Chapter 4
Topics on Fourier Series

In this chapter we go deeper into the theory of Fourier series and we study topics such as convergence in norm and the conjugate function, divergence of Fourier series and Bochner–Riesz summability. We also study transference of multipliers on the torus and of maximal multipliers. This is a powerful technique that allows one to infer results concerning Fourier series from corresponding results about Fourier integrals and vice versa.

   We also take a quick look at applications of Fourier series such as the isoperimetric inequality problem, the distribution of lattice points in a ball, and the heat equation. The power of Fourier series techniques manifests itself in the study of these problems which represent only a small part of the wide and vast range of applications of the subject known today.

4.1 Convergence in Norm, Conjugate Function, and Bochner–Riesz Means

In this section we address the following fundamental question: Do Fourier series converge in norm? We begin with some abstract necessary and sufficient conditions that guarantee such a convergence. In one dimension, we are able to reduce matters to the study of the so-called conjugate function on the circle, a sister operator of the Hilbert transform, which is the center of study of the next chapter. In higher dimensions the situation is more complicated, but we are able to give a positive answer in the case of square summability.

4.1.1 Equivalent Formulations of Convergence in Norm

The question we pose is for which indices p, with $1 \le p < \infty$, we have

$$\left\| D_N^n * f - f \right\|_{L^p(\mathbf{T}^n)} \to 0 \qquad \text{as } N \to \infty, \tag{4.1.1}$$

and similarly for the circular Dirichlet kernel $\overset{\circ}{D}_N^n$. We tackle this question by looking at an equivalent formulation of it.

Theorem 4.1.1. *For $R > 0$ and $m \in \mathbf{Z}^n$, let $a(m,R)$ be complex numbers such that*

(i) For every $R > 0$ there is a q_R such that $a(m,R) = 0$ if $|m| > q_R$.
(ii) There is an $M_0 < \infty$ such that $|a(m,R)| \le M_0$ for all $m \in \mathbf{Z}^n$ and all $R > 0$.
(iii) For each $m \in \mathbf{Z}^n$, the limit of $a(m,R)$ exists as $R \to \infty$ and $\lim_{R\to\infty} a(m,R) = a_m$.

Let $1 \le p < \infty$. For $f \in L^p(\mathbf{T}^n)$ and $x \in \mathbf{T}^n$ define

$$S_R(f)(x) = \sum_{m \in \mathbf{Z}^n} a(m,R)\widehat{f}(m)e^{2\pi i m \cdot x}$$

noting that the sum is well defined because of (i). Also, for $h \in \mathscr{C}^\infty(\mathbf{T}^n)$ define

$$A(h)(x) = \sum_{m \in \mathbf{Z}^n} a_m \widehat{h}(m)e^{2\pi i m \cdot x}.$$

Then for all $f \in L^p(\mathbf{T}^n)$ the sequence $S_R(f)$ converges in L^p as $R \to \infty$ if and only if there exists a constant $K < \infty$ such that

$$\sup_{R>0} \left\| S_R \right\|_{L^p \to L^p} \le K. \tag{4.1.2}$$

Furthermore, if (4.1.2) holds, then for the same constant K we have

$$\sup_{\substack{h \in \mathscr{C}^\infty \\ h \ne 0}} \frac{\left\| A(h) \right\|_{L^p}}{\|h\|_{L^p}} \le K, \tag{4.1.3}$$

and then A extends to a bounded operator $\widetilde{A}$ from $L^p(\mathbf{T}^n)$ to itself; moreover, for every $f \in L^p(\mathbf{T}^n)$ we have that $S_R(f) \to \widetilde{A}(f)$ in L^p as $R \to \infty$.

Proof. If $S_R(f)$ converges in L^p, then $\|S_R(f)\|_{L^p} \le C_f$ for some constant C_f that depends on $f \in L^p(\mathbf{T}^n)$. Moreover, each S_R is a bounded operator from $L^p(\mathbf{T}^n)$ to itself with norm at most $\#\{m \in \mathbf{Z}^n : |m| \le q_R\} M_0$. Thus $\{S_R\}_{R>0}$ is a family of L^p-bounded linear operators that satisfy $\sup_{R>0} \|S_R(f)\|_{L^p} \le C_f$ for each $f \in L^p(\mathbf{T}^n)$. The uniform boundedness theorem applies and yields that the operator norms of S_R from L^p to L^p are bounded uniformly in R. This proves (4.1.2).

Conversely, assume (4.1.2). For $h \in \mathscr{C}^\infty(\mathbf{T}^n)$, we have that

$$\lim_{R\to\infty} \sum_{m \in \mathbf{Z}^n} a(m,R)\widehat{h}(m)e^{2\pi i m \cdot x} = \sum_{m \in \mathbf{Z}^n} a_m \widehat{h}(m)e^{2\pi i m \cdot x}$$

in view of property (ii) and of the Lebesgue dominated convergence theorem, since $\sum_{m \in \mathbf{Z}^n} |\widehat{h}(m)| < \infty$. Fatou's lemma now gives

$$\big\|A(h)\big\|_{L^p} = \big\| \lim_{R \to \infty} S_R(h)\big\|_{L^p} \le \liminf_{R \to \infty} \big\|S_R(h)\big\|_{L^p} \le K\big\|h\big\|_{L^p};$$

hence (4.1.3) holds. Thus A extends to a bounded operator $\widetilde{A}$ on $L^p(\mathbf{T}^n)$ by density.

We show that for all $f \in L^p(\mathbf{T}^n)$ we have $S_R(f) \to \widetilde{A}(f)$ in L^p as $R \to \infty$. Fix f in $L^p(\mathbf{T}^n)$ and let $\varepsilon > 0$ be given. Pick a trigonometric polynomial P satisfying $\|f - P\|_{L^p} \le \varepsilon$. Let d be the degree of P. Then there is an $R_0 > 0$ such that for all $R > R_0$ we have

$$\sum_{|m_1| + \cdots + |m_n| \le d} |a(m,R) - a_m| |\widehat{P}(m)| \le \varepsilon$$

since $a(m,R) \to a_m$ for every m with $|m_1| + \cdots + |m_n| \le d$. We deduce that

$$\begin{aligned}
\big\|S_R(P) - A(P)\big\|_{L^p} &\le \big\|S_R(P) - A(P)\big\|_{L^\infty} \\
&\le \sum_{|m_1| + \cdots + |m_n| \le d} |a(m,R) - a_m| |\widehat{P}(m)| \\
&\le \varepsilon,
\end{aligned}$$

whenever $R > R_0$. Then

$$\begin{aligned}
\big\|S_R(f) - \widetilde{A}(f)\big\|_{L^p} &\le \big\|S_R(f) - S_R(P)\big\|_{L^p} + \big\|S_R(P) - \widetilde{A}(P)\big\|_{L^p} + \big\|\widetilde{A}(P) - \widetilde{A}(f)\big\|_{L^p} \\
&\le K\varepsilon + \varepsilon + K\varepsilon = (2K+1)\varepsilon
\end{aligned}$$

for $R > R_0$. This proves that $S_R(f)$ converges to $\widetilde{A}(f)$ in L^p as $R \to \infty$. $\qquad\square$

The most interesting situation arises, of course, when $a(m,R) \to a_m = 1$ for all $m \in \mathbf{Z}^n$. In this case A (and $\widetilde{A}$) is the identity operator, and thus we expect the operators $S_R(f)$ to converge back to f as $R \to \infty$. We should keep in mind the following three examples:

(a) The sequence $a(m,R) = 1$ when $|m_j| \le R$ for all $j \in \{1,2,\ldots,n\}$ and zero otherwise, in which case the operator S_R of Theorem 4.1.1 is

$$S_R(f) = f * D_R^n; \tag{4.1.4}$$

(b) The sequence $a(m,R) = 1$ when $|m| \le R$ and zero otherwise, in which case the S_R of Theorem 4.1.1 is

$$\overset{\circ}{S}_R(f) = f * \overset{\circ}{D}_R^n; \tag{4.1.5}$$

(c) The sequence $a(m,R) = \left(1 - \frac{|m|^2}{R^2}\right)_+^\alpha$, for some $\alpha > 0$, in which case we denote S_R by B_R^α.

Definition 4.1.2. The *Bochner–Riesz operator* or *Bochner–Riesz means* of order $\alpha \geq 0$ is the operator

$$B_R^{\alpha}(f)(x) = \sum_{\substack{m \in \mathbf{Z}^n \\ |m| \leq R}} \left(1 - \frac{|m|^2}{R^2}\right)^{\alpha} \widehat{f}(m) e^{2\pi i m \cdot x} \tag{4.1.6}$$

defined on integrable functions f on $\mathbf{T}^n$.

Corollary 4.1.3. *Let $1 \leq p < \infty$ and $\alpha \geq 0$. Let S_R and $\mathring{S}_R$ be as in (4.1.4) and (4.1.5), respectively, and let B_R^{α} be the Bochner–Riesz means as defined in (4.1.6). Then*

$$\forall f \in L^p(\mathbf{T}^n), \quad \lim_{R \to \infty} \left\|D_R^n * f - f\right\|_{L^p} = 0 \quad \Longleftrightarrow \quad \sup_{R \geq 0} \left\|S_R\right\|_{L^p \to L^p} < \infty,$$

$$\forall f \in L^p(\mathbf{T}^n), \quad \lim_{R \to \infty} \left\|\mathring{D}_R^n * f - f\right\|_{L^p} = 0 \quad \Longleftrightarrow \quad \sup_{R \geq 0} \left\|\mathring{S}_R\right\|_{L^p \to L^p} < \infty,$$

$$\forall f \in L^p(\mathbf{T}^n), \quad \lim_{R \to \infty} \left\|B_R^{\alpha} * f - f\right\|_{L^p} = 0 \quad \Longleftrightarrow \quad \sup_{R \geq 0} \left\|B_R^{\alpha}\right\|_{L^p \to L^p} < \infty.$$

Example 4.1.4. We investigate the one-dimensional case in some detail. We take $n = 1$, and we define $a(m, N) = 1$ for all $-N \leq m \leq N$, and zero otherwise. Then $S_N(f) = \mathring{S}_N(f) = D_N * f$, where D_N is the Dirichlet kernel. Clearly, the expressions $\|S_N\|_{L^p \to L^p}$ are bounded above by the L^1 norm of D_N, but this estimation yields a bound that blows up as $N \to \infty$. We later show, via a more delicate argument, that the expressions $\|S_N\|_{L^p \to L^p}$ are uniformly bounded in N when $1 < p < \infty$.

This reasoning, however, allows us to deduce that for some function $g \in L^1(\mathbf{T}^1)$, $S_N(g)$ may not converge in L^1. This is also a consequence of the proof of Theorem 4.2.1; see (4.2.13). Note that since the Fejér kernel F_M has L^1 norm 1, we have

$$\left\|S_N\right\|_{L^1 \to L^1} \geq \lim_{M \to \infty} \left\|D_N * F_M\right\|_{L^1} = \left\|D_N\right\|_{L^1}.$$

This implies that the expressions $\|S_N\|_{L^1 \to L^1}$ are not uniformly bounded in N, and therefore Corollary 4.1.3 gives that for some $f_0 \in L^1(\mathbf{T}^1)$, $S_N(f_0)$ does not converge to f_0 in L^1.

Although the partial sums of Fourier series fail to convergence in $L^1(\mathbf{T}^n)$, it is a consequence of Plancherel's theorem that they converge in $L^2(\mathbf{T}^n)$. More precisely, if $f \in L^2(\mathbf{T}^n)$, then

$$\left\|\mathring{D}_N^n * f - f\right\|_{L^2}^2 = \sum_{|m| > N} |\widehat{f}(m)|^2 \to 0$$

as $N \to \infty$ and the same result is true for $D_N^n * f$ and for $B_R^{\alpha} * f$; for the latter, we apply Theorem 4.1.1, noting that

$$\left\|B_R^{\alpha}(f)\right\|_{L^2}^2 = \sum_{m \in \mathbf{Z}^n} \left|\left(1 - \frac{|m|^2}{R^2}\right)_+^{\alpha}\right|^2 |\widehat{f}(m)|^2 \leq \sum_{m \in \mathbf{Z}^n} |\widehat{f}(m)|^2 = \|f\|_{L^2}^2$$

and thus $\sup_{R > 0} \|B_R^{\alpha}\|_{L^2 \to L^2} \leq 1$.

Motivated by the preceding discussion for $p = 2$, it is natural to pose the following question. Can 2 be replaced by $p \neq 2$ in the preceding results? This question has an affirmative answer in dimension one for D_N. In higher dimensions an interesting dichotomy appears. As a consequence of the one-dimensional result, the square partial sums $D_N^n * f$ converge in L^p to a given f in $L^p(\mathbf{T}^n)$, but for the circular partial sums this may not be the case.

We begin the discussion with the one-dimensional situation.

Definition 4.1.5. For $f \in \mathscr{C}^\infty(\mathbf{T}^1)$ define the *conjugate function* $\widetilde{f}$ by

$$\widetilde{f}(x) = -i \sum_{m \in \mathbf{Z}^1} \operatorname{sgn}(m) \widehat{f}(m) e^{2\pi i m x},$$

where $\operatorname{sgn}(m) = 1$ for $m > 0$, -1 for $m < 0$, and 0 for $m = 0$. Also define the *Riesz projections* P_+ and P_- by

$$P_+(f)(x) = \sum_{m=1}^{\infty} \widehat{f}(m) e^{2\pi i m x}, \tag{4.1.7}$$

$$P_-(f)(x) = \sum_{m=-\infty}^{-1} \widehat{f}(m) e^{2\pi i m x}. \tag{4.1.8}$$

Observe that $f = P_+(f) + P_-(f) + \widehat{f}(0)$, while $\widetilde{f} = -iP_+(f) + iP_-(f)$, when f is in $\mathscr{C}^\infty(\mathbf{T}^1)$. Consequently, one has

$$P_+(f) = \frac{1}{2}(f + i\widetilde{f}) - \frac{1}{2}\widehat{f}(0) \tag{4.1.9}$$

and therefore the L^p boundedness of the operator $f \mapsto \widetilde{f}$ is equivalent to that of the operator $f \mapsto P_+(f)$, since the identity and the operator $f \mapsto \widehat{f}(0)$ are obviously L^p bounded. Clearly, these statements are also valid for the other Riesz projection $f \mapsto P_-(f)$. The following is a consequence of Theorem 4.1.1.

Proposition 4.1.6. *Let $1 \leq p < \infty$. Then the expressions $S_N(f) = D_N * f$ converge to f in $L^p(\mathbf{T}^1)$ as $N \to \infty$ if and only if there exists a constant $C_p > 0$ such that for all smooth functions f on $\mathbf{T}^n$ we have $\|\widetilde{f}\|_{L^p(\mathbf{T}^1)} \leq C_p \|f\|_{L^p(\mathbf{T}^1)}$.*

Proof. In view of Corollary 4.1.3, the fact that for all $f \in L^p(\mathbf{T}^1)$, $S_N(f) \to f$ in L^p as $N \to \infty$ is equivalent to the uniform (in N) L^p boundedness of S_N.

We note the validity of the identity

$$e^{-2\pi i N x} \sum_{m=0}^{2N} \big(f(\cdot) e^{2\pi i N(\cdot)}\big)^{\widehat{}}(m) e^{2\pi i m x} = \sum_{m=-N}^{N} \widehat{f}(m) e^{2\pi i m x}.$$

Since multiplication by exponentials does not affect L^p norms, this identity implies that the norm of the operator $S_N(f) = D_N * f$ from L^p to L^p is equal to that of the operator

$$S'_N(g)(x) = \sum_{m=0}^{2N} \widehat{g}(m)e^{2\pi imx}$$

from L^p to L^p. Therefore,

$$\sup_{N\geq 0}\left\|S_N\right\|_{L^p\to L^p} < \infty \iff \sup_{N\geq 0}\left\|S'_N\right\|_{L^p\to L^p} < \infty, \tag{4.1.10}$$

and both of these statements are equivalent to the fact that for all $f \in L^p(\mathbf{T}^1)$, $S_N(f) \to f$ in L^p as $N \to \infty$.

We have already observed that the L^p boundedness of the conjugate function is equivalent to that of P_+. Therefore, it suffices to show that the L^p boundedness of P_+ is equivalent to the uniform L^p boundedness of S'_N.

Suppose first that $\sup_{N\geq 0}\|S'_N\|_{L^p\to L^p} < \infty$. Theorem 4.1.1 applied to the sequence $a(m,R)=1$ for $0 \leq m \leq R$ and $a(m,R)=0$ otherwise gives that the operator $A(f) = P_+(f) + \widehat{f}(0)$ is bounded on $L^p(\mathbf{T}^1)$. Hence so is P_+.

Conversely, suppose that P_+ extends to a bounded operator from $L^p(\mathbf{T}^1)$ to itself. For all h in $\mathscr{C}^\infty(\mathbf{T}^n)$ we can write

$$
\begin{aligned}
S'_N(h)(x) &= \sum_{m=0}^{\infty} \widehat{h}(m)e^{2\pi imx} - \sum_{m=2N+1}^{\infty} \widehat{h}(m)e^{2\pi imx} \\
&= \sum_{m=1}^{\infty} \widehat{h}(m)e^{2\pi imx} + \widehat{h}(0) - e^{2\pi i(2N)x}\sum_{m=1}^{\infty} \widehat{h}(m+2N)e^{2\pi imx} \\
&= P_+(h)(x) - e^{2\pi i(2N)x}P_+(e^{-2\pi i(2N)(\cdot)}h) + \widehat{h}(0).
\end{aligned}
$$

This identity implies that

$$\sup_{N\geq 0}\left\|S'_N(f)\right\|_{L^p} \leq \left(2\|P_+\|_{L^p\to L^p} + 1\right)\|f\|_{L^p} \tag{4.1.11}$$

for all f smooth, and by density for all $f \in L^p(\mathbf{T}^1)$. Note that S'_N is well defined on $L^p(\mathbf{T}^1)$. Thus the operators S'_N are uniformly bounded on $L^p(\mathbf{T}^n)$.

Thus the uniform L^p boundedness of S_N is equivalent to the uniform L^p boundedness of S'_N, which is equivalent to the L^p boundedness of P_+, which in turn is equivalent to the L^p boundedness of the conjugate function. $\square$

4.1.2 The L^p Boundedness of the Conjugate Function

We know now that convergence of Fourier series in L^p is equivalent to the L^p boundedness of the conjugate function or either of the two Riesz projections. It is natural to ask whether these operators are L^p bounded.

Theorem 4.1.7. *Given* $1 < p < \infty$, *there is a constant* $A_p > 0$ *such that for all* f *in* $\mathscr{C}^\infty(\mathbf{T}^1)$ *we have*

$$\|\widetilde{f}\|_{L^p} \le A_p\|f\|_{L^p}. \tag{4.1.12}$$

Thus the operator $f \mapsto \widetilde{f}$ *has a bounded extension on* $L^p(\mathbf{T}^1)$ *that also satisfies* (4.1.12).

Consequently, the Fourier series of L^p functions on the circle converge back to the functions in the L^p norm for $1 < p < \infty$.

Proof. In proving the inequality (4.1.12), we make the following reductions:
(a) We assume that f is trigonometric polynomial.
(b) We assume that $\widehat{f}(0) = 0$.
(c) We assume that f is real valued.

Since f is a real-valued function, we have that $\widehat{f}(-m) = \overline{\widehat{f}(m)}$ for all m, and since $\widehat{f}(0) = 0$, we can write

$$\widetilde{f}(t) = -i\sum_{m=1}^\infty \widehat{f}(m)e^{2\pi imt} + i\sum_{m=1}^\infty \widehat{f}(-m)e^{-2\pi imt} = 2\mathrm{Re}\left[-i\sum_{m=1}^\infty \widehat{f}(m)e^{2\pi imt}\right],$$

which implies that $\widetilde{f}$ is also real-valued (see also Exercise 4.1.4(b)). Therefore the polynomial $f + i\widetilde{f}$ contains only positive frequencies. Thus for $k \in \mathbf{Z}^+$ we have

$$\int_{\mathbf{T}^1}(f(t) + i\widetilde{f}(t))^{2k}\,dt = 0.$$

Expanding the $2k$ power and taking real parts, we obtain

$$\sum_{j=0}^k (-1)^{k-j}\binom{2k}{2j}\int_{\mathbf{T}^1}\widetilde{f}(t)^{2k-2j}f(t)^{2j}\,dt = 0,$$

where we used that f is real-valued. Therefore,

$$\|\widetilde{f}\|_{L^{2k}}^{2k} \le \sum_{j=1}^k \binom{2k}{2j}\int_{\mathbf{T}^1}\widetilde{f}(t)^{2k-2j}f(t)^{2j}\,dt$$

$$\le \sum_{j=1}^k \binom{2k}{2j}\|\widetilde{f}\|_{L^{2k}}^{2k-2j}\|f\|_{L^{2k}}^{2j},$$

by applying Hölder's inequality with exponents $2k/(2k-2j)$ and $2k/(2j)$ to the jth term of the sum. Dividing the last inequality by $\|f\|_{L^{2k}}^{2k}$, we obtain

$$R^{2k} \le \sum_{j=1}^k \binom{2k}{2j}R^{2k-2j}, \tag{4.1.13}$$

where $R = \|\widetilde{f}\|_{L^{2k}}/\|f\|_{L^{2k}}$. If $R > 0$ satisfies (4.1.13), then $R \leq C_{2k}$, where C_{2k} is the largest real root of the polynomial $g(t) = t^{2k} - \sum_{j=1}^{k} \binom{2k}{2j} t^{2k-2j}$. (Since $g(0) < 0$ and $\lim_{t\to\infty} g(t) = \infty$, g has at least one real root.) We conclude that if f satisfies (a), (b), and (c), then we have for all $k = 1, 2, \ldots$

$$\|\widetilde{f}\|_{L^{2k}} \leq C_{2k}\|f\|_{L^{2k}}. \tag{4.1.14}$$

We now remove assumptions (a), (b), and (c). We first remove assumption (c). Given a complex-valued trigonometric polynomial f with $\widehat{f}(0) = 0$, we write

$$f(t) = \sum_{j=-N}^{N} c_j e^{2\pi ijt} = \left[\sum_{j=-N}^{N} \frac{c_j + \overline{c_{-j}}}{2} e^{2\pi ijt}\right] + i\left[\sum_{j=-N}^{N} \frac{c_j - \overline{c_{-j}}}{2i} e^{2\pi ijt}\right]$$

(with $c_0 = 0$) and we note that the expressions inside the square brackets are real-valued trigonometric polynomials. Thus we can express f as $P + iQ$, where P and Q are real-valued trigonometric polynomials, and applying (4.1.14) to P and Q we obtain the inequality

$$\|\widetilde{f}\|_{L^{2k}} \leq 2C_{2k}\|f\|_{L^{2k}} \tag{4.1.15}$$

for all trigonometric polynomials f with $\widehat{f}(0) = 0$.

Next, we remove the assumption that $\widehat{f}(0) = 0$. We write $f = (f - \widehat{f}(0)) + \widehat{f}(0)$, we observe that the conjugate function of a constant is zero, and we apply (4.1.15) to obtain

$$\|\widetilde{f}\|_{L^{2k}} \leq 2C_{2k}\|f - \widehat{f}(0)\|_{L^{2k}} \leq 2C_{2k}\left[\|f\|_{L^{2k}} + \|f\|_{L^1}\right] \leq 4C_{2k}\|f\|_{L^{2k}}.$$

Since trigonometric polynomials are dense in L^p, it follows that the operator $f \mapsto \widetilde{f}$ has a bounded extension on L^{2k} that satisfies (4.1.12) for all $f \in L^{2k}$, and in particular for all $f \in \mathscr{C}^\infty(\mathbf{T}^1)$.

Every real number $p \geq 2$ lies in an interval of the form $[2k, 2k+2]$, for some $k \in \mathbf{Z}^+$. Theorem 1.3.4 gives that for all $2 \leq p < \infty$ there is a constant A_p such that

$$\|\widetilde{f}\|_{L^p} \leq A_p\|f\|_{L^p} \tag{4.1.16}$$

when f is a simple function. Thus the conjugate function has a bounded extension on L^p that satisfies (4.1.16) when $p \geq 2$.

To extend this result for $p < 2$ we use duality. We observe that the adjoint operator of $f \mapsto \widetilde{f}$ is $f \mapsto -\widetilde{f}$. Indeed, for f, g in $\mathscr{C}^\infty(\mathbf{T}^1)$ we have

$$\langle \widetilde{f} \,|\, g \rangle = \sum_{m\in\mathbf{Z}} -i\,\mathrm{sgn}\,(m)\widehat{f}(m)\overline{\widehat{g}(m)} = -\sum_{m\in\mathbf{Z}} \widehat{f}(m)\overline{-i\,\mathrm{sgn}\,(m)\widehat{g}(m)} = -\langle f \,|\, \widetilde{g} \rangle.$$

By duality, estimate (4.1.16) is also valid for $1 < p \leq 2$ with constant $A_{p'} = A_p$. $\quad\square$

We extend the preceding result to higher dimensions.

Theorem 4.1.8. *Let $1 < p < \infty$ and $f \in L^p(\mathbf{T}^n)$. Then $D_N^n * f$ converges to f in L^p as $N \to \infty$.*

Proof. As a consequence of Corollary 4.1.3, Proposition 4.1.6, and Theorem 4.1.7, it suffices to show that for all f trigonometric polynomials on $\mathbf{T}^n$ we have

$$\sup_{N>0} \int_0^1 \cdots \int_0^1 \left| (D_N^n * f)(x) \right|^p dx_1 \cdots dx_n \leq K^{np} \|f\|_{L^p(\mathbf{T}^n)}^p .$$

Obviously, this inequality is valid in dimension $n = 1$. We extend it by induction to all dimensions. We assume that it is valid in dimension $n - 1$ and we prove it in dimension n.

Let $x' = (x_2, \ldots, x_n) \in \mathbf{T}^{n-1}$. For a fixed trigonometric polynomial f, and for fixed $N \geq 0$ and $x' \in \mathbf{T}^{n-1}$, define a trigonometric polynomial $g_{N,x'}$ on $\mathbf{T}^1$ by setting

$$g_{N,x'}(x_1) = \sum_{m_1 \in \mathbf{Z}} \left[\sum_{|m_2|,\ldots,|m_n| \leq N} e^{2\pi i m' \cdot x'} \widehat{f}(m_1, m') \right] e^{2\pi i m_1 x_1}$$

where $m' = (m_2, \ldots, m_n)$. Then we have

$$\begin{aligned}
g_{N,x'}(x_1) &= \sum_{|m_2|,\ldots,|m_n| \leq N} e^{2\pi i m' \cdot x'} \left[\sum_{m_1 \in \mathbf{Z}} e^{2\pi i m_1 x_1} \widehat{f}(m_1, m') \right] \\
&= \sum_{|m_2|,\ldots,|m_n| \leq N} e^{2\pi i m' \cdot x'} \left[\int_{\mathbf{T}^{n-1}} f(x_1, y') e^{-2\pi i m' \cdot y'} dy' \right] \\
&= \sum_{|m_2|,\ldots,|m_n| \leq N} e^{2\pi i m' \cdot x'} \widehat{f_{x_1}}(m') \\
&= (D_N^{n-1} * f_{x_1})(x'),
\end{aligned}$$

where f_{x_1} is the trigonometric polynomial of $n - 1$ variables defined by $f_{x_1}(x') = f(x_1, x')$. We also have that

$$(D_N * g_{N,x'})(x_1) = (D_N^n * f)(x_1, x').$$

Combining this information, we write

$$\begin{aligned}
\int_{\mathbf{T}^{n-1}} \int_0^1 & \left| (D_N^n * f)(x_1, x') \right|^p dx_1 \, dx' \\
&= \int_{\mathbf{T}^{n-1}} \int_0^1 \left| (D_N * g_{N,x'})(x_1) \right|^p dx_1 \, dx' \\
&\leq K^p \int_{\mathbf{T}^{n-1}} \int_0^1 \left| g_{N,x'}(x_1) \right|^p dx_1 \, dx' \\
&= K^p \int_0^1 \int_{\mathbf{T}^{n-1}} \left| (D_N^{n-1} * f_{x_1})(x') \right|^p dx' \, dx_1
\end{aligned}$$

$$\leq K^p K^{(n-1)p} \int_{\mathbf{T}^{n-1}} \int_0^1 \left| f_{x_1}(x') \right|^p dx' \, dx_1$$
$$= K^{np} \left\| f \right\|_{L^p(\mathbf{T}^n)}^p,$$

where the penultimate inequality follows from the induction hypothesis.   □

4.1.3 Bochner–Riesz Summability

In dimension 1 the Fejér means of an integrable function are better behaved than the Dirichlet means. We investigate whether there is a similar phenomenon in higher dimensions. Recall that the circular (or spherical) partial sums of the Fourier series of f are given by

$$(f * \mathring{D}_R^n)(x) = \sum_{\substack{m \in \mathbf{Z}^n \\ |m| \leq R}} \widehat{f}(m) e^{2\pi i m \cdot x},$$

where $R \geq 0$. Taking the averages of these expressions, we obtain

$$\frac{1}{R} \int_0^R (f * \mathring{D}_r^n)(x) \, dr = \sum_{\substack{m \in \mathbf{Z}^n \\ |m| \leq R}} \left(1 - \tfrac{|m|}{R}\right) \widehat{f}(m) e^{2\pi i m \cdot x} = B_R^1(f)(x),$$

and we call these expressions the *circular Cesàro means* (or *circular Fejér means*) of f. It turns out that the circular Cesàro means of integrable functions on $\mathbf{T}^2$ always converge in L^1, but in dimension 3, this may fail. Theorem 4.2.5 gives an example of an integrable function f on $\mathbf{T}^3$ whose circular Cesàro means diverge a.e. However, we show below that this is not the case if the circular Cesàro means of a function f in $L^1(\mathbf{T}^3)$ are replaced by the only slightly different-looking means

$$\sum_{\substack{m \in \mathbf{Z}^n \\ |m| \leq R}} \left(1 - \tfrac{|m|}{R}\right)^{1+\varepsilon} \widehat{f}(m) e^{2\pi i m \cdot x},$$

for some $\varepsilon > 0$. This discussion suggests that the preceding expressions behave better as ε increases, but for a fixed ε they get worse as the dimension increases. The need to understand the behavior of these operators for different values of $\alpha \geq 0$ led to introduction of the operators B_R^α given in Definition 4.1.2.

The family of operators B_R^α forms a natural "spherical" analogue of the Cesàro–Fejér sums. It turns out that there is no significant difference in the behavior of these means if the expression $\left(1 - \tfrac{|m|^2}{R^2}\right)^\alpha$ in (4.1.6) is replaced by the expression $\left(1 - \tfrac{|m|}{R}\right)^\alpha$; see Exercise 4.3.1. The advantage of the quadratic expression in (4.1.6) is that it has an easily computable kernel and yields the elegant reproducing formula

$$B_R^\alpha(f) = \frac{2\Gamma(\alpha+1)}{\Gamma(\alpha-\beta)\Gamma(\beta+1)} \frac{1}{R} \int_0^R \left(1 - \frac{r^2}{R^2}\right)^{\alpha-\beta-1} \left(\frac{r^2}{R^2}\right)^{\beta+\frac{1}{2}} B_r^\beta(f) \, dr, \quad (4.1.17)$$

which precisely quantifies the way in which B_R^α is smoother than B_R^β when $\alpha > \beta$. Identity (4.1.17) also says that when $\alpha > \beta$, the operator $B_R^\alpha(f)$ is an average of the operators $B_r^\beta(f)$, $0 < r < R$, with respect to a certain density.

Note that the Bochner–Riesz means of order zero coincide with the circular (or spherical) Dirichlet means, and, as we have seen, these converge in $L^2(\mathbf{T}^n)$. We address an analogous question on $L^p(\mathbf{T}^n)$ for $p \neq 2$.

Proposition 4.1.9. *Let $1 \leq p < \infty$ and $f \in L^p(\mathbf{T}^n)$. Then the Bochner–Riesz means $B_R^\alpha(f)$ converge to f in $L^p(\mathbf{T}^n)$ as $R \to \infty$ when $\alpha > (n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$. Moreover, if f is continuous on $\mathbf{T}^n$ and $\alpha > \frac{n-1}{2}$, then $B_R^\alpha(f)$ converges to f uniformly as $R \to \infty$.*

Proof. For $z \in \mathbf{C}$ with $\mathrm{Re}\, z \geq 0$, consider the function

$$m_z(\xi) = (1 - |\xi|^2)_+^z$$

defined for ξ in $\mathbf{R}^n$. Note that $\|m_z\|_{L^\infty} = 1$. Using an identity proved in Appendix B.5, we have that

$$(m_z)^\vee(y) = K^z(y) = \frac{\Gamma(z+1)}{\pi^z} \frac{J_{\frac{n}{2}+z}(2\pi|y|)}{|y|^{\frac{n}{2}+z}}, \qquad (4.1.18)$$

where $y \in \mathbf{R}^n$ and J_ν is the Bessel function of order ν. The estimates in Appendices B.6 and B.7 imply that there is a constant $C(\mathrm{Re}\,\nu)$ such that

$$|J_\nu(r)| \leq C(\mathrm{Re}\,\nu) e^{10|\mathrm{Im}\,\nu|^2}(1+r)^{-\frac{1}{2}}$$

whenever $\mathrm{Re}\,\nu > 0$. This yields that if $\mathrm{Re}\, z > \frac{n-1}{2}$, then there is a constant $C'(t)$ such that the function K^z obeys the estimate

$$|K^z(y)| \leq C'(\tfrac{n}{2} + \mathrm{Re}\, z)\, e^{10|\mathrm{Im}\, z|^2}(1+|y|)^{-n-(\mathrm{Re}\, z - \frac{n-1}{2})}, \qquad (4.1.19)$$

and hence it lies in $L^1(\mathbf{R}^n)$. Using identity (3.1.10), whenever $\mathrm{Re}\, z > \frac{n-1}{2}$, we define for an integrable function f on $\mathbf{T}^n$ and $x \in \mathbf{T}^n$ the operator

$$B_R^z(f)(x) = \sum_{\ell \in \mathbf{Z}^n} m_z(\tfrac{\ell}{R})\widehat{f}(\ell)e^{2\pi i \ell \cdot x} = (f * L^{z,R})(x),$$

where $L^{z,R}$ is a function whose sequence of Fourier coefficients is $\{m_z(\frac{\ell}{R})\}_{\ell \in \mathbf{Z}^n}$. But the function $L^{z,R}$ can be precisely identified. By the Poisson summation formula (Theorem 3.2.8), which applies since both $K_z(x)$ and $m_z(x)$ are bounded by a constant multiple of $(1+|x|)^{-n-\delta}$ for some $\delta > 0$, we have

$$L^{z,R}(x) = \sum_{k \in \mathbf{Z}^n} m_z(\tfrac{k}{R})e^{2\pi i x \cdot k} = R^n \sum_{\ell \in \mathbf{Z}^n} K^z((x+\ell)R)$$

for all $x \in \mathbf{T}^n$. We show that the family $\{L^{z,R}\}_{R>0}$ is an approximate identity on $\mathbf{T}^n$ when $\operatorname{Re} z > \frac{n-1}{2}$; on this see the related Exercise 3.1.3. Obviously, using (4.1.19) we have that

$$\int_{\mathbf{T}^n} |L^{z,R}(x)| \, dx = \int_{\mathbf{R}^n} |K^z(y)| \, dy = C''(n, \operatorname{Re} z) e^{10|\operatorname{Im} z|^2} < \infty \qquad (4.1.20)$$

for some constant $C''(n, \operatorname{Re} z)$, and also

$$\int_{\mathbf{T}^n} L^{z,R}(x) \, dx = \int_{\mathbf{R}^n} K^z(y) \, dy = m_z(0) = 1$$

for all $R > 0$ when $\operatorname{Re} z > \frac{n-1}{2}$. Moreover, for $\delta < \frac{1}{2}$ using (4.1.19) we have

$$\int_{\delta \leq \sup_j |x_j| \leq \frac{1}{2}} |L^{z,R}(x)| \, dx \leq \frac{C_{n,z}}{R^{\operatorname{Re} z - \frac{n-1}{2}}} \int_{\delta \leq \sup_j |x_j| \leq \frac{1}{2}} \sum_{\ell \in \mathbf{Z}^n} \frac{1}{|x+\ell|^{n+\operatorname{Re} z - \frac{n-1}{2}}} \, dx \to 0,$$

thus the integral of $L^{z,R}$ over $[-1/2, 1/2]^n \setminus [-\delta, \delta]^n$ tends to zero as $R \to \infty$.

Using Theorem 1.2.19, we obtain these conclusions for $\operatorname{Re} z > \frac{n-1}{2}$:

(a) For $f \in L^1(\mathbf{T}^n)$, $B_R^z(f)$ converge to f in L^1 as $R \to \infty$.
(b) For f continuous on $\mathbf{T}^n$, $B_R^z(f)$ converge to f uniformly as $R \to \infty$.

We turn to the corresponding results for $1 < p < \infty$. We have that

$$\operatorname{Re} z > \frac{n-1}{2} \implies \sup_{R>0} \|B_R^z\|_{L^1(\mathbf{T}^n) \to L^1(\mathbf{T}^n)} = C''(n, \operatorname{Re} z) e^{10|\operatorname{Im} z|^2} \qquad (4.1.21)$$

$$\operatorname{Re} z = 0 \implies \sup_{R>0} \|B_R^z\|_{L^2(\mathbf{T}^n) \to L^2(\mathbf{T}^n)} = \|m_z\|_{L^\infty} = 1. \qquad (4.1.22)$$

The family of operators $f \mapsto B_R^z(f)$ is of admissible growth for all $\operatorname{Re} z \geq 0$, since for all measurable subsets A, B of $\mathbf{T}^n$ we have

$$\left| \int_{\mathbf{T}^n} B_R^z(\chi_A) \chi_B \, dx \right| = \left| \sum_{k \in \mathbf{Z}^n} \widehat{\chi_A}(k) m^z(k) \overline{\widehat{\chi_B}(k)} \right| \leq \sum_{|k| \leq R} 1 \leq C_n R^n,$$

thus condition (1.3.23) holds. Moreover, hypothesis (1.3.24) of Theorem 1.3.7 holds in view of (4.1.21) and (4.1.22). Applying Theorem 1.3.7 (or rather Exercise 1.3.4 in which the strip $[0,1] \times \mathbf{R}$ is replaced by the more general strip $[a,b] \times \mathbf{R}$) we obtain that when $\alpha = \operatorname{Re} z > (n-1)|\frac{1}{p} - \frac{1}{2}|$, we have

$$\sup_{R>0} \|B_R^\alpha\|_{L^p(\mathbf{T}^n) \to L^p(\mathbf{T}^n)} < \infty.$$

Finally, using Corollary 4.1.3, we deduce that $B_R^\alpha(f) \to f$ in $L^p(\mathbf{T}^n)$ as $R \to \infty$ for all $f \in L^p(\mathbf{T}^n)$.   □

The preceding result is sharp in the case $p = 1$ (Theorem 4.2.5). For this reason, the number $\alpha = (n-1)/2$ is referred to as the *critical index of Bochner–Riesz summability*.

Exercises

4.1.1. If $f \in \mathscr{C}^\infty(\mathbf{T}^n)$, then show that $D_N^n * f$ and $\mathring{D}_N^n * f$ converge to f uniformly and in L^p for $1 \le p \le \infty$.

4.1.2. Prove that

$$\|P_+\|_{L^2(\mathbf{T}^1) \to L^2(\mathbf{T}^1)} = \|P_-\|_{L^2(\mathbf{T}^1) \to L^2(\mathbf{T}^1)} = \|W\|_{L^2(\mathbf{T}^1) \to L^2(\mathbf{T}^1)} = 1,$$

where $W(f) = \widetilde{f}$ is the conjugate function on the circle. Moreover, show that the mappings $f \mapsto W(f) + \widehat{f}(0)$ and $f \mapsto W(f) - \widehat{f}(0)$ are isometries on $L^2(\mathbf{T}^1)$.

4.1.3. Let $-\infty \le a_j < b_j \le +\infty$ for $1 \le j \le n$. Consider the rectangular projection operator defined on $\mathscr{C}^\infty(\mathbf{T}^n)$ by

$$P(f)(x) = \sum_{a_j \le m_j \le b_j} \widehat{f}(m) e^{2\pi i (m_1 x_1 + \cdots + m_n x_n)}.$$

Prove that when $1 < p < \infty$, P extends to a bounded operator from $L^p(\mathbf{T}^n)$ to itself with bounds independent of the a_j, b_j.
[Hint: Express P in terms of the Riesz projection P_+.]

4.1.4. Let $P_r(t)$ be the Poisson kernel on $\mathbf{T}^1$ as defined in Exercise 3.1.7. For $0 < r < 1$, define the *conjugate Poisson kernel* $Q_r(t)$ on the circle by

$$Q_r(t) = -i \sum_{m=-\infty}^{+\infty} \text{sgn}\,(m)\, r^{|m|} e^{2\pi i m t}.$$

(a) For $0 < r < 1$, prove the identity

$$Q_r(t) = \frac{2r \sin(2\pi t)}{1 - 2r \cos(2\pi t) + r^2}.$$

(b) Prove that $\widetilde{f}(t) = \lim_{r \to 1}(Q_r * f)(t)$ whenever f is smooth. Conclude that if f is real-valued, then so is $\widetilde{f}$.
(c) Let $f \in L^1(\mathbf{T}^1)$. Prove that the function

$$z \mapsto (P_r * f)(t) + i(Q_r * f)(t)$$

is analytic in $z = r e^{2\pi i t}$ on the open unit disc $\{z \in \mathbf{C} : |z| < 1\}$.
(d) Let $f \in L^1(\mathbf{T}^1)$. Conclude that the functions $z \mapsto (P_r * f)(t)$ and $z \mapsto (Q_r * f)(t)$ are *conjugate harmonic functions* of $z = r e^{2\pi i t}$ in the region $|z| < 1$. The term conjugate Poisson kernel stems from this property.

4.1.5. Let f be in $\dot{\Lambda}_\alpha(\mathbf{T}^1)$ for some $0 < \alpha < 1$. Prove that the conjugate function $\widetilde{f}$ is well defined and can be written as

$$\widetilde{f}(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |t| \leq 1/2} f(x-t)\cot(\pi t)\, dt$$

$$= \int_{|t| \leq 1/2} \big(f(x-t) - f(x)\big)\cot(\pi t)\, dt.$$

$\big[$*Hint:* Use part (b) of Exercise 4.1.4 and the fact that Q_r has integral zero over the circle to write $(f * Q_r)(x) = \big((f - f(x)) * Q_r\big)(x)$, allowing use of the Lebesgue dominated convergence theorem.$\big]$

4.1.6. Suppose that f is a real-valued function on $\mathbf{T}^1$ with $|f| \leq 1$ and $0 \leq \lambda < \pi/2$.
(a) Prove that

$$\int_{\mathbf{T}^1} e^{\lambda \widetilde{f}(t)}\, dt \leq \frac{1}{\cos(\lambda)}\,.$$

(b) Conclude that for $0 \leq \lambda < \pi/2$ we have

$$\int_{\mathbf{T}^1} e^{\lambda |\widetilde{f}(t)|}\, dt \leq \frac{2}{\cos(\lambda)}\,.$$

$\big[$*Hint:* Part (a): Consider the analytic function $F(z)$ on the disk $|z| < 1$ defined by $F(z) = -i(P_r * f)(\theta) + (Q_r * f)(\theta)$, where $z = re^{2\pi i\theta}$. Then $\mathrm{Re}\, e^{\lambda F(z)}$ is harmonic and its average over the circle $|z| = r$ is equal to its value at the origin, which is $\cos(\lambda f(0)) \leq 1$. Let $r \uparrow 1$ and use that for $z = e^{2\pi i t}$ on the circle we have $\mathrm{Re}\, e^{\lambda F(z)} \geq e^{\lambda \widetilde{f}(t)}\cos(\lambda)$.$\big]$

4.1.7. Prove that for $0 < \alpha < 1$ there is a constant C_α such that

$$\big\|\widetilde{f}\big\|_{\dot{\Lambda}_\alpha(\mathbf{T}^1)} \leq C_\alpha \|f\|_{\dot{\Lambda}_\alpha(\mathbf{T}^1)}\,.$$

$\big[$*Hint:* Using Exercise 4.1.5, for $|h| \leq 1/10$ write $\widetilde{f}(x+h) - \widetilde{f}(x)$ as

$$\int_{|t| \leq 5|h|} \big(f(x-t) - f(x+h)\big)\cot(\pi(t+h))\, dt$$

$$- \int_{|t| \leq 5|h|} \big(f(x-t) - f(x)\big)\cot(\pi t)\, dt$$

$$+ \int_{5|h| \leq |t| \leq 1/2} \big(f(x-t) - f(x)\big)\big(\cot(\pi(t+h)) - \cot(\pi t)\big)\, dt$$

$$+ \big(f(x) - f(x+h)\big)\int_{5|h| \leq |t| \leq 1/2} \cot(\pi(t+h))\, dt\,.$$

You may use the fact that $\cot(\pi t) = \frac{1}{\pi t} + b(t)$, where $b(t)$ is a bounded function when $|t| \leq 1/2$. The case $|h| \geq 1/10$ is easy.$\big]$

4.1.8. The beta function is defined in Appendix A.2. Derive the identity

$$t^\alpha = \frac{1}{B(\alpha-\beta,\beta+1)} \int_0^t (t-s)^{\alpha-\beta-1} s^\beta \, ds$$

and show that the function $K_R^\alpha(x) = \sum_{|m|\le R} \left(1 - \frac{|m|^2}{R^2}\right)^\alpha e^{2\pi i m\cdot x}$ satisfies (4.1.17).
$\big[$*Hint:* Take $t = 1 - \frac{|m|^2}{R^2}$ and change variables $s = \frac{r^2-|m|^2}{R^2}$ in the displayed identity.$\big]$

4.2 A. E. Divergence of Fourier Series and Bochner–Riesz means

We saw in Proposition 3.4.6 that the Fourier series of a continuous function may diverge at a point. As expected, the situation can only get worse as the functions get worse. In this section we present an example, due to A. N. Kolmogorov, of an integrable function on $\mathbf{T}^1$ whose Fourier series diverges almost everywhere. We also prove an analogous result for the Bochner–Riesz means at the critical index.

4.2.1 Divergence of Fourier Series of Integrable Functions

It is natural to start our investigation with the case $n = 1$. We begin with the following important result:

Theorem 4.2.1. *There exists an integrable function on the circle $\mathbf{T}^1$ whose Fourier series diverges almost everywhere.*

Proof. The proof of this theorem is a bit involved, and we need a sequence of lemmas, which we prove first.

Lemma 4.2.2. *(**Kronecker**) Suppose that $N \in \mathbf{Z}^+$ and*

$$\{x_1, x_2, \ldots, x_N, 1\}$$

is a linearly independent set over the rationals. Then for any $\varepsilon > 0$ and any complex numbers $z_1, z_2, \ldots, z_N$ with $|z_j| = 1$, there exists an integer $L \in \mathbf{Z}$ such that

$$|e^{2\pi i L x_j} - z_j| < \varepsilon \qquad \text{for all} \quad 1 \le j \le N.$$

Proof. Suppose that the assertion claimed is false. Then there is an $\varepsilon > 0$ and complex numbers $z_j = e^{2\pi i \theta_j}$, $j = 1, \ldots, N$, with $0 \le \theta_j < 1$, such that

$$\{m(x_1, \ldots, x_N) : m \in \mathbf{Z}\} \cap B\big((\theta_1, \ldots, \theta_N), \varepsilon\big) = \emptyset,$$

where $B\big((\theta_1,\ldots,\theta_N),\varepsilon\big)$ denotes a neighborhood in $\mathbf{T}^N$ of radius ε centered at the point $(\theta_1,\ldots,\theta_N)$. Pick a smooth, nonzero, and nonnegative function f on $\mathbf{T}^N$ supported in $B\big((\theta_1,\ldots,\theta_N),\varepsilon\big)$. Then $f(m(x_1,\ldots,x_N))=0$ for all $m\in\mathbf{Z}$, but

$$\widehat{f}(0)=\int_{\mathbf{T}^N} f(y)\,dy>0. \tag{4.2.1}$$

Set $x=(x_1,\ldots,x_N)$. Then we have

$$
\begin{aligned}
0=\frac{1}{M}\sum_{m=0}^{M-1}f(mx) &=\frac{1}{M}\sum_{m=0}^{M-1}\left(\sum_{\ell\in\mathbf{Z}^N}\widehat{f}(\ell)e^{2\pi i\ell\cdot mx}\right)\\
&=\sum_{\ell\in\mathbf{Z}^N}\widehat{f}(\ell)\left(\frac{1}{M}\sum_{m=0}^{M-1}e^{2\pi im(\ell\cdot x)}\right)\\
&=\widehat{f}(0)+\sum_{\ell\in\mathbf{Z}^N\setminus\{0\}}\widehat{f}(\ell)\left(\frac{1}{M}\frac{e^{2\pi iM(\ell\cdot x)}-1}{e^{2\pi i(\ell\cdot x)}-1}\right).
\end{aligned}
$$

Note that $e^{2\pi i(\ell\cdot x)}-1\neq 0$ because $\ell\cdot x=\ell_1x_1+\cdots+\ell_Nx_N\notin\mathbf{Z}$, since by assumption the set $\{x_1,x_2,\ldots,x_N,1\}$ is linearly independent over the rationals. Observe that

$$\widehat{f}(\ell)\frac{1}{M}\frac{e^{2\pi iM(\ell\cdot x)}-1}{e^{2\pi i(\ell\cdot x)}-1}=\widehat{f}(\ell)\frac{1}{M}\sum_{m=0}^{M-1}e^{2\pi im(\ell\cdot x)}$$

tends to 0 as $M\to\infty$ for every fixed $\ell\in\mathbf{Z}^N$ and is bounded uniformly in M by $|\widehat{f}(\ell)|$ which satisfies $\sum_{\ell\in\mathbf{Z}^N}|\widehat{f}(\ell)|<\infty$. Using the Lebesgue dominated convergence theorem, we obtain that

$$
\begin{aligned}
0&=\widehat{f}(0)+\lim_{M\to\infty}\sum_{\ell\in\mathbf{Z}^N\setminus\{0\}}\widehat{f}(\ell)\left(\frac{1}{M}\frac{e^{2\pi iM(\ell\cdot x)}-1}{e^{2\pi i(\ell\cdot x)}-1}\right)\\
&=\widehat{f}(0)+\sum_{\ell\in\mathbf{Z}^N\setminus\{0\}}\widehat{f}(\ell)\lim_{M\to\infty}\left(\frac{1}{M}\frac{e^{2\pi iM(\ell\cdot x)}-1}{e^{2\pi i(\ell\cdot x)}-1}\right)\\
&=\widehat{f}(0)+0,
\end{aligned}
$$

which contradicts (4.2.1). Therefore the claimed L exists. $\qquad\square$

Lemma 4.2.3. *There exists a positive constant $c>0$ such that given any integer $N\geq 2$ there exists a positive measure μ_N on $\mathbf{T}^1$ with $\mu_N(\mathbf{T}^1)=1$ such that*

$$\sup_{L\geq 1}\big|(\mu_N*D_L)(x)\big|=\sup_{L\geq 1}\left|\sum_{k=-L}^{L}\widehat{\mu_N}(k)e^{2\pi ikx}\right|\geq c\log N \tag{4.2.2}$$

for almost all $x\in\mathbf{T}^1$ (c is a fixed constant).

Proof. Given irrational real numbers $x_1, \ldots, x_N$ such that the set $\{x_1, \ldots, x_N, 1\}$ is linearly independent over the rationals, we define $\mathbf{Q}[x_1, \ldots, x_N]$ to be the field extension of $\mathbf{Q}$ consisting of all linear combinations of the form $q_0 + q_1 x_1 + \cdots + q_N x_N$, where q_j are rational numbers. Obviously $\mathbf{Q}[x_1, \ldots, x_N]$ is a countable set. Fix $N \geq 100$ and choose points x_j as follows:

$$0 < x_1 < \frac{1}{N} < x_2 < \frac{2}{N} < x_3 < \frac{3}{N} < \cdots < \frac{N-1}{N} < x_N < 1 \qquad (4.2.3)$$

and such that $x_1 \notin \mathbf{Q}$, $x_2 \notin \mathbf{Q}[x_1]$, $\ldots$, $x_N \notin \mathbf{Q}[x_1, \ldots, x_{N-1}]$. Then obviously the set $\{x_1, \ldots, x_N, 1\}$ is linearly independent over the rationals. Let

$$E_N = \left\{ x \in [0,1] : \{x - x_1, \ldots, x - x_N, 1\} \quad \text{is linearly independent over } \mathbf{Q} \right\}$$

and observe that every x in $[0,1] \setminus \mathbf{Q}[x_1, \ldots, x_N]$ belongs to E_N. Indeed, if $x \notin E_N$, then there are rational numbers q_j such that

$$q_0 + q_1(x - x_1) + \cdots + q_N(x - x_N) = 0.$$

Then $q = q_1 + \cdots + q_N \neq 0$, since $\{x_1, \ldots, x_N, 1\}$ are linearly independent over $\mathbf{Q}$. It follows that

$$x = -q^{-1} q_0 + q^{-1} q_1 x_1 + \cdots + q^{-1} q_N x_N,$$

thus $x \in \mathbf{Q}[x_1, \ldots, x_N]$. We conclude that E_N has full measure.

Next, we define the probability measure

$$\mu_N = \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j},$$

where δ_{x_j} are Dirac delta masses at the points x_j. For this measure we have

$$
\left| \sum_{k=-L}^{L} \widehat{\mu_N}(k) e^{2\pi i k x} \right| = \left| \sum_{k=-L}^{L} \left(\frac{1}{N} \sum_{j=1}^{N} e^{-2\pi i k x_j} \right) e^{2\pi i k x} \right|
$$

$$
= \left| \frac{1}{N} \sum_{j=1}^{N} D_L(x - x_j) \right|
$$

$$
= \left| \frac{1}{N} \sum_{j=1}^{N} \frac{\sin(2\pi(L + \frac{1}{2})(x - x_j))}{\sin(\pi(x - x_j))} \right| \qquad (4.2.4)
$$

$$
= \left| \frac{1}{N} \sum_{j=1}^{N} \frac{\mathrm{Im}\left[e^{2\pi i (L + \frac{1}{2})(x - x_j)} \right] \mathrm{sgn}\left(\sin(\pi(x - x_j)) \right)}{|\sin(\pi(x - x_j))|} \right|,
$$

where the signum function is defined as $\mathrm{sgn}\, a = 1$ for $a > 0$, -1 for $a < 0$, and zero if $a = 0$. By Lemma 4.2.2, for all $x \in E_N$ there exists an $L \in \mathbf{Z}^+$ such that

$$\left| e^{2\pi i L(x - x_j)} - i e^{-2\pi i \frac{1}{2}(x - x_j)} \mathrm{sgn}\left(\sin(\pi(x - x_j)) \right) \right| < \frac{1}{2},$$

which can be equivalently written as

$$\left|e^{2\pi i(L+\frac{1}{2})(x-x_j)}\text{sgn}\big(\sin(\pi(x-x_j))\big) - i\right| < \frac{1}{2}. \tag{4.2.5}$$

It follows from (4.2.5) that

$$\text{Im}\left[e^{2\pi i(L+\frac{1}{2})(x-x_j)}\right]\text{sgn}\big(\sin(\pi(x-x_j))\big) > \frac{1}{2}.$$

Combining this with the result of the calculation in (4.2.4), we obtain that

$$\left|\sum_{k=-L}^{L}\widehat{\mu_N}(k)e^{2\pi ikx}\right| > \frac{1}{2N}\sum_{j=1}^{N}\frac{1}{|\sin(\pi(x-x_j))|} \geq \frac{1}{2\pi N}\sum_{j=1}^{N}\frac{1}{|x-x_j|}.$$

But for every $x \in [0,1)$, there exists a j_0 such that $x \in [x_{j_0}, x_{j_0+1})$. It follows from (4.2.3) that $|x - x_j| \leq C(|j - j_0| + 1)N^{-1}$, and thus

$$\sum_{j=1}^{N}\frac{1}{|x-x_j|} \geq c'N\log N.$$

Thus for every $x \in E_N$ there exists an $L \in \mathbf{Z}^+$ such that

$$\left|D_L * \mu_N(x)\right| = \left|\sum_{k=-L}^{L}\widehat{\mu_N}(k)e^{2\pi ikx}\right| > c\log N,$$

which proves the required conclusion since E_N is a set of measure 1. $\square$

Lemma 4.2.4. *For each $0 < M < \infty$ there exists a trigonometric polynomial g_M and a measurable subset A_M of $\mathbf{T}^1$ with measure $|A_M| > 1 - 2^{-M}$ such that $\|g_M\|_{L^1} = 1$, and such that*

$$\inf_{x \in A_M}\sup_{L \geq 1}\left|(D_L * g_M)(x)\right| = \inf_{x \in A_M}\sup_{L \geq 1}\left|\sum_{k=-L}^{L}\widehat{g_M}(k)e^{2\pi ikx}\right| > 2^M. \tag{4.2.6}$$

Proof. Given an $M \in \mathbf{Z}^+$, we pick an integer $N(M)$ such that $c\log N(M) > 2^{M+2}$, where c is as in (4.2.2), and we also pick the measure $\mu_{N(M)}$, which satisfies (4.2.2). By Fatou's lemma we have

$$\begin{aligned}
1 &= \left|\left\{x \in \mathbf{T}^1 : \sup_{L \geq 1}|(D_L * \mu_{N(M)})(x)| \geq 2^{M+2}\right\}\right| \\
&= \left|\bigcup_{L \geq 1}\left\{x \in \mathbf{T}^1 : \sup_{1 \leq j \leq L}|(D_j * \mu_{N(M)})(x)| \geq 2^{M+2}\right\}\right| \\
&= \int_{\mathbf{T}^1}\lim_{L \to \infty}\chi_{\{x \in \mathbf{T}^1 : \sup_{1 \leq j \leq L}|(D_j * \mu_{N(M)})(x)| \geq 2^{M+2}\}}dx \\
&\leq \liminf_{L \to \infty}\left|\left\{x \in \mathbf{T}^1 : \sup_{1 \leq j \leq L}|(D_j * \mu_{N(M)})(x)| \geq 2^{M+2}\right\}\right|,
\end{aligned}$$

and thus we can find a positive integer $L(M)$ such that the set

$$A_M = \left\{ x \in \mathbf{T}^1 : \sup_{1 \leq L \leq L(M)} |(D_L * \mu_{N(M)})(x)| \geq 2^{M+2} \right\}$$

has measure greater than $1 - 2^{-M}$. We pick a positive integer $K(M)$ such that

$$\sup_{1 \leq j \leq L(M)} \left\| F_{K(M)} * D_j - D_j \right\|_{L^\infty} \leq 1,$$

where F_K is the Fejér kernel. This is possible, since the Fejér kernel is an approximate identity and $\{D_j : 1 \leq j \leq L(M)\}$ is a finite family of continuous functions. Then we define $g_M = \mu_{N(M)} * F_{K(M)}$. Since $\mu_{N(M)}$ is a probability measure, we obtain

$$|(D_j * g_M)(x) - (D_j * \mu_{N(M)})(x)| \leq \left\| D_j * F_{K(M)} - D_j \right\|_{L^\infty} \leq 1$$

for all $x \in [0,1]$ and $1 \leq j \leq L(M)$. But given $x \in A_M$ there exists an L in $\{1, \ldots, L(M)\}$ such that $|(D_L * \mu_{N(M)})(x)| \geq 2^{M+2}$ and for this L we have

$$|(D_L * g_M)(x)| \geq |(D_L * \mu_{N(M)})(x)| - 1 \geq 2^{M+2} - 1 \geq 2^{M+1} > 2^M.$$

Therefore, (4.2.6) is satisfied for this g_M and A_M. Since μ_N is a nonnegative measure and $F_{K(M)}$ is nonnegative and has L^1 norm 1, we have that

$$\left\| g_M \right\|_{L^1} = \left\| \mu_{N(M)} * F_{K(M)} \right\|_{L^1} = \left\| \mu_{N(M)} \right\|_{\mathcal{M}} \left\| F_{K(M)} \right\|_{L^1} = 1,$$

showing that g_M has L^1 norm equal to one. $\qquad\qquad\square$

We now have the tools needed to construct an example of a function whose Fourier series diverges almost everywhere. The example is given as a series of functions whose behavior worsens as its index becomes bigger. The function we wish to construct is a sum of the form

$$g = \sum_{j=1}^{\infty} \varepsilon_j g_{M_j}, \tag{4.2.7}$$

for a choice of sequences $\varepsilon_j \to 0$ and $M_j \to \infty$, where g_M are as in Lemma 4.2.4.

Let us be specific. First, we set $d_0 = 1$ and for $N \geq 1$

$$d_N = \max_{1 \leq s \leq N} \text{degree}\,(g_{M_s}), \tag{4.2.8}$$

where g_M is the trigonometric polynomial of Lemma 4.2.4. We set $\varepsilon_0 = M_0 = 1$. Assume that we have defined ε_j and M_j for all $1 \leq j < N$ for some $N \geq 2$. We set

$$\varepsilon_N = 2^{-N}(3d_{N-1})^{-1} \tag{4.2.9}$$

and then we pick M_N such that

$$2^{M_N} \geq (2^N + d_{N-1} + 1)\varepsilon_N^{-1}. \tag{4.2.10}$$

This defines ε_N and M_N for a given positive integer N, provided ε_j and M_j are known for all $j < N$. This way we define ε_N and M_N for all natural numbers N.

We observe that the selections of ε_j and M_j force the inequalities $\varepsilon_j \leq 2^{-j}$ and $d_j \leq d_{j+1}$ for all $j \geq 1$. Since each g_{M_j} has L^1 norm 1 and $\varepsilon_j \leq 2^{-j}$, the function g in (4.2.7) is integrable and has L^1 norm at most 1.

For a given $j \geq 1$ and $x \in A_{M_j}$, by Lemma 4.2.4 there exists an $L \geq 1$ such that $|(D_L * g_{M_j})(x)| > 2^{M_j}$. Set

$$k = k(x) = \min(L, d_j).$$

Then we have

$$|(D_k * g)(x)| \geq \varepsilon_j |(D_k * g_{M_j})(x)| - \sum_{1 \leq s < j} \varepsilon_s |(D_k * g_{M_s})(x)| - \sum_{s > j} \varepsilon_s |(D_k * g_{M_s})(x)|.$$

We make the following observations:

(i) $|(D_k * g_{M_j})(x)| = |(D_L * g_{M_j})(x)| > 2^{M_j}.$

(ii) $|(D_k * g_{M_s})(x)| = |(D_{\min(d_s, k)} * g_{M_s})(x)| \leq \|D_{\min(d_s, L)}\|_{L^\infty} \leq 3d_s$, when $s < j$.

(iii) $|(D_k * g_{M_s})(x)| = |(D_{\min(d_s, k)} * g_{M_s})(x)| \leq \|D_{\min(d_j, L)}\|_{L^\infty} \leq 3d_j$, when $s > j$.

In these estimates we have used that $k = \min(L, d_j)$, $\|D_m\|_{L^\infty} \leq 2m + 1 \leq 3m$, and that

$$D_r * g_{M_s} = D_{\min(r, d_s)} * g_{M_s},$$

which follows easily by examining the corresponding Fourier coefficients.

Using the estimates in (i), (ii), and (iii), for a fixed $x \in A_{M_j}$ and $k = k(x)$ we obtain

$$|(D_k * g)(x)| \geq \varepsilon_j 2^{M_j} - 3 \sum_{1 \leq s < j} \varepsilon_s d_s - 3 \sum_{s > j} \varepsilon_s d_j. \tag{4.2.11}$$

Our selection of ε_j and M_j now ensures that (4.2.11) is a large number. In fact, we have

$$3 \sum_{s > j} \varepsilon_s d_j = \sum_{s > j} 2^{-s} d_j (d_{s-1})^{-1} \leq \sum_{s > j} 2^{-s} \leq 1$$

and

$$3 \sum_{1 \leq s < j} \varepsilon_s d_s \leq 3 d_{j-1} \sum_{1 \leq s < j} \varepsilon_s \leq d_{j-1} \sum_{1 \leq s < j} 2^{-s} (d_{s-1})^{-1} \leq d_{j-1}.$$

Therefore, the expression in (4.2.11) is at least $\varepsilon_j 2^{M_j} - d_{j-1} - 1 \geq 2^j$. It follows that for every $j \geq 1$ and every $x \in A_{M_j}$ there exists a $k = k(x) \in \mathbf{Z}^+$ such that

$$|(D_k * g)(x)| \geq 2^j.$$

We conclude that for every $r \geq 1$ and $x \in \bigcup_{j=r}^{\infty} A_{M_j}$,

$$\sup_{k \geq 1} |(D_k * g)(x)| \geq 2^j \geq 2^r, \qquad (4.2.12)$$

since x belongs to some A_{M_j} with $j \geq r$. For given $r \geq 1$, Lemma 4.2.4 yields that

$$1 \geq \left| \bigcup_{j=r}^{\infty} A_{M_j} \right| \geq \liminf_{j \to \infty} |A_{M_j}| \geq \lim_{j \to \infty} (1 - 2^{-M_j}) = 1.$$

Then the set

$$A = \bigcap_{r=0}^{\infty} \bigcup_{j=r}^{\infty} A_{M_j}$$

has measure 1, since it is a countable intersection of subsets of $\mathbf{T}^1$ of full measure. In view of (4.2.12) we have that for all x in A

$$\sup_{k \geq 1} |(D_k * g)(x)| \geq \sup_{r \geq 1} 2^r = \infty \qquad (4.2.13)$$

and thus the required conclusion follows. $\qquad \square$

4.2.2 Divergence of Bochner–Riesz Means of Integrable Functions

We now turn to the corresponding n-dimensional problem for spherical summability of Fourier series. The situation here is quite similar at the critical index $\alpha = \frac{n-1}{2}$.

Theorem 4.2.5. *Let $n > 1$. There exists an integrable function f on $\mathbf{T}^n$ such that*

$$\limsup_{R \to \infty} \left| B_R^{\frac{n-1}{2}} (f)(x) \right| = \limsup_{R \to \infty} \left| \sum_{\substack{m \in \mathbf{Z}^n \\ |m| \leq R}} \left(1 - \frac{|m|^2}{R^2} \right)^{\frac{n-1}{2}} \widehat{f}(m) e^{2\pi i m \cdot x} \right| = \infty$$

for almost all $x \in \mathbf{T}^n$. Furthermore, such a function can be constructed such that it is supported in an arbitrarily small given neighborhood of the origin.

We will need a couple of lemmas.

Lemma 4.2.6. *Let $n \geq 2$. The complement of the set*

$$S = \left\{ x \in \mathbf{R}^n : \{1\} \cup \{|x - m| : m \in \mathbf{Z}^n\} \text{ is linearly independent over } \mathbf{Q} \right\}$$

has n-dimensional Lebesgue measure zero.

Proof. Recall that a function g defined on an open subset Ω of $\mathbf{R}^n$ is called *real analytic* if for every point x_0 in Ω there is a ball $B(x_0, \varepsilon)$ is contained in Ω and there exist coefficients $c_\beta(x_0)$ such that $g(x) = \sum_\beta c_\beta(x_0)(x - x_0)^\beta$ for all $|x - x_0| < \varepsilon$, where the sum is taken over all multiindices. We will need two facts about real analytic functions. First, the function $x \to |x|$ is real analytic on $\mathbf{R}^n \setminus \{0\}$. Indeed, given $x_0 \neq 0$, for $|x - x_0| < |x_0|/3$ we have that

$$\frac{|x - x_0|^2}{|x_0|^2} + 2(x - x_0) \cdot \frac{x_0}{|x_0|^2} < 1.$$

This allows us to write

$$|x| = |x_0| \left(1 + \frac{|x - x_0|^2}{|x_0|^2} + 2(x - x_0) \cdot \frac{x_0}{|x_0|^2} \right)^{\frac{1}{2}}$$

$$= |x_0| \sum_{k=0}^\infty \binom{1/2}{k} \left(\frac{|x - x_0|^2}{|x_0|^2} + 2(x - x_0) \cdot \frac{x_0}{|x_0|^2} \right)^k,$$

which is a power series of the form $\sum_\beta c_\beta(x_0)(x - x_0)^\beta$.

Secondly, we need the fact that a real analytic function defined on an open connected subset of $\mathbf{R}^n$ cannot vanish on a set of positive measure, unless it is identically equal to zero; a proof of this in dimension one and an outline of the proof in higher dimensions is contained in [205].

We return to the proof of the lemma which requires us to show that S has full measure in $\mathbf{R}^n$. Indeed, if $x \in \mathbf{R}^n \setminus S$, then there exist $k \in \mathbf{Z}^+$, $m_1, \dots m_k \in \mathbf{Z}^n$, and $q_0, q_1, \dots, q_k$ nonzero rational numbers such that

$$q_0 + \sum_{j=1}^k q_j |x - m_j| = 0. \tag{4.2.14}$$

Since the function

$$y \mapsto q_0 + \sum_{j=1}^k q_j |y - m_j|$$

is nonzero and real analytic on $\mathbf{R}^n \setminus \mathbf{Z}^n$, it must vanish only on a set of Lebesgue measure zero. Therefore, there exists a set $A_{m_1, \dots, m_k, q_0, q_1, \dots, q_k}$ of Lebesgue measure zero such that (4.2.14) holds exactly when x is in this set. Then

$$\mathbf{R}^n \setminus S \subseteq \bigcup_{k=1}^\infty \bigcup_{m_1, \dots, m_k \in \mathbf{Z}^n} \bigcup_{q_0, q_1, \dots, q_k \in \mathbf{Q}} A_{m_1, \dots, m_k, q_0, q_1, \dots, q_k},$$

from which it follows that $\mathbf{R}^n \setminus S$ has Lebesgue measure zero. $\square$

Let us denote the Bochner–Riesz kernel by

$$K_R^\alpha(x) = \sum_{|m| \le R} \left(1 - \frac{|m|^2}{R^2}\right)^\alpha e^{2\pi i m \cdot x}$$

when $x \in \mathbf{T}^n$. We need the following lemma regarding K_R^α:

Lemma 4.2.7. *Let $n \ge 2$. For almost every $x \in \mathbf{T}^n$ we have*

$$\limsup_{R \to \infty} |K_R^{\frac{n-1}{2}}(x)| = \infty.$$

It is noteworthy to compare the result of this lemma with the analogous one-dimensional statement

$$\limsup_{R \to \infty} |D_R(x)| = \infty$$

for the Dirichlet kernel, which holds exactly when $x = 0$. Thus the uniform ill behavior of the kernel $K_R^{\frac{n-1}{2}}$ reflects in some sense its lack of localization.

Proof. Fix $n \ge 2$ and fix $x_0 \in \big([-1/2, 1/2)^n \setminus \{0\}\big) \cap S$, where S is as in Lemma 4.2.6. Using (4.1.18) and the Poisson summation formula (Theorem 3.2.8), for each $\alpha > \frac{n-1}{2}$ we obtain the identity

$$K_R^\alpha(x_0) = \frac{\Gamma(\alpha+1)}{\pi^\alpha} R^n \sum_{m \in \mathbf{Z}^n} \frac{J_{\frac{n}{2}+\alpha}(2\pi R|x_0 - m|)}{(R|x_0 - m|)^{\frac{n}{2}+\alpha}} \tag{4.2.15}$$

and the sum converges absolutely because of the asymptotics for the Bessel functions in Appendix B.8. The term with $m = 0$ in the sum in (4.2.15) is a finite constant since by Appendix B.6 the function

$$y \mapsto \frac{J_{\frac{n}{2}+\alpha}(2\pi R|y|)}{|y|^{\frac{n}{2}+\alpha}}$$

is smooth and therefore bounded. But for $m \ne 0$ in (4.2.15) we have $|m - x_0| \ge 1/2$. The asymptotics in Appendix B.8 imply that for $R \ge 2$ we have

$$J_{\frac{n}{2}+\alpha}(2\pi R|x_0 - m|) = \frac{e^{2\pi i R|x_0 - m|}e^{-i\frac{\pi}{2}(\frac{n}{2}+\alpha)-i\frac{\pi}{4}} + e^{-2\pi i R|x_0-m|}e^{i\frac{\pi}{2}(\frac{n}{2}+\alpha)+i\frac{\pi}{4}}}{\pi\sqrt{R|x_0-m|}}$$
$$+ O\big((R|x_0 - m|)^{-\frac{3}{2}}\big)$$

for all $\alpha > \frac{n-1}{2}$. We insert this expression in (4.2.15), we multiply by $e^{2\pi i \lambda R}$ for some λ real, and then we average in R from 1 to T, for some $T > 10$. We obtain

$$\frac{1}{T}\int_1^T K_R^\alpha(x_0)e^{2\pi i\lambda R}\,dR$$

$$= \frac{\Gamma(\alpha+1)}{\pi^\alpha}\sum_{m\in\mathbf{Z}^n\setminus\{0\}}\frac{e^{-i\frac{\pi}{2}(\frac{n}{2}+\alpha)-i\frac{\pi}{4}}}{|x_0-m|^{\frac{n+1}{2}+\alpha}}\frac{1}{T}\int_1^T e^{2\pi iR(\lambda+|x_0-m|)}R^{\frac{n-1}{2}-\alpha}\,dR$$

$$+\frac{\Gamma(\alpha+1)}{\pi^\alpha}\sum_{m\in\mathbf{Z}^n\setminus\{0\}}\frac{e^{i\frac{\pi}{2}(\frac{n}{2}+\alpha)+i\frac{\pi}{4}}}{|x_0-m|^{\frac{n+1}{2}+\alpha}}\frac{1}{T}\int_1^T e^{2\pi iR(\lambda-|x_0-m|)}R^{\frac{n-1}{2}-\alpha}\,dR \quad (4.2.16)$$

$$+\frac{\Gamma(\alpha+1)}{\pi^\alpha}\sum_{m\in\mathbf{Z}^n\setminus\{0\}}O\left(\frac{1}{|x_0-m|^{\frac{n+3}{2}+\alpha}}\right)\frac{1}{T}\int_1^T R^{\frac{n-3}{2}-\alpha}\,dR$$

$$+\frac{\Gamma(\alpha+1)}{\pi^\alpha}\frac{1}{T}\int_1^T\frac{J_{\frac{n}{2}+\alpha}(2\pi R|x_0|)}{(R|x_0|)^{\frac{n}{2}+\alpha}}e^{2\pi i\lambda R}R^n\,dR.$$

Assume that we are able to pass the limit as $\alpha\to\frac{n-1}{2}+$ through the sums and integrals in the preceding identity; we justify this step momentarily. Then we obtain

$$\frac{1}{T}\int_1^T K_R^{\frac{n-1}{2}}(x_0)e^{2\pi i\lambda R}\,dR$$

$$= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n-1}{2}}}\sum_{m\in\mathbf{Z}^n\setminus\{0\}}\frac{e^{-i\frac{\pi}{2}(\frac{2n-1}{2})-i\frac{\pi}{4}}}{|x_0-m|^n}\frac{1}{T}\int_1^T e^{2\pi iR(\lambda+|x_0-m|)}\,dR$$

$$+\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n-1}{2}}}\sum_{m\in\mathbf{Z}^n\setminus\{0\}}\frac{e^{i\frac{\pi}{2}(\frac{2n-1}{2})+i\frac{\pi}{4}}}{|x_0-m|^n}\frac{1}{T}\int_1^T e^{2\pi iR(\lambda-|x_0-m|)}\,dR \quad (4.2.17)$$

$$+\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n-1}{2}}}\sum_{m\in\mathbf{Z}^n\setminus\{0\}}O\left(\frac{1}{|x_0-m|^{n+1}}\right)\frac{1}{T}\int_1^T\frac{dR}{R}$$

$$+\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n-1}{2}}}\frac{1}{T}\int_1^T\frac{J_{n-\frac{1}{2}}(2\pi R|x_0|)}{(R|x_0|)^{n-\frac{1}{2}}}e^{2\pi i\lambda R}R^n\,dR.$$

We now justify the passage of the limit in α inside the sums and the integrals in (4.2.16) to obtain (4.2.17). First, when $|m|\le R\le T$ and $\alpha>\frac{n-1}{2}$, the mean value theorem gives

$$\left|\left(1-\frac{|m|^2}{R^2}\right)^{\frac{n-1}{2}}e^{2\pi ix_0\cdot m}-\left(1-\frac{|m|^2}{R^2}\right)^\alpha e^{2\pi ix_0\cdot m}\right|$$

$$\le\left(\alpha-\frac{n-1}{2}\right)\left(1-\frac{|m|^2}{R^2}\right)^{\frac{n-1}{2}}\log\frac{1}{1-\frac{|m|^2}{R^2}}$$

$$\le\left(\alpha-\frac{n-1}{2}\right)\sup_{0<t\le 1}t^{\frac{n-1}{2}}\log\frac{1}{t},$$

thus $K_R^\alpha(x_0)$ converges to $K_R^{\frac{n-1}{2}}(x_0)$ uniformly in $R \in [1,T]$ as $\alpha \to \frac{n-1}{2}$ and therefore the integral over $[1,T]$ of the former converges to the integral over $[1,T]$ of the latter. Next, an integration by parts shows that the integral

$$\int_1^T e^{2\pi i R(\lambda + |x_0 - m|)} R^{\frac{n-1}{2} - \alpha} \, dR$$

is bounded by a constant multiple of $(\lambda + |x_0 - m|)^{-1}$, which makes the first infinite sum in (4.2.16) converge absolutely and uniformly in $\alpha \geq \frac{n-1}{2}$, thus one may pass the limit in α inside the sum. Also, the integral

$$\int_1^T e^{2\pi i R(\lambda - |x_0 - m|)} R^{\frac{n-1}{2} - \alpha} \, dR$$

is bounded by a constant multiple of $(\lambda - |x_0 - m|)^{-1}$ whenever λ is not in the set

$$\Lambda_{x_0} = \{|x_0 - m| : \ m \in \mathbf{Z}^n\} = \{\lambda_1, \lambda_2, \lambda_3, \dots\},$$

where $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$. Thus for $\lambda \notin \Lambda_{x_0}$, the preceding argument explains the passage of the limit in α inside the second infinite sum in (4.2.16). If λ happens to be in Λ_{x_0}, then there is at most one $m_0 \neq 0$, such that $\lambda = |x_0 - m_0|$ and the second sum in (4.2.16) restricted to $m \in \mathbf{Z}^n \setminus \{0, m_0\}$ converges absolutely, while for the single term with $m = m_0$, letting $\alpha \to \frac{n-1}{2}+$ is trivial. Finally, for the term involving the Bessel function $J_{n - \frac{1}{2}}$, the passage of the limit in α inside the integral is straightforward since the function

$$(\alpha, R) \mapsto \frac{J_{\frac{n}{2} + \alpha}(2\pi R |x_0|)}{(R|x_0|)^{\frac{n}{2} + \alpha}}$$

is continuous on the compact set $[\frac{n-1}{2}, \frac{n}{2}] \times [1,T]$. This completes the proof of (4.2.17).

There are four terms to the right of (4.2.17) and we observe that if $\lambda \neq \pm|x_0 - m_0|$ for any $m_0 \in \mathbf{Z}^n$, then all these terms converge to zero as $T \to \infty$. This assertion is trivial for the first three of these four terms, while for the last we assume that $T > |x_0|^{-1}$. We split the integral

$$\frac{1}{T} \int_1^T \frac{J_{n-\frac{1}{2}}(2\pi R |x_0|)}{(R|x_0|)^{n - \frac{1}{2}}} e^{2\pi i \lambda R} R^n \, dR \tag{4.2.18}$$

as a sum of the integral over $[1, |x_0|^{-1}]$, which obviously converges to zero as $T \to \infty$ by Appendix B.6, and of the integral over $[|x_0|^{-1}, T]$. For the latter, we use the asymptotics in Appendix B.8 to write

$$J_{n-\frac{1}{2}}(2\pi R |x_0|) = \frac{e^{2\pi i R |x_0|} e^{-i \frac{\pi}{2}(\frac{2n-1}{2}) - i \frac{\pi}{4}} + e^{-2\pi i R |x_0|} e^{i \frac{\pi}{2}(\frac{2n-1}{2}) + i \frac{\pi}{4}}}{\pi \sqrt{R|x_0|}} + O\big((R|x_0|)^{-\frac{3}{2}}\big).$$

The part of the integral in (4.2.18) over $\left[|x_0|^{-1}, T\right]$ corresponding to $O\left((R|x_0|)^{-\frac{3}{2}}\right)$ grows like $\log(T|x_0|)$ which divided by T obviously tends to zero. The part of the integral in (4.2.18) over $\left[|x_0|^{-1}, T\right]$ corresponding to the main term is

$$\frac{1}{T}\frac{1}{\pi\sqrt{|x_0|}}\int_{|x_0|^{-1}}^{T} e^{2\pi iR(\lambda+|x_0|)}e^{-i\frac{\pi}{2}\left(\frac{2n-1}{2}\right)-i\frac{\pi}{4}}+e^{2\pi iR(\lambda-|x_0|)}e^{i\frac{\pi}{2}\left(\frac{2n-1}{2}\right)+i\frac{\pi}{4}}\,dR$$

which tends to zero as $T \to \infty$ by an integration by parts, since $\lambda \neq |x_0|$ because we are considering the case where $\lambda \neq \pm|x_0 - m|$ for any $m \in \mathbf{Z}^n$.

Now consider the case where $\lambda = \pm|x_0 - m_0|$ for some $m_0 \in \mathbf{Z}^n$. In this case the expression to the right in (4.2.17) converges to

$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n-1}{2}}}\frac{e^{\pm i\left(\frac{\pi}{2}\left(\frac{2n-1}{2}\right)+\frac{\pi}{4}\right)}}{|x_0-m_0|^n}=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n-1}{2}}}\frac{e^{\pm i\frac{\pi n}{2}}}{|x_0-m_0|^n}$$

as $T \to \infty$. Next observe that

$$\sum_{j=1}^{\infty}\frac{1}{\lambda_j^n}=\infty.\tag{4.2.19}$$

We have now shown that

$$\lim_{T\to\infty}\frac{1}{T}\int_1^T K_t^{\frac{n-1}{2}}(x_0)e^{2\pi i\lambda t}\,dt=\begin{cases}\dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n-1}{2}}}\dfrac{e^{i\frac{\pi n}{2}}}{\lambda_j^n} & \text{if } \lambda=\lambda_j,\\[2mm] 0 & \text{if } \lambda\neq\pm\lambda_j,\\[2mm] \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n-1}{2}}}\dfrac{e^{-i\frac{\pi n}{2}}}{\lambda_j^n} & \text{if } \lambda=-\lambda_j.\end{cases}\tag{4.2.20}$$

Since x_0 lies in S, the set $\{1\}\cup\{\lambda_1,\lambda_2,\lambda_3,\dots\}$ is linearly independent over the rationals and thus no expression of the form $\pm\lambda_{j_1}\pm\cdots\pm\lambda_{j_s}$ is equal to an integer. It follows from this fact and (4.2.20) that

$$\lim_{T\to\infty}\frac{1}{T}\int_1^T K_t^{\frac{n-1}{2}}(x_0)\prod_{j=1}^{N}\left[1+\frac{e^{-i\frac{\pi n}{2}}e^{2\pi i\lambda_j t}+e^{i\frac{\pi n}{2}}e^{-2\pi i\lambda_j t}}{2}\right]dt=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n-1}{2}}}\sum_{j=1}^{N}\frac{1}{\lambda_j^n}.$$

Suppose we had that

$$\sup_{R\geq 1}|K_R^{\frac{n-1}{2}}(x_0)|\leq A_{x_0}<\infty.$$

Then, setting $c_n=\dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n-1}{2}}}$, we would have

$$c_n\sum_{j=1}^{N}\frac{1}{\lambda_j^n}=\lim_{T\to\infty}\frac{1}{T}\int_1^T K_t^{\frac{n-1}{2}}(x_0)\prod_{j=1}^{N}\left[1+\frac{e^{-i\frac{\pi n}{2}}e^{2\pi i\lambda_j t}+e^{i\frac{\pi n}{2}}e^{-2\pi i\lambda_j t}}{2}\right]dt$$

$$=\limsup_{T\to\infty}\frac{1}{T}\int_1^T |K_t^{\frac{n-1}{2}}(x_0)|\prod_{j=1}^{N}\left|1+\frac{e^{-i\frac{\pi n}{2}}e^{2\pi i\lambda_j t}+e^{i\frac{\pi n}{2}}e^{-2\pi i\lambda_j t}}{2}\right|dt$$

$$\leq A_{x_0} \limsup_{T\to\infty} \frac{1}{T} \int_1^T \prod_{j=1}^N \left[1 + \frac{e^{-i\frac{\pi n}{2}} e^{2\pi i \lambda_j t} + e^{i\frac{\pi n}{2}} e^{-2\pi i \lambda_j t}}{2} \right] dt$$

$$= A_{x_0},$$

which contradicts (4.2.19) by letting $N \to \infty$. Here, once again, we used the fact that no expression of the form $\pm\lambda_{j_1} \pm \cdots \pm \lambda_{j_s}$ is equal to an integer and thus the preceding $\limsup$ is a limit and is equal to 1, since the integral of all the exponentials produces another exponential which remains bounded.

We deduce that $\sup_{R\geq 1} |K_R^{\frac{n-1}{2}}(x_0)| = \infty$ for every point $x_0 \in S \cap [-\frac{1}{2}, \frac{1}{2})^n \setminus \{0\}$ and this concludes the proof of Lemma 4.2.7. $\qquad\square$

Proof. We now prove Theorem 4.2.5. This part of the proof is similar to the proof of Theorem 4.2.1. Lemma 4.2.7 says that the means $B_R^{\frac{n-1}{2}}(\delta_0)(x)$, where δ_0 is the Dirac mass at 0, do not converge for almost all $x \in \mathbf{T}^n$. Our goal is to replace this Dirac mass by a series of integrable functions on $\mathbf{T}^n$ that have a peak at the origin.

Let us fix a nonnegative $\mathscr{C}^\infty$ radial function $\widehat{\Phi}$ on $\mathbf{R}^n$ that is supported in the unit ball $|\xi| \leq 1$ and has integral equal to 1. We set

$$\varphi_\varepsilon(x) = \sum_{m\in\mathbf{Z}^n} \frac{1}{\varepsilon^n} \widehat{\Phi}\left(\frac{x+m}{\varepsilon}\right) = \sum_{m\in\mathbf{Z}^n} \Phi(\varepsilon m) e^{2\pi i m \cdot x},$$

where the identity is valid because of the Poisson summation formula. It follows that the mth Fourier coefficient of φ_ε is $\Phi(\varepsilon m)$. Therefore, we have the estimate

$$\sup_{x\in\mathbf{T}^n} \sup_{R>0} |B_R^{\frac{n-1}{2}}(\varphi_\varepsilon)(x)| \leq \sum_{m\in\mathbf{Z}^n} |\Phi(\varepsilon m)| \leq \sum_{m\in\mathbf{Z}^n} \frac{C_n'}{(1+\varepsilon|m|)^{n+1}} \leq \frac{C_n}{\varepsilon^n}. \qquad (4.2.21)$$

For any $k \geq 1$, we construct measurable subsets E_k of $\mathbf{T}^n$ with $|E_k| \geq 1 - \frac{1}{k}$, a sequence of positive numbers $R_1 < R_2 < \cdots$, with $R_k \uparrow \infty$, and two sequences of positive numbers $\varepsilon_k \downarrow 0$ and $\gamma_k \downarrow 0$ such that $\varepsilon_k \leq \gamma_k$ for all k and

$$\sup_{R\leq R_k} \left| B_R^{\frac{n-1}{2}} \left(\sum_{s=1}^\infty 2^{-s} (\varphi_{\varepsilon_s} - \varphi_{\gamma_s}) \right)(x) \right| \geq k \qquad \text{for } x \in E_k. \qquad (4.2.22)$$

We pick $E_1 = \emptyset$, $R_1 = 1$, and $\varepsilon_1 = \gamma_1 = 1$. Let $k > 1$ and suppose that we have selected E_j, R_j, γ_j, and ε_j for all $1 \leq j \leq k-1$ such that (4.2.22) is satisfied. We construct E_k, R_k, γ_k, and ε_k such that (4.2.22) is satisfied with $j = k$. We begin by choosing γ_k. Let B be a constant such that

$$|\Phi(x) - \Phi(y)| \leq B|x-y|$$

for all $x, y \in \mathbf{R}^n$. Define γ_k such that

$$B\gamma_k \sum_{|m|\leq R_{k-1}} |m| = 1. \qquad (4.2.23)$$

Then define

$$A_k = C_n 2^{-k}\gamma_k^{-n} + C_n \sum_{j=1}^{k-1} 2^{-j}(\varepsilon_j^{-n} + \gamma_j^{-n}),$$

where C_n is the constant in (4.2.21), and observe that in view of (4.2.21) we have

$$\sup_{x\in\mathbf{T}^n}\sup_{R>0}\left|B_R^{\frac{n-1}{2}}\left(-2^{-k}\varphi_{\gamma_k} + \sum_{j=1}^{k-1} 2^{-j}(\varphi_{\varepsilon_j} - \varphi_{\gamma_j})\right)(x)\right| \le A_k. \qquad (4.2.24)$$

Let δ_0 be the Dirac mass at the origin in $\mathbf{T}^n$. Since by Fatou's lemma and Lemma 4.2.7 we have

$$\liminf_{R'\to\infty}\left|\left\{x\in\mathbf{T}^n: \sup_{0<R\le R'}\left|B_R^{\frac{n-1}{2}}(\delta_0)(x)\right| > 2^k(A_k+k+2)\right\}\right| = 1,$$

there exists an $R_k > \max(R_{k-1}, k)$ such that the set

$$E_k = \left\{x\in\mathbf{T}^n: \sup_{0<R\le R_k}\left|B_R^{\frac{n-1}{2}}(2^{-k}\delta_0)(x)\right| > A_k+k+2\right\}$$

has measure at least $1 - \frac{1}{k}$. Note that since R_k is increasing and tends to infinity, (4.2.23) yields that γ_k is decreasing and tends to zero.

We now choose ε_k such that $\varepsilon_k \le \gamma_k$, $\varepsilon_k \le \varepsilon_{k-1}$, and that

$$\sup_{x\in\mathbf{T}^n}\sup_{R\le R_k} 2^{-k}\left|B_R^{\frac{n-1}{2}}(\delta_0)(x) - B_R^{\frac{n-1}{2}}(\varphi_{\varepsilon_k})(x)\right| \le \sum_{|m|\le R_k} 2^{-k}\left(1 - \frac{|m|^2}{R_k^2}\right)^{\frac{n-1}{2}}\left|1 - \widehat{\varphi_{\varepsilon_k}}(m)\right| \le 1.$$

This is possible, since for a fixed R_k, the preceding sum tends to zero as $\varepsilon_k \to 0$. Then for $x\in E_k$ we have

$$\inf_{x\in E_k}\sup_{R\le R_k} 2^{-k}\left|B_R^{\frac{n-1}{2}}(\varphi_{\varepsilon_k})(x)\right| \ge A_k+k+1. \qquad (4.2.25)$$

The inductive selection of the parameters can be schematically described as follows:

$$\{\gamma_{k-1}, R_{k-1}, E_{k-1}, \varepsilon_{k-1}\} \implies \gamma_k \implies A_k \implies \{R_k, E_k\} \implies \varepsilon_k \implies \{\gamma_k, R_k, E_k, \varepsilon_k\}.$$

Observe that the construction of γ_k gives for all $s \ge k+1$ the estimate

$$\begin{aligned}
\sup_{x\in\mathbf{T}^n}\sup_{R\le R_k}\left|B_R^{\frac{n-1}{2}}(\varphi_{\varepsilon_s} - \varphi_{\gamma_s})(x)\right| &\le \sum_{|m|\le R_k}\left|\Phi(\varepsilon_s m) - \Phi(\gamma_s m)\right| \\
&\le B(\gamma_s - \varepsilon_s)\sum_{|m|\le R_k}|m| \\
&\le B\gamma_s\sum_{|m|\le R_k}|m| \\
&\le B\gamma_{k+1}\sum_{|m|\le R_k}|m| = 1,
\end{aligned} \qquad (4.2.26)$$

using (4.2.23) and the fact that the sequence γ_k is decreasing.

We now prove (4.2.22). For $x \in \mathbf{T}^n$ write

$$B_R^{\frac{n-1}{2}}\left(\sum_{s=1}^{\infty} 2^{-s}(\varphi_{\varepsilon_s} - \varphi_{\gamma_s})\right)(x) = B_R^{\frac{n-1}{2}}\left(-2^{-k}\varphi_{\gamma_k} + \sum_{s=1}^{k-1} 2^{-s}(\varphi_{\varepsilon_s} - \varphi_{\gamma_s})\right)(x)$$
$$+ B_R^{\frac{n-1}{2}}\left(2^{-k}\varphi_{\varepsilon_k}\right)(x)$$
$$+ B_R^{\frac{n-1}{2}}\left(\sum_{s=k+1}^{\infty} 2^{-s}(\varphi_{\varepsilon_s} - \varphi_{\gamma_s})\right)(x),$$

from which it follows that

$$\sup_{R \leq R_k}\left|B_R^{\frac{n-1}{2}}\left(\sum_{s=1}^{\infty} 2^{-s}(\varphi_{\varepsilon_s} - \varphi_{\gamma_s})\right)(x)\right| \geq \sup_{R \leq R_k}\left|B_R^{\frac{n-1}{2}}\left(2^{-k}\varphi_{\varepsilon_k}\right)(x)\right|$$
$$- \sup_{R \leq R_k}\left|B_R^{\frac{n-1}{2}}\left(-2^{-k}\varphi_{\gamma_k} + \sum_{s=1}^{k-1} 2^{-s}(\varphi_{\varepsilon_s} - \varphi_{\gamma_s})\right)(x)\right|$$
$$- \sup_{R \leq R_k}\left|B_R^{\frac{n-1}{2}}\left(\sum_{s=k+1}^{\infty} 2^{-s}(\varphi_{\varepsilon_s} - \varphi_{\gamma_s})\right)(x)\right|.$$

In view of (4.2.25), (4.2.24), and (4.2.26) for all $x \in E_k$, we obtain

$$\sup_{R \leq R_k}\left|B_R^{\frac{n-1}{2}}\left(\sum_{s=1}^{\infty} 2^{-s}(\varphi_{\varepsilon_s} - \varphi_{\gamma_s})\right)(x)\right| \geq (A_k + k + 1) - A_k - \sum_{s=k+1}^{\infty} 2^{-s} \geq k,$$

which clearly implies (4.2.22). Setting

$$f = \sum_{s=1}^{\infty} 2^{-s}(\varphi_{\varepsilon_s} - \varphi_{\gamma_s}) \in L^1(\mathbf{T}^n),$$

we deduce that $\sup_{R>0}\left|B_R^{\frac{n-1}{2}}(f)(x)\right| \geq k$ for all x in $\bigcup_{r=k}^{\infty} E_r$, and thus

$$\sup_{R>0}\left|B_R^{\frac{n-1}{2}}(f)(x)\right| = \infty$$

for all x in

$$\bigcap_{k=1}^{\infty} \bigcup_{r=k}^{\infty} E_r.$$

Since this set has full measure in $\mathbf{T}^n$, the required conclusion follows.

By taking ε_1 arbitrarily small (instead of picking $\varepsilon_1 = 1$), we force f to be supported in an arbitrarily small neighborhood of the origin. $\qquad\square$

The previous argument shows that the Bochner–Riesz means B_R^{α} are badly behaved on $L^1(\mathbf{T}^n)$ when $\alpha = \frac{n-1}{2}$. It follows that the "rougher" spherical Dirichlet means $\overset{\circ}{D}_N^n * f$ (which correspond to $\alpha = 0$) are also ill behaved on $L^1(\mathbf{T}^n)$. See Exercise 4.2.2.

Exercises

4.2.1. Using Theorem 4.2.1 construct a function F on $\mathbf{T}^n$ such that

$$\limsup_{N \to \infty} |(D_N^n * F)(x_1, \ldots, x_n)| = \infty$$

for almost all $(x_1, \ldots, x_n) \in \mathbf{T}^n$.

4.2.2. For any $0 \le \alpha < \infty$ and $R > 0$ consider the Bochner–Riesz kernel

$$K_R^\alpha(x) = \sum_{|m| \le R} \left(1 - \frac{|m|^2}{R^2}\right)^\alpha e^{2\pi i m \cdot x}.$$

Use Exercise 4.1.8 to obtain that if for some $x_0 \in \mathbf{T}^n$ we have

$$\limsup_{R \to \infty} |K_R^\alpha(x_0)| < \infty,$$

then for all $\beta > \alpha$ we have

$$\sup_{R > 0} |K_R^\beta(x_0)| < \infty.$$

Conclude that whenever $0 \le \alpha \le \frac{n-1}{2}$, the Bochner–Riesz means of order α of the function f constructed in the proof of Theorem 4.2.5, in particular the circular (spherical) Dirichlet means of this function, diverge a.e.

4.2.3. (a) Show that for M, N positive integers we have

$$(F_M * D_N)(x) = \begin{cases} F_M(x) & \text{for } M \le N, \\ F_N(x) + \frac{M-N}{(M+1)(N+1)} \sum_{|k| \le N} |k| e^{2\pi i k x} & \text{for } M > N. \end{cases}$$

(b) Prove that for some constant $c > 0$ we have

$$\int_{\mathbf{T}^1} \left| \sum_{|k| \le N} |k| e^{2\pi i k x} \right| dx \ge c N \log N$$

as $N \to \infty$.
$\big[$*Hint:* Part (b): Show that for $x \in [-\frac{1}{2}, \frac{1}{2}]$ we have

$$\sum_{|k| \le N} |k| e^{2\pi i k x} = (N+1)(D_N(x) - F_N(x))$$

and use the result of Exercise 3.1.5.$\big]$

4.2.4. Given the integrable functions

$$f_1(x) = \sum_{j=0}^{\infty} 2^{-j} F_{2^{2^j}}(x), \qquad f_2(x) = \sum_{j=1}^{\infty} \frac{1}{j^2} F_{2^{2^j}}(x), \qquad x \in \mathbf{T}^1,$$

show that $\|f_1 * D_N\|_{L^1} \to \infty$ and $\|f_2 * D_N\|_{L^1} \to \infty$ as $N \to \infty$.

$\big[$*Hint:* Let $M_j = 2^{2^{2^j}}$ or $M_j = 2^{2^j}$ depending on the situation. For fixed N let j_N be the least integer j such that $M_j > N$. Then for $j \geq j_N + 1$ we have $M_j \geq M_{j_N}^2 > N^2 \geq 2N+1$, hence $\frac{M_j - N}{M_j + 1} \geq \frac{1}{2}$. Split the summation indices into the sets $j \geq j_N$ and $j < j_N$. Conclude that $\|f_1 * D_N\|_{L^1}$ and $\|f_2 * D_N\|_{L^1}$ tend to infinity as $N \to \infty$ using Exercise 4.2.3.$\big]$

4.3 Multipliers, Transference, and Almost Everywhere Convergence

In Chapter 2 we saw that bounded operators from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ that commute with translations are given by convolution with tempered distributions on $\mathbf{R}^n$. In particular, when $p = q$, these tempered distributions have bounded Fourier transforms, called Fourier multipliers. Convolution operators that commute with translations can also be defined on the torus. These lead to Fourier multipliers on the torus.

4.3.1 Multipliers on the Torus

In analogy with the nonperiodic case, we could identify convolution operators on $\mathbf{T}^n$ with appropriate distributions on the torus; see Exercise 4.3.2 for an introduction to this topic. However, it is simpler to avoid this point of view and consider the study of multipliers directly, bypassing the discussion of distributions on the torus.

For $h \in \mathbf{T}^n$ we define the *translation operator* τ^h acting on a periodic function f as follows: $\tau^h(f)(x) = f(x - h)$ for $x \in \mathbf{T}^n$. We say that a linear operator T acting on functions on the torus *commutes with translations* if for all $h \in \mathbf{T}^n$ we have $\tau^h(T(f))(x) = T(\tau^h f)(x)$ for almost all $x \in \mathbf{T}^n$.

Theorem 4.3.1. *Suppose that T is a linear operator that commutes with translations and maps $L^p(\mathbf{T}^n)$ to $L^q(\mathbf{T}^n)$ for some $1 \leq p, q \leq \infty$. Then there exists a bounded sequence $\{a_m\}_{m \in \mathbf{Z}^n}$ such that*

$$T(f)(x) = \sum_{m \in \mathbf{Z}^n} a_m \widehat{f}(m) e^{2\pi i m \cdot x} \tag{4.3.1}$$

for all $f \in \mathscr{C}^\infty(\mathbf{T}^n)$. Moreover, we have

$$\left\|\{a_m\}\right\|_{\ell^\infty} \le \left\|T\right\|_{L^p \to L^q}.$$

Proof. Consider the functions $e_m(x) = e^{2\pi i m \cdot x}$ defined on $\mathbf{T}^n$ for m in $\mathbf{Z}^n$. Since T commutes with translations, for every $h \in \mathbf{T}^n$ there is a subset F_h of $\mathbf{T}^n$ of full measure such that

$$T(e_m)(x - h) = T(\tau^h(e_m))(x) = e^{-2\pi i m \cdot h} T(e_m)(x)$$

for every $x \in F_h$. Note that

$$\begin{aligned}
\int_{\mathbf{T}^n} |\{h \in \mathbf{T}^n : x \in F_h\}| \, dx &= \int_{\mathbf{T}^n} \int_{\mathbf{T}^n} \chi_{\{(h,x) \in \mathbf{T}^n \times \mathbf{T}^n : x \in F_h\}} \, dh \, dx \\
&= \int_{\mathbf{T}^n} \int_{\mathbf{T}^n} \chi_{\{(h,x) \in \mathbf{T}^n \times \mathbf{T}^n : x \in F_h\}} \, dx \, dh \\
&= \int_{\mathbf{T}^n} |F_h| \, dh = 1.
\end{aligned}$$

Therefore there exists an $x_0 \in \mathbf{T}^n$ such that $|\{h \in \mathbf{T}^n : x_0 \in F_h\}| = 1$. It follows that for almost all $h \in \mathbf{T}^n$ we have $T(e_m)(x_0 - h) = e^{-2\pi i m \cdot h} T(e_m)(x_0)$. Replacing $x_0 - h$ by x, we obtain

$$T(e_m)(x) = e^{2\pi i m \cdot x}\left(e^{-2\pi i m \cdot x_0} T(e_m)(x_0)\right) = a_m e_m(x) \qquad (4.3.2)$$

for almost all $x \in \mathbf{T}^n$, where we set $a_m = e^{-2\pi i m \cdot x_0} T(e_m)(x_0)$, for $m \in \mathbf{Z}^n$. Taking L^q norms in (4.3.2), we deduce $|a_m| = \|T(e_m)\|_{L^q} \le \|T\|_{L^p \to L^q}$, and thus a_m is bounded. Moreover, since $T(e_m) = a_m e_m$ for all m in $\mathbf{Z}^n$, it follows that (4.3.1) holds for all trigonometric polynomials. By density this extends to all $f \in \mathscr{C}^\infty(\mathbf{T}^n)$ and the theorem is proved. $\qquad \square$

Definition 4.3.2. Let $1 \le p, q \le \infty$. We call a bounded sequence $\{a_m\}_{m \in \mathbf{Z}^n}$ an (L^p, L^q) *multiplier* if the corresponding operator given by (4.3.1) maps $L^p(\mathbf{T}^n)$ to $L^q(\mathbf{T}^n)$. If $p = q$, (L^p, L^p) multipliers are simply called L^p multipliers. When $1 \le p < \infty$, the space of all L^p multipliers on $\mathbf{T}^n$ is denoted by $\mathscr{M}_p(\mathbf{Z}^n)$. This notation follows the convention that $\mathscr{M}_p(\widehat{G})$ denote the space of L^p multipliers on $L^p(G)$, where G is a locally compact group and $\widehat{G}$ is its dual group. The norm of an element $\{a_m\}$ in $\mathscr{M}_p(\mathbf{Z}^n)$ is the norm of the operator T given by (4.3.1) from $L^p(\mathbf{T}^n)$ to itself. This norm is denoted by $\left\|\{a_m\}\right\|_{\mathscr{M}_p}$.

We now examine some special cases. We begin with the case $p = q = 2$. As expected, it turns out that $\mathscr{M}_2(\mathbf{Z}^n) = \ell^\infty(\mathbf{Z}^n)$.

Theorem 4.3.3. *A linear operator T that commutes with translations maps $L^2(\mathbf{T}^n)$ to itself if and only if there exists a sequence $\{a_m\}_{m \in \mathbf{Z}^n}$ in $\ell^\infty(\mathbf{Z}^n)$ such that*

$$T(f)(x) = \sum_{m \in \mathbf{Z}^n} a_m \widehat{f}(m) e^{2\pi i m \cdot x} \qquad (4.3.3)$$

for all $f \in \mathscr{C}^\infty(\mathbf{T}^n)$. Moreover, in this case we have $\left\|T\right\|_{L^2 \to L^2} = \left\|\{a_m\}_m\right\|_{\ell^\infty}$.

Proof. The existence of such a sequence is guaranteed by Theorem 4.3.1, which also gives $\|\{a_m\}_m\|_{\ell^\infty} \leq \|T\|_{L^2 \to L^2}$. Conversely, any operator of the form (4.3.3) satisfies

$$\|T(f)\|_{L^2}^2 = \sum_{m \in \mathbf{Z}^n} |a_m \widehat{f}(m)|^2 \leq \|\{a_m\}_m\|_{\ell^\infty}^2 \sum_{m \in \mathbf{Z}^n} |\widehat{f}(m)|^2,$$

and thus $\|T\|_{L^2 \to L^2} \leq \|\{a_m\}_m\|_{\ell^\infty}$. $\square$

We continue with the case $p = q = 1$. Recall the definition of a finite Borel measure on $\mathbf{T}^n$. Given such a measure μ, its Fourier coefficients are defined by

$$\widehat{\mu}(m) = \int_{\mathbf{T}^n} e^{-2\pi i x \cdot m} d\mu(x), \qquad m \in \mathbf{Z}^n.$$

Clearly, all the Fourier coefficients of the measure μ are bounded by the total variation $\|\mu\|$ of μ. See Exercise 4.3.3 for basic properties of Fourier transforms of distributions on the torus.

Theorem 4.3.4. *A linear operator T that commutes with translations maps $L^1(\mathbf{T}^n)$ to itself if and only if there exists a finite Borel measure μ on the torus such that*

$$T(f)(x) = \sum_{m \in \mathbf{Z}^n} \widehat{\mu}(m) \widehat{f}(m) e^{2\pi i m \cdot x} \tag{4.3.4}$$

for all $f \in \mathscr{C}^\infty(\mathbf{T}^n)$. Moreover, in this case we have $\|T\|_{L^1 \to L^1} = \|\mu\|$. In other words, $\mathscr{M}_1(\mathbf{Z}^n)$ is the set of all sequences given by Fourier coefficients of finite Borel measures on $\mathbf{T}^n$.

Proof. Fix $f \in L^1(\mathbf{T}^n)$. If (4.3.4) is valid, then $\widehat{T(f)}(m) = \widehat{f}(m)\widehat{\mu}(m)$ for all $m \in \mathbf{Z}^n$. But Exercise 4.3.3 gives that $\widehat{f * \mu}(m) = \widehat{f}(m)\widehat{\mu}(m)$ for all $m \in \mathbf{Z}^n$; therefore, the integrable functions $f * \mu$ and $T(f)$ have the same Fourier coefficients, and they must be equal. Thus $T(f) = f * \mu$, which implies that T is bounded on L^1 and $\|T(f)\|_{L^1} \leq \|\mu\| \|f\|_{L^1}$.

To prove the converse direction, we suppose that T commutes with translations and maps $L^1(\mathbf{T}^n)$ to itself. We recall the Poisson kernel P_ε defined on $\mathbf{T}^n$, which can be expressed in the following two ways:

$$P_\varepsilon(x) = \sum_{m \in \mathbf{Z}^n} e^{-2\pi |m| \varepsilon} e^{2\pi i m \cdot x} = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{m \in \mathbf{Z}^n} \frac{\varepsilon^{-n}}{(1 + |\frac{x+m}{\varepsilon}|^2)^{\frac{n+1}{2}}} \geq 0 \tag{4.3.5}$$

for all $x \in \mathbf{T}^n$, in view of the identity obtained in (3.2.4). The preceding identity says that $P_\varepsilon \geq 0$; hence, $\|P_\varepsilon\|_{L^1} = \int_{\mathbf{T}^n} P_\varepsilon(x)\, dx$. Integrating the first series in (4.3.5) over $\mathbf{T}^n$ we conclude that $\|P_\varepsilon\|_{L^1(\mathbf{T}^n)} = 1$. The boundedness of T now gives

$$\|T(P_\varepsilon)\|_{L^1(\mathbf{T}^n)} \leq \|T\|_{L^1 \to L^1}$$

for all $\varepsilon > 0$. The Banach–Alaoglu theorem implies that there exist a sequence $\varepsilon_j \downarrow 0$ and a finite Borel measure μ on $\mathbf{T}^n$ such that $T(P_{\varepsilon_j})$ tends to μ weakly as $j \to \infty$. This means that for all continuous functions g on $\mathbf{T}^n$ we have

$$\lim_{j\to\infty} \int_{\mathbf{T}^n} g(x) T(P_{\varepsilon_j})(x)\,dx = \int_{\mathbf{T}^n} g(x)\,d\mu(x). \qquad (4.3.6)$$

It follows from (4.3.6) that for all g continuous on $\mathbf{T}^n$ we have

$$\left| \int_{\mathbf{T}^n} g(x)\,d\mu(x) \right| \leq \sup_j \left\| T(P_{\varepsilon_j}) \right\|_{L^1} \|g\|_{L^\infty} \leq \left\| T \right\|_{L^1 \to L^1} \|g\|_{L^\infty}.$$

Since, by the Riesz representation theorem we have that the norm of the linear functional

$$g \mapsto \int_{\mathbf{T}^n} g(x)\,d\mu(x)$$

on the space of continuous functions $\mathscr{C}(\mathbf{T}^n)$ is $\|\mu\|$, it follows that

$$\|\mu\| \leq \left\| T \right\|_{L^1 \to L^1}. \qquad (4.3.7)$$

It remains to prove that T has the form given in (4.3.4). By Theorem 4.3.1 we have that there exists a bounded sequence $\{a_m\}$ on $\mathbf{Z}^n$ such that (4.3.1) is satisfied. Taking $g(x) = e^{-2\pi i k \cdot x}$ in (4.3.6) and using the representation for T in (4.3.1), we obtain

$$\begin{aligned}
\widehat{\mu}(k) &= \int_{\mathbf{T}^n} e^{-2\pi i k \cdot x}\,d\mu(x) \\
&= \lim_{j\to\infty} \int_{\mathbf{T}^n} e^{-2\pi i k \cdot x} \sum_{m\in\mathbf{Z}^n} a_m e^{-2\pi \varepsilon_j |m|} e^{2\pi i m \cdot x}\,dx \\
&= \lim_{j\to\infty} \sum_{m\in\mathbf{Z}^n} \int_{\mathbf{T}^n} e^{-2\pi i k \cdot x} a_m e^{-2\pi \varepsilon_j |m|} e^{2\pi i m \cdot x}\,dx \\
&= \lim_{j\to\infty} a_k e^{-2\pi \varepsilon_j |k|} = a_k.
\end{aligned}$$

This proves assertion (4.3.4). It follows from (4.3.4) that $T(f) = f * \mu$ and thus $\|T\|_{L^1 \to L^1} \leq \|\mu\|$. This fact combined with (4.3.7) gives $\|T\|_{L^1 \to L^1} = \|\mu\|$. $\qquad \square$

Remark 4.3.5. It is not hard to see that most basic properties of the space $\mathscr{M}_p(\mathbf{R}^n)$ of L^p Fourier multipliers on $\mathbf{R}^n$ are also valid for $\mathscr{M}_p(\mathbf{Z}^n)$. In particular, $\mathscr{M}_p(\mathbf{Z}^n)$ is a closed subspace of $\ell^\infty(\mathbf{Z}^n)$ and thus a Banach space itself. Moreover, sums, scalar multiples, and products of elements of $\mathscr{M}_p(\mathbf{Z}^n)$ are also in $\mathscr{M}_p(\mathbf{Z}^n)$, which makes this space a Banach algebra. As in the nonperiodic case, we also have $\mathscr{M}_p(\mathbf{Z}^n) = \mathscr{M}_{p'}(\mathbf{Z}^n)$ when $1 < p < \infty$.

4.3.2 Transference of Multipliers

It is clear by now that multipliers on $L^1(\mathbf{T}^n)$ and $L^1(\mathbf{R}^n)$ are very similar, and the same is true for $L^2(\mathbf{T}^n)$ and $L^2(\mathbf{R}^n)$. These similarities became obvious when we characterized L^1 and L^2 multipliers on both $\mathbf{R}^n$ and $\mathbf{T}^n$. So far, it is not known if a nontrivial characterization of $\mathscr{M}_p(\mathbf{R}^n)$ exists, but we might ask whether this space is related to $\mathscr{M}_p(\mathbf{Z}^n)$. There are several connections of this type and there are general ways to produce multipliers on the torus from multipliers on $\mathbf{R}^n$ and vice versa. General methods of this sort are called *transference of multipliers*.

We begin with a useful definition.

Definition 4.3.6. Let $t_0 \in \mathbf{R}^n$. A bounded function b on $\mathbf{R}^n$ is called *regulated at the point t_0* if

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} \int_{|t| \leq \varepsilon} \big(b(t_0 - t) - b(t_0)\big)\, dt = 0. \qquad (4.3.8)$$

The function b is called *regulated* if it is regulated at every $t_0 \in \mathbf{R}^n$.

Clearly, if t_0 is a Lebesgue point of b, then b is regulated at t_0. In particular, this is the case if b is continuous at t_0. If $b(t_0) = 0$, condition (4.3.8) also holds when $b(t_0 - t) = -b(t_0 + t)$ whenever $|t| \leq \varepsilon$ for some $\varepsilon > 0$; for instance the function $b(t) = -i \operatorname{sgn}(t - t_0)$ has this property.

An example of a regulated function is the following modification of the characteristic function of the cube $[-1,1]^n$

$$\widetilde{\chi}_{[-1,1]^n}(x_1, \ldots, x_n) = \begin{cases} 1 & \text{when all } |x_j| < 1, \\ 2^{k-n} & \text{if } (x_1, \ldots, x_n) \text{ belongs to some } k\text{-dimensional} \\ & \text{face of the boundary of } [-1,1]^n, \\ 0 & \text{when some } |x_j| > 1, \end{cases}$$

with the understanding that points are zero-dimensional.

The first transference result we discuss is the following.

Theorem 4.3.7. *Suppose that b is a regulated function at every point $m \in \mathbf{Z}^n$ and that b lies in $\mathscr{M}_p(\mathbf{R}^n)$ for some $1 < p < \infty$. Then the sequence $\{b(m)\}_{m \in \mathbf{Z}^n}$ is in $\mathscr{M}_p(\mathbf{Z}^n)$ and moreover,*

$$\big\|\{b(m)\}_{m \in \mathbf{Z}^n}\big\|_{\mathscr{M}_p(\mathbf{Z}^n)} \leq \|b\|_{\mathscr{M}_p(\mathbf{R}^n)}.$$

If b is regulated everywhere, then for all $R > 0$ the sequences $\{b(m/R)\}_{m \in \mathbf{Z}^n}$ are in $\mathscr{M}_p(\mathbf{Z}^n)$ and we have

$$\sup_{R > 0} \big\|\{b(m/R)\}_{m \in \mathbf{Z}^n}\big\|_{\mathscr{M}_p(\mathbf{Z}^n)} \leq \|b\|_{\mathscr{M}_p(\mathbf{R}^n)}.$$

The second conclusion of the theorem is a consequence of the first, since for a given $R > 0$ the function $b(\xi/R)$ is regulated on $\mathbf{Z}^n$ and has the same $\mathscr{M}_p(\mathbf{R}^n)$ norm as $b(\xi)$. Before we prove this result, we state and prove a couple of lemmas.

Lemma 4.3.8. *Suppose that the function b on $\mathbf{R}^n$ is regulated at the point x_0. Let $K_\varepsilon(x) = \varepsilon^{-n} e^{-\pi|x/\varepsilon|^2}$ for $\varepsilon > 0$. Then we have that $(b * K_\varepsilon)(x_0) \to b(x_0)$ as $\varepsilon \to 0$.*

Proof. For $r > 0$ define the function

$$F_{x_0}(r) = \frac{1}{r^n} \int_{|t| \le r} \left(b(x_0 - t) - b(x_0) \right) dt = \frac{1}{r^n} \int_0^r \int_{\mathbf{S}^{n-1}} \left(b(x_0 - s\theta) - b(x_0) \right) d\theta \, s^{n-1} ds.$$

Let $\eta > 0$. Since b is regulated at x_0 there is a $\delta > 0$ such that for $r \le \delta$ we have $|F_{x_0}(r)| \le \eta$. Fix such a δ and write

$$(b * K_\varepsilon)(x_0) - b(x_0) = \int_{y \in \mathbf{R}^n} \left(b(x_0 - y) - b(x_0) \right) K_\varepsilon(y) \, dy = A_1^\varepsilon + A_2^\varepsilon,$$

where

$$A_1^\varepsilon = \int_{|y| \ge \delta} \left(b(x_0 - y) - b(x_0) \right) K_\varepsilon(y) \, dy$$

and

$$A_2^\varepsilon = \int_{|y| < \delta} \left(b(x_0 - y) - b(x_0) \right) K_\varepsilon(y) \right) dy$$

$$= \int_0^\delta \frac{1}{\varepsilon^n} e^{-\pi(r/\varepsilon)^2} \int_{\mathbf{S}^{n-1}} \left(b(x_0 - r\theta) - b(x_0) \right) r^n dr$$

$$= \int_0^\delta \frac{1}{\varepsilon^n} e^{-\pi(r/\varepsilon)^2} \frac{d}{dr} \left(r^n F_{x_0}(r) \right) dr.$$

For our given $\eta > 0$ there is an $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ we have

$$|A_1^\varepsilon| \le 2\|b\|_{L^\infty} \int_{|y| \ge \frac{\delta}{\varepsilon}} e^{-\pi|y|^2} dy < \eta.$$

Via an integration by parts A_2^ε can be written as

$$|A_2^\varepsilon| = \left| \delta^n F_{x_0}(\delta) \frac{1}{\varepsilon^n} e^{-\pi(\delta/\varepsilon)^2} - 0 + 2\pi \int_0^\delta \frac{r}{\varepsilon^{n+2}} e^{-\pi(r/\varepsilon)^2} r^n F_{x_0}(r) \, dr \right|$$

$$= \left| F_{x_0}(\delta) \frac{\delta^n}{\varepsilon^n} e^{-\pi(\delta/\varepsilon)^2} + 2\pi \int_0^{\delta/\varepsilon} r^{n+1} F_{x_0}(\varepsilon r) e^{-\pi r^2} dr \right|$$

$$\le |F_{x_0}(\delta)| \frac{\delta^n}{\varepsilon^n} e^{-\pi(\delta/\varepsilon)^2} + \sup_{0 < r \le \frac{\delta}{\varepsilon}} |F_{x_0}(\varepsilon r)| 2\pi \int_0^{\delta/\varepsilon} r^{n+1} e^{-\pi r^2} dr$$

$$\le |F_{x_0}(\delta)| C_n + \sup_{0 < r \le \delta} |F_{x_0}(r)| C_n'$$

$$\le (C_n + C_n') \eta,$$

where we set $C_n = \sup_{t>0} t^n e^{-\pi t^2}$ and $C_n' = 2\pi \int_0^\infty r^{n+1} e^{-\pi r^2} dr$. Then for $\varepsilon < \varepsilon_0$ we have $|(b * K_\varepsilon)(x_0 - b(x_0)| < (C_n + C_n' + 1)\eta$, thus $(b * K_\varepsilon)(x_0) \to b(x_0)$ as $\varepsilon \to 0$. $\square$

Lemma 4.3.9. *Let T be the operator on $\mathbf{R}^n$ whose multiplier is $b(\xi)$, and let S be the operator on $\mathbf{T}^n$ whose multiplier is the sequence $\{b(m)\}_{m\in\mathbf{Z}^n}$. Assume that $b(\xi)$ is regulated at every point $\xi = m \in \mathbf{Z}^n$. Suppose that P and Q are trigonometric polynomials on $\mathbf{T}^n$ and let $L_\varepsilon(x) = e^{-\pi\varepsilon|x|^2}$ for $x \in \mathbf{R}^n$ and $\varepsilon > 0$. Then the following identity is valid whenever $\alpha, \beta > 0$ and $\alpha + \beta = 1$:*

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{n}{2}} \int_{\mathbf{R}^n} T(PL_{\varepsilon\alpha})(x)\overline{Q(x)L_{\varepsilon\beta}(x)}\,dx = \int_{\mathbf{T}^n} S(P)(x)\overline{Q(x)}\,dx. \tag{4.3.9}$$

Proof. It suffices to prove the required assertion for $P(x) = e^{2\pi i m\cdot x}$ and $Q(x) = e^{2\pi i k\cdot x}$, $k, m \in \mathbf{Z}^n$, since the general case follows from this case by linearity. In view of Parseval's relation (Proposition 3.2.7 (3)), we have

$$\int_{\mathbf{T}^n} S(P)(x)\overline{Q(x)}\,dx = \sum_{r\in\mathbf{Z}^n} b(r)\widehat{P}(r)\overline{\widehat{Q}(r)} = \begin{cases} b(m) & \text{when } k = m, \\ 0 & \text{when } k \neq m. \end{cases} \tag{4.3.10}$$

On the other hand, using the identity in Theorem 2.2.14 (3), we obtain

$$\begin{aligned}
\varepsilon^{\frac{n}{2}} \int_{\mathbf{R}^n} &T(PL_{\varepsilon\alpha})(x)\overline{Q(x)L_{\varepsilon\beta}(x)}\,dx \\
&= \varepsilon^{\frac{n}{2}} \int_{\mathbf{R}^n} b(\xi)\widehat{PL_{\varepsilon\alpha}}(\xi)\overline{\widehat{QL_{\varepsilon\beta}}(\xi)}\,d\xi \\
&= \varepsilon^{\frac{n}{2}} \int_{\mathbf{R}^n} b(\xi)(\varepsilon\alpha)^{-\frac{n}{2}}e^{-\pi\frac{|\xi-m|^2}{\varepsilon\alpha}}(\varepsilon\beta)^{-\frac{n}{2}}e^{-\pi\frac{|\xi-k|^2}{\varepsilon\beta}}\,d\xi \\
&= (\varepsilon\alpha\beta)^{-\frac{n}{2}} \int_{\mathbf{R}^n} b(\xi)e^{-\pi\frac{|\xi-m|^2}{\varepsilon\alpha}}e^{-\pi\frac{|\xi-k|^2}{\varepsilon\beta}}\,d\xi. \tag{4.3.11}
\end{aligned}$$

Now if $m = k$, since $\alpha + \beta = 1$, the expression in (4.3.11) is equal to

$$(\varepsilon\alpha\beta)^{-\frac{n}{2}} \int_{\mathbf{R}^n} b(\xi)e^{-\pi\frac{|\xi-m|^2}{\varepsilon\alpha\beta}}\,d\xi, \tag{4.3.12}$$

which tends to $b(m)$ in view of Lemma 4.3.8, since b is regulated at every point $m \in \mathbf{Z}^n$.

We now consider the case $m \neq k$ in (4.3.11). Since $|m - k| \geq 1$, then every ξ in $\mathbf{R}^n$ must satisfy either $|\xi - m| \geq 1/2$ or $|\xi - k| \geq 1/2$. Therefore, the expression in (4.3.11) is controlled by

$$(\varepsilon\alpha\beta)^{-\frac{n}{2}} \left(\int_{|\xi-m|\geq\frac{1}{2}} b(\xi)e^{-\frac{\pi}{4\varepsilon\alpha}}e^{-\pi\frac{|\xi-k|^2}{\varepsilon\beta}}\,d\xi + \int_{|\xi-k|\geq\frac{1}{2}} b(\xi)e^{-\frac{\pi}{4\varepsilon\beta}}e^{-\pi\frac{|\xi-m|^2}{\varepsilon\alpha}}\,d\xi \right),$$

which is in turn controlled by

$$\|b\|_{L^\infty}\left(\alpha^{-\frac{n}{2}}e^{-\frac{\pi}{4\varepsilon\alpha}} + \beta^{-\frac{n}{2}}e^{-\frac{\pi}{4\varepsilon\beta}}\right),$$

which tends to zero as $\varepsilon \to 0$. This proves that the expression in (4.3.10) is equal to the limit of the expression in (4.3.11) as $\varepsilon \to 0$. This completes the proof of Lemma 4.3.9 □

Proof (Theorem 4.3.7). We are assuming that T maps $L^p(\mathbf{R}^n)$ to itself and we need to show that S maps $L^p(\mathbf{T}^n)$ to itself. We prove this using duality. For P and Q trigonometric polynomials, using Lemma 4.3.9, we have

$$\left| \int_{\mathbf{T}^n} S(P)(x)\overline{Q(x)}\,dx \right|$$

$$= \left| \lim_{\varepsilon \to 0} \varepsilon^{\frac{n}{2}} \int_{\mathbf{R}^n} T(PL_{\varepsilon/p})(x)\overline{Q(x)L_{\varepsilon/p'}(x)}\,dx \right|$$

$$\leq \|T\|_{L^p \to L^p} \limsup_{\varepsilon \to 0} \varepsilon^{\frac{n}{2}} \|PL_{\varepsilon/p}\|_{L^p(\mathbf{R}^n)} \|QL_{\varepsilon/p'}\|_{L^{p'}(\mathbf{R}^n)}$$

$$= \|T\|_{L^p \to L^p} \limsup_{\varepsilon \to 0} \left(\varepsilon^{\frac{n}{2}} \int_{\mathbf{R}^n} |P(x)|^p e^{-\varepsilon\pi|x|^2}\,dx \right)^{\frac{1}{p}} \left(\varepsilon^{\frac{n}{2}} \int_{\mathbf{R}^n} |Q(x)|^{p'} e^{-\varepsilon\pi|x|^2}\,dx \right)^{\frac{1}{p'}}$$

$$= \|T\|_{L^p \to L^p} \left(\int_{\mathbf{T}^n} |P(x)|^p\,dx \right)^{\frac{1}{p}} \left(\int_{\mathbf{T}^n} |Q(x)|^{p'}\,dx \right)^{\frac{1}{p'}},$$

provided for all continuous 1-periodic functions g on $\mathbf{R}^n$ we have that

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{n}{2}} \int_{\mathbf{R}^n} g(x)e^{-\varepsilon\pi|x|^2}\,dx = \int_{\mathbf{T}^n} g(x)\,dx. \qquad (4.3.13)$$

Assuming (4.3.13) for the moment, we take the supremum over all trigonometric polynomials Q on $\mathbf{T}^n$ with $L^{p'}$ norm at most 1 to obtain that S maps $L^p(\mathbf{T}^n)$ to itself with norm at most $\|T\|_{L^p \to L^p}$, yielding the required conclusion.

We now prove (4.3.13). Use the Poisson summation formula to write the left-hand side of (4.3.13) as

$$\varepsilon^{\frac{n}{2}} \sum_{k\in\mathbf{Z}^n} \int_{\mathbf{T}^n} g(x-k)e^{-\varepsilon\pi|x-k|^2}\,dx = \int_{\mathbf{T}^n} g(x)\varepsilon^{\frac{n}{2}} \sum_{k\in\mathbf{Z}^n} e^{-\varepsilon\pi|x-k|^2}\,dx$$

$$= \int_{\mathbf{T}^n} g(x) \sum_{k\in\mathbf{Z}^n} e^{-\pi|k|^2/\varepsilon} e^{2\pi ix\cdot k}\,dx$$

$$= \int_{\mathbf{T}^n} g(x)\,dx + A_\varepsilon,$$

where

$$|A_\varepsilon| \leq \|g\|_{L^\infty} \sum_{|k|\geq 1} e^{-\pi|k|^2/\varepsilon} \to 0$$

as $\varepsilon \to 0$. This completes the proof of Theorem 4.3.7. □

We now obtain a converse of Theorem 4.3.7. If $b(\xi)$ is a bounded function on $\mathbf{R}^n$ and the sequence $\{b(m)\}_{m\in\mathbf{Z}^n}$ is in $\mathscr{M}_p(\mathbf{Z}^n)$, then we cannot necessarily obtain that

b is in $\mathscr{M}_p(\mathbf{R}^n)$, since such a conclusion would depend on the values of b on the integer lattice, which is a set of measure zero. However, a converse can be formulated if we assume that for all $R > 0$, the sequences $\{b(m/R)\}_{m\in\mathbf{Z}^n}$ are in $\mathscr{M}_p(\mathbf{Z}^n)$ uniformly in R. Then we obtain that $b(\xi/R)$ is in $\mathscr{M}_p(\mathbf{R}^n)$ uniformly in $R > 0$, which is equivalent to saying that $b \in \mathscr{M}_p(\mathbf{R}^n)$, since dilations of multipliers on $\mathbf{R}^n$ do not affect their norms (see Proposition 2.5.14). These remarks can be precisely expressed in the following theorem.

Theorem 4.3.10. *Suppose that $b(\xi)$ is a bounded function defined on $\mathbf{R}^n$ which is Riemann integrable over any cube. Suppose that the sequences $\{b(\frac{m}{R})\}_{m\in\mathbf{Z}^n}$ are in $\mathscr{M}_p(\mathbf{Z}^n)$ uniformly in $R > 0$ for some $1 < p < \infty$. Then b is in $\mathscr{M}_p(\mathbf{R}^n)$ and we have*

$$\|b\|_{\mathscr{M}_p(\mathbf{R}^n)} \le \sup_{R>0}\left\|\{b(\tfrac{m}{R})\}_{m\in\mathbf{Z}^n}\right\|_{\mathscr{M}_p(\mathbf{Z}^n)}. \tag{4.3.14}$$

Proof. Suppose that f and g are smooth functions with compact support on $\mathbf{R}^n$. Then there is an $R_0 > 0$ such that for $R \ge R_0$, the functions $x \mapsto f(Rx)$ and $x \mapsto g(Rx)$ are supported in $[-1/2, 1/2]^n$. We define periodic functions

$$F_R(x) = \sum_{k\in\mathbf{Z}^n} f(R(x-k)) \quad\text{and}\quad G_R(x) = \sum_{k\in\mathbf{Z}^n} g(R(x-k))$$

on $\mathbf{T}^n$. Observe that the mth Fourier coefficient of F_R is $\widehat{F_R}(m) = R^{-n}\widehat{f}(m/R)$ and that of G_R is $\widehat{G_R}(m) = R^{-n}\widehat{g}(m/R)$.

Now for $R \ge R_0$ we have

$$\left|\sum_{m\in\mathbf{Z}^n} b(m/R)\widehat{f}(m/R)\overline{\widehat{g}(m/R)}\,\text{Volume}\left(\tfrac{m}{R} + [0,\tfrac{1}{R}]^n\right)\right| \tag{4.3.15}$$

$$= \left|R^n \sum_{m\in\mathbf{Z}^n} b(m/R)\widehat{F_R}(m)\overline{\widehat{G_R}(m)}\right|$$

$$= \left|R^n \int_{\mathbf{T}^n}\left(\sum_{m\in\mathbf{Z}^n} b(m/R)\widehat{F_R}(m)e^{2\pi im\cdot x}\right)\overline{G_R(x)}\,dx\right|$$

$$\le R^n\left\|\{b(m/R)\}_m\right\|_{\mathscr{M}_p(\mathbf{Z}^n)}\|F_R\|_{L^p(\mathbf{T}^n)}\|G_R\|_{L^{p'}(\mathbf{T}^n)}$$

$$\le \sup_{R>0}\left\|\{b(m/R)\}_{m\in\mathbf{Z}^n}\right\|_{\mathscr{M}_p(\mathbf{Z}^n)}R^n\|F_R\|_{L^p(\mathbf{R}^n)}\|G_R\|_{L^{p'}(\mathbf{R}^n)}$$

$$= \sup_{R>0}\left\|\{b(m/R)\}_{m\in\mathbf{Z}^n}\right\|_{\mathscr{M}_p(\mathbf{Z}^n)}\|f\|_{L^p(\mathbf{R}^n)}\|g\|_{L^{p'}(\mathbf{R}^n)}. \tag{4.3.16}$$

Since b is bounded and Riemann integrable over any cube in $\mathbf{R}^n$, the function $b(\xi)\widehat{f}(\xi)\overline{\widehat{g}(\xi)}$ is Riemann integrable over $\mathbf{R}^n$. The expressions in (4.3.15) are sums associated with the partition $\{[\frac{m}{R},\frac{m+1}{R})^n\}_{m\in\mathbf{Z}^n}$ of $\mathbf{R}^n$ which tend to

$$\left|\int_{\mathbf{R}^n} b(\xi)\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi\right|$$

as $R \to \infty$ by the definition of the Riemann integral. We deduce that the absolute value of

$$\int_{\mathbf{R}^n} b(\xi)\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi = \int_{\mathbf{R}^n} (b\widehat{f})^{\vee}(x)\,\overline{g(x)}\,dx$$

is bounded by the expression in (4.3.16). This proves the theorem via duality.   □

4.3.3 Applications of Transference

Having established two main transference theorems, we turn to an application.

Corollary 4.3.11. *Let* $1 < p < \infty$, $f \in L^p(\mathbf{T}^n)$, *and* $\alpha \geq 0$. *Then*

(a) $\left\|D_R^n * f - f\right\|_{L^p(\mathbf{T}^n)} \to 0$ *as* $R \to \infty$ *if and only if* $\chi_{[-1,1]^n} \in \mathcal{M}_p(\mathbf{R}^n)$.
(b) $\left\|\mathring{D}_R^n * f - f\right\|_{L^p(\mathbf{T}^n)} \to 0$ *as* $R \to \infty$ *if and only if* $\chi_{B(0,1)} \in \mathcal{M}_p(\mathbf{R}^n)$.
(c) $\left\|B_R^{\alpha}(f) - f\right\|_{L^p(\mathbf{T}^n)} \to 0$ *as* $R \to \infty$ *if and only if* $(1 - |\xi|^2)_+^{\alpha} \in \mathcal{M}_p(\mathbf{R}^n)$.

Proof. First observe that in view of Corollary 4.1.3, the assertions on the left in (a), (b), and (c) are equivalent to the statements

$$\sup_{R>0}\left\|D_R^n * f\right\|_{L^p(\mathbf{T}^n)} \leq C_p\|f\|_{L^p(\mathbf{T}^n)},$$

$$\sup_{R>0}\left\|\mathring{D}_R^n * f\right\|_{L^p(\mathbf{T}^n)} \leq C_p\|f\|_{L^p(\mathbf{T}^n)},$$

$$\sup_{R>0}\left\|B_R^{\alpha}(f)\right\|_{L^p(\mathbf{T}^n)} \leq C_p\|f\|_{L^p(\mathbf{T}^n)},$$

for some constant $0 < C_p < \infty$ and all f in $L^p(\mathbf{T}^n)$. These statements can be rephrased as

$$\sup_{R>0}\left\|\{\chi_{[-1,1]^n}(m/R)\}_{m\in\mathbf{Z}^n}\right\|_{\mathcal{M}_p(\mathbf{Z}^n)} < \infty,$$

$$\sup_{R>0}\left\|\{\chi_{B(0,1)}(m/R)\}_{m\in\mathbf{Z}^n}\right\|_{\mathcal{M}_p(\mathbf{Z}^n)} < \infty,$$

$$\sup_{R>0}\left\|\{(1 - |m/R|^2)_+^{\alpha}\}_{m\in\mathbf{Z}^n}\right\|_{\mathcal{M}_p(\mathbf{Z}^n)} < \infty.$$

If these statements hold, then Theorem 4.3.10 gives that the functions $\chi_{[-1,1]^n}(\xi)$, $\chi_{B(0,1)}(\xi)$, and $(1 - |\xi|^2)_+^{\alpha}$ lie in $\mathcal{M}_p(\mathbf{R}^n)$.

To prove the converse implication, for any given $R' \in \mathbf{R}^+ \setminus \{|m| : m \in \mathbf{Z}^n\}$, the functions $\chi_{[-1,1]^n}(\xi/R')$, $\chi_{B(0,1)}(\xi/R')$ are Riemann integrable over $\mathbf{R}^n$ and are regulated (actually continuous) at every point in $\mathbf{Z}^n$. Moreover, the function $(1 - |\xi|^2)_+^{\alpha}$ is continuous, regulated, and Riemann integrable over $\mathbf{R}^n$. Then the hypotheses of Theorem 4.3.7 are satisfied and its conclusion yields that

$$\left\|\{\chi_{[-1,1]^n}(m/R')\}_{m\in\mathbf{Z}^n}\right\|_{\mathcal{M}_p(\mathbf{Z}^n)} \leq \left\|\chi_{[-1,1]^n}\right\|_{\mathcal{M}_p}, \tag{4.3.17}$$

$$\big\|\{\chi_{B(0,1)}(m/R')\}_{m\in\mathbf{Z}^n}\big\|_{\mathcal{M}_p(\mathbf{Z}^n)} \leq \big\|\chi_{B(0,1)}\big\|_{\mathcal{M}_p}, \tag{4.3.18}$$

$$\sup_{R>0}\big\|\{(1-|m/R|^2)_+^\alpha\}_{m\in\mathbf{Z}^n}\big\|_{\mathcal{M}_p(\mathbf{Z}^n)} \leq \big\|(1-|\cdot|^2)_+^\alpha\big\|_{\mathcal{M}_p}.$$

Notice that the first and second estimates are uniform in R', so one may insert a supremum over $R' \in \mathbf{R}^+ \setminus \{|m| : m \in \mathbf{Z}^n\}$ in (4.3.17) and (4.3.18). To replace R' by a general $R \in \mathbf{Z}^+$ simply notice that for any $R > 0$ there is an $R' \in \mathbf{R}^+ \setminus \{|m| : m \in \mathbf{Z}^n\}$ such that

$$D_R^n * f = D_{R'}^n * f \qquad \text{and} \qquad \overset{\circ}{D}_R^n * f = \overset{\circ}{D}_{R'}^n * f$$

for any $f \in L^p(\mathbf{T}^n)$. Then using (4.3.17) we obtain

$$\sup_{R>0}\big\|D_R^n * f\big\|_{L^p} = \sup_{R>0}\big\|D_{R'}^n * f\big\|_{L^p} = \sup_{R'>0}\big\|D_{R'}^n * f\big\|_{L^p} \leq \big\|\chi_{[-1,1]^n}\big\|_{\mathcal{M}_p}\big\|f\big\|_{L^p}$$

and likewise for $\overset{\circ}{D}_R^n$. $\qquad\qquad\square$

4.3.4 Transference of Maximal Multipliers

We now prove a theorem concerning maximal multipliers analogous to Theorems 4.3.7 and 4.3.10. This enables us to reduce problems related to almost everywhere convergence of Fourier series on the torus to problems of boundedness of maximal operators on $\mathbf{R}^n$.

Let b be a bounded function defined on all of $\mathbf{R}^n$. For $R > 0$, we introduce the multiplier operators

$$S_{b,R}(F)(x) = \sum_{m\in\mathbf{Z}^n} b(m/R)\widehat{F}(m)e^{2\pi i m \cdot x}, \tag{4.3.19}$$

$$T_{b,R}(f)(x) = \int_{\mathbf{R}^n} b(\xi/R)\widehat{f}(\xi)e^{2\pi i \xi \cdot x}\,d\xi, \tag{4.3.20}$$

initially defined for smooth functions with compact support f on $\mathbf{R}^n$ and smooth functions F on $\mathbf{T}^n$.

We introduce the maximal operators

$$M_b(F)(x) = \sup_{R>0}\big|S_{b,R}(F)(x)\big|, \tag{4.3.21}$$

$$N_b(f)(x) = \sup_{R>0}\big|T_{b,R}(f)(x)\big|, \tag{4.3.22}$$

defined for smooth functions F on $\mathbf{T}^n$ and smooth functions with compact support f on $\mathbf{R}^n$. Let $\tau^y(b)(\xi) = b(\xi - y)$ be a translation operator defined for $y \in \mathbf{R}^n$. We have the following result concerning these operators.

Theorem 4.3.12. *Let b be a function defined on $\mathbf{R}^n$. Suppose that b is bounded, regulated, Riemann integrable over any cube, and assume that for all $\xi \in \mathbf{R}^n$ the*

function $t \mapsto b(\xi/t)$ *has only countably many discontinuities on* $\mathbf{R}^+$. *Let* $1 < p < \infty$ *and* $C_p < \infty$, *and suppose that* b *lies in* $\mathscr{M}_p(\mathbf{R}^n)$. *Let* M_b *and* N_b *be as in (4.3.21) and (4.3.22). Then the following assertions are equivalent:*

$$\left\|M_b(F)\right\|_{L^p(\mathbf{T}^n)} \le C_p \left\|b\right\|_{\mathscr{M}_p} \left\|F\right\|_{L^p(\mathbf{T}^n)}, \qquad F \in \mathscr{C}^\infty(\mathbf{T}^n), \qquad (4.3.23)$$

$$\left\|N_b(f)\right\|_{L^p(\mathbf{R}^n)} \le C_p \left\|b\right\|_{\mathscr{M}_p} \left\|f\right\|_{L^p(\mathbf{R}^n)}, \qquad f \in \mathscr{C}_0^\infty(\mathbf{R}^n). \qquad (4.3.24)$$

Proof. Let $\mathscr{F} = \{t_1, \ldots, t_k\}$ be a finite subset of $\mathbf{R}^+$. We prove the claimed equivalences for the maximal operators

$$M_b^{\mathscr{F}}(G)(x) = \sup_{t \in \mathscr{F}} \left|S_{b,t}(G)(x)\right|,$$

$$N_b^{\mathscr{F}}(g)(x) = \sup_{t \in \mathscr{F}} \left|T_{b,t}(g)(x)\right|,$$

with constants that are uniform in the finite set $\mathscr{F}$. Then $M_b^{\mathscr{F}}$ may be viewed as an operator defined on the dense subspace $\mathscr{C}^\infty(\mathbf{T}^n)$ of $L^p(\mathbf{T}^n)$ and taking values in $L^p(\mathbf{T}^n, \ell^\infty(\mathscr{F}))$, which is the dual space of $L^{p'}(\mathbf{T}^n, \ell^1(\mathscr{F}))$. Likewise, $N_b^{\mathscr{F}}$ is defined on the dense subspace $\mathscr{C}_0^\infty(\mathbf{R}^n)$ of $L^p(\mathbf{R}^n)$ and takes values in $L^p(\mathbf{R}^n, \ell^\infty(\mathscr{F}))$, which is the dual space of $L^{p'}(\mathbf{R}^n, \ell^1(\mathscr{F}))$. Using duality, with respect to the complex inner product, estimates (4.3.23) and (4.3.24) are equivalent to the pair of inequalities

$$\left| \sum_{m \in \mathbf{Z}^n} \overline{\widehat{G}(m)} \sum_{j=1}^{k} b\left(\frac{m}{t_j}\right) \widehat{F}_j(m) \right| \le C_p \left\|b\right\|_{\mathscr{M}_p} \left\|G\right\|_{L^p(\mathbf{T}^n)} \left\| \sum_{j=1}^{k} |F_j| \right\|_{L^{p'}(\mathbf{T}^n)}, \quad (4.3.25)$$

$$\left| \int_{\mathbf{R}^n} \overline{\widehat{g}(\xi)} \sum_{j=1}^{k} b\left(\frac{\xi}{t_j}\right) \widehat{f}_j(\xi) \, d\xi \right| \le C_p \left\|b\right\|_{\mathscr{M}_p} \left\|g\right\|_{L^p(\mathbf{R}^n)} \left\| \sum_{j=1}^{k} |f_j| \right\|_{L^{p'}(\mathbf{R}^n)}, \quad (4.3.26)$$

where $g, f_j \in \mathscr{C}_0^\infty(\mathbf{R}^n)$, and $G, F_j \in \mathscr{C}^\infty(\mathbf{T}^n)$. In proving the equivalence of (4.3.25) and (4.3.26), by density, we work with smooth functions with compact support g, f_j and trigonometric polynomials G, F_j.

Suppose that (4.3.25) holds and let $f_1, \ldots, f_k, g$ be smooth functions with compact support on $\mathbf{R}^n$. Then there is an $R_0 > 0$ such that for $R \ge R_0$ the functions $F_{j,R}(x) = f_j(Rx)$ and $G_R(x) = g(Rx)$ are supported in $[-1/2, 1/2]^n$ and thus they can be viewed as functions on $\mathbf{T}^n$ once they are periodized. Also, the mth Fourier coefficient of $F_{j,R}$ is $\widehat{F_{j,R}}(m) = R^{-n} \widehat{f}_j(m/R)$ and that of G_R is $\widehat{G_R}(m) = R^{-n} \widehat{g}(m/R)$. Since b lies in $\mathscr{M}_p(\mathbf{R}^n)$ we have

$$\left\|b\right\|_{\mathscr{M}_p(\mathbf{R}^n)} = \sup_{R > 0} \left\|\{b(m/R)\}_{m \in \mathbf{Z}^n}\right\|_{\mathscr{M}_p(\mathbf{Z}^n)}$$

in view of Theorems 4.3.7 and 4.3.10, which are both applicable in view of the hypotheses of b.

As in the proof of Theorem 4.3.10, for $R \geq R_0$ we have

$$\left| \sum_{m\in\mathbf{Z}^n} \sum_{j=1}^{k} b(m/Rt_j)\widehat{f}_j(m/R)\overline{\widehat{g}(m/R)} \, \text{Volume} \left(\tfrac{m}{R} + [0,\tfrac{1}{R}]^n \right) \right| \tag{4.3.27}$$

$$= \left| R^n \sum_{m\in\mathbf{Z}^n} \sum_{j=1}^{k} b(m/Rt_j)\widehat{F_{j,R}}(m)\overline{\widehat{G_R}(m)} \right|$$

$$\leq C_p \|b\|_{\mathscr{M}_p} R^n \left\| \sum_{j=1}^{k} |F_{j,R}| \right\|_{L^{p'}(\mathbf{T}^n)} \|G_R\|_{L^p(\mathbf{T}^n)}$$

$$= C_p \|b\|_{\mathscr{M}_p} \left\| \sum_{j=1}^{k} |f_j| \right\|_{L^{p'}(\mathbf{R}^n)} \|g\|_{L^p(\mathbf{R}^n)} ,$$

where we applied (4.3.25) in the first inequality above for the function $\xi \mapsto b(\xi/R)$, which has the same $\mathscr{M}_p$ norm as b.

Since b is bounded and Riemann integrable over any cube in $\mathbf{R}^n$, the functions $b(\xi/t_j)\widehat{f}_j(\xi)\overline{\widehat{g}_j(\xi)}$ are Riemann integrable over $\mathbf{R}^n$. Realizing the limit of the partial sums in (4.3.27) when $R \to \infty$ as a Riemann integral, we obtain

$$\left| \int_{\mathbf{R}^n} \sum_{j=1}^{k} b(\xi/t_j)\widehat{f}_j(\xi)\overline{\widehat{g}(\xi)} \, d\xi \right| \leq C_p \|b\|_{\mathscr{M}_p} \left\| \sum_{j=1}^{k} |f_j| \right\|_{L^{p'}(\mathbf{R}^n)} \|g\|_{L^p(\mathbf{R}^n)}$$

and thus we showed that (4.3.25) implies (4.3.26).

We now turn to the converse. Assume that (4.3.26) holds. We will prove (4.3.25) for trigonometric polynomials and then by density we extend it to all $\mathscr{C}^\infty$ functions on $\mathbf{T}^n$. Expressing $\widehat{g}$ in terms of g in (4.3.26) and taking the supremum in (4.3.26) over all $\mathscr{C}_0^\infty$ functions g with L^p norm 1 we deduce that

$$\left\| \int_{\mathbf{R}^n} \sum_{j=1}^{k} b\left(\frac{\xi}{t_j}\right)\widehat{f}_j(\xi)e^{2\pi i(\cdot)\cdot\xi} \, d\xi \right\|_{L^{p'}} \leq C_p \|b\|_{\mathscr{M}_p} \left\| \sum_{j=1}^{k} |f_j| \right\|_{L^{p'}(\mathbf{R}^n)} . \tag{4.3.28}$$

Let $P_1,\ldots,P_k$ and Q be trigonometric polynomials on $\mathbf{T}^n$. Set $L_\varepsilon(x) = e^{-\pi\varepsilon|x|^2}$. Since b is regulated at every point in $\mathbf{R}^n$, Lemma 4.3.9 gives

$$\left| \sum_{m\in\mathbf{Z}^n} \sum_{j=1}^{k} \overline{\widehat{Q}(m)} b(m/t_j)\widehat{P}_j(m) \right|$$

$$= \left| \int_{\mathbf{T}^n} \left(\sum_{m\in\mathbf{Z}^n} \sum_{j=1}^{k} \widehat{P}_j(m)b(m/t_j)e^{2\pi i m\cdot x} \right)\overline{Q(x)} \, dx \right|$$

$$= \left| \lim_{\varepsilon\to 0} \varepsilon^{\frac{n}{2}} \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} \sum_{j=1}^{k} \widehat{P_j L_{\varepsilon/p'}}(\xi)b(\xi/t_j)e^{2\pi i\xi\cdot x} d\xi \right)\overline{Q(x)L_{\varepsilon/p}(x)} \, dx \right|$$

$$\leq C_p \|b\|_{\mathscr{M}_p} \limsup_{\varepsilon \to 0} \left[\varepsilon^{\frac{n}{2}} \Big\| \sum_{j=1}^k |P_j L_{\varepsilon/p'}| \Big\|_{L^{p'}(\mathbf{R}^n)} \|Q L_{\varepsilon/p}\|_{L^p(\mathbf{R}^n)} \right]$$

$$= C_p \|b\|_{\mathscr{M}_p} \limsup_{\varepsilon \to 0} \left[\varepsilon^{\frac{n}{2p'}} \Big\| \Big(\sum_{j=1}^k |P_j| \Big) L_{\varepsilon/p'} \Big\|_{L^{p'}(\mathbf{R}^n)} \Big(\varepsilon^{\frac{n}{2}} \int_{\mathbf{R}^n} |Q(x)|^p e^{-\varepsilon \pi |x|^2} dx \Big)^{\frac{1}{p}} \right]$$

$$= C_p \|b\|_{\mathscr{M}_p} \Big\| \sum_{j=1}^k |P_j| \Big\|_{L^{p'}(\mathbf{R}^n)} \|Q\|_{L^p(\mathbf{T}^n)},$$

where we used Hölder's inequality and (4.3.28) in the only inequality above and (4.3.13) in the last equality. Thus we obtain that (4.3.26) implies (4.3.25), and this completes the equivalence of boundedness of $M_b^{\mathscr{F}}$ and $N_b^{\mathscr{F}}$.

We now prove the claimed equivalence for the operators M_b and N_b. We first show that if $M_b^{\mathscr{F}}$ is bounded on $(\mathscr{C}^\infty(\mathbf{T}^n), \|\cdot\|_{L^p})$ with bound independent of the finite set $\mathscr{F}$, then M_b is bounded on $(\mathscr{C}^\infty(\mathbf{T}^n), \|\cdot\|_{L^p})$.

For each $\xi \in \mathbf{R}^n$, let A_ξ be the null subset of $\mathbf{R}^+$ such that $t \mapsto b(\xi/t)$ is continuous on $\mathbf{R}^+ \setminus A_\xi$. We fix a function F in $\mathscr{C}^\infty(\mathbf{T}^n)$, and we note that for each $x \in \mathbf{T}^n$ the function

$$t \mapsto S_{b,t}(F)(x) = \sum_{m \in \mathbf{Z}^n} b(m/t) \widehat{F}(m) e^{2\pi i m \cdot x} \tag{4.3.29}$$

is continuous on the set $\mathbf{R}^+ \setminus \bigcup_{m \in \mathbf{Z}^n} A_m$. We pick a countable dense subset D' of $\mathbf{R}^+ \setminus \bigcup_{m \in \mathbf{Z}^n} A_m$, and we let $D = D' \cup \bigcup_{m \in \mathbf{Z}^n} A_m$. Then D is a countable set and the Lebesgue monotone convergence theorem gives that

$$\Big\| \sup_{t \in D} |S_{b,t}(F)| \Big\|_{L^p(\mathbf{T}^n)} = \lim_{k \to \infty} \big\| M_b^{\mathscr{F}_k}(F) \big\|_{L^p(\mathbf{T}^n)} \leq C_p \|b\|_{\mathscr{M}_p} \|F\|_{L^p(\mathbf{T}^n)}, \tag{4.3.30}$$

where $\mathscr{F}_k$ is an increasing sequence of finite sets whose union is D. Using that the function in (4.3.29) is continuous on $\mathbf{R}^+ \setminus D$, we conclude that the supremum over $t \in D$ in (4.3.30) can be replaced by the supremum over $t \in \mathbf{Z}^+$ (Exercise 4.3.7).

Assume now that $N_b^{\mathscr{F}}$ is bounded on $(\mathscr{C}_0^\infty(\mathbf{R}^n), \|\cdot\|_{L^p})$ with bound independent of the finite set $\mathscr{F}$. We show that N_b is bounded on $(\mathscr{C}_0^\infty(\mathbf{R}^n), \|\cdot\|_{L^p})$. Let f be in $\mathscr{C}_0^\infty(\mathbf{R}^n)$. We have that the map

$$t \mapsto T_{b,t}(f)(x) = \int_{\mathbf{R}^n} b(\xi/t) \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi = t^n \int_{\mathbf{R}^n} b(\xi) \widehat{f}(t\xi) e^{2\pi i \xi \cdot tx} d\xi \tag{4.3.31}$$

is a continuous function on $\mathbf{R}^+$ since $\widehat{f}$ is continuous. Thus the estimate

$$\Big\| \sup_{t \in D} |T_{b,t}(f)| \Big\|_{L^p(\mathbf{R}^n)} \leq C_p \|b\|_{\mathscr{M}_p} \|f\|_{L^p(\mathbf{R}^n)} \tag{4.3.32}$$

for a countable dense subset D of $\mathbf{R}^+$ (such as $D = \mathbf{Q}^+$) can be easily extended by replacing the supremum over D by the supremum over $\mathbf{R}^+$. And estimate (4.3.32) for $D = \mathbf{Q}^+$ follows from the corresponding estimate on finite sets via the Lebesgue monotone convergence theorem. $\square$

Remark 4.3.13. Under the hypotheses of Theorem 4.3.12, the following two inequalities are also equivalent:

$$\big\|M_b(G)\big\|_{L^{p,\infty}(\mathbf{T}^n)} \le C_p\|b\|_{\mathscr{M}_p}\|G\|_{L^p(\mathbf{T}^n)}, \qquad G \in \mathscr{C}^\infty(\mathbf{T}^n), \qquad (4.3.33)$$

$$\big\|N_b(g)\big\|_{L^{p,\infty}(\mathbf{R}^n)} \le C_p'\|b\|_{\mathscr{M}_p}\|g\|_{L^p(\mathbf{R}^n)}, \qquad g \in \mathscr{C}_0^\infty(\mathbf{R}^n), \qquad (4.3.34)$$

with $C_p' \le C_p \le C(n,p)C_p'$ for some other constant $C(n,p)$. Indeed, Exercise 1.4.12 gives that the pair of inequalities (4.3.33) and (4.3.34) is equivalent to the pair of inequalities

$$\left|\sum_{m\in\mathbf{Z}^n}\sum_{j=1}^k \widehat{F}_j(m)b(m/t_j)\overline{\widehat{G}(m)}\right| \le C_p\|b\|_{\mathscr{M}_p}\|G\|_{L^p(\mathbf{T}^n)}\left\|\sum_{j=1}^k |F_j|\right\|_{L^{p',1}(\mathbf{T}^n)}, \quad (4.3.35)$$

$$\left|\int_{\mathbf{R}^n}\sum_{j=1}^k \widehat{f}_j(\xi)b(\xi/t_j)\overline{\widehat{g}(\xi)}\,d\xi\right| \le C_p'\|b\|_{\mathscr{M}_p}\|g\|_{L^p(\mathbf{R}^n)}\left\|\sum_{j=1}^k |f_j|\right\|_{L^{p',1}(\mathbf{R}^n)}, \quad (4.3.36)$$

where $L^{p',1}$ is the Lorentz space and $f_j \in \mathscr{C}_0^\infty(\mathbf{R}^n)$ and $F_j \in \mathscr{C}^\infty(\mathbf{T}^n)$.

Now (4.3.36) follows from (4.3.35) just like (4.3.26) follows from (4.3.25) with the only exception being that Hölder's inequality for L^p and $L^{p'}$ is replaced by Hölder's inequality for $L^{p,\infty}$ and $L^{p',1}$ and we use that $\|g\|_{L^{p,\infty}} \le \|g\|_{L^p}$. Conversely, assuming (4.3.36), in order to prove (4.3.35) it will suffice to know that

$$\sup_{0<\varepsilon<1}\varepsilon^{\frac{n}{2q}}\left\|\left(\sum_{j=1}^k |P_j|\right)L_{\varepsilon/q}\right\|_{L^{q,1}(\mathbf{R}^n)} \le C(n,q)\left\|\sum_{j=1}^k |P_j|\right\|_{L^{q,1}(\mathbf{T}^n)}. \qquad (4.3.37)$$

For this we refer to Exercise 4.3.6.

4.3.5 Applications to Almost Everywhere Convergence

As an application of the preceding results, we relate the almost everywhere convergence of Fourier series of functions on $\mathbf{T}^1$ with the almost everywhere convergence of Fourier integrals of functions on $\mathbf{R}$. In this subsection we show that the following two results are equivalent:

Theorem 4.3.14. *For every $1 < p < \infty$ there exists a finite constant C_p such that for all $F \in \mathscr{C}^\infty(\mathbf{T}^1)$ we have*

$$\left\|\sup_{N\in\mathbf{Z}^+} |F * D_N|\right\|_{L^p} \le C_p\|F\|_{L^p}. \qquad (4.3.38)$$

Theorem 4.3.15. *For every $1 < p < \infty$ there exists a finite constant C_p such that for all $f \in \mathscr{C}_0^\infty(\mathbf{R})$ we have*

$$\big\|\mathscr{C}_{**}(f)\big\|_{L^p(\mathbf{R})} \le C_p\|f\|_{L^p(\mathbf{R})} \qquad (4.3.39)$$

where

$$\mathscr{C}_{**}(f)(x) = \sup_{R>0} \left| \int_{|\xi| \leq R} \widehat{f}(\xi) e^{2\pi i x \xi} \, d\xi \right|$$

is the Carleson operator.

As a consequence of Theorem 4.3.14, we obtain that for any $F \in L^p(\mathbf{T}^1)$, we have

$$\lim_{N \to \infty} \sum_{|m| \leq N} \widehat{F}(m) e^{2\pi i m x} = F(x)$$

for almost every $x \in [0,1]$.

Theorem 4.3.14 can be proved directly, but we do not pursue this here. Instead, we show the equivalence of the two theorems and refer the interested reader to [131], which contains the proof of Theorem 4.3.15.

We observe that both operators $F \mapsto \mathscr{C}_*(F) = \sup_{N>0} |F * D_N|$ and $f \mapsto \mathscr{C}_{**}(f)$ are sublinear and take nonnegative values. Thus they satisfy the inequalities

$$|\mathscr{C}_*(F) - \mathscr{C}_*(G)| \leq \mathscr{C}_*(F - G) \qquad |\mathscr{C}_{**}(f) - \mathscr{C}_{**}(g)| \leq \mathscr{C}_{**}(f - g)$$

for all F, G in $\mathscr{C}^\infty(\mathbf{T}^1)$ and f, g in $\mathscr{C}_0^\infty(\mathbf{R})$. Then, by density (see the argument in the proof of Theorem 1.4.19 or Exercise 1.4.17), they admit bounded extensions to $L^p(\mathbf{T}^1)$ and $L^p(\mathbf{R})$, respectively, so that (4.3.38) and (4.3.39) hold for all $F \in L^p(\mathbf{T}^1)$ and $f \in L^p(\mathbf{R})$.

Next, we discuss the details of the transference argument that claims the equivalence of Theorems 4.3.14 and 4.3.15.

Consider the following function defined on $\mathbf{R}$:

$$b(x) = \begin{cases} 1 & \text{when } |x| < 1, \\ 1/2 & \text{when } |x| = 1, \\ 0 & \text{when } |x| > 1. \end{cases} \qquad (4.3.40)$$

Then b is bounded and Riemann integrable over any interval, and is easily seen to be regulated; also, given any $x \in \mathbf{R}$, the function $t \mapsto b(x/t)$ is discontinuous only for $t \in \{x, -x\}$.

Let $S_{b,R}$ be as in (4.3.19), where b is defined in (4.3.40). We note that inequality (4.3.38) is equivalent to

$$\left\| \sup_{R>0} |S_{b,R}(F)| \right\|_{L^p} \leq C_p' \|F\|_{L^p} \qquad (4.3.41)$$

for all $F \in \mathscr{C}^\infty(\mathbf{T}^1)$, where $\{D_R\}_{R>0}$ is the family of Dirichlet kernels as defined in (3.1.16), depending on the continuous parameter R. Indeed, we have

$$S_{b,R}(F)(x) = \begin{cases} \sum_{|m|\leq[R]} \widehat{F}(m)e^{2\pi imx} & \text{if } R \notin \mathbf{Z}^+, \\ \\ (D_{R-1}*F)(x) + \dfrac{\widehat{F}(R)e^{2\pi ixR}+\widehat{F}(-R)e^{-2\pi ixR}}{2} & \text{if } R \in \mathbf{Z}^+. \end{cases} \quad (4.3.42)$$

Since $\sup_{R>0}|\widehat{F}(\pm R)| \leq \|F\|_{L^1} \leq \|F\|_{L^p}$, it follows that if (4.3.38) holds, then (4.3.41) also holds with $C_p' = C_p + 1$.

The only hypothesis of Theorem 4.3.12 missing is that b lies in $\mathscr{M}_p(\mathbf{R})$. We obtain this from the fact that $\sup_{R>0}\|b(\cdot/R)\|_{\mathscr{M}_p(\mathbf{Z})} < \infty$ via Theorem 4.3.10, since

$$\sup_{R>0}\|F*D_R\|_{L^p(\mathbf{T}^1)} = \sup_{N\in\mathbf{Z}^+}\|F*D_N\|_{L^p(\mathbf{T}^1)} \leq C_p''\|F\|_{L^p(\mathbf{T}^1)}, \quad (4.3.43)$$

where the last estimate follows from Proposition 4.1.6, Theorem 4.1.7, and Corollary 4.1.3. The preceding equality is due to the fact that $D_R = D_{R+\varepsilon}$ whenever $0 < \varepsilon < 1$.

Now all hypotheses of Theorem 4.3.12 are valid. As a consequence we obtain the equivalence of the boundedness of the the maximal operator

$$N_b(f)(x) = \mathscr{C}_{**}(f)(x) = \sup_{R>0}\left|\int_{-R}^{+R} \widehat{f}(\xi)e^{2\pi ix\xi}\,d\xi\right|$$

on $L^p(\mathbf{R})$ and of

$$M_b(F)(x) = \sup_{R>0}\left|\sum_{m\in\mathbf{Z}} \widehat{F}(m)e^{2\pi imx}b\left(\frac{m}{R}\right)\right| = \sup_{R>0}|S_{b,R}(F)(x)|,$$

on $L^p(\mathbf{T}^1)$. But in view of (4.3.42) and of the fact that $\sup_{R>0}|\widehat{F}(\pm R)| \leq \|F\|_{L^p}$, the L^p boundedness of M_b is equivalent to the L^p boundedness of $\mathscr{C}_*$ on $L^p(\mathbf{T}^1)$. This discussion concludes the equivalence of Theorems 4.3.14 and 4.3.15.

4.3.6 Almost Everywhere Convergence of Square Dirichlet Means

The extension of Theorem 4.3.14 to higher dimensions is a rather straightforward consequence of the one-dimensional result.

Theorem 4.3.16. *For every $1 < p < \infty$, there exists a finite constant $C_{p,n}$ such that for all $f \in L^p(\mathbf{T}^n)$ we have*

$$\left\|\sup_{N>0}|D_N^n*f|\right\|_{L^p(\mathbf{T}^n)} \leq C_{p,n}\|f\|_{L^p(\mathbf{T}^n)} \quad (4.3.44)$$

and consequently

$$\lim_{N\to\infty} \sum_{\substack{m\in\mathbf{Z}^n \\ |m_j|\leq N}} \widehat{f}(m)e^{2\pi im\cdot x} = f(x)$$

for almost every $x \in \mathbf{T}^n$ and $f \in L^p(\mathbf{T}^n)$.

Proof. We prove Theorem 4.3.16 when $n = 2$. Fix a p with $1 < p < \infty$. Since the Riesz projection P_+ is bounded on $L^p(\mathbf{T}^1)$ (Theorem 4.1.7 and identity (4.1.9)), applying Theorem 4.3.10 with $b(\xi) = \chi_{(0,\infty)}$, we obtain that that the function $\chi_{(0,\infty)}$ is in $\mathscr{M}_p(\mathbf{R})$. It follows that the characteristic function of the half-space $\xi_1 > 0$ in $\mathbf{R}^2$ lies in $\mathscr{M}_p(\mathbf{R}^2)$. Since rotations and translations of multipliers preserve their $\mathscr{M}_p$ norms (Proposition 2.5.14), it follows that the characteristic function of any half space created by a line in $\mathbf{R}^2$ lies in $\mathscr{M}_p(\mathbf{R}^2)$ with a fixed norm. The product of three multipliers is a multiplier (Proposition 2.5.13); thus the characteristic function of the triangle T created by the lines $\xi_2 = \xi_1 - \frac{1}{4}$, $\xi_2 = -\xi_1 - \frac{1}{4}$, $\xi_2 = L + \frac{1}{4}$ lies also in $\mathscr{M}_p(\mathbf{R}^2)$ with norm independent of $L \in \mathbf{Z}^+$. The regulated function

$$
\sigma(\xi_1, \xi_2) = \begin{cases} 1 & \text{if } (\xi_1, \xi_2) \in \overline{T} \setminus \partial T \\ 0 & \text{if } (\xi_1, \xi_2) \notin \overline{T} \\ \frac{1}{2} & \text{if } (\xi_1, \xi_2) \in \partial T \setminus \{(0, -\frac{1}{4}), (L + \frac{1}{2}, L + \frac{1}{4}), (-L - \frac{1}{2}, L + \frac{1}{4})\} \\ \frac{1}{8} & \text{if } (\xi_1, \xi_2) \in \{(L + \frac{1}{2}, L + \frac{1}{4}), (-L - \frac{1}{2}, L + \frac{1}{4})\} \\ \frac{1}{4} & \text{if } (\xi_1, \xi_2) = (0, -\frac{1}{4}) \end{cases}
$$

is a.e. equal to the characteristic function of T. Thus Theorem 4.3.7 gives that the restriction of σ on $\mathbf{Z}^2$, i.e., the sequence $\{a_{m_1, m_2}\}_{m_1, m_2}$ defined by $a_{m_1, m_2} = 1$ when $|m_1| \le |m_2| \le L$ and zero otherwise, lies in $\mathscr{M}_p(\mathbf{Z}^2)$ with norm independent of L in $\mathbf{Z}^+$. This means that for some constant B_p we have the following inequality for all f in $L^p(\mathbf{T}^2)$:

$$
\int_{\mathbf{T}^2} \left| \sum_{\substack{m_2 \in \mathbf{Z} \\ |m_2| \le L}} \sum_{\substack{m_1 \in \mathbf{Z} \\ |m_1| \le |m_2|}} \widehat{f}(m_1, m_2) e^{2\pi i (m_1 x_1 + m_2 x_2)} \right|^p dx_2\, dx_1 \le B_p^p \|f\|_{L^p(\mathbf{T}^2)}^p, \quad (4.3.45)
$$

where B_p is independent of $L \in \mathbf{Z}^+$. There is also a a version of (4.3.45), proved similarly, in which $|m_1| \le |m_2|$ is replaced by the strict inequality $|m_1| < |m_2|$.

Now let $1 < p < \infty$, $L \in \mathbf{Z}^+$, and $f \in L^p(\mathbf{T}^2)$. For fixed $x_1 \in \mathbf{T}^1$ define

$$
f_{x_1}^L(x_2) = \sum_{\substack{m_2 \in \mathbf{Z} \\ |m_2| \le L}} \left[\sum_{\substack{m_1 \in \mathbf{Z} \\ |m_1| \le |m_2|}} \widehat{f}(m_1, m_2) e^{2\pi i m_1 x_1} \right] e^{2\pi i m_2 x_2} = \sum_{\substack{m_2 \in \mathbf{Z} \\ |m_2| \le L}} \widehat{f_{x_1}^L}(m_2) e^{2\pi i m_2 x_2}
$$

and for fixed $x_2 \in \mathbf{T}^1$ define

$$
f_L^{x_2}(x_1) = \sum_{\substack{m_1 \in \mathbf{Z} \\ |m_1| \le L}} \left[\sum_{\substack{m_2 \in \mathbf{Z} \\ |m_2| < |m_1|}} \widehat{f}(m_1, m_2) e^{2\pi i m_2 x_2} \right] e^{2\pi i m_1 x_1} = \sum_{\substack{m_1 \in \mathbf{Z} \\ |m_1| \le L}} \widehat{f_L^{x_2}}(m_1) e^{2\pi i m_1 x_1}.
$$

We have

$$
\int_{\mathbf{T}^1} \int_{\mathbf{T}^1} \sup_{0<N\leq L} \left| \sum_{|m_1|\leq N} \sum_{|m_2|\leq N} \widehat{f}(m_1,m_2) e^{2\pi i m_1 x_1} e^{2\pi i m_2 x_2} \right|^p dx_2\, dx_1
$$

$$
\leq 2^{p-1} \int_{\mathbf{T}^1} \int_{\mathbf{T}^1} \sup_{0<N\leq L} \left| \sum_{|m_2|\leq N} \left[\sum_{|m_1|\leq|m_2|} \widehat{f}(m_1,m_2) e^{2\pi i m_1 x_1} \right] e^{2\pi i m_2 x_2} \right|^p
$$

$$
+ \sup_{0<N\leq L} \left| \sum_{|m_1|\leq N} \left[\sum_{|m_2|<|m_1|} \widehat{f}(m_1,m_2) e^{2\pi i m_2 x_2} \right] e^{2\pi i m_1 x_1} \right|^p dx_1\, dx_2
$$

$$
= 2^{p-1} \left[\int_{\mathbf{T}^1} \int_{\mathbf{T}^1} \sup_{0<N\leq L} \left| (D_N * f^L_{x_1})(x_2) \right|^p dx_2\, dx_1 \right.
$$

$$
\left. + \int_{\mathbf{T}^1} \int_{\mathbf{T}^1} \sup_{0<N\leq L} \left| (D_N * f^{x_2}_L)(x_1) \right|^p dx_1\, dx_2 \right]
$$

$$
\leq 2^{p-1} \left[\int_{\mathbf{T}^1} \int_{\mathbf{T}^1} \sup_{N\in\mathbf{Z}^+} \left| (D_N * f^L_{x_1})(x_2) \right|^p dx_2 dx_1 \right.
$$

$$
\left. + \int_{\mathbf{T}^1} \int_{\mathbf{T}^1} \sup_{N\in\mathbf{Z}^+} \left| (D_N * f^{x_2}_L)(x_1) \right|^p dx_1\, dx_2 \right]
$$

$$
\leq 2^{p-1} C_p^p \int_{\mathbf{T}^1} \int_{\mathbf{T}^1} |f^L_{x_1}(x_2)|^p dx_2\, dx_1 + 2^{p-1} C_p^p \int_{\mathbf{T}^1} \int_{\mathbf{T}^1} |f^{x_2}_L(x_1)|^p dx_2\, dx_1
$$

$$
\leq 2^p C_p^p B_p^p \|f\|_{L^p(\mathbf{T}^2)}^p,
$$

where we used Theorem 4.3.14 in the penultimate inequality and estimate (4.3.45) in the last inequality. Since the last estimate we obtained is independent of $L \in \mathbf{Z}^+$, letting $L \to \infty$ and applying Fatou's lemma, we obtain the conclusion (4.3.44) for $n = 2$. When $n \geq 3$ the idea of the proof is similar, but the notation a bit more cumbersome. □

Exercises

4.3.1. Let $\alpha \geq 0$. Prove that the function $(1-|\xi|^2)^\alpha_+$ is in $\mathscr{M}_p(\mathbf{R}^n)$ if and only if the function $(1-|\xi|)^\alpha_+$ is in $\mathscr{M}_p(\mathbf{R}^n)$.
[*Hint:* Use that smooth functions with compact support lie in $\mathscr{M}_p(\mathbf{R}^n)$.]

4.3.2. The purpose of this exercise is to introduce distributions on the torus. The set of test functions on the torus is $\mathscr{C}^\infty(\mathbf{T}^n)$ equipped with the following topology. Given f_j, f in $\mathscr{C}^\infty(\mathbf{T}^n)$, we say that $f_j \to f$ in $\mathscr{C}^\infty(\mathbf{T}^n)$ if

$$
\left\| \partial^\alpha f_j - \partial^\alpha f \right\|_{L^\infty(\mathbf{T}^n)} \to 0 \quad \text{as } j \to \infty,\ \forall\, \alpha.
$$

Under this notion of convergence, $\mathscr{C}^\infty(\mathbf{T}^n)$ is a topological vector space with topology induced by the family of seminorms $\rho_\alpha(\varphi) = \sup_{x\in\mathbf{T}^n} |(\partial^\alpha f)(x)|$, where α

ranges over all multi-indices. The dual space of $\mathscr{C}^\infty(\mathbf{T}^n)$ under this topology is the set of all distributions on $\mathbf{T}^n$ and is denoted by $\mathscr{D}'(\mathbf{T}^n)$. The definition implies that for u_j and u in $\mathscr{D}'(\mathbf{T}^n)$ we have $u_j \to u$ in $\mathscr{D}'(\mathbf{T}^n)$ if and only if

$$\langle u_j, f \rangle \to \langle u, f \rangle \quad \text{as } j \to \infty \text{ for all } f \in \mathscr{C}^\infty(\mathbf{T}^n).$$

The following operations can be defined on elements of $\mathscr{D}'(\mathbf{T}^n)$: differentiation (as in Definition 2.3.6), translation and reflection (as in Definition 2.3.11), convolution with a $\mathscr{C}^\infty$ function (as in Definition 2.3.13), multiplication by a $\mathscr{C}^\infty$ function (as in Definition 2.3.15), the support of a distribution (as in Definition 2.3.16). Use the same ideas as in $\mathbf{R}^n$ to prove the following:

(a) Prove that if $u \in \mathscr{D}'(\mathbf{T}^n)$ and $f \in \mathscr{C}^\infty(\mathbf{T}^n)$, then $(f*u)(x) = \langle u, \tau^x(\widetilde{f}) \rangle$ is a $\mathscr{C}^\infty$ function.

(b) In contrast to $\mathbf{R}^n$, the convolution of two distributions on $\mathbf{T}^n$ can be defined. For $u, v \in \mathscr{D}'(\mathbf{T}^n)$ and $f \in \mathscr{C}^\infty(\mathbf{T}^n)$ define

$$\langle u*v, f \rangle = \langle u, f*\widetilde{v} \rangle.$$

Check that convolution of distributions on $\mathscr{D}'(\mathbf{T}^n)$ is associative, commutative, and distributive.

(c) Prove the analogue of Proposition 2.3.23, i.e., that $\mathscr{C}^\infty(\mathbf{T}^n)$ is dense in $\mathscr{D}'(\mathbf{T}^n)$.

4.3.3. For $u \in \mathscr{D}'(\mathbf{T}^n)$ and $m \in \mathbf{Z}^n$ define the Fourier coefficient $\widehat{u}(m)$ by

$$\widehat{u}(m) = u(e^{-2\pi i m \cdot (\,\cdot\,)}) = \langle u, e^{-2\pi i m \cdot (\,\cdot\,)} \rangle.$$

Prove properties (1), (2), (4), (5), (6), (8), (9), (11), and (12) of Proposition 2.3.22 regarding the Fourier coefficients of distributions on the circle. Moreover, prove that for any u, v in $\mathscr{D}'(\mathbf{T}^n)$ we have $(u*v)^\wedge(m) = \widehat{u}(m)\widehat{v}(m)$. In particular, this is valid for finite Borel measures.

4.3.4. Let μ be a finite Borel measure on $\mathbf{R}^n$ and let ν be the periodization of μ, that is, ν is a measure on $\mathbf{T}^n$ defined by

$$\nu(A) = \sum_{m \in \mathbf{Z}^n} \mu(A+m)$$

for all measurable subsets A of $\mathbf{T}^n$. Prove that the restriction of the Fourier transform of μ on $\mathbf{Z}^n$ coincides with the sequence of the Fourier coefficients of the measure ν.

4.3.5. Let v_n be the volume of the unit ball in $\mathbf{R}^n$ and $e_1 = (1,0,\dots,0)$. Prove that

$$\lim_{\varepsilon \to 0} \frac{1}{v_n \varepsilon^n} \int_{|x-e_1| \le \varepsilon} \chi_{|x| \le 1} \, dx = \frac{1}{2}.$$

Conclude that the function

$$\widetilde{\chi}_{B(0,1)}(x) = \begin{cases} 1 & \text{when } |x| < 1, \\ 1/2 & \text{when } |x| = 1, \\ 0 & \text{when } |x| > 1 \end{cases}$$

is regulated.

4.3.6. Let $L_\varepsilon(x) = e^{-\pi\varepsilon|x|^2}$ be defined for $\varepsilon > 0$ and $x \in \mathbf{R}^n$ and let $1 < q < \infty$. Prove that there is a constant $C(n,q) < \infty$ such that for any 1-periodic continuous function g on $\mathbf{R}^n$ we have

$$\sup_{0<\varepsilon<1} \varepsilon^{\frac{n}{2q}} \big\| g\, L_{\varepsilon/q} \big\|_{L^{q,1}(\mathbf{R}^n)} \le C(n,q) \|g\|_{L^{q,1}(\mathbf{T}^n)}.$$

[*Hint:* Reduce matters to the situation where $g = \sum_{k\in\mathbf{Z}^n} \chi_{k+E}$, where E is a measurable subset of $[-1/2, 1/2)^n$. Express the $L^{q,1}$ norm of $gL_{\varepsilon/q}$ in terms of its distribution function and for $0 < \lambda < 1$ estimate the measure

$$\Big|\{L_{\varepsilon/q} > \lambda\} \cap \bigcup_{k\in\mathbf{Z}^n} (k+E)\Big| = \Big| B\Big(0, \Big(\frac{q}{\pi\varepsilon}\log\frac{1}{\lambda}\Big)^{\frac{1}{2}}\Big) \cap \bigcup_{k\in\mathbf{Z}^n} (k+E)\Big|$$

by $C_n\big(\sqrt{n} + \big(\frac{q}{\pi\varepsilon}\log\frac{1}{\lambda}\big)^{1/2}\big)^n |E|.$]

4.3.7. Let $0 < C_0 < \infty$. Suppose that $\{f_t\}_{t\in\mathbf{R}^+}$ is a family of measurable functions on a measure space X that satisfies

$$\Big\|\sup_{t\in F} |f_t|\Big\|_{L^p} \le C_0$$

for every finite subset F of $\mathbf{R}^+$.
(a) Suppose that for each $x \in X$, the function $t \mapsto f_t(x)$ is continuous. Show that

$$\Big\|\sup_{t>0} |f_t|\Big\|_{L^p} \le C_0.$$

(b) Prove that for any $t > 0$ there is a measurable function $\widetilde{f}_t$ on X that is a.e. equal to f_t such that

$$\Big\|\sup_{t\in\mathbf{R}^+} |\widetilde{f}_t|\Big\|_{L^p} \le C_0.$$

[*Hint:* Part (a): Notice that in view of the Lebesgue monotone convergence theorem, we have $\big\|\sup_{t\in\mathbf{Q}} |f_t|\big\|_{L^p} \le C_0$. Also, for each $x \in X$ we have $\sup_{t\in\mathbf{R}^+} |f_t(x)| = \sup_{t\in\mathbf{R}} |f_t(x)|$ by continuity. Part (b): Let $a = \sup_F \big\|\sup_{t\in F} |f_t|\big\|_{L^p} \le C_0$, where the supremum is taken over all finite subsets F of $\mathbf{R}$. Pick an increasing sequence of finite sets F_n such that $\big\|\sup_{t\in F_n} |f_t|\big\|_{L^p} \to a$ as $n \to \infty$. Let $g = \sup_n \sup_{t\in F_n} |f_t|$ and note that $\|g\|_{L^p} = a$. Then for any $s \in \mathbf{R}$ we have

$$\left\|\max(|f_s|,\sup_{t\in F_k}|f_t|)\right\|_{L^p}\leq a\,.$$

This implies $\|\max(|f_s|,g)\|_{L^p}\leq a=\|g\|_{L^p}$, so that $|f_s|\leq g$ a.e. for all $s\in\mathbf{R}$.]

4.3.8. (*E. Prestini*) Show that for $f\in L^2(\mathbf{T}^2)$ we have that

$$\sum_{\substack{|m_1|\leq N\\|m_2|\leq N^2}}\widehat{f}(m_1,m_2)e^{2\pi i(m_1x_1+m_2x_2)}\to f(x_1,x_2)$$

for almost all (x_1,x_2) in $\mathbf{T}^2$.
$\bigl[$*Hint:* Use the splitting $\widehat{f}(m_1,m_2)=\widehat{f}(m_1,m_2)\chi_{|m_2|\leq|m_1|^2}+\widehat{f}(m_1,m_2)\chi_{|m_2|>|m_1|^2}$ and apply the idea of the proof of Theorem 4.3.16.$\bigr]$

4.4 Applications to Geometry and Partial Differential Equations

In this section we discuss two applications of Fourier series. The first concerns a classical result in planar geometry and the other the heat equation.

4.4.1 The Isoperimetric Inequality

Suppose we are given a closed positively oriented nonself intersecting $\mathscr{C}^1$ curve C in the (x,y) plane of length L that encloses a region R of area A. The curve can be described in terms of its parametric equations $x=x(t)$ and $y=y(t)$, where $t\in[0,1]$. Since the curve is closed, we have $(x(0),y(0))=(x(1),y(1))$ and the $\mathscr{C}^1$ functions $x(t),y(t)$ can be thought of as 1-periodic functions on the circle. The perimeter L of the curve is given by the equation

$$L=\int_0^1\sqrt{|x'(t)|^2+|y'(t)|^2}\,dt$$

while the area of the region R enclosed by the curve is equal to

$$\begin{aligned}
A&=\iint_R 1\,dxdy\\
&=\iint_R\frac{\partial}{\partial x}\frac{x}{2}-\frac{\partial}{\partial y}\left(-\frac{y}{2}\right)dxdy\\
&=\frac{1}{2}\oint_C xdy-ydx\\
&=\frac{1}{2}\int_0^1 x(t)y'(t)-x'(t)y(t)\,dt\,,
\end{aligned}$$

where we made use of Green's theorem in the third equality above.

A $\mathscr{C}^1$ curve $\gamma(t)$ is regular if $\gamma'(t) \neq 0$ for all t. We have the following result relating the perimeter and the enclosed area of a region enclosed by a closed $\mathscr{C}^1$ curve.

Theorem 4.4.1. *Given a closed, positively oriented, nonself intersecting, regular, $\mathscr{C}^1$ planar curve of length L that encloses a region of area A, we have that*

$$A \leq L^2/4\pi \tag{4.4.1}$$

with equality holding if and only if the curve is a circle.

Proof. Assume that the curve has parametric equations $x = x(t)$, $y = y(t)$, $0 \leq t \leq 1$. We may assume that the curve has constant speed, i.e., it satisfies

$$\sqrt{|x'(t)|^2 + |y'(t)|^2} = L$$

for all $t \in [0,1]$. This is achieved via the reparametrization of the curve in terms of the inverse function $s^{-1}(t) = \gamma(t)$ of the normalized arc length function

$$s(t) = \frac{1}{L} \int_0^t \sqrt{|x'(u)|^2 + |y'(u)|^2} \, du \, .$$

Since $|(x'(t), y'(t))| \neq 0$, $t \mapsto s(t)$ is a one-to-one and onto continuous map from $[0,1]$ to $[0,1]$. Then the curve $t \mapsto (x(\gamma(t)), y(\gamma(t)))$ has constant speed, since

$$|x'(\gamma(t))|^2 |\gamma'(t)|^2 + |y'(\gamma(t))|^2 |\gamma'(t)|^2 = \frac{|x'(s^{-1}(t))|^2 + |y'(s^{-1}(t))|^2}{|s'(s^{-1}(t))|^2} = L^2 \, .$$

So we can replace the map $(x(t), y(t))$ by $(x(\gamma(t)), y(\gamma(t)))$ which produces the same curve. Let

$$f(t) = x(t) + iy(t)$$

for $t \in [0,1]$. Then in view of the preceding discussion, we may assume that the function $f(t) = x(t) + iy(t)$ satisfies $|f'(t)| = L$ for all $t \in [0,1]$.

Under the assumption $|f'(t)| = L$ for all $t \in [0,1]$, we now show that (4.4.1) holds, with equality if and only if $f(t) = c_0 e^{2\pi it} + C_0$ for some $c_0, C_0 \in \mathbf{C}$ with $|c_0| = \frac{L}{2\pi}$. To prove this claim we argue as follows:

$$\begin{aligned} A &= \frac{1}{2} \operatorname{Im} \int_0^1 f'(t)\overline{f(t)} \, dt \\ &= \frac{1}{2} \operatorname{Im} \int_0^1 f'(t) \overline{\left(f(t) - \widehat{f}(0)\right)} \, dt \\ &\leq \frac{1}{2} L \left\| f - \widehat{f}(0) \right\|_{L^2} \\ &\leq \frac{L}{2} \frac{1}{2\pi} \left\| f' \right\|_{L^2} \\ &= \frac{L^2}{4\pi}, \end{aligned}$$

establishing (4.4.1), but we need to explain why the inequality

$$\|f - \widehat{f}(0)\|_{L^2} \le \frac{1}{2\pi}\|f'\|_{L^2} \tag{4.4.2}$$

is valid. Indeed, we have

$$f'(t) = \sum_{m \in \mathbf{Z}} 2\pi i m \widehat{f}(m) e^{2\pi i m t}$$

where the series converges in L^2. Thus we have

$$\|f'\|_{L^2} = 2\pi \left[\sum_{m \in \mathbf{Z}} |m\widehat{f}(m)|^2 \right]^{\frac{1}{2}} \ge 2\pi \left[\sum_{m \in \mathbf{Z}\setminus\{0\}} |\widehat{f}(m)|^2 \right]^{\frac{1}{2}} = 2\pi\|f - \widehat{f}(0)\|_{L^2}, \tag{4.4.3}$$

which proves (4.4.2).

Now suppose that equality holds in (4.4.1), then we must have equality in (4.4.2) and thus in (4.4.3), which implies that $\widehat{f}(m) = 0$ when $|m| \ge 2$; hence for all $t \in [0,1]$ we must have

$$f(t) = ce^{2\pi i t} + c'e^{-2\pi i t} + \widehat{f}(0) \tag{4.4.4}$$

where c, c' are complex numbers. But since $\|f'\|_{L^2} = L$, it follows that

$$4\pi^2(|c|^2 + |c'|^2) = L^2, \tag{4.4.5}$$

and since $|f'(t)| = L$ for all $t \in [0,1]$, it follows that

$$\left(\frac{L}{2\pi}\right)^2 = |c|^2 + |c'|^2 - 2\operatorname{Re}\left[c\overline{c'}e^{2\pi i 2t}\right] \tag{4.4.6}$$

for all $t \in [0,1]$. Combining (4.4.5) and (4.4.6) we obtain

$$\operatorname{Re}\left[c\overline{c'}e^{2\pi i 2t}\right] = 0. \tag{4.4.7}$$

Inserting $t = 0$ and $t = 1/8$ in (4.4.7) and using that $\operatorname{Im}(iz) = -\operatorname{Re}z$, we deduce that $\operatorname{Re}c\overline{c'} = \operatorname{Im}c\overline{c'} = 0$. This implies that either c or c' is zero. In either case (4.4.4) and (4.4.5) imply that $f(t)$ is a circle of radius $L/2\pi$ centered at the point $\widehat{f}(0)$. $\qquad\square$

4.4.2 The Heat Equation with Periodic Boundary Condition

Let $k > 0$ be a fixed quantity. Consider the partial differential equation

$$\frac{\partial}{\partial t}F(x,t) = k \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}F(x,t) \qquad\qquad t \in (0,\infty), \quad x \in \mathbf{R}^n, \tag{4.4.8}$$

which is called the *heat equation*. Assume that there is an initial condition

$$F(0,x) = f(x) \qquad x \in \mathbf{R}^n \qquad (4.4.9)$$

for a given $\mathscr{C}^\infty$ function f on $\mathbf{R}^n$ which is assumed to be 1-periodic in every variable.

We would like to find a continuous function $F(t,x)$ on $[0,\infty) \times \mathbf{R}^n$ which is $\mathscr{C}^\infty$ on $(0,\infty) \times \mathbf{R}^n$ such that

$$F(t,x+e_j) = F(t,x)$$

for all $t \geq 0$ and all $e_j = (0,\ldots,0,1,0,\ldots,0)$, so that F solves the equation (4.4.8).

The function $F(t,x)$ represents the temperature of a body at time $t > 0$ at the location $(x_1,\ldots,x_n)$. Since the initial temperature f is 1-periodic in each variable, we expect $F(t,\cdot)$ to also be periodic in each variable. For example, $F(t,x)$ is a good model for the temperature of the torus $\{(e^{2\pi i x_1},\ldots,e^{2\pi i x_n}) : x_j \in \mathbf{R}\}$ at time $t > 0$, given that its temperature at time $t = 0$ is $f(x)$. When $n = 1$, $F(t,x)$ models the temperature of a infinitesimally thin ring, thought of as the unit circle, at time $t > 0$ at the location $e^{2\pi i x}$.

Let us suppose there is a continuous function $F(t,x)$ on $[0,\infty) \times \mathbf{R}^n$ which is $\mathscr{C}^\infty$ on $(0,\infty) \times \mathbf{R}^n$ that solves the equation (4.4.8) and satisfies $F(t,x+e_j) = F(t,x)$ for all $x \in \mathbf{R}^n$ and $t \geq 0$. Denote by $c_m(t)$ the Fourier coefficient of the function $x \mapsto F(t,x)$ defined by

$$c_m(t) = \int_{\mathbf{T}^n} F(t,x) e^{-2\pi i m \cdot x} dx.$$

Then $c_m(t)$ is a continuous function on $[0,\infty)$ since F is continuous in the variable t. For the same reason, c_m is a smooth function on $(0,\infty)$ whose jth derivative is given by

$$\frac{d^j}{dt^j} c_m(t) = \int_{\mathbf{T}^n} \frac{\partial^j}{\partial t^j} F(t,x) e^{-2\pi i m \cdot x} dx$$

for any $j = 1,2,\ldots$. Using equation (4.4.8) we obtain that

$$c'_m(t) = \int_{\mathbf{T}^n} \frac{\partial}{\partial t} F(t,x) e^{-2\pi i m \cdot x} dx = \int_{\mathbf{T}^n} k \frac{\partial^2}{\partial^2 x} F(t,x) e^{-2\pi i m \cdot x} dx = -4\pi^2 |m|^2 k c_m(t),$$

where the last identity is due to an integration by parts in which the boundary terms cancel each other in view of the periodicity of the integrand in x. Also $c_m(0) = \widehat{f}(m)$. The ordinary differential equation $c'_m(t) = -4\pi^2 |m|^2 k c_m(t)$ with initial condition $c_m(0) = \widehat{f}(m)$ is easily solved by separating the variables

$$\frac{dc_m(t)}{c_m(t)} = -4\pi^2 |m|^2 k dt, \qquad (4.4.10)$$

yielding the solution

$$c_m(t) = \widehat{f}(m) e^{-4\pi^2 |m|^2 k t}.$$

We may therefore define the function

$$F(t,x) = \sum_{m \in \mathbf{Z}^n} \widehat{f}(m) e^{-4\pi^2 |m|^2 kt} e^{2\pi i m \cdot x} \tag{4.4.11}$$

on $[0,\infty) \times \mathbf{R}^n$ and observe the following:

(a) F is continuous on $[0,\infty) \times \mathbf{R}^n$ and $\mathscr{C}^\infty$ on $(0,\infty) \times \mathbf{R}^n$.
(b) F satisfies the heat equation (4.4.8) and the initial condition (4.4.9).
(c) F is 1-periodic in each of the last n variables.

These statements can be easily proved by passing the differentiation inside the sum, in view of the rapid convergence of the series in (4.4.11) due to the fact that the periodic function f is $\mathscr{C}^\infty(\mathbf{R}^n)$. Furthermore, F is unique with properties (a), (b), and (c), since any other function $G(x,t)$ with these properties is derived in the preceding way, and so it has to be equal to $F(x,t)$.

Definition 4.4.2. Define the *heat kernel*

$$H_t(x) = \sum_{m \in \mathbf{Z}^n} e^{-4\pi^2 |m|^2 kt} e^{2\pi i m \cdot x}$$

for $t > 0$. Notice that the series defining H_t is absolutely convergent for any $t > 0$. The importance of the heat kernel lies in the fact that one can express the solution $F(x,t)$ of (4.4.8) in terms of the convolution $F(x,t) = (f * H_t)(x)$.

We summarize these facts in the following proposition.

Proposition 4.4.3. *Let $k > 0$ be fixed and let f be in $\mathscr{C}^\infty(\mathbf{R}^n)$. Assume that f is 1-periodic function in each variable. Then the heat equation*

$$\frac{\partial}{\partial t} F(x,t) = k \Delta_x F(x,t) \qquad t \in (0,\infty), \quad x \in \mathbf{R}^n \tag{4.4.12}$$

under the initial condition

$$F(0,x) = f(x) \qquad x \in \mathbf{R}^n \tag{4.4.13}$$

has a unique solution which is continuous on $[0,\infty) \times \mathbf{R}^n$ and $\mathscr{C}^\infty$ on $(0,\infty) \times \mathbf{R}^n$ given by

$$F(x,t) = (f * H_t)(x) = \sum_{m \in \mathbf{Z}^n} \widehat{f}(m) e^{-4\pi^2 |m|^2 kt} e^{2\pi i m \cdot x}. \tag{4.4.14}$$

Proof. Since f is $\mathscr{C}^\infty$, the series in (4.4.14) is rapidly convergent in m and thus it gives a continuous function on $[0,\infty) \times \mathbf{R}^n$. Moreover, the series can be differentiated term by term in the variable $t > 0$, and thus it produces a $\mathscr{C}^\infty$ function on $(0,\infty) \times \mathbf{R}^n$. By Fourier inversion (Proposition 3.2.5), F satisfies the initial condition (4.4.13). Finally, to verify (4.4.12), we simply notice that

$$\frac{\partial}{\partial t}F(x,t) = \sum_{m\in\mathbf{Z}^n} \frac{\partial}{\partial t}\widehat{f}(m)e^{-4\pi^2|m|^2kt}e^{2\pi im\cdot x}$$

$$= -4\pi^2 k \sum_{m\in\mathbf{Z}^n} \widehat{f}(m)e^{-4\pi^2|m|^2kt}|m|^2 e^{2\pi im\cdot x}$$

$$= k \sum_{m\in\mathbf{Z}^n} \widehat{f}(m)e^{-4\pi^2|m|^2kt}\left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right)e^{2\pi im\cdot x}$$

$$= k\left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right)\sum_{m\in\mathbf{Z}^n}\widehat{f}(m)e^{-4\pi^2|m|^2kt}e^{2\pi im\cdot x}$$

$$= k\frac{\partial^2}{\partial^2 x}F(x,t),$$

where the rapid convergence of the series in m makes it possible to pass the differentiations in and out of the sum. Finally, to show uniqueness, assume that there is another solution $G(t,x)$, continuous on $[0,\infty)\times\mathbf{R}^n$ and $\mathscr{C}^\infty$ on $(0,\infty)\times\mathbf{R}^n$, that can be expanded in Fourier series as follows:

$$G(t,x) = \sum_{m\in\mathbf{Z}^n} c_m(t)e^{2\pi im\cdot x}.$$

Conditions (4.4.12) and the rapid decay of the coefficients $c_m(t)$ yield the ordinary differential equation (4.4.10) with initial condition $c_m(0) = \widehat{f}(m)$, which has the solution $c_m(t) = \widehat{f}(m)e^{-4\pi^2|m|^2kt}$. Thus $G = F$ on $[0,\infty)\times\mathbf{R}^n$. □

It is important to observe that the family $\{H_t\}_{t>0}$ is an approximate identity on $\mathbf{T}^1$. Indeed, the Poisson summation formula (Theorem 3.2.8) and the fact that the inverse Fourier transform of $e^{-4\pi^2 kt|\xi|^2}$ is $e^{-|x|^2/4kt}/(2\sqrt{\pi kt})^n$ [Example 2.2.9 and Proposition 2.2.11 (8)] yield that for all $x\in[0,1]^n$ we have

$$H_t(x) = \frac{1}{(2\sqrt{\pi kt})^n}\sum_{\ell\in\mathbf{Z}} e^{-\frac{|x+\ell|^2}{4kt}}.$$

This identity implies that $H_t(x)\geq 0$ for all $t>0$ and that

$$\int_{\mathbf{T}^n} H_t(x)\,dx = \int_{\mathbf{R}^n} \frac{1}{(2\sqrt{\pi kt})^n}e^{-\frac{|x|^2}{4kt}}\,dx = \int_{\mathbf{R}^n} e^{-\pi|x|^2}\,dx = 1$$

for all $t>0$ and that

$$\int_{\delta\leq|u|} H_t(u)\,du \leq \int_{|x|\geq\delta}\frac{1}{(2\sqrt{\pi kt})^n}e^{-\frac{|x|^2}{4kt}}\,dx = \int_{|x|\geq\frac{\delta}{2\sqrt{k\pi t}}} e^{-\pi|x|^2}\,dx$$

which tends to zero as $t\to 0$ for any $\delta>0$ in view of the Lebesgue differentiation theorem. Thus properties (i), (ii), and (iii) of approximate identities hold.

As a consequence, we have that $\|F(t,\cdot)-f\|_{L^p(\mathbf{T}^n)}\to 0$ as $t\to 0$ for $1\leq p<\infty$ and $F(t,\cdot)$ converges to f uniformly on $\mathbf{T}^n$; see Theorem 1.2.19.

Exercises

4.4.1. Let f, F be as in Proposition 4.4.3. Prove that the total heat on the torus remains constant in time by showing that for all $t \geq 0$ we have

$$\int_{\mathbf{T}^n} F(t,x)\,dx = \int_{\mathbf{T}^n} f(x)\,dx.$$

Moreover, show that the temperature at any fixed point $x \in \mathbf{T}^n$ on the torus tends to the average initial temperature, i.e., it satisfies

$$\lim_{t \to \infty} F(t,x) = \int_{\mathbf{T}^n} f(y)\,dy.$$

4.4.2. Derive the following property of the heat kernel,

$$H_t * H_s = H_{t+s}$$

for all $t, s > 0$.

4.4.3. Consider the heat equation on $[0, \infty) \times \mathbf{R}$

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)$$

without a boundary condition. Show that $u = 2t + x^2$ and $u(t,x) = e^{-q^2 t} e^{iqx}$, as well as constant functions, are solutions of this equation. Prove that the set of solutions is a vector space over the field of complex numbers $\mathbf{C}$.

4.4.4. Suppose that a square-integrable function $g(x)$ on $\mathbf{R}^n$ is supported in a cube $[-A, A]^n$ for some $A > 0$. Then we have the following representation:

$$g(x) = \sum_{m \in \mathbf{Z}^n} \left(\frac{1}{(2A)^n} \int_{\mathbf{R}^n} g(y) e^{-2\pi i \frac{m \cdot y}{2A}}\,dy \right) e^{2\pi i \frac{x \cdot m}{2A}} \chi_{[-A,A]^n},$$

where the series converges in L^2

4.4.5. This exercise provides an application of Fourier series in complex analysis. Let $z \in \mathbf{C} \setminus \mathbf{Z}$. Consider the function $h_z(x) = \cos(2\pi z x)$ defined on $[-\frac{1}{2}, \frac{1}{2}]$ extended periodically on the entire line [notice $h_z(-\frac{1}{2}) = h_z(\frac{1}{2})$].
(a) Compute the Fourier coefficients of h_z.
(b) Obtain a Fourier series expansion of h_z noticing that it is a Lipschitz function.
(c) Plug in $x = 1/2$ to prove that

$$\cot(\pi z) = \frac{1}{\pi z} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{2z}{z^2 - m^2}.$$

4.5 Applications to Number theory and Ergodic theory

In this section we discuss three applications of Fourier series techniques to number theory and ergodic theory.

4.5.1 Evaluation of the Riemann Zeta Function at even Natural numbers

Definition 4.5.1. We define the *Bernoulli polynomials* $\{B_k\}_{k=0}^{\infty}$ on $[0,1]$ recursively as follows:

$$B_0(x) = 1$$

$$B'_k(x) = kB_{k-1}(x)$$

for $k = 1, 2, \ldots$, and

$$\int_0^1 B_k(x)\,dx = 0.$$

In view of this definition we find the first few polynomials $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$, $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$, etc. Unlike orthogonal polynomials, the Bernoulli polynomials have the remarkable property that their number of zeros in the unit interval does not increase as the degree of the polynomials increases; in fact all Bernoulli polynomials have at most three zeros in $[0,1]$.

Notice that for $k \geq 2$ we have

$$B_k(1) - B_k(0) = \int_0^1 B'_k(x)\,dx = k\int_0^1 B_{k-1}(x)\,dx = 0,$$

thus we may think of these polynomials as functions on the circle $\mathbf{T}^1$. We extend the Bernoulli polynomials to the whole line periodically by setting $B_k(x+l) = B_k(x)$ for $x \in [0,1]$. We now compute the Fourier coefficients of B_k. We have

$$\widehat{B_1}(m) = \int_0^1 (t - \tfrac{1}{2})e^{-2\pi imt}\,dt = \begin{cases} 0 & \text{if } m = 0 \\ -\frac{1}{2\pi im} & \text{if } m \neq 0. \end{cases}$$

Therefore, using Corollary 3.4.10, we can write

$$B_1(x) = \sum_{m \neq 0} -\frac{1}{2\pi im} e^{2\pi imx}$$

where the series converges at every $x \in (0,1)$.

We have the following result concerning the Fourier expansion of the Bernoulli polynomials.

Theorem 4.5.2. *For each $k \geq 2$ we have*

$$B_k(x) = -k! \sum_{m \in \mathbf{Z} \setminus \{0\}} \frac{1}{(2\pi i m)^k} e^{2\pi i m x}, \tag{4.5.1}$$

where the series converges absolutely and uniformly on $[0,1]$. When $k = 1$ we have

$$B_1(x) = - \sum_{m \in \mathbf{Z} \setminus \{0\}} \frac{1}{2\pi i m} e^{2\pi i m x} \tag{4.5.2}$$

for all $x \in (0,1)$ and the series converges conditionally.

Proof. We have already proved (4.5.2) and we focus attention to the case $k \geq 2$. As a consequence of $B_k' = k B_{k-1}$ we obtain

$$B_k(x) = k \int_0^x B_{k-1}(t) \, dt + C_k \, .$$

Using the property that B_k has integral zero over $[0,1]$ we evaluate the constant C_k. We have

$$\begin{aligned}
0 &= \int_0^1 \left[k \int_0^x B_{k-1}(t) \, dt + C_k \right] dx \\
&= k \int_0^1 \left(\int_t^1 dx \right) B_{k-1}(t) \, dt + C_k \\
&= -k \int_0^1 t B_{k-1}(t) \, dt + C_k \, .
\end{aligned}$$

Thus

$$C_k = k \int_0^1 t B_{k-1}(t) \, dt \, .$$

The Fourier series of $B_k(x)$ can be obtained by integrating the one for $B_{k-1}(x)$ for all $k \geq 2$ by induction via the identity

$$B_k(x) = k \int_0^x B_{k-1}(t) \, dt + k \int_0^1 t B_{k-1}(t) \, dt \, . \tag{4.5.3}$$

Indeed, assume that (4.5.1) holds for some $k \geq 2$. Then using (4.5.3) we obtain

$$\begin{aligned}
B_k(x) &= \int_0^x \lim_{N \to \infty} \sum_{\substack{|m| \leq N \\ m \neq 0}} -k \frac{(k-1)!}{(2\pi i m)^{k-1}} e^{2\pi i m t} \, dt - \int_0^1 t \lim_{N \to \infty} \sum_{\substack{|m| \leq N \\ m \neq 0}} k \frac{(k-1)!}{(2\pi i m)^{k-1}} e^{2\pi i m t} \, dt \\
&= \lim_{N \to \infty} \int_0^x \sum_{\substack{|m| \leq N \\ m \neq 0}} -k \frac{(k-1)!}{(2\pi i m)^{k-1}} e^{2\pi i m t} \, dt - \lim_{N \to \infty} \int_0^1 t \sum_{\substack{|m| \leq N \\ m \neq 0}} k \frac{(k-1)!}{(2\pi i m)^{k-1}} e^{2\pi i m t} \, dt
\end{aligned}$$

$$= -\lim_{N\to\infty} \sum_{\substack{|m|\le N \\ m\ne 0}} \frac{k!}{(2\pi im)^{k-1}} \frac{e^{2\pi imx}-1}{2\pi im} - \lim_{N\to\infty} \sum_{\substack{|m|\le N \\ m\ne 0}} \frac{k!}{(2\pi im)^{k-1}} \left[t\frac{e^{2\pi imt}}{2\pi im} \right]_{t=0}^{t=1}$$

$$= -k! \sum_{m\ne 0} \frac{e^{2\pi imx}}{(2\pi im)^k}.$$

Passing the limit from inside the integral to outside is allowed due to the uniform convergence of the series when $k \ge 3$. In the case $k = 2$, one may use Exercise 3.5.6 which says that for all $[a,b] \subseteq \mathbf{T}^1$ and g integrable functions over $[a,b]$ one has

$$\lim_{N\to\infty} \int_a^b (g * D_N)(t)\,dt = \int_a^b \lim_{N\to\infty} (g * D_N)(t)\,dt = \int_a^b g(t)\,dt.$$

This argument proves identity (4.5.1) for all $k \ge 2$ by induction and concludes the proof.   □

We recall the following definition from number theory.

Definition 4.5.3. For $s > 1$ we define

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

called the *Riemann zeta function*.

We use the Fourier expansions of the Bernoulli polynomials to obtain the values of the Riemann zeta function for integers. When k is an even integer, identity (4.5.1) can also be written as

$$B_k(x) = 2(-1)^{1+\frac{k}{2}} k! \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{(2\pi n)^k}$$

and inserting $x = 0$ yields

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = \frac{B_k(0)(2\pi)^k}{2(-1)^{1+\frac{k}{2}}k!}.$$

The polynomial $B_1(x) = x - 1/2$ has rational coefficients and thus so do all the B_k by a straightforward inductive argument that uses the identity (4.5.3). Thus $B_k(0)$ is a rational number for all $k \ge 1$. We conclude that

$$\zeta(2m) = \frac{B_{2m}(0)(2\pi)^{2m}}{2(-1)^{1+m}(2m)!} \tag{4.5.4}$$

which is a rational multiple of $(2\pi)^{2m}$, hence transcendental, since π is a transcendental number. We have therefore obtained the following.

Corollary 4.5.4. *(Euler) The value of the Riemann zeta function $\zeta(2m)$, $m = 1, 2, \ldots$, is equal to a rational multiple of $(2\pi)^{2m}$; hence it is a transcendental number.*

The corresponding statement for odd integers remains unresolved in general, as of this writing.

4.5.2 Equidistributed sequences

Here we discuss Weyl's theorem on equidistributed sequences.

Definition 4.5.5. A sequence $\{a_k\}_{k=0}^{\infty}$ with values in $\mathbf{T}^n$ is called *equidistributed* if for every cube Q in $\mathbf{T}^n$ we have

$$\lim_{N \to \infty} \frac{\#\{k : 0 \le k \le N-1, \quad a_k \in Q\}}{N} = |Q|.$$

Theorem 4.5.6. *The following statements are equivalent:*
(a) The sequence $\{a_k\}_{k=0}^{\infty}$ is equidistributed.
(b) For every smooth function f on $\mathbf{T}^n$ we have that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} f(a_k) = \int_{\mathbf{T}^n} f(x)\, dx.$$

(c) For every $m \in \mathbf{Z}^n \setminus \{0\}$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i m \cdot a_k} = 0.$$

Proof. We first prove the equivalence of (a) and (b). We begin by observing that (b) is a restatement of (a) if $f = \chi_Q$ and Q is a cube in $\mathbf{T}^n$. Thus, if (a) holds, then (b) holds for all step functions, i.e., finite linear combinations of characteristic functions of cubes. We prove that (a) implies (b) for smooth functions. Given a smooth function f on $\mathbf{T}^n$ and given $\varepsilon > 0$, by the uniform continuity of f, there is a step function $g = \sum_{j=1}^{m} c_j \chi_{Q_j}$ ($c_j \in \mathbf{C}$ and Q_j are cubes in $\mathbf{T}^n$) such that $\|f - g\|_{L^\infty} < \frac{\varepsilon}{3}$. Since g is a finite linear combination of step functions, there is an N_0 such that for $N \ge N_0$ we have

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} g(a_k) - \int_{\mathbf{T}^n} g(x)\, dx \right| < \frac{\varepsilon}{3}.$$

Since

$$\left| \int_{\mathbf{T}^n} f(x)\, dx - \int_{\mathbf{T}^n} g(x)\, dx \right| \le \|f - g\|_{L^\infty} < \frac{\varepsilon}{3}$$

and

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} g(a_k) - \frac{1}{N} \sum_{k=0}^{N-1} f(a_k) \right| \le \|f - g\|_{L^\infty} < \frac{\varepsilon}{3},$$

it follows that for $N \ge N_0$ we have

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} f(a_k) - \int_{\mathbf{T}^n} f(x)\,dx \right| < \varepsilon,$$

thus (b) holds.

To prove that (b) implies (a) given a cube Q in $\mathbf{T}^n$ pick two smooth functions g and h such that

$$0 \le h \le \chi_Q \le g$$

and such that g is equal to 1 on Q and vanishes off $(1+\varepsilon)Q$ while h is equal to 1 on $(1-\varepsilon)Q$ and vanishes off Q. Observe that

$$|Q| - c_n \varepsilon \le \int_{\mathbf{T}^n} h(x)\,dx \le |Q| \le \int_{\mathbf{T}^n} g(x)\,dx \le |Q| + c_n \varepsilon.$$

for some $c_n > 0$. Since

$$\frac{1}{N} \sum_{k=0}^{N-1} h(a_k) \le \frac{1}{N} \sum_{k=0}^{N-1} \chi_Q(a_k) \le \frac{1}{N} \sum_{k=0}^{N-1} g(a_k),$$

the sandwich theorem implies that

$$|Q| - c_n \varepsilon \le \liminf_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \chi_Q(a_k) \le \limsup_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \chi_Q(a_k) \le |Q| + c_n \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary the conclusion follows.

The implication $(b) \implies (c)$ is trivial.

We now prove that $(c) \implies (b)$.

Given a smooth function f on $\mathbf{T}^n$ we write

$$\frac{1}{N} \sum_{k=0}^{N-1} f(a_k) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{m\in\mathbf{Z}^n} \widehat{f}(m) e^{2\pi i m \cdot a_k} = \widehat{f}(0) + \sum_{m\in\mathbf{Z}^n\setminus\{0\}} \widehat{f}(m) \left(\frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i m \cdot a_k} \right).$$

Because of the rapid decay of the Fourier coefficients of f we can pass the limit as $N \to \infty$ inside the sum in m. It follows that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(a_k) = \widehat{f}(0) = \int_{\mathbf{T}^n} f(x)\,dx.$$

$\square$

Example 4.5.7. The sequence $\{k\sqrt{2}-[k\sqrt{2}]\}_{k=0}^{\infty}$ is equidistributed on $\mathbf{T}^1$. We check this by verifying condition (c) of Theorem 4.5.6. Indeed if $m \in \mathbf{Z} \setminus \{0\}$ then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi im(k\sqrt{2}-[k\sqrt{2}])} = \lim_{N\to\infty} \frac{1}{N} \frac{e^{2\pi iN(m\sqrt{2})}-1}{e^{2\pi i(m\sqrt{2})}-1} = 0,$$

since $m\sqrt{2}$ is never a rational and thus the denominator never vanishes.

Naturally, the same conclusion is valid for any other irrational number in place of $\sqrt{2}$.

Example 4.5.8. We examine the sequence of the first digits of powers of 2. Consider the following sequence of numbers defined for $m = 1, 2, \ldots$

$$d_m = \text{first digit of } 2^m.$$

For instance we have $d_1 = 2, d_2 = 4, d_3 = 8, d_4 = 1, d_5 = 3, \ldots$.

Fix an integer $k \in \{1,2,3,4,5,6,7,8,9\}$. We would like to find the frequency in which k appears as a first digit of 2^m, precisely, we would like to compute

$$\lim_{N\to\infty} \frac{\#\{m \in \{1,2,\ldots,N\} : \ d_m = k\}}{N}.$$

The crucial observation is that the first digit of 2^m is equal to k if and only if there is a nonnegative integer s such that

$$k10^s \leq 2^m < (k+1)\,10^s.$$

Taking logarithms with base 10 we obtain

$$s + \log_{10}(k) \leq m\log_{10} 2 < s + \log_{10}(k+1),$$

but since $0 \leq \log_{10}(k)$ and $\log_{10}(k+1) \leq 1$, taking fractional parts we obtain that

$$s = [m\log_{10} 2]$$

and that

$$\log_{10}(k) \leq m\log_{10} 2 - [m\log_{10} 2] < \log_{10}(k+1).$$

Since the number $\log_{10} 2$ is irrational, it follows from Example 4.5.7 that the sequence

$$\{m\log_{10} 2 - [m\log_{10} 2]\}_{m=1}^{\infty}$$

is equidistributed in $[0, 1)$. Using Definition 4.5.5 in dimension $n = 1$ with

$$Q = [a,b] = \left[\log_{10}(k), \log_{10}(k+1)\right]$$

we obtain that

$$\lim_{N\to\infty} \frac{\#\{m\in\{1,2,\ldots,N\}:\ d_m=k\}}{N} = \log_{10}(k+1)-\log_{10}(k) = \log_{10}(1+\tfrac{1}{k}).$$

This gives the frequency in which k appears as first digit of 2^m. Notice that

$$\sum_{k=1}^{9}\log_{10}(1+\tfrac{1}{k}) = \sum_{k=1}^{9}\big(\log_{10}(k+1)-\log_{10}(k)\big) = 1,$$

as expected, and that the digit with the highest frequency that appears first in a term of the sequence $\{1,2,4,8,16,32,64,\ldots\}$ is 1, while the one with the lowest frequency is 9.

4.5.3 The Number of Lattice Points inside a Ball

Points in $\mathbf{Z}^n$ are called *lattice points*. In this subsection we obtain the number of lattice points $N(R)$ inside a closed ball of radius R in $\mathbf{R}^n$ centered at the origin, precisely, we compute the asymptotic behavior of

$$N(R) = |\overline{B(0,R)}\cap\mathbf{Z}^n|$$

as $R\to\infty$. We denote by v_n the volume of the closed unit ball in $\mathbf{R}^n$. We have the following result.

Theorem 4.5.9. *Let $n\geq 2$. If $N(R)$ is the number of lattice points inside the closed ball of radius R centered at zero in $\mathbf{R}^n$, then we have that*

$$N(R) = v_n R^n + O(R^{n\frac{n-1}{n+1}}),$$

as $R\to\infty$.

Proof. Let B be the closed unit ball in $\mathbf{R}^n$ and χ_B its characteristic function. Using the result in Appendix B.5 we have $\widehat{\chi_B}(\xi) = \frac{J_{n/2}(2\pi|\xi|)}{|\xi|^{n/2}}$. Now in view of the behavior of the Bessel function given in Appendix B.6 for $|\xi| < \frac{1}{2\pi}$ we have $J_{n/2}(2\pi|\xi|) \leq C|\xi|^{\frac{n}{2}}$. Also for $|\xi| \geq \frac{1}{2\pi}$ we have $J_{n/2}(2\pi|\xi|) \leq C|\xi|^{-\frac{1}{2}}$, in view of the result in Appendix B.7. Consequently, there is a constant C_n such that for all $\xi \in \mathbf{R}^n$ we have

$$|\widehat{\chi_B}(\xi)| \leq C_n(1+|\xi|)^{-\frac{n+1}{2}}.$$

Fix a smooth nonnegative radial function ζ supported in $|x|\leq\frac{1}{2}$ with integral equal to 1 and define $\zeta_\varepsilon(x) = \frac{1}{\varepsilon^n}\zeta(\frac{x}{\varepsilon})$ for $\varepsilon > 0$. For $0 < \varepsilon < \frac{1}{10}$, define functions

$$\Phi^\varepsilon = \chi_{(1-\frac{\varepsilon}{2})B}*\zeta_\varepsilon$$
$$\Psi^\varepsilon = \chi_{(1+\frac{\varepsilon}{2})B}*\zeta_\varepsilon.$$

These functions are even, hence their Fourier transforms are real-valued. We observe that

$$\Phi^{\varepsilon}(x) = 1 \text{ when } |x| \le 1 - \varepsilon \quad \text{and} \quad \Phi^{\varepsilon}(x) = 0 \text{ when } |x| \ge 1. \tag{4.5.5}$$

Indeed, we have

$$\Phi^{\varepsilon}(x) = \int_{\mathbf{R}^n} \chi_{(1-\frac{\varepsilon}{2})B}(y) \zeta_{\varepsilon}(x-y) dy$$

$$= \int_{|y| \le 1 - \frac{\varepsilon}{2}} \frac{1}{\varepsilon^n} \zeta(\tfrac{x-y}{\varepsilon}) dy$$

$$= \int_{|y| \le \frac{1}{\varepsilon} - \frac{1}{2}} \zeta(\tfrac{x}{\varepsilon} - y) dy.$$

For $|x| \le 1 - \varepsilon$, $|\frac{x}{\varepsilon}| \le \frac{1}{\varepsilon} - 1$, so $|\frac{x}{\varepsilon} - t| \le \frac{1}{\varepsilon} - 1 + \frac{1}{2} = \frac{1}{\varepsilon} - \frac{1}{2}$ for $|t| \le \frac{1}{2}$, which means

$$\Phi^{\varepsilon}(x) = \int_{|y| \le \frac{1}{\varepsilon} - \frac{1}{2}} \zeta(\tfrac{x}{\varepsilon} - y) dy = \int_{|t| \le \frac{1}{2}} \zeta(t) dt = 1.$$

For $|x| \ge 1$, $|\frac{x}{\varepsilon} - y| \ge \frac{1}{\varepsilon} - \frac{1}{\varepsilon} + \frac{1}{2} = \frac{1}{2}$, so

$$\Phi^{\varepsilon}(x) = \int_{|y| \le \frac{1}{\varepsilon} - \frac{1}{2}} \zeta(\tfrac{x}{\varepsilon} - y) dy = 0,$$

proving (4.5.5). Likewise one can show that

$$\Psi^{\varepsilon}(x) = 1 \text{ when } |x| \le 1 \quad \text{and} \quad \Psi^{\varepsilon}(x) = 0 \text{ when } |x| \ge 1 + \varepsilon. \tag{4.5.6}$$

Next we claim that

$$\left|\widehat{\Phi^{\varepsilon}}(\xi)\right| + \left|\widehat{\Psi^{\varepsilon}}(\xi)\right| \le C_{n,N}(1+|\xi|)^{-\frac{n+1}{2}}(1+\varepsilon|\xi|)^{-N} \tag{4.5.7}$$

for every $\xi \in \mathbf{R}^n$ and N a large positive number. Indeed to show (4.5.7) for Φ^{ε} we write

$$|\widehat{\Phi^{\varepsilon}}(\xi)| = |\widehat{\chi_{(1-\frac{\varepsilon}{2})B}}(\xi)\widehat{\zeta}(\xi\varepsilon)|$$

$$\le (1-\tfrac{\varepsilon}{2})^n C_n (1+|\xi|(1-\tfrac{\varepsilon}{2}))^{-\frac{n+1}{2}} |\widehat{\zeta}(\xi\varepsilon)| \tag{4.5.8}$$

$$\le C_{n,N}(1+|\xi|)^{-\frac{n+1}{2}}(1+\varepsilon|\xi|)^{-N}$$

since $\zeta \in \mathscr{S}(\mathbf{R})$ and $0 < \varepsilon < \frac{1}{10}$. The proof for Ψ^{ε} is completely similar.

We now notice that for $R > 0$, $m \in \mathbf{Z}^n \setminus \{0\}$, and $x \in [0,1]^n$ we have

$$1 + |m + xR| \le 1 + (\sqrt{n} + |m|)R \le 2\sqrt{n}(1 + |m|R).$$

This implies that for $\varepsilon < 1/10$ we have

$$\sum_{m \in \mathbf{Z}^n \setminus \{0\}} R^n (1 + R|m|)^{-\frac{n+1}{2}} (1 + \varepsilon R|m|)^{-N}$$

$$\leq C' \int_{[0,1]^n} \sum_{m \in \mathbf{Z}^n \setminus \{0\}} R^n (1 + R|m+x|)^{-\frac{n+1}{2}} (1 + \varepsilon R|m+x|)^{-N} dx$$

$$\leq C' \int_{\mathbf{R}^n} R^n (1 + R|x|)^{-\frac{n+1}{2}} (1 + \varepsilon R|x|)^{-N} dx$$

$$\leq C'' \varepsilon^{-\frac{n-1}{2}}, \tag{4.5.9}$$

where the proof of (4.5.9) is easily deduced by considering the cases (a) $|x| \leq R^{-1}$ which yields a constant, (b) $R^{-1} \leq |x| \leq (R\varepsilon)^{-1}$ which yields a constant multiple of $\varepsilon^{-\frac{n-1}{2}}$, and (c) $(R\varepsilon)^{-1} \leq |x|$ which also produces a constant multiple of $\varepsilon^{-\frac{n-1}{2}}$ if we pick $N > \frac{n-1}{2}$.

Using (4.5.5) and the Poisson summation formula we write

$$\sum_{m \in \mathbf{Z}^n} \chi_B\left(\frac{m}{R}\right) \geq \sum_{m \in \mathbf{Z}^n} \Phi^\varepsilon\left(\frac{m}{R}\right)$$

$$= R^n \widehat{\Phi^\varepsilon}(0) + \sum_{m \in \mathbf{Z}^n \setminus \{0\}} R^n \widehat{\Phi^\varepsilon}(Rm)$$

$$\geq v_n R^n (1-\varepsilon)^n - C_{n,N} \sum_{m \in \mathbf{Z}^n \setminus \{0\}} R^n (1 + R|m|)^{-\frac{n+1}{2}} (1 + \varepsilon R|m|)^{-N}$$

$$\geq v_n R^n - n v_n R^n \varepsilon - C'_{n,N} \varepsilon^{-\frac{n-1}{2}},$$

where we used that $(1-\varepsilon)^n \geq 1 - n\varepsilon$, (4.5.8), and (4.5.9). Now pick ε such that $\varepsilon R^n = \varepsilon^{-\frac{n-1}{2}}$, or equivalently $\varepsilon = R^{-\frac{2n}{n+1}}$ to deduce the estimate

$$N(R) \geq v_n R^n - O(R^{n\frac{n-1}{n+1}})$$

as $R \to \infty$.

Finally, making use of (4.5.6), and via a similar argument we write

$$\sum_{m \in \mathbf{Z}^n} \chi_B\left(\frac{m}{R}\right) \leq \sum_{m \in \mathbf{Z}^n} \Psi^\varepsilon\left(\frac{m}{R}\right) = R^n \widehat{\Psi^\varepsilon}(0) + \sum_{m \in \mathbf{Z}^n \setminus \{0\}} R^n \widehat{\Psi^\varepsilon}(Rm)$$

$$\leq v_n R^n (1+\varepsilon)^n + C_{n,N} \sum_{m \in \mathbf{Z}^n \setminus \{0\}} R^n (1 + R|m|)^{-\frac{n+1}{2}} (1 + \varepsilon R|m|)^{-N}$$

$$\leq v_n R^n + v_n 2^n R^n \varepsilon + C_{n,N} \varepsilon^{-\frac{n-1}{2}}.$$

The same choice of $\varepsilon = R^{-\frac{2n}{n+1}}$, yields the upper estimate for $N(R)$.

Combining the upper and lower estimates for $N(R)$ we obtain

$$N(R) = v_n R^n + O(R^{n\frac{n-1}{n+1}}),$$

as $R \to \infty$.   □

Exercise 4.5.8 contains an application of Theorem 4.5.9.

Exercises

4.5.1. Prove that for all $x \in [0,1]$ we have

$$\sum_{j=1}^{\infty} \frac{\sin(2\pi jx)}{j^{2m+1}} = \frac{(-1)^{m+1}}{2} \frac{(2\pi)^{2m+1}}{(2m+1)!} B_{2m+1}(x).$$

4.5.2. Show that for all $z \in \mathbf{C}$ with $|z| < 1$ we have

$$\pi z \cot(\pi z) = 1 - 2 \sum_{k=0}^{\infty} z^{2k+2} \zeta(2k+2).$$

$\big[$*Hint:* Use the result of Exercise 4.4.5.$\big]$

4.5.3. Suppose that a point $x = (x_1, \ldots, x_n) \in [0,1]^n$ has the property that $m \cdot x$ is irrational for all $m \in \mathbf{Z}^n \setminus \{0\}$. Show that the sequence $\{(kx_1 - [kx_1], \ldots, kx_n - [kx_n])\}_{k=0}^{\infty}$ is equidistributed in $\mathbf{T}^n$.

4.5.4. ([191]) Let $N(x,R)$ be the number of lattice points inside the closed ball of radius $R > 0$ centered at $x \in \mathbf{R}^n$. Show that

$$\int_{\mathbf{T}^n} |N(x,R) - v_n R^n|^2 \, dx = O(R^{n-1})$$

as $R \to \infty$, where v_n is the volume of the unit ball on $\mathbf{R}^n$.

4.5.5. (*Minkowski*) Let S be an open convex symmetric set in $\mathbf{R}^n$ and assume that the Fourier transform of its characteristic function satisfies the decay estimate

$$|\widehat{\chi}_S(\xi)| \le C(1 + |\xi|)^{-\frac{n+1}{2}}.$$

(This is the case if the boundary of S has nonzero Gaussian curvature.) Assume that $|S| > 2^n$. Prove that S contains at least one lattice point other than the origin. $\big[$*Hint:* Assume the contrary, set $f = \chi_{\frac{1}{2}S} * \chi_{\frac{1}{2}S}$, and apply the Poisson summation formula to f to prove that $f(0) \ge \widehat{f}(0)$.$\big]$

4.5.6. For $t \in [0, \infty)$ let

$$N(t) = \#\{m \in \mathbf{Z}^n : |m| \le t\}.$$

Let $0 = r_0 < r_1 < r_2 < \cdots$ be the sequence all of numbers r for which there exist $m \in \mathbf{Z}^n$ such that $|m| = r$.
(a) Observe that N is right continuous and constant on intervals of the form $[r_j, r_{j+1})$.
(b) Show that the distributional derivative of N is the measure

$$\mu(t) = \#\{m \in \mathbf{Z}^n : |m| = t\},$$

defined via the identity $\langle \mu, \varphi \rangle = \sum_{j=0}^{\infty} \#\{m \in \mathbf{Z}^n : |m| = r_j\} \varphi(r_j)$.

4.5.7. Let $f \in \mathscr{C}^1((0, \infty))$, and let $0 < a < b < \infty$. Derive the identity

$$\sum_{\substack{m \in \mathbf{Z}^n \\ a < |m| \le b}} f(|m|) = \int_a^b f(t) \, dN(t) = f(b)N(b) - f(a)N(a) - \int_a^b f'(x)N(x) \, dx,$$

where N is defined in Exercise 4.5.6 and $\int_a^b f(t) \, dN(t)$ is the Riemann-Stieltjes integral of f with respect to N.

4.5.8. Let $n \in \mathbf{Z}^+$ and $0 < \lambda < \infty$.
(a) Prove that for $k \in \mathbf{Z}^+ \cup \{0\}$ we have

$$\sum_{\substack{m \in \mathbf{Z}^n \\ a < |m| \le b}} \frac{e^{i|m|}}{|m|^\lambda} = \frac{-i\, \omega_{n-1}\, e^{ib}}{b^{\lambda - (n-1)}} - \frac{-i\, \omega_{n-1}\, e^{ia}}{a^{\lambda - (n-1)}} + O(a^{-\lambda + (n-1) - \frac{n-1}{n+1}})$$

for all $0 < a < b < \infty$ with $b - a \le 1$, where ω_{n-1} is the volume of $\mathbf{S}^{n-1}$.
(b) Show that when $\lambda > n - \frac{n-1}{n+1}$, the limit

$$\lim_{R \to \infty} \sum_{\substack{m \in \mathbf{Z}^n \setminus \{0\} \\ |m| \le R}} \frac{e^{i|m|}}{|m|^\lambda}$$

exists.
(c) Prove, however, that when $n - 1 - \frac{n-1}{n+1} < \lambda \le n - 1$, the limit in part (b) does not exist.
$\big[$*Hint:* Use Exercise 4.5.7 and Theorem 4.5.9. Part (b): For $R > 1$ use the identity

$$\sum_{\substack{m \in \mathbf{Z}^n \setminus \{0\} \\ |m| \le R}} \frac{e^{i|m|}}{|m|^\lambda} = \sum_{k=0}^{[R - \frac{1}{2}] - 1} \left(\sum_{\substack{m \in \mathbf{Z}^n \\ k + \frac{1}{2} < |m| \le k + \frac{3}{2}}} \frac{e^{i|m|}}{|m|^\lambda} \right) + \left\{ \sum_{\substack{m \in \mathbf{Z}^n \\ [R - \frac{1}{2}] + \frac{1}{2} < |m| \le R}} \frac{e^{i|m|}}{|m|^\lambda} \right\}.$$

Notice that the main term in the first sum on the right is telescoping. Part (c): Show that $\lim_{R \to \infty} \sum_{\substack{m \in \mathbf{Z}^n \setminus \{0\} \\ R < |m| \le R+1}} \frac{e^{i|m|}}{|m|^\lambda}$ does not tend to zero.$\big]$

HISTORICAL NOTES

The boundedness of the conjugate function on the circle (Theorem 4.1.7) and, hence, the L^p convergence of one-dimensional Fourier series was announced by Riesz in [292], but its proof appeared a little later in [293]. The proof of Theorem 4.1.7 in the text is attributed to S. Bochner. Luzin's conjecture [235] on almost everywhere convergence of the Fourier series of continuous functions was announced in 1913 and settled by Carleson [54] in 1965 for the more general class of square summable functions (Theorem 4.3.14). Carleson's theorem was later extended by Hunt [165] for the class of L^p functions for all $1 < p < \infty$ (Theorem 4.3.15). Sjölin [325] sharpened this result by showing that the Fourier series of functions f with $|f|(\log^+ |f|)(\log^+ \log^+ |f|)$ integrable over $\mathbf{T}^1$ converge almost everywhere. Antonov [5] improved Sjölin's result by extending it to functions f with $|f|(\log^+ |f|)(\log^+ \log^+ \log^+ |f|)$ integrable over $\mathbf{T}^1$. One should also consult the related results of Soria [330] and Arias de Reyna [9]. The book [10] of Arias de Reyna contains a historically motivated comprehensive study of topics related to the Carleson–Hunt theorem. Counterexamples due to Konyagin [200] show that Fourier series of functions f with $|f|(\log^+ |f|)^{\frac{1}{2}}(\log^+ \log^+ |f|)^{-\frac{1}{2}-\varepsilon}$ integrable over $\mathbf{T}^1$ may diverge when $\varepsilon > 0$. Examples of continuous functions whose Fourier series diverge exactly on given sets of measure zero are given in Katznelson [189] and Kahane and Katznelson [183].

The extension of the Carleson–Hunt theorem to higher dimensions for square summability of Fourier series (Theorem 4.3.16) is a rather straightforward consequence of the one-dimensional result and was independently obtained by Fefferman [112], Sjölin [325], and Tevzadze [359]. An example showing that the circular partial sums of a Fourier series may not converge in $L^p(\mathbf{T}^n)$ for $n \geq 2$ and $p \neq 2$ was obtained by Fefferman [113]. This example also shows that there exist L^p functions on $\mathbf{T}^n$ for $n \geq 2$ whose circular partial sums do not converge almost everywhere when $1 \leq p < 2$. Indeed, if the opposite happened, then the maximal operator $f \to \sup_{N \geq 0} |\widetilde{D}(n, N) * f|$ would have to be finite a.e. for all $f \in L^p(\mathbf{T}^n)$, and by Stein's theorem [335] it would have to be of weak type (p, p) for some $1 < p < 2$. But this would contradict Fefferman's counterexample on L^{p_1} for some $p < p_1 < 2$. On the other hand, almost everywhere is valid for the square partial sums of functions f with $|f|(\log^+ |f|)^n(\log^+ \log^+ \log^+ |f|)$ integrable over $\mathbf{T}^n$, as shown by Antonov [6]; see also Sjölin and Soria [327].

The development of the complex methods in the study of Fourier series was pioneered by the Russian school, especially Luzin and his students Kolmogorov, Menshov, and Privalov. The existence of an integrable function on $\mathbf{T}^1$ whose Fourier series diverges almost everywhere (Theorem 4.2.1) is due to Kolmogorov [195]. An example of an integrable function whose Fourier series diverges everywhere was also produced by Kolmogorov [198] three years later. Localization of the Bochner–Riesz means at the critical exponent $\alpha = \frac{n-1}{2}$ fails for L^1 functions on $\mathbf{T}^n$ (see Bochner [30]) but holds for functions f such that $|f| \log^+ |f|$ is integrable over $\mathbf{T}^n$ (see Stein [333]). The latter article also contains the L^p boundedness of the maximal Bochner–Riesz operator $\sup_{R>0} |B_R^\alpha(f)|$ for $1 < p < \infty$ when $\alpha > |\frac{1}{p} - \frac{1}{2}|$. Proposition 4.1.9 is due to Stein [331] and Theorem 4.2.5 is also due to Stein [335]. The technique that involves the points for which the set $\{|x - m| : m \in \mathbf{Z}^n\}$ is linearly independent over the rationals was introduced by Bochner [30].

Transference of regulated multipliers originated in the article of de Leeuw [94]. The methods of transference in Section 4.3 were beautifully placed into the framework of a general theory by Coifman and Weiss [70]. The key Lemma 4.3.8 is attributed to G. Weiss. Transference of maximal multipliers (Theorem 4.3.12) was first obtained by Kenig and Tomas [192] and later elaborated by Asmar, Berkson, and Gillespie [12], [13].

Paraphrasing Pappus of Alexandria, *bees know than a hexagon will hold more honey than a triangle or square of the same length, but people claim a greater share of wisdom knowing that the circle of a given length holds the maximum area among all geometric shapes of equal perimeter.* This reflection captures the isoperimetric inequality, which was first recorded by Pappus in the fourth century A.D. and was credited it to Zenodorus (second century B.C.). Archimedes also studied the problem, but his work on the subject, like the original writings of Zenodorus, has been lost. Rigorous modern-day proofs of this inequality can be traced to J. Steiner, K. Weierstrass, and

F. Edler, whose methods are based in geometry and calculus. The proof in the text is due to A. Hurwitz. On the history of the isoperimetric inequality see [322].

The mean square error for lattice points (Exercise 4.5.4) is due to Kendall [191] while the more delicate pointwise asymptotic formula of Theorem 4.5.9 was obtained by Landau [212]. Using Landau's formula Pinsky, Stanton, and Trapa [284] showed that the spherical partial sums of the Fourier series of the characteristic function of a sufficiently small ball in $\mathbf{T}^n$ converge at the center of the ball if and only if the dimension n is strictly less than three; this property is valid for the characteristic function of any ball as shown in Pinsky [283].