y,\mathscr{B}_2)} \le 9A_0^{1-\theta}A_1^{\theta}$ by picking $\lambda = (1+\sqrt{2})^{a\theta}$.
$\big[$*Hint:* Part (d): Split the sum according to whether $\lambda^{ja} > \frac{g_1(y)}{g_0(y)}$ and $\lambda^{ja} \le \frac{g_1(y)}{g_0(y)}$.$\big]$

5.5.2. Prove the following version of the Riesz–Thorin interpolation theorem. Let (X,μ) and (Y,ν) be σ-finite measure spaces. Let $1 < p_0, q_0, , p_1, q_1, r_0, s_0, r_1, s_1 < \infty$ and $0 < \theta < 1$ satisfy

$$\begin{aligned} \frac{1-\theta}{p_0} + \frac{\theta}{p_1} &= \frac{1}{p}\,, & \qquad \frac{1-\theta}{q_0} + \frac{\theta}{q_1} &= \frac{1}{q}\,, \\ \frac{1-\theta}{r_0} + \frac{\theta}{r_1} &= \frac{1}{r}\,, & \qquad \frac{1-\theta}{s_0} + \frac{\theta}{s_1} &= \frac{1}{s}\,. \end{aligned}$$

Suppose that T is a linear operator that maps $L^{p_0}(X)$ to $L^{q_0}(Y)$ and $L^{p_1}(X)$ to $L^{q_1}(Y)$. Define a vector-valued operator $\vec{T}$ by setting $\vec{T}(\{f_j\}_j) = \{T(f_j)\}_j$ acting on sequences of complex-valued functions defined on X. Suppose that $\vec{T}$ maps $L^{p_0}(X,\ell^{r_0}(\mathbf{C}))$ to $L^{q_0}(Y,\ell^{s_0}(\mathbf{C}))$ with norm M_0 and $L^{p_1}(X,\ell^{r_1}(\mathbf{C}))$ to $L^{q_1}(Y,\ell^{s_1}(\mathbf{C}))$ with norm M_1. Prove that $\vec{T}$ maps $L^p(X,\ell^r(\mathbf{C}))$ to $L^q(Y,\ell^s(\mathbf{C}))$ with norm at most $M_0^{1-\theta}M_1^{\theta}$.
$\big[$*Hint:* Use the idea of the proof of Theorem 1.3.4. Apply Lemma 1.3.5 to the function

$$F(z) = \sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{l\in\mathbf{Z}} \frac{a_{k,l}^{P(z)}\,e^{i\alpha_{k,l}}}{\|\{a_{k,l}\}_l\|_{\ell^r}^{R(z)-P(z)}}\,\frac{b_{j,l}^{Q(z)}\,e^{i\beta_{j,l}}}{\|\{b_{k,l}\}_l\|_{\ell^{s'}}^{S(z)-Q(z)}}\int_Y T(\chi_{A_k})(y)\,\chi_{B_j}(y)\,d\nu(y)$$

where $P(z) = \frac{p}{p_0}(1-z) + \frac{p}{p_1}z$, $Q(z) = \frac{q'}{q'_0}(1-z) + \frac{q'}{q'_1}z$, $R(z) = \frac{r}{r_0}(1-z) + \frac{r}{r_1}z$, and $S(z) = \frac{s'}{s'_0}(1-z) + \frac{s'}{s'_1}z$, $\{a_{k,l}\}_{l\in\mathbf{Z}}, \{b_{j,l}\}_{l\in\mathbf{Z}}$ are finitely supported sequences of positive reals, $\alpha_{k,l}, \beta_{k,l} \in \mathbf{R}$, A_k are pairwise disjoint subsets of X with finite measure, and B_j are pairwise disjoint subsets of Y with finite measure.$\big]$

5.5.3. Prove the following version of the Marcinkiewicz interpolation theorem. Let (X,μ) and (Y,ν) be σ-finite measure spaces and let $0 < p_0 < p < p_1 \le \infty$ and $0 < \theta < 1$ satisfy

$$\frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{p}.$$

Let $\mathscr{B}_1, \mathscr{B}_2$ be Banach spaces and $\vec{T}$ be defined on $L^{p_0}(X,\mathscr{B}_1) + L^{p_1}(X,\mathscr{B}_1)$ such that for every $y \in Y$ and for all $F, G \in L^{p_0}(X,\mathscr{B}_1) + L^{p_1}(X,\mathscr{B}_1)$ we have

$$\big\|\vec{T}(F+G)(y)\big\|_{\mathscr{B}_2} \le \big\|\vec{T}(F)(y)\big\|_{\mathscr{B}_2} + \big\|\vec{T}(G)(y)\big\|_{\mathscr{B}_2}.$$

(a) Suppose that $\vec{T}$ maps $L^{p_0}(X,\mathscr{B}_1)$ to $L^{p_0,\infty}(Y,\mathscr{B}_2)$ with norm A_0 and $L^{p_1}(X,\mathscr{B}_1)$ to $L^{p_1,\infty}(Y,\mathscr{B}_2)$ with norm A_1. Show that $\vec{T}$ maps $L^p(X,\mathscr{B}_1)$ to $L^p(Y,\mathscr{B}_2)$ with norm at most $2\big(\frac{p}{p-p_0} + \frac{p}{p_1-p}\big)^{\frac{1}{p}} A_0^{1-\theta} A_1^{\theta}$.
(b) Let $p_0 = 1$. If $\vec{T}$ is linear and maps $L^1(X,\mathscr{B}_1)$ to $L^{1,\infty}(Y,\mathscr{B}_2)$ with norm A_0 and $L^{p_1}(X,\mathscr{B}_1)$ to $L^{p_1}(Y,\mathscr{B}_2)$ with norm A_1, show that $\vec{T}$ maps $L^p(X,\mathscr{B}_1)$ to $L^p(Y,\mathscr{B}_2)$ with norm at most $72\big(p-1\big)^{-1/p} A_0^{1-\theta} A_1^{\theta}$.
$\big[$*Hint:* Part (a): Copy the proof of Theorem 1.3.2. Part (b): Use Exercise 5.5.1.$\big]$

5.5.4. For all $x \in \mathbf{R}^n$ let $\vec{K}(x)$ be a bounded linear operator from $\mathscr{B}_1$ to $\mathscr{B}_2$ and let $\mathscr{Q} \otimes \mathscr{B}_1$ the space of all finite linear combinations of elements of the form $F = \sum_{i=1}^m \chi_{E_i} u_i$, where E_i are disjoint measurable subsets of $\mathbf{R}^n$ of finite measure, u_i in $\mathscr{B}_1$, and $m \in \mathbf{Z}^+$.
(a) Suppose that $\vec{K}$ satisfies

$$\int_{\mathbf{R}^n} \big\|\vec{K}(x)\big\|_{\mathscr{B}_1\to\mathscr{B}_2}\, dx = C_1 < \infty.$$

Prove that the operator

$$\vec{T}(F)(x) = \int_{\mathbf{R}^n} \vec{K}(x-y)(F(y))\, dy,$$

initially defined on $\mathscr{Q} \otimes \mathscr{B}_1$, has an extension that maps $L^p(\mathbf{R}^n,\mathscr{B}_1)$ to $L^p(\mathbf{R}^n,\mathscr{B}_2)$ with norm at most C_1 for $1 \le p < \infty$.
(b) (*Young's inequality*) Let $1 \le p, q, s \le \infty$ be such that $p < \infty$, $s > 1$, $1/q + 1 = 1/s + 1/p$. Suppose that $\vec{K}$ satisfies

$$\Big\|\big\|\vec{K}(\cdot)\big\|_{\mathscr{B}_1\to\mathscr{B}_2}\Big\|_{L^s(\mathbf{R}^n)} = C_2 < \infty.$$

Prove that the $\vec{T}$ defined in part (a) has an extension that maps $L^p(\mathbf{R}^n,\mathscr{B}_1)$ to $L^q(\mathbf{R}^n,\mathscr{B}_2)$ with norm at most C_2.
(c) (*Young's inequality for weak type spaces*) Suppose that $1<p,s<\infty$, $1/q+1=1/s+1/p$, and that $\vec{K}$ satisfies

$$\Big\| \|\vec{K}(\,\cdot\,)\|_{\mathscr{B}_1\to\mathscr{B}_2} \Big\|_{L^{s,\infty}(\mathbf{R}^n)} = C_3 < \infty .$$

Prove that the $\vec{T}$ defined in part (a) has an extension that maps $L^p(\mathbf{R}^n,\mathscr{B}_1)$ to $L^q(\mathbf{R}^n,\mathscr{B}_2)$.
(d) Prove the following (slight) generalization of the assertion in part (a). Suppose that $\vec{K}$ satisfies

$$\int_{\mathbf{R}^n} \big\|\vec{K}(x)(u)\big\|_{\mathscr{B}_2}\,dx \le C_1\|u\|_{\mathscr{B}_1}$$

for all $u\in\mathscr{B}_1$. Then $\vec{T}$ has an extension that maps $L^1(\mathbf{R}^n,\mathscr{B}_1)$ to $L^1(\mathbf{R}^n,\mathscr{B}_2)$ with norm at most C_1.

5.5.5. Use the inequality for the Rademacher functions in Appendix C.2 instead of Lemma 5.5.2 to prove part (a) of Theorem 5.5.1 in the special case $p=q$. Notice that this approach does not yield a sharp constant.

5.5.6. Let $0<p\neq 2\le\infty$ and suppose that T_j are uniformly bounded linear operators from $L^p(\mathbf{R})$ to $L^p(\mathbf{R})$. Show that the inequality

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}|T_j(f_j)|^2\Big)^{\frac12}\Big\|_{L^p} \le C_p\Big\|\Big(\sum_{j\in\mathbf{Z}}|f_j|^2\Big)^{\frac12}\Big\|_{L^p}$$

may fail.
$\big[$*Hint:* Let $T_j(g)(x)=g(x-j)$. When $p>2$ take $f_j(x)=\chi_{[-j,1-j]}$ for $j=1,2,\dots,N$. When $p<2$ take $f_j=\chi_{[0,1]}$ for $j=1,2,\dots,N.\big]$

5.5.7. Suppose that T is a linear operator that takes real-valued functions to real-valued functions. Use Theorem 5.5.1(a) with $p=q$ to prove that

$$\sup_{\substack{f \text{ real-valued}\\ f\neq 0}} \frac{\|T(f)\|_{L^p}}{\|f\|_{L^p}} = \sup_{\substack{f \text{ complex-valued}\\ f\neq 0}} \frac{\|T(f)\|_{L^p}}{\|f\|_{L^p}} .$$

5.6 Vector-Valued Singular Integrals

We now discuss some results about vector-valued singular integrals. By this we mean singular integral operators acting on functions defined on $\mathbf{R}^n$ and taking values in Banach spaces.

5.6.1 Banach-Valued Singular Integral Operators

Suppose that $\mathscr{B}_1$ and $\mathscr{B}_2$ are Banach spaces. We denote by $L(\mathscr{B}_1,\mathscr{B}_2)$ the space of all bounded linear operators from $\mathscr{B}_1$ to $\mathscr{B}_2$. We consider a kernel $\vec{K}$ defined on $\mathbf{R}^n\setminus\{0\}$ that takes values in $L(\mathscr{B}_1,\mathscr{B}_2)$. In other words, for all $x\in\mathbf{R}^n\setminus\{0\}$, $\vec{K}(x)$ is a bounded linear operator from $\mathscr{B}_1$ to $\mathscr{B}_2$ with norm $\|\vec{K}(x)\|_{\mathscr{B}_1\to\mathscr{B}_2}$. Thus for any $v\in\mathscr{B}_1$ and any $x\in\mathbf{R}^n\setminus\{0\}$ we have

$$\big\|\vec{K}(x)(v)\big\|_{\mathscr{B}_2}\le\|\vec{K}(x)\|_{\mathscr{B}_1\to\mathscr{B}_2}\|v\|_{\mathscr{B}_1}\,.$$

We assume that there is a constant $A<\infty$ such that the size condition holds

$$\big\|\vec{K}(x)\big\|_{\mathscr{B}_1\to\mathscr{B}_2}\le A\,|x|^{-n}\,, \tag{5.6.1}$$

and also the regularity condition

$$\sup_{y\in\mathbf{R}^n\setminus\{0\}}\int_{|x|\ge 2|y|}\big\|\vec{K}(x-y)-\vec{K}(x)\big\|_{\mathscr{B}_1\to\mathscr{B}_2}\,dx\le A<\infty\,. \tag{5.6.2}$$

Moreover, we assume that there is a sequence $\varepsilon_k\downarrow 0$ as $k\to\infty$ and an element $\vec{K}_0$ of $L(\mathscr{B}_1,\mathscr{B}_2)$ such that

$$\lim_{k\to\infty}\bigg\|\int_{\varepsilon_k\le|y|\le 1}\vec{K}(y)\,dy-\vec{K}_0\bigg\|_{\mathscr{B}_1\to\mathscr{B}_2}=0\,. \tag{5.6.3}$$

Given these assumptions, we define an operator $\vec{T}$ on $\mathscr{C}_0^\infty\otimes\mathscr{B}_1$ as follows:

For functions $f_i\in\mathscr{C}_0^\infty(\mathbf{R}^n)$ and $u_i\in\mathscr{B}_1$ we define

$$\vec{T}\Big(\sum_{i=1}^m f_iu_i\Big)(x)=\lim_{k\to\infty}\int_{\varepsilon_k\le|y|}\vec{K}(y)\Big(\sum_{i=1}^m f_i(x-y)u_i\Big)\,dy \tag{5.6.4}$$

$$=\sum_{i=1}^m\int_{|y|\le 1}(f_i(x-y)-f_i(x))\vec{K}(y)(u_i)\,dy+\sum_{i=1}^m f_i(x)\vec{K}_0(u_i)$$
$$+\int_{|y|>1}\sum_{i=1}^m f_i(x-y)\vec{K}(y)(u_i)\,dy\,. \tag{5.6.5}$$

Notice that for each $i\in\{1,\dots,m\}$ we have

$$\int_{|y|\le 1}|f_i(x-y)-f_i(x)|\,\|\vec{K}(y)(u_i)\|_{\mathscr{B}_2}dy\le\|\nabla f_i\|_{L^\infty}\|u_i\|_{\mathscr{B}_1}\int_{|y|\le 1}|y|\,\|\vec{K}(y)\|_{\mathscr{B}_1\to\mathscr{B}_2}dy$$

and this is a finite integral in view of (5.6.1). Thus the function

$$(f_i(x-y)-f_i(x))\,\vec{K}(y)(u_i)$$

is $\mathscr{B}_2$-integrable and the expression

$$\int_{|y|\le 1} (f_i(x-y)-f_i(x))\,\|\vec{K}(y)(u_i)\|_{\mathscr{B}_2}\,dy$$

is a well-defined element of $\mathscr{B}_2$. Also the integral in (5.6.5) is over the compact set $1 \le |y| \le |x|+M$, where the ball $B(0,M)$ contains the supports of all f_i, and thus it also converges in $\mathscr{B}_2$, using (5.6.1).

The following vector-valued extension of Theorem 5.3.3 is the main result of this section.

Theorem 5.6.1. *Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be Banach spaces. Suppose that $\vec{K}(x)$ satisfies (5.6.1), (5.6.2), and (5.6.3) for some $A>0$ and $\vec{K}_0 \in L(\mathscr{B}_1,\mathscr{B}_2)$. Let $\vec{T}$ be the operator associated with $\vec{K}$ as in (5.6.4). Assume that $\vec{T}$ is a bounded linear operator from $L^r(\mathbf{R}^n,\mathscr{B}_1)$ to $L^r(\mathbf{R}^n,\mathscr{B}_2)$ with norm $B_\star$ for some $1<r\le\infty$. Then $\vec{T}$ has well-defined extensions on $L^p(\mathbf{R}^n,\mathscr{B}_1)$ for all $1\le p<\infty$. Moreover, there exist dimensional constants C_n and C_n' such that*

$$\big\|\vec{T}(F)\big\|_{L^{1,\infty}(\mathbf{R}^n,\mathscr{B}_2)} \le C_n'(A+B_\star)\big\|F\big\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)} \tag{5.6.6}$$

for all F in $L^1(\mathbf{R}^n,\mathscr{B}_1)$ and

$$\big\|\vec{T}(F)\big\|_{L^p(\mathbf{R}^n,\mathscr{B}_2)} \le C_n \max\big(p,(p-1)^{-1}\big)(A+B_\star)\big\|F\big\|_{L^p(\mathbf{R}^n,\mathscr{B}_1)} \tag{5.6.7}$$

whenever $1<p<\infty$ and F is in $L^p(\mathbf{R}^n,\mathscr{B}_1)$.

Proof. Although $\vec{T}$ is defined on the entire $L^1(\mathbf{R}^n,\mathscr{B}_1)\cap L^r(\mathbf{R}^n,\mathscr{B}_1)$, it will be convenient to work with its restriction to a smaller dense subspace of $L^1(\mathbf{R}^n,\mathscr{B}_1)$. We make the observation that the space $\mathscr{Q}\otimes\mathscr{B}_1$ of all functions of the form $\sum_{i=1}^m \chi_{R_i}u_i$, where R_i are disjoint dyadic cubes and $u_i\in\mathscr{B}_1$, is dense in $L^1(\mathbf{R}^n,\mathscr{B}_1)$. Indeed, by Proposition 5.5.6 (b) it suffices to approximate a $\mathscr{C}_0^\infty\otimes\mathscr{B}_1$-valued function with a $\mathscr{Q}\otimes\mathscr{B}_1$-valued function. But this is immediate since any function in $\mathscr{C}_0^\infty(\mathbf{R}^n)$ can be approximated in $L^1(\mathbf{R}^n)$ by finite linear combinations of characteristic functions of disjoint dyadic cubes.

Case 1: $r=\infty$. We fix $F=\sum_{i=1}^m \chi_{R_i}u_i$ in $\mathscr{Q}\otimes\mathscr{B}_1$ and we notice that for each $x\in\mathbf{R}^n$ we have $\|F(x)\|_{\mathscr{B}_1}=\sum_{i=1}^m \chi_{R_i}(x)\|u_i\|_{\mathscr{B}_1}$, which is also a finite linear combination of characteristic functions of dyadic cubes. Apply the Calderón-Zygmund decomposition to $\|F\|_{\mathscr{B}_1}$ at height $\gamma\alpha$, where $\gamma=2^{-n-1}B_\star^{-1}$ as in the proof of Theorem 6.3.1. We extract a finite collection of closed dyadic cubes $\{Q_j\}_j$ satisfying $\sum_j|Q_j|\le(\gamma\alpha)^{-1}\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)}$ and we define the good function of the decomposition

$$G(x)=\begin{cases} F(x) & \text{for } x\notin\cup_j Q_j \\ |Q_j|^{-1}\int_{Q_j}F(x)\,dx & \text{for } x\in Q_j. \end{cases}$$

Also define the bad function $B(x) = F(x) - G(x)$. Then $B(x) = \sum_j B_j(x)$, where each B_j is supported in Q_j and has mean value zero over Q_j. Moreover,

$$\|G\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)} \le \|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)} \tag{5.6.8}$$

$$\|G\|_{L^\infty(\mathbf{R}^n,\mathscr{B}_1)} \le 2^n\gamma\alpha \tag{5.6.9}$$

and $\|B_j\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)} \le 2^{n+1}\gamma\alpha|Q_j|$, by an argument similar to that given in the proof of Theorem 5.3.1. We only verify (5.6.9). On the cube Q_j, G is equal to the constant $|Q_j|^{-1}\int_{Q_j} F(x)\,dx$, and this is bounded by $2^n\gamma\alpha$. For each $x \in \mathbf{R}^n \setminus \bigcup_j Q_j$ and for each $k = 0,1,2,\dots$ there exists a unique nonselected dyadic cube $Q_x^{(k)}$ of generation k that contains x. Then for each $k \ge 0$, we have

$$\left\| \frac{1}{|Q_x^{(k)}|}\int_{Q_x^{(k)}} F(y)\,dy \right\|_{\mathscr{B}_1} \le \frac{1}{|Q_x^{(k)}|}\int_{Q_x^{(k)}} \|F(y)\|_{\mathscr{B}_1}\,dy \le \gamma\alpha.$$

The intersection of the closures of the cubes $Q_x^{(k)}$ is the singleton $\{x\}$. Using Corollary 2.1.16, we deduce that for almost all $x \in \mathbf{R}^n \setminus \bigcup_j Q_j$ we have

$$F(x) = \sum_{i=1}^m \chi_{R_i}(x)u_i = \sum_{i=1}^m \lim_{k\to\infty}\left(\frac{1}{|Q_x^{(k)}|}\int_{Q_x^{(k)}} \chi_{R_i}(y)\,dy\right)u_i = \lim_{k\to\infty}\frac{1}{|Q_x^{(k)}|}\int_{Q_x^{(k)}} F(y)\,dy\,.$$

Since these averages are at most $\gamma\alpha$, we conclude that $\|F\|_{\mathscr{B}_1} \le \gamma\alpha$ almost everywhere on $\mathbf{R}^n \setminus \bigcup_j Q_j$, hence $\|G\|_{\mathscr{B}_1} \le \gamma\alpha$ a.e. on this set. This proves (5.6.9).

By assumption we have

$$\|\vec{T}(G)\|_{L^\infty(\mathbf{R}^n,\mathscr{B})} \le B_\star\|G\|_{L^\infty(\mathbf{R}^n,\mathscr{B})} \le 2^n\gamma\alpha B_\star = \alpha/2\,.$$

Then the set $\big\{x \in \mathbf{R}^n : \|\vec{T}(G)(x)\|_{\mathscr{B}_2} > \alpha/2\big\}$ is null and we have

$$\big|\big\{x \in \mathbf{R}^n : \|\vec{T}(F)(x)\|_{\mathscr{B}_2} > \alpha\big\}\big| \le \big|\big\{x \in \mathbf{R}^n : \|\vec{T}(B)(x)\|_{\mathscr{B}_2} > \alpha/2\big\}\big|.$$

Let $Q_j^* = 2\sqrt{n}\,Q_j$. We have

$$\begin{aligned}
&\big|\{x \in\mathbf{R}^n : \|T(B)(x)\|_{\mathscr{B}_2} > \alpha/2\}\big| \\
&\quad\le \Big|\bigcup_j Q_j^*\Big| + \Big|\big\{x \notin \bigcup_j Q_j^* : \|\vec{T}(B)(x)\|_{\mathscr{B}_2} > \alpha/2\big\}\Big| \\
&\quad\le \frac{(2\sqrt{n})^n}{\gamma}\frac{\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)}}{\alpha} + \frac{2}{\alpha}\int_{(\cup_j Q_j^*)^c} \|\vec{T}(B)(x)\|_{\mathscr{B}_2}\,dx \\
&\quad\le \frac{(2\sqrt{n})^n}{\gamma}\frac{\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)}}{\alpha} + \frac{2}{\alpha}\sum_j\int_{(Q_j^*)^c} \|T(B_j)(x)\|_{\mathscr{B}_2}\,dx,
\end{aligned}$$

since $B = \sum_j B_j$. It suffices to estimate the last sum. Denoting by y_j is the center of the cube Q_j and using the fact that B_j has mean value zero over Q_j, we write

$$
\begin{aligned}
&\sum_j \int_{(Q_j^*)^c} \big\|\vec{T}(B_j)(x)\big\|_{\mathscr{B}_2}\,dx \\
&\qquad = \sum_j \int_{(Q_j^*)^c} \bigg\| \int_{Q_j} \big(\vec{K}(x-y)-\vec{K}(x-y_j)\big)(B_j(y))\,dy\bigg\|_{\mathscr{B}_2} dx \\
&\qquad \le \sum_j \int_{Q_j} \big\|B_j(y)\big\|_{\mathscr{B}_1} \int_{(Q_j^*)^c} \big\|\vec{K}(x-y)-\vec{K}(x-y_j)\big\|_{\mathscr{B}_1\to\mathscr{B}_2}\,dxdy \\
&\qquad \le \sum_j \int_{Q_j} \big\|B_j(y)\big\|_{\mathscr{B}_1} \int_{|x-y_j|\ge 2|y-y_j|} \big\|\vec{K}(x-y)-\vec{K}(x-y_j)\big\|_{\mathscr{B}_1\to\mathscr{B}_2}\,dxdy \\
&\qquad \le A\sum_j \|B_j\|_{L^1(Q_j,\mathscr{B}_1)} \\
&\qquad \le 2^{n+1}A\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)},
\end{aligned}
$$

where we used the fact that $|x-y_j|\ge 2|y-y_j|$ for all $x\notin Q_j^*$ and $y\in Q_j$ and (5.6.2). Consequently,

$$
\begin{aligned}
\big|\big\{x\in\mathbf{R}^n:\ \|\vec{T}(f)(x)\|_{\mathscr{B}_2}>\alpha\big\}\big| &\le \frac{(2\sqrt{n})^n}{\gamma}\frac{\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)}}{\alpha}+\frac{2}{\alpha}2^{n+1}A\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)} \\
&= \big((2\sqrt{n})^n 2^{n+1}B_\star+2^{n+1}A\big)\frac{\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)}}{\alpha} \\
&\le C_n'\,(A+B_\star)\frac{\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)}}{\alpha},
\end{aligned}
$$

where $C_n'=(2\sqrt{n})^n 2^{n+1}+2^{n+1}$. Thus $\vec{T}$ has an extension that maps $L^1(\mathbf{R}^n,\mathscr{B}_1)$ to $L^{1,\infty}(\mathbf{R}^n,\mathscr{B}_2)$ with constant $C_n(A+B_\star)$. By interpolation (Exercise 5.5.3 (b)) it has an extension that satisfies (5.6.7).

Case 2: $1<r<\infty$. We fix $F=\sum_{i=1}^m \chi_{R_i}u_i$ in $\mathscr{Q}\otimes\mathscr{B}_1$ and we notice that for each $x\in\mathbf{R}^n$ we have $\|F(x)\|_{\mathscr{B}_1}=\sum_{i=1}^m \chi_{R_i}(x)\|u_i\|_{\mathscr{B}_1}$. Thus the function $x\mapsto\|F(x)\|_{\mathscr{B}_1}$ is a finite linear combination of characteristic functions of disjoint dyadic cubes. We prove the weak type estimate (5.6.6) by applying the Calderón–Zygmund decomposition to the function $x\mapsto\|F(x)\|_{\mathscr{B}_1}$ defined on $\mathbf{R}^n$. Then we decompose $F=G+B$, where G and B satisfy properties analogous to the case $r=\infty$. The new ingredient in this case is that the set $\big\{x\in\mathbf{R}^n:\ \|\vec{T}(G)(x)\|_{\mathscr{B}_2}>\alpha/2\big\}$ is not null but its measure can be estimated as follows:

$$
\big|\big\{x\in\mathbf{R}^n:\ \|\vec{T}(G)(x)\|_{\mathscr{B}_2}>\alpha/2\big\}\big| \le \left(\frac{2B_\star}{\alpha}\right)^r\|G\|^r_{L^2(\mathbf{R}^n,\mathscr{B}_1)} \le \frac{2}{\alpha}\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_1)},
$$

where the first inequality is a consequence of the boundedness of $\vec{T}$ on L^r and the second is obtained by combining (5.6.8) and (5.6.9). Combining this estimate for the good function with the one for the bad function obtained in the preceding case, it follows that $\vec{T}$ has an extension that satisfies (5.6.6), i.e., it maps $\vec{T}:\ L^1(\mathbf{R}^n,\mathscr{B}_1)$ to $L^{1,\infty}(\mathbf{R}^n,\mathscr{B}_2)$ with constant $C_n'(A+B_\star)$, where $C_n'=2+(2\sqrt{n})^n 2^{n+1}+2^{n+1}$.

Next we interpolate between the estimates $\vec{T} : L^1(\mathbf{R}^n, \mathscr{B}_1) \to L^{1,\infty}(\mathbf{R}^n, \mathscr{B}_2)$ and $\vec{T} : L^r(\mathbf{R}^n, \mathscr{B}_1) \to L^r(\mathbf{R}^n, \mathscr{B}_2)$. Using Exercise 5.5.3 (b) and the fact that $(p-1)^{-1/p} \le (p-1)^{-1}$ when $1 < p < 2$, we obtain

$$\big\|\vec{T}(F)\big\|_{L^p(\mathbf{R}^n,\mathscr{B}_2)} \le C_n(p-1)^{-1}(A+B_\star)\|F\|_{L^p(\mathbf{R}^n,\mathscr{B}_1)}, \tag{5.6.10}$$

when $1 < p < \min(r,2)$, where C_n is independent of r, p, $\mathscr{B}_1$, and $\mathscr{B}_2$.

We prove (5.6.7) for $p > r$ via duality. Since $\vec{K}(x)$ is an operator from $\mathscr{B}_1$ to $\mathscr{B}_2$, its adjoint $\vec{K}^*(x)$ is an operator from $\mathscr{B}_2^*$ to $\mathscr{B}_1^*$. Obviously $\vec{K}^*(x)$ and $\vec{K}(x)$ have the same norm, so (5.6.1) holds. For the same reason, condition (5.6.2) also holds for $\vec{K}^*$, Finally condition (5.6.3) also holds since for any $\varepsilon_k \downarrow$ as $k \to \infty$ we have

$$\left\| \int_{\varepsilon_k \le |y| \le 1} \vec{K}^*(y)\,dy - \vec{K}_0^* \right\|_{\mathscr{B}_2^* \to \mathscr{B}_1^*} = \left\| \int_{\varepsilon_k \le |y| \le 1} \vec{K}(y)\,dy - \vec{K}_0 \right\|_{\mathscr{B}_1 \to \mathscr{B}_2} \to 0\,.$$

Let $\vec{T}'$ be the Banach-valued operator with kernel $\vec{K}^*(-x)$. Clearly $\vec{T}'$ is well defined on $\mathscr{C}_0^\infty \otimes \mathscr{B}_2^*$. For $F(y) = \sum_{i=1}^m f_i(y)w_i^*$ in $\mathscr{C}_0^\infty \otimes \mathscr{B}_2^*$ and $G(z) = \sum_{j=1}^l g_j(z)v_j$ in $\mathscr{C}_0^\infty \otimes \mathscr{B}_1$ we prove the following duality relation

$$\int_{\mathbf{R}^n} \big\langle \vec{T}'(F)(x), G(x) \big\rangle\,dx = \int_{\mathbf{R}^n} \big\langle F(z), \vec{T}(G)(z) \big\rangle\,dz\,. \tag{5.6.11}$$

Indeed, for each index $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,l\}$ we have

$$\begin{aligned}
&\int_{\mathbf{R}^n} \Big\langle \lim_{k\to\infty} \int_{|y|\ge\varepsilon_k} \vec{K}^*(-y)(f_i(x-y)w_i^*)\,dy, g_j(x)v_j \Big\rangle\,dx \\
&\qquad = \lim_{k\to\infty} \int_{|y|\ge\varepsilon_k} \int_{\mathbf{R}^n} \Big\langle \vec{K}^*(-y)(f_i(x-y)w_i^*), g_j(x)v_j \Big\rangle\,dx\,dy \\
&\qquad = \lim_{k\to\infty} \int_{|y|\ge\varepsilon_k} \int_{\mathbf{R}^n} \Big\langle \vec{K}^*(-y)(f_i(z)w_i^*), g_j(z+y)v_j \Big\rangle\,dz\,dy \\
&\qquad = \lim_{k\to\infty} \int_{|y|\ge\varepsilon_k} \int_{\mathbf{R}^n} \Big\langle f_i(z)w_i^*, \vec{K}(-y)(g_j(z+y)v_j) \Big\rangle\,dz\,dy \\
&\qquad = \int_{\mathbf{R}^n} \Big\langle f_i(z)w_i^*, \lim_{k\to\infty} \int_{|y|\ge\varepsilon_k} \vec{K}(y)(g_j(z-y)v_j)\,dy \Big\rangle\,dz\,,
\end{aligned}$$

proving (5.6.11), provided we can justify the interchange the x (or z)-integral with the y-integral paired with the limit. These justifications can be given using the definition in (5.6.4). For the part of the y-integral where $|y| \ge 1$ the interchange is easily justified in view of the absolute convergence of the double integral and (5.5.21). For the part of the y-integral where $|y| \le 1$ we introduce the operators $\vec{K}_0^*$ and $\vec{K}_0$ and we use the facts that $|g_j(z-y) - g_j(z)| \le \|\nabla g_j\|_{L^\infty}|y|$ and $|f_i(x-y) - f_i(x)| \le \|\nabla f_i\|_{L^\infty}|y|$ together with the assumption $\|\vec{K}(y)\|_{\mathscr{B}_1\to\mathscr{B}_2} \le A|y|^{-n}$ and (5.5.21) to obtain the absolute convergence of the double integral, and thus justify the interchange.

We claim that $\vec{T}'$ is bounded from $L^{r'}(\mathbf{R}^n,\mathscr{B}_2^*)$ to $L^{r'}(\mathbf{R}^n,\mathscr{B}_1^*)$. Indeed, to verify this assertion, we fix F in $\mathscr{C}_0^\infty\otimes\mathscr{B}_2^*$ and use Proposition 5.5.7 (b). Using (5.6.11), for each $G\in\mathscr{C}_0^\infty\otimes\mathscr{B}_1$, we write

$$\begin{aligned}\Big|\int_{\mathbf{R}^n}\big\langle\vec{T}'(F)(x),G(x)\big\rangle\,dx\Big| &= \Big|\int_{\mathbf{R}^n}\big\langle F(x),\vec{T}(G)(x)\big\rangle\,dx\Big|\\ &\le \int_{\mathbf{R}^n}\big\|F(x)\big\|_{\mathscr{B}_2^*}\big\|\vec{T}(G)(x)\big\|_{\mathscr{B}_2}\,dx\\ &\le \|F\|_{L^{r'}(\mathbf{R}^n,\mathscr{B}_2^*)}\big\|\vec{T}(G)\big\|_{L^r(\mathbf{R}^n,\mathscr{B}_2)}\\ &\le \|F\|_{L^{r'}(\mathbf{R}^n,\mathscr{B}_2^*)}B_\star\,\|G\|_{L^r(\mathbf{R}^n,\mathscr{B}_1)}\,,\end{aligned}$$

so taking the supremum over all $G\in\mathscr{C}_0^\infty\otimes\mathscr{B}_1$ with $\|G\|_{L^r(\mathbf{R}^n,\mathscr{B}_1)}\le 1$ we deduce that

$$\big\|\vec{T}'(F)\big\|_{L^{r'}(\mathbf{R}^n,\mathscr{B}_1^*)}\le B_\star\,\|F\|_{L^{r'}(\mathbf{R}^n,\mathscr{B}_2^*)}\,.$$

Collecting these facts, we have that $\vec{T}'$ is associated with a kernel $\vec{K}^*(-x)$ which satisfies (5.6.1), (5.6.2), and (5.6.3) (with $\vec{K}_0^*$ in place of $\vec{K}_0$), and moreover it has a bounded extension that maps $L^{r'}(\mathbf{R}^n,\mathscr{B}_2^*)$ to $L^{r'}(\mathbf{R}^n,\mathscr{B}_1^*)$. The Calderón-Zygmund decomposition in the vector-valued setting (discussed in the first paragraph of the proof) yields that $\vec{T}'$ has an extension that satisfies

$$\big\|\vec{T}'(F)\big\|_{L^{1,\infty}(\mathbf{R}^n,\mathscr{B}_1^*)}\le C_n'(A+B_\star)\|F\|_{L^1(\mathbf{R}^n,\mathscr{B}_2^*)}\,.$$

Using interpolation (Exercise 5.5.3 (b)) and the fact that $(p'-1)^{1/p'}\le p$, we obtain that for $1<p'<r'$, $\vec{T}'$ has an extension on $L^{p'}(\mathbf{R}^n,\mathscr{B}_2^*)$ that satisfies

$$\big\|\vec{T}'(F)\big\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}_1^*)}\le C_n\,p\,(A+B_\star)\|F\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}_2^*)}\,. \tag{5.6.12}$$

Let $F=\sum_{i=1}^m\varphi_i\,u_i$ be in the dense subspace $\mathscr{C}_0^\infty\otimes\mathscr{B}_1$ of $L^p(\mathbf{R}^n,\mathscr{B}_1)$. We observe that $\|\vec{T}(F)\|_{L^p(\mathbf{R}^n,\mathscr{B}_2)}<\infty$. Indeed, all φ_i are supported in $|x|\le R$, then for $|x|\ge 2R$ we have

$$\bigg\|\int_{|y|\le R}\vec{K}(x-y)\Big(\sum_{i=1}^m\varphi_i\,u_i\Big)\,dy\bigg\|_{\mathscr{B}_2}\le A\Big(\frac{|x|}{2}\Big)^{-n}\sum_{i=1}^m\|\varphi_i\|_{L^1}\|u_i\|_{\mathscr{B}_1} \tag{5.6.13}$$

which is integrable to the power $p>1$ in the region $|x|\ge 2R$. Also using the definition in (5.6.4) we see that the expression on the left in (5.6.13) is bounded, hence integrable to the power p in the region $|x|\le 2R$. For a fixed $r<p<\infty$, we are now able to apply Proposition 5.5.7 (a) to write

$$\begin{aligned}\big\|\vec{T}(F)\big\|_{L^p(\mathbf{R}^n,\mathscr{B}_2)} &\le \sup_{\|G\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}_2^*)}\le 1}\Big|\int_{\mathbf{R}^n}\big\langle G(x),\vec{T}(F)(x)\big\rangle\,dx\Big|\\ &= \sup_{\|G\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}_2^*)}\le 1}\Big|\int_{\mathbf{R}^n}\big\langle\vec{T}'(G)(x),F(x)\big\rangle\,dx\Big|\end{aligned}$$

$$\leq \sup_{\|G\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}_2^*)}\leq 1} \big\|\vec{T}'(G)\big\|_{L^{p'}(\mathbf{R}^n,\mathscr{B}_1^*)} \|F\|_{L^p(\mathbf{R}^n,\mathscr{B}_1)}$$
$$\leq C_n\, p\,(A+B_\star)\|F\|_{L^p(\mathbf{R}^n,\mathscr{B}_1)}\,,$$

where we used (5.6.12). This combined with (5.6.10) implies the required conclusion whenever $r<\infty$ and $p\in\big(1,\min(r,2)\big)\cup(r,\infty)$. The remaining p's follow by interpolation (Exercise 5.5.3 (a)). □

5.6.2 Applications

We proceed with some applications. An important consequence of Theorem 5.6.1 is the following:

Corollary 5.6.2. *Let $A,B>0$ and let W_j be a sequence of tempered distributions on $\mathbf{R}^n$ whose Fourier transforms are uniformly bounded functions (i.e., $|\widehat{W_j}|\leq B$). Suppose that for each j, W_j coincides with a function K_j on $\mathbf{R}^n\setminus\{0\}$ that satisfies*

$$|K_j(x)|\leq A\,|x|^{-n}\,, \qquad x\neq 0, \tag{5.6.14}$$

$$\lim_{\varepsilon_k\to 0}\int_{|x|\geq\varepsilon_k} K_j(x)\,dx = L_j\,, \tag{5.6.15}$$

for some complex constant L_j, and

$$\sup_{y\in\mathbf{R}^n\setminus\{0\}}\int_{|x|\geq 2|y|}\sup_j |K_j(x-y)-K_j(x)|\,dx\leq A\,. \tag{5.6.16}$$

Then there are constants $C_n,C_n'>0$ such that for all $1<p,r<\infty$ we have

$$\Big\|\Big(\sum_j |W_j*f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^{1,\infty}} \leq C_n'\max(r,(r-1)^{-1})(A+B)\Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^1},$$
$$\Big\|\Big(\sum_j |W_j*f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p} \leq C_n\,c(p,r)(A+B)\Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p},$$

where $c(p,r)=\max(p,(p-1)^{-1})\max(r,(r-1)^{-1})$.

Proof. Let T_j be the operator given by convolution with the distribution W_j. Clearly T_j is L^2 bounded with norm at most B. It follows from Theorem 5.3.3 that the T_j's are of weak type $(1,1)$ and also bounded on L^r with bounds at most a dimensional constant multiple of $\max(r,(r-1)^{-1})(A+B)$, uniformly in j. We set $\mathscr{B}_1=\mathscr{B}_2=\ell^r$ and define

$$\vec{T}(\{f_j\}_j)=\{W_j*f_j\}_j$$

for $\{f_j\}_j \in L^r(\mathbf{R}^n,\ell^r)$. It is immediate to verify that $\vec{T}$ maps $L^r(\mathbf{R}^n,\ell^r)$ to itself with norm at most a dimensional constant multiple of $\max(r,(r-1)^{-1})(A+B)$. The kernel of $\vec{T}$ is $\vec{K}$ in $L(\ell^r,\ell^r)$ defined by

$$\vec{K}(x)(\{t_j\}_j) = \{K_j(x)t_j\}_j, \qquad \{t_j\}_j \in \ell^r.$$

Obviously, we have

$$\big\|\vec{K}(x-y)-\vec{K}(x)\big\|_{\ell^r\to\ell^r} \le \sup_j |K_j(x-y)-K_j(x)|\,,$$

and therefore condition (5.6.3) holds for $\vec{K}$ as a consequence of (5.6.16). Moreover, (5.6.1) and (5.6.2) with $\vec{K}_0 = \{L_j\}_j$ are also valid for this $\vec{K}$, in view of assumptions (5.6.14) and (5.6.15). The desired conclusion follows from Theorem 5.6.1. $\square$

If all the W_j's are equal, we obtain the following corollary, which contains in particular the inequality (5.5.16) mentioned earlier.

Corollary 5.6.3. *Let W be an element of $\mathscr{S}'(\mathbf{R}^n)$ whose Fourier transform is a function bounded in absolute value by some $B>0$. Suppose that W coincides with some locally integrable function K on $\mathbf{R}^n\setminus\{0\}$ that satisfies*

$$|K(x)| \le A\,|x|^{-n}\,, \qquad x\neq 0,$$

$$\lim_{\varepsilon_k\to 0}\int_{\varepsilon_k\le|x|\le 1} K(x)\,dx = L,$$

and

$$\sup_{y\in\mathbf{R}^n\setminus\{0\}}\int_{|x|\ge 2|y|}|K(x-y)-K(x)|\,dx \le A\,. \tag{5.6.17}$$

Let T be the operator given by convolution with W. Then there exist constants $C_n, C_n'>0$ such that for all $1<p,r<\infty$ we have that

$$\Big\|\Big(\sum_j|T(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^{1,\infty}} \le C_n'\max(r,(r-1)^{-1})(A+B)\Big\|\Big(\sum_j|f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^1},$$

$$\Big\|\Big(\sum_j|T(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^p} \le C_n c(p,r)\,(A+B)\Big\|\Big(\sum_j|f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p},$$

where $c(p,r)=\max(p,(p-1)^{-1})\max(r,(r-1)^{-1})$. In particular, these inequalities are valid for the Hilbert transform and the Riesz transforms.

Interestingly enough, we can use the very statement of Theorem 5.6.1 to obtain its corresponding vector-valued version.

Proposition 5.6.4. *Let let $1<p,r<\infty$ and let $\mathscr{B}_1$ and $\mathscr{B}_2$ be two Banach spaces. Suppose that $\vec{T}$ given by (5.6.4) is a bounded linear operator from $L^r(\mathbf{R}^n,\mathscr{B}_1)$ to $L^r(\mathbf{R}^n,\mathscr{B}_2)$ with norm $B=B(r)$. Also assume that for all $x\in\mathbf{R}^n\setminus\{0\}$, $\vec{K}(x)$ is a* *bounded linear operator from $\mathscr{B}_1$ to $\mathscr{B}_2$ that satisfies conditions (5.6.1) , (5.6.2), (5.6.3) for some $A > 0$ and $\vec{K}_0 \in L(\mathscr{B}_1, \mathscr{B}_2)$. Then there exist positive constants C_n, C_n' such that for all $\mathscr{B}_1$-valued functions F_j we have*

$$\Big\|\Big(\sum_j \|\vec{T}(F_j)\|_{\mathscr{B}_2}^r\Big)^{\frac{1}{r}}\Big\|_{L^{1,\infty}(\mathbf{R}^n)} \le C_n'(A+B)\Big\|\Big(\sum_j \|F_j\|_{\mathscr{B}_1}^r\Big)^{\frac{1}{r}}\Big\|_{L^1(\mathbf{R}^n)},$$

$$\Big\|\Big(\sum_j \|\vec{T}(F_j)\|_{\mathscr{B}_2}^r\Big)^{\frac{1}{r}}\Big\|_{L^p(\mathbf{R}^n)} \le C_n(A+B)c(p)\Big\|\Big(\sum_j \|F_j\|_{\mathscr{B}_1}^r\Big)^{\frac{1}{r}}\Big\|_{L^p(\mathbf{R}^n)},$$

where $c(p) = \max(p,(p-1)^{-1})$.

Proof. Let us denote by $\ell^r(\mathscr{B}_1)$ the Banach space of all $\mathscr{B}_1$-valued sequences $\{u_j\}_j$ that satisfy

$$\|\{u_j\}_j\|_{\ell^r(\mathscr{B}_1)} = \big(\sum_j \|u_j\|_{\mathscr{B}_1}^r\big)^{\frac{1}{r}} < \infty.$$

Now consider the operator $\vec{S}$ defined on $L^r(\mathbf{R}^n, \ell^r(\mathscr{B}_1))$ by

$$\vec{S}(\{F_j\}_j) = \{\vec{T}(F_j)\}_j\,.$$

It is obvious that $\vec{S}$ maps $L^r(\mathbf{R}^n, \ell^r(\mathscr{B}_1))$ to $L^r(\mathbf{R}^n, \ell^r(\mathscr{B}_2))$ with norm at most B. Moreover, $\vec{S}$ has kernel $\widetilde{K}(x) \in L(\ell^r(\mathscr{B}_1), \ell^r(\mathscr{B}_2))$ given by

$$\widetilde{K}(x)(\{u_j\}_j) = \{\vec{K}(x)(u_j)\}_j,$$

where $\vec{K}$ is the kernel of $\vec{T}$. It is not hard to verify that for $x \in \mathbf{R}^n \setminus \{0\}$ we have

$$\|\widetilde{K}(x)\|_{\ell^r(\mathscr{B}_1)\to\ell^r(\mathscr{B}_2)} = \|\vec{K}(x)\|_{\mathscr{B}_1\to\mathscr{B}_2}\,,$$

hence for $x \neq y \in \mathbf{R}^n$ we also have

$$\|\widetilde{K}(x-y) - \widetilde{K}(x)\|_{\ell^r(\mathscr{B}_1)\to\ell^r(\mathscr{B}_2)} = \|\vec{K}(x-y) - \vec{K}(x)\|_{\mathscr{B}_1\to\mathscr{B}_2}\,.$$

Moreover, if we define $\widetilde{K_0} \in L\big(\ell^r(\mathscr{B}_1), \ell^r(\mathscr{B}_2)\big)$ by

$$\widetilde{K_0}(\{u_j\}_j) = \big\{\vec{K}_0(y)(u_j)\big\}_j\,.$$

for $\{u_j\}_j \in \ell^r(\mathscr{B}_1)$, then we have

$$\lim_{k\to\infty}\int_{\varepsilon_k\le|y|\le 1} \widetilde{K}(y)\,dy = \widetilde{K_0},$$

in $L\big(\ell^r(\mathscr{B}_1), \ell^r(\mathscr{B}_2)\big)$.

We conclude that $\widetilde{K}$ satisfies conditions (5.6.1) , (5.6.2), (5.6.3). Hence the operator $\vec{S}$ associate with $\widetilde{K}$ satisfies the conclusion of Theorem 5.6.1, that is, the desired inequalities for $\vec{T}$. □

5.6.3 Vector-Valued Estimates for Maximal Functions

Next, we discuss applications of vector-valued inequalities to some nonlinear operators. We fix an integrable function Φ on $\mathbf{R}^n$ and for $t>0$ define $\Phi_t(x)=t^{-n}\Phi(t^{-1}x)$. We suppose that Φ satisfies the following *regularity* condition:

$$\sup_{y\in\mathbf{R}^n\setminus\{0\}}\int_{|x|\geq 2|y|}\sup_{t>0}|\Phi_t(x-y)-\Phi_t(x)|\,dx=A_\Phi<\infty. \tag{5.6.18}$$

We consider the maximal operator

$$M_\Phi(f)(x)=\sup_{t>0}|(f*\Phi_t)(x)|$$

defined for f in L^1+L^∞. We are interested in obtaining L^p estimates for M_Φ. We observe that the trivial estimate

$$\big\|M_\Phi(f)\big\|_{L^\infty}\leq\|\Phi\|_{L^1}\|f\|_{L^\infty} \tag{5.6.19}$$

holds when $p=\infty$. It is natural to set

$$\mathscr{B}_1=\mathbf{C}\qquad\text{and}\qquad\mathscr{B}_2=L^\infty(\mathbf{R}^+)$$

and view M_Φ as the linear operator $f\mapsto\{f*\Phi_\delta\}_{\delta>0}$ that maps $\mathscr{B}_1$-valued functions to $\mathscr{B}_2$-valued functions.

To do this precisely, for each $x\in\mathbf{R}^n$ we define a bounded linear operator $\vec{K}_\Phi(x)$ from $\mathscr{B}_1$ to $\mathscr{B}_2$ by setting for $c\in\mathbf{C}$

$$\vec{K}_\Phi(x)(c)=\{c\,\Phi_\delta(x)\}_{\delta\in\mathbf{R}^+}.$$

Clearly we have

$$\big\|\vec{K}_\Phi(x)\big\|_{\mathbf{C}\to L^\infty(\mathbf{R}^+)}=\sup_{\delta>0}|\Phi_\delta(x)|.$$

Now (5.6.18) implies condition (5.6.2) for the kernel $\vec{K}_\Phi$. Also condition (5.6.1) holds (for some $A<$ depending on n) since

$$\sup_{\delta>0}|\Phi_\delta(x)|\leq A\,|x|^{-n}$$

and also condition (5.6.3) holds since for every $\delta>0$ we have

$$\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|y|\leq 1}\Phi_\delta(y)\,dy=\int_{|y|\leq 1}\Phi_\delta(y)\,dy.$$

We also define a $\mathscr{B}_2$-valued linear operator acting on complex-valued functions on $\mathbf{R}^n$ by

$$\vec{M}_\Phi(f)=f*\vec{K}_\Phi=\{f*\Phi_\delta\}_{\delta\in\mathbf{R}^+}.$$

Obvisouly $\vec{M}_\Phi$ maps $L^\infty(\mathbf{R}^n,\mathscr{B}_1)$ to $L^\infty(\mathbf{R}^n,\mathscr{B}_2)$ with norm at most $\|\Phi\|_{L^1}$.

Applying Theorem 5.6.1 with $r=\infty$ we obtain for $1<p<\infty$,

$$\big\|\vec{M}_{\Phi}(f)\big\|_{L^p(\mathbf{R}^n,\mathscr{B}_2)} \le C_n \max(p,(p-1)^{-1})\big(A_\Phi + \|\Phi\|_{L^1}\big)\big\|f\big\|_{L^p(\mathbf{R}^n)}, \tag{5.6.20}$$

which can be immediately improved to

$$\big\|\vec{M}_{\Phi}(f)\big\|_{L^r(\mathbf{R}^n,\mathscr{B}_2)} \le C_n \max(1,(r-1)^{-1})\big(A_\Phi + \|\Phi\|_{L^1}\big)\big\|f\big\|_{L^r(\mathbf{R}^n)} \tag{5.6.21}$$

via interpolation with estimate (5.6.19) for all $1<r<\infty$.

Next we use estimate (5.6.21) to obtain vector-valued estimates for the sublinear operator M_Φ.

Corollary 5.6.5. *Let Φ be an integrable function on $\mathbf{R}^n$ that satisfies (5.6.18). Then there exist dimensional constants C_n and C_n' such that for all $1<p,r<\infty$ the following vector-valued inequalities are valid:*

$$\Big\|\Big(\sum_j |M_\Phi(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^{1,\infty}} \le C_n' c(r)\big(A_\Phi + \|\Phi\|_{L^1}\big)\Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^1}, \tag{5.6.22}$$

where $c(r)= \ 1+(r-1)^{-1}$, and

$$\Big\|\Big(\sum_j |M_\Phi(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^p} \le C_n c(p,r)\big(A_\Phi + \|\Phi\|_{L^1}\big)\Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p}, \tag{5.6.23}$$

where $c(p,r)=\big(1+(r-1)^{-1}\big)\big(p+(p-1)^{-1}\big)$.

Proof. We set $\mathscr{B}_1=\mathbf{C}$ and $\mathscr{B}_2=L^\infty(\mathbf{R}^+)$. We use estimate (5.6.21) as a starting point in Proposition 5.6.4, which immediately yields the required conclusions (5.6.22) and (5.6.23). □

Similar estimates hold for the Hardy–Littlewood maximal operator.

Theorem 5.6.6. *For $1<p,r<\infty$ the Hardy–Littlewood maximal function M satisfies the vector-valued inequalities*

$$\Big\|\Big(\sum_j |M(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^{1,\infty}} \le C_n'\big(1+(r-1)^{-1}\big)\Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^1}, \tag{5.6.24}$$

$$\Big\|\Big(\sum_j |M(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^p} \le C_n c(p,r)\Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p}, \tag{5.6.25}$$

where $c(p,r)=\big(1+(r-1)^{-1}\big)\big(p+(p-1)^{-1}\big)$.

Proof. Let us fix a positive radial symmetrically decreasing Schwartz function Φ on $\mathbf{R}^n$ that satisfies $\Phi(x)\ge 1$ when $|x|\le 1$. Then the Hardy–Littlewood maximal function $M(f)$ is pointwise controlled by a constant multiple of the function $M_\Phi(|f|)$.

In view of Corollary 5.6.5, it suffices to check that for such a Φ, (5.6.18) holds. First observe that in view of the decreasing character of Φ, we have

$$\sup_j |f|*\Phi_{2^j} \le M_\Phi(|f|) \le 2^n \sup_j |f|*\Phi_{2^j}\,,$$

and for this reason we choose to work with the easier dyadic maximal operator

$$M_\Phi^d(f) = \sup_j |f*\Phi_{2^j}|\,.$$

We observe the validity of the simple inequalties

$$2^{-n}M(f) \le \mathcal{M}(f) \le M_\Phi(|f|) \le 2^n M_\Phi^d(|f|)\,. \tag{5.6.26}$$

If we can show that

$$\sup_{y\in\mathbf{R}^n\setminus\{0\}} \int_{|x|\ge 2|y|} \sup_{j\in\mathbf{Z}} |\Phi_{2^j}(x-y)-\Phi_{2^j}(x)|\,dx = C_n < \infty\,, \tag{5.6.27}$$

then (5.6.22) and (5.6.23) are satisfied with M_Φ^d replacing M_Φ. We therefore turn our attention to (5.6.27). We have

$$\begin{aligned}
&\int_{|x|\ge 2|y|} \sup_{j\in\mathbf{Z}} |\Phi_{2^j}(x-y)-\Phi_{2^j}(x)|\,dx \\
&\le \sum_{j\in\mathbf{Z}} \int_{|x|\ge 2|y|} |\Phi_{2^j}(x-y)-\Phi_{2^j}(x)|\,dx \\
&\le \sum_{2^j>|y|} \int_{|x|\ge 2|y|} \frac{|y|\,|\nabla\Phi\big(\frac{x-\theta y}{2^j}\big)|}{2^{(n+1)j}}\,dx + \sum_{2^j\le|y|} \int_{|x|\ge 2|y|} \big(|\Phi_{2^j}(x-y)|+|\Phi_{2^j}(x)|\big)\,dx \\
&\le \sum_{2^j>|y|} \int_{|x|\ge 2|y|} \frac{|y|}{2^{(n+1)j}} \frac{C_N\,dx}{(1+|2^{-j}(x-\theta y)|)^N} + 2\sum_{2^j\le|y|} \int_{|x|\ge|y|} |\Phi_{2^j}(x)|\,dx \\
&\le \sum_{2^j>|y|} \int_{|x|\ge 2|y|} \frac{|y|}{2^{(n+1)j}} \frac{C_N}{(1+|2^{-j-1}x|)^N}\,dx + 2\sum_{2^j\le|y|} \int_{|x|\ge 2^{-j}|y|} |\Phi(x)|\,dx \\
&\le \sum_{2^j>|y|} \int_{|x|\ge 2^{-j}|y|} \frac{|y|}{2^j} \frac{C_N}{(1+|x|)^N}\,dx + 2\sum_{2^j\le|y|} C_N(2^{-j}|y|)^{-N} \\
&\le C_N \sum_{2^j>|y|} \frac{|y|}{2^j} + C_N \\
&\le 3C_N\,,
\end{aligned}$$

where $C_N>0$ depends on $N>n$, $\theta\in[0,1]$, and $|x-\theta y|\ge |x|/2$ when $|x|\ge 2|y|$.

Now apply (5.6.22) and (5.6.23) to M_Φ^d and use (5.6.26) to obtain the desired vector-valued inequalities. $\square$

Remark 5.6.7. Observe that (5.6.24) and (5.6.25) also hold for $r = \infty$. These endpoint estimates can be proved directly by observing that

$$\sup_j M(f_j) \le M(\sup_j |f_j|)\,.$$

The same is true for estimates (5.6.22) and (5.6.23). Finally, estimates (5.6.25) and (5.6.23) also hold for $p = \infty$.

Exercises

5.6.1. (a) For all $j \in \mathbf{Z}$, let I_j be an interval in $\mathbf{R}$ and let T_j be the operator given on the Fourier transform by multiplication by the characteristic function of I_j. Prove that there exists a constant $C > 0$ such that for all $1 < p, r < \infty$ and for all square integrable functions f_j on $\mathbf{R}$ we have

$$\Big\|\Big(\sum_j |T_j(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^p(\mathbf{R})} \le C \max\Big(r, \frac{1}{r-1}\Big) \max\Big(p, \frac{1}{p-1}\Big) \Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p(\mathbf{R})},$$

$$\Big\|\Big(\sum_j |T_j(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^{1,\infty}(\mathbf{R})} \le C \max\Big(r, \frac{1}{r-1}\Big) \Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^1(\mathbf{R})}.$$

(b) Let R_j be arbitrary rectangles on $\mathbf{R}^n$ with sides parallel to the axes and let S_j be the operators given on the Fourier transform by multiplication by the characteristic functions of R_j. Prove that there exists a dimensional constant $C_n < \infty$ such that for all indices $1 < p, r < \infty$ and for all square integrable functions f_j in $L^p(\mathbf{R}^n)$ we have

$$\Big\|\Big(\sum_j |S_j(f_j)|^r\Big)^{\frac{1}{r}}\Big\|_{L^p(\mathbf{R}^n)} \le C_n \max\Big(r, \frac{1}{r-1}\Big)^n \max\Big(p, \frac{1}{p-1}\Big)^n \Big\|\Big(\sum_j |f_j|^r\Big)^{\frac{1}{r}}\Big\|_{L^p(\mathbf{R}^n)}.$$

[*Hint:* Part (a): Use Theorem 5.5.1 and the identity $T_j = \frac{i}{2}\big(M^a H M^{-a} - M^b H M^{-b}\big)$, if I_j is $\chi_{(a,b)}$, where $M^a(f)(x) = f(x)e^{2\pi i a x}$ and H is the Hilbert transform. Part (b): Apply the result in part (a) in each variable.]

5.6.2. Let $(T, d\mu)$ be a σ-finite measure space. For every $t \in T$, let $R(t)$ be a rectangle in $\mathbf{R}^n$ with sides parallel to the axes such that the map $t \mapsto R(t)$ is measurable. Then there is a constant $C_n > 0$ such that for all $1 < p < \infty$ and for all families of square integrable functions $\{f_t\}_{t\in T}$ on $\mathbf{R}^n$ such that $t \mapsto f_t(x)$ is measurable for all $x \in \mathbf{R}^n$ we have

$$\Big\|\Big(\int_T |(\widehat{f_t}\chi_{R(t)})^\vee|^2 \, d\mu(t)\Big)^{\frac{1}{2}}\Big\|_{L^p} \le C_n \max(p, (p-1)^{-1})^n \Big\|\Big(\int_T |f_t|^2 \, d\mu(t)\Big)^{\frac{1}{2}}\Big\|_{L^p},$$

$\big[$*Hint:* When $n=1$ reduce matters to an $L^p(L^2(T,d\mu),L^2(T,d\mu))$ inequality for the Hilbert transform, via the hint in the preceding exercise. Verify the inequality $p=2$ and then use Theorem 5.6.1 for the other p's. Obtain the n-dimensional inequality by iterating the one-dimensional.$\big]$

5.6.3. Let Φ be a function on $\mathbf{R}^n$ that satisfies $\sup_{x\in\mathbf{R}^n}|x|^n|\Phi(x)|\le A$ and

$$\int_{\mathbf{R}^n}|\Phi(x-y)-\Phi(x)|\,dx\le\eta(y),\qquad\int_{|x|\ge R}|\Phi(x)|\,dx\le\eta(R^{-1}),$$

for all $R\ge 1$, where η is a continuous increasing function on $[0,2]$ that satisfies $\eta(0)=0$ and $\int_0^2\frac{\eta(t)}{t}\,dt<\infty$.
(a) Prove that (5.6.27) holds.
(b) Show that if Φ lies in $L^1(\mathbf{R}^n)$, then the maximal function $f\mapsto\sup_{j\in\mathbf{Z}}|f*\Phi_{2^j}|$ maps $L^p(\mathbf{R}^n)$ to itself for $1<p\le\infty$.
$\big[$*Hint:* Part (a): Modify the calculation in the proof of Theorem 5.6.6. Part (b): Use Theorem 5.6.1 with $r=\infty$.$\big]$

5.6.4. (a) On $\mathbf{R}$, take $f_j=\chi_{[2^{j-1},2^j]}$ to prove that inequality (5.6.25) fails when $p=\infty$ and $1<r<\infty$.
(b) Again on $\mathbf{R}$, take $N>2$ and $f_j=\chi_{[\frac{j-1}{N},\frac{j}{N}]}$ for $j=1,2,\dots,N$ to prove that (5.6.25) fails when $1<p<\infty$ and $r=1$.

5.6.5. Let K be an integrable function on the real line and assume that the operator $f\mapsto f*K$ is bounded on $L^p(\mathbf{R})$ for some $1<p<\infty$. Prove that the vector-valued inequality

$$\Big\|\Big(\sum_j|K*f_j|^q\Big)^{\frac1q}\Big\|_{L^p}\le C_{p,q}\Big\|\Big(\sum_j|f_j|^q\Big)^{\frac1q}\Big\|_{L^p}$$

may fail in general when $q<1$.
$\big[$*Hint:* Take $K=\chi_{[-1,1]}$ and $f_j=\chi_{[\frac{j-1}{N},\frac{j}{N}]}$ for $1\le j\le N$.$\big]$

5.6.6. Let $\{Q_j\}_j$ be a countable collection of cubes in $\mathbf{R}^n$ with disjoint interiors. Let c_j be the center of the cube Q_j and d_j its diameter. For $\varepsilon>0$, define the *Marcinkiewicz function* associated with the family $\{Q_j\}_j$ as follows:

$$M_\varepsilon(x)=\sum_j\frac{d_j^{n+\varepsilon}}{|x-c_j|^{n+\varepsilon}+d_j^{n+\varepsilon}}.$$

Prove that for some constants $C_{n,\varepsilon,p}$ and $C_{n,\varepsilon}$ one has

$$\begin{aligned}\big\|M_\varepsilon\big\|_{L^p}&\le C_{n,\varepsilon,p}\Big(\sum_j|Q_j|\Big)^{\frac1p}, &\qquad p>\frac{n}{n+\varepsilon},\\ \big\|M_\varepsilon\big\|_{L^{\frac{n}{n+\varepsilon},\infty}}&\le C_{n,\varepsilon}\Big(\sum_j|Q_j|\Big)^{\frac{n+\varepsilon}{n}}, &\end{aligned}$$

and consequently $\int_{\mathbf{R}^n}M_\varepsilon(x)\,dx\le C_{n,\varepsilon}\sum_j|Q_j|$.

[*Hint:* Verify that

$$\frac{d_j^{n+\varepsilon}}{|x-c_j|^{n+\varepsilon}+d_j^{n+\varepsilon}} \le CM(\chi_{Q_j})(x)^{\frac{n+\varepsilon}{n}}$$

and use Theorem 5.6.6.]

HISTORICAL NOTES

The L^p boundedness of the conjugate function on the circle was announced in 1924 by Riesz [292], but its first proof appeared three years later in [294]. In view of the identification of the Hilbert transform with the conjugate function, the L^p boundedness of the Hilbert transform is also attributed to M. Riesz. Riesz's proof was first given for $p = 2k$, $k \in \mathbf{Z}^+$, via an argument similar to that in the proof of Theorem 4.1.7. For $p \neq 2k$ this proof relied on interpolation and was completed with the simultaneous publication of Riesz's article on interpolation of bilinear forms [293]. The weak type $(1,1)$ property of the Hilbert transform is due to Kolmogorov [197]. Additional proofs of the boundedness of the Hilbert transform have been obtained by Stein [350], Loomis [230], and Calderón [41]. The proof of Theorem 5.1.7, based on identity (5.1.23), is a refinement of a proof given by Cotlar [75].

The norm of the conjugate function on $L^p(\mathbf{T}^1)$, and consequently that of the Hilbert transform on $L^p(\mathbf{R})$, was shown by Gohberg and Krupnik [129] to be $\cot(\pi/2p)$ when p is a power of 2. Duality gives that this norm is $\tan(\pi/2p)$ for $1 < p \le 2$ whenever p' is a power of 2. Pichorides [282] extended this result to all $1 < p < \infty$ by refining Calderón's proof of Riesz's theorem. This result was also independently obtained by B. Cole (unpublished). The direct and simplified proof for the Hilbert transform given in Exercise 5.1.12 is in Grafakos [130]. The norm of the operators $\frac{1}{2}(I \pm iH)$ for real-valued functions was found to be $\frac{1}{2}\big[\min(\cos(\pi/2p), \sin(\pi/2p))\big]^{-1}$ by Verbitsky [366] and later independently by Essén [108]. The norm of the same operators for complex-valued functions was shown to be equal to $[\sin(\pi/p)]^{-1}$ by Hollenbeck and Verbitsky [156]. Exact formulas for the L^p norm, $1 \le p < \infty$ of the Hilbert transform acting on a characteristic function were obtained by Laeng [211]. The best constant in the weak type $(1,1)$ estimate for the Hilbert transform is equal to $(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots)(1-\frac{1}{3^2}+\frac{1}{5^2}-\cdots)^{-1}$ as shown by Davis [91] using Brownian motion; an alternative proof was later obtained by Baernstein [17]. Iwaniec and Martin [175] showed that the norms of the Riesz transforms on $L^p(\mathbf{R}^n)$ coincide with that of the Hilbert transform on $L^p(\mathbf{R})$ for $1 < p < \infty$.

Operators of the kind T_Ω as well as the stopping-time decomposition of Theorem 5.3.1 were introduced by Calderón and Zygmund [46]. In the same article, Calderón and Zygmund used this decomposition to prove Theorem 5.3.3 for operators of the form T_Ω when Ω satisfies a certain weak smoothness condition. The more general condition (5.3.12) first appeared in Hörmander's article [159]. A more flexible condition sufficient to yield weak type $(1,1)$ bounds is contained in the article of Duong and McIntosh [104]. Theorems 5.2.10 and 5.2.11 are also due to Calderón and Zygmund [48]. The latter article contains the method of rotations. Algebras of operators of the form T_Ω were studied in [49]. For more information on algebras of singular integrals see the article of Calderón [44]. Theorem 5.4.1 is due to Benedek, Calderón, and Panzone [22], while Example 5.4.2 is taken from Muckenhoupt [259]. Theorem 5.4.5 is due to Riviere [296]. A weaker version of this theorem, applicable for smoother singular integrals such as the maximal Hilbert transform, was obtained by Cotlar [75] (Theorem 5.3.4). Improvements of the main inequality in Theorem 5.3.4 for homogeneous singular integrals were obtained by Mateu and Verdera [245] and Mateu, Orobitg, and Verdera [244]. For a general overview of singular integrals and their applications, one may consult the expository article of Calderón [43].

Part (a) of Theorem 5.5.1 is due to Marcinkiewicz and Zygmund [242], although the case $p = q$ was proved earlier by Paley [273] with a larger constant. The values of r for which a general linear operator of weak or strong type (p,q) admits bounded ℓ^r extensions are described in Rubio de Francia and Torrea [304]. The L^p and weak L^p spaces in Theorem 5.5.1 can be replaced by general Banach lattices, as shown by Krivine [206] using Grothendieck's inequality. Hilbert-space-valued estimates for singular integrals were obtained by Benedek, Calderón, and Panzone [22]. Other operator-valued singular integral operators were studied by Rubio de Francia, Ruiz, and Torrea [303]. Banach-valued singular integrals are studied in great detail in the book of García-Cuerva and Rubio de Francia [122], which provides an excellent presentation of the subject. The ℓ^r-valued estimates (5.5.16) for the Hilbert transform were first obtained by Boas and Bochner [28]. The corresponding vector-valued estimates for the Hardy–Littlewood maximal function in Theorem 5.6.6 are due to Fefferman and Stein [115]. Conditions of the form (5.6.18) have been applied to several situations and can be traced in Zo [386].

The sharpness of the logarithmic condition (5.2.24) was indicated by Weiss and Zygmund [372], who constructed an example of an integrable function Ω with vanishing integral on $\mathbf{S}^1$ satisfying $\int_{\mathbf{S}^{n-1}} |\Omega(\theta)| \log^+ |\Omega(\theta)| \big(\log(2+\log(2+|\Omega(\theta)|))\big)^{-\delta} d\theta = \infty$ for all $\delta > 0$ and of a continuous function in $L^p(\mathbf{R}^2)$ for all $1 < p < \infty$ such that $\limsup_{\varepsilon \to 0} |T_\Omega^{(\varepsilon)}(f)(x)| = \infty$ for almost all $x \in \mathbf{R}^2$. The proofs of Theorems 5.2.10 and 5.2.11 can be modified to give that if Ω is in the Hardy space H^1 of $\mathbf{S}^{n-1}$, then T_Ω and $T_\Omega^{(*)}$ map L^p to L^p for $1 < p < \infty$. For T_Ω this fact was proved by Connett [72] and independently by Ricci and Weiss [289]; for $T_\Omega^{(*)}$ this was proved by Fan and Pan [110] and independently by Grafakos and Stefanov [139]. The latter authors [138] also obtained that the logarithmic condition $\operatorname{ess.sup}_{|\xi|=1} \int_{\mathbf{S}^{n-1}} |\Omega(\theta)| (\log \frac{1}{|\xi \cdot \theta|} |)^{1+\alpha} d\theta < \infty$, $\alpha > 0$, implies L^p boundedness for T_Ω and $T_\Omega^{(*)}$ for some $p \neq 2$. See also Fan, Guo, and Pan [109] as well as Ryabogin and Rubin [308] for extensions. Examples of functions Ω for which T_Ω maps L^p to L^p for a certain range of p's but not for other ranges of p's is given in Grafakos, Honzík, and Ryabogin [132]. A different example of this sort was provided later by Honzík [158]; the range of p's for which boundedness holds are different for these examples. Honzík [157] also constructed a delicate example of an integrable function Ω with mean value zero over $\mathbf{S}^1$ such that T_Ω is bounded on $L^2(\mathbf{R})$ but $T_\Omega^{(*)}$ is not.

The relatively weak condition $|\Omega| \log^+ |\Omega| \in L^1(\mathbf{S}^{n-1})$ also implies weak type $(1,1)$ boundedness for operators T_Ω. This was obtained by Seeger [317] and later extended by Tao [355] to situations in which there is no Fourier transform structure. Earlier partial results are in Christ and Rubio de Francia [63] and in the simultaneous work of Hofmann [155], both inspired by the work of Christ [60]. Soria and Sjögren [324] showed that for arbitrary Ω in $L^1(\mathbf{S}^{n-1})$, T_Ω is weak type $(1,1)$ when restricted to radial functions. Examples due to Christ (published in [139]) indicate that even for bounded functions Ω on $\mathbf{S}^{n-1}$, T_Ω may not map the endpoint Hardy space $H^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$. However, Seeger and Tao [318] have showed that T_Ω always maps the Hardy space $H^1(\mathbf{R}^n)$ to the Lorentz space $L^{1,2}(\mathbf{R}^n)$ when $|\Omega|(\log^+ |\Omega|)^2$ is integrable over $\mathbf{S}^{n-1}$. This result is sharp in the sense that for such Ω, T_Ω may not map $H^1(\mathbf{R}^n)$ to $L^{1,q}(\mathbf{R}^n)$ when $q < 2$ in general. If T_Ω maps $H^1(\mathbf{R}^n)$ to itself, Daly and Phillips [87] (in dimension $n = 2$) and Daly [86] (in dimensions $n \geq 3$) showed that Ω must lie in the Hardy space $H^1(\mathbf{S}^{n-1})$. There are also results concerning the singular maximal operator $M_\Omega(f)(x) = \sup_{r>0} \frac{1}{v_n r^n} \int_{|y| \leq r} |f(x-y)| |\Omega(y)| dy$, where Ω is an integrable function on $\mathbf{S}^{n-1}$ of not necessarily vanishing integral. Such operators were studied by Fefferman [116], Christ [60], and Hudson [162]. An excellent treatment of several kinds of singular integral operators with rough kernels is contained in the book of Lu, Ding, and Yan [234].