Chapter 7

Beams, Frames, Arches

7

7 Beams, Frames, Arches

Objectives: Beams are among the most important elements in structural engineering. In this chapter it is explained how the internal forces in a beam can be made accessible to calculation.

The *normal force*, the *shear force* and the *bending moment* are introduced. Students will learn how to determine these quantities with the aid of the conditions of equilibrium. In addition, they will learn how to correctly apply the differential relationships between external loading and internal forces.

7.1 Stress Resultants

Beams are slender structural members that offer resistance to bending. They are among the most important elements in engineering. In this chapter the internal forces in structures composed of beams are analyzed. Knowledge of these internal forces is important in order to be able to determine the load-bearing capacity of a beam, to compute the area of the cross-section required to sustain a given load, or to compute the deformation (see Volume 2). For the sake of simplicity, the following discussion is limited to statically determinate plane problems, as indicated in Fig. 7.1a.

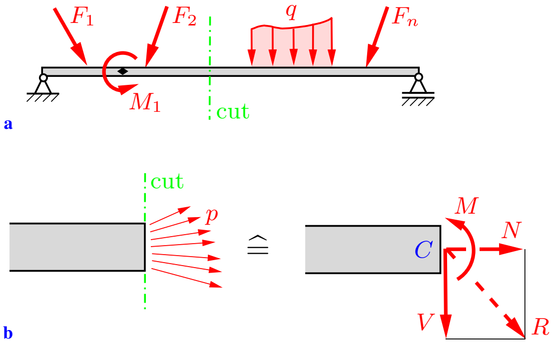


Fig. 7.1

According to Section 1.4, the internal forces in a beam can be made visible and thus accessible to calculation with the aid of a free-body diagram. Accordingly, we pass an imaginary section perpendicularly to the axis of the beam. The internal forces p (forces per unit area) acting at the cross-section are distributed across the cross-sectional area (Fig. 7.1b). Their intensity is called *stress* (see Volume 2). The actual distribution of the forces across the cross-section is unknown; it will be determined in Volume 2, Chapter 4. However, it was shown in Section 3.1.3 that any force system can be replaced by a resultant force R acting at an arbitrary point C and a corresponding couple $M^{(C)}$. When carrying this out, we choose the centroid C of the cross-sectional area as the reference point of the reduction. The reason for this particular choice will become apparent in Volume 2. In the following, we adopt the common practice of omitting the superscript C that refers to the reference point: instead of $M^{(C)}$, we simply write M. The resultant force R is resolved into its components N (normal to the cross-section, in the direction of the axis of the beam) and V (in the cross section, orthogonal to the axis of the beam). The quantities N, V and M are called the *stress resultants*. In particular,

> N is called the *normal force*, V is the *shear force* and M is the *bending moment*.

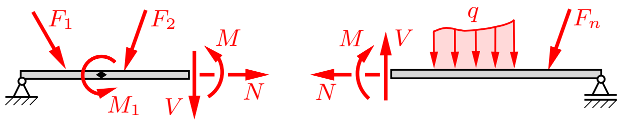


Fig. 7.2

In order to determine the stress resultants, the beam may be divided by a cut into two segments (method of sections). A free-body diagram of each part of the beam will include all of the forces acting on the respective part, i.e., the applied loads (forces and couples), the support reactions and the stress resultants acting at the cut sections. Because of Newton's third law (action equals reaction) they act in opposite directions at the two faces of the segments of the beam (compare Fig. 7.2). Since each part of the beam is in equilibrium, the three conditions of equilibrium for either part can be used to compute the three unknown stress resultants.

Before we can provide examples for the determination of the stress resultants, a sign convention must be introduced. Consider the two adjoining portions of the same beam shown in Fig. 7.3. The coordinate x coincides with the direction of the axis of the beam and points to the right; the coordinate z points downward. Accordingly, the y-axis is directed out of the x, z-plane (right-hand system, see Appendix A.1). By cutting the beam, a left-hand face and a right-hand face are obtained (see Fig. 7.3). They are characterized by a normal vector $\boldsymbol{n}$ that points outward from the interior of the beam. If the vector $\boldsymbol{n}$ points in the positive (negati-

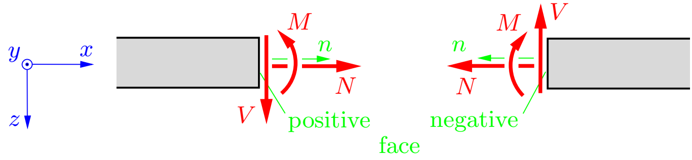


Fig. 7.3

ve) direction of the x-axis, the corresponding face is called positive (negative). The following sign convention is adopted:

> *Positive* stress resultants at a *positive (negative)* face point in the *positive (negative)* directions of the coordinates.

Here, the bending moment M has to be interpreted as a moment vector pointing in the direction of the y-axis (positive direction according to the right-hand rule). Fig. 7.3 shows the stress resultants with their positive directions. In the following examples, we shall strictly adhere to this sign convention. It should be noted, however, that different sign conventions exist.

In the case of a horizontal beam, very often only the x-coordinate is given. Then it is understood that the z-axis points downward. Sometimes it is convenient to use a coordinate system where the x-axis points to the left (instead of to the right) with the z-axis again pointing downward (compare Example 7.4). Then the y-axis is directed into the plane of the paper. In this case, only the positive direction of the shear force V according to Fig. 7.3 is reversed, the positive directions of M and N remain unchanged.

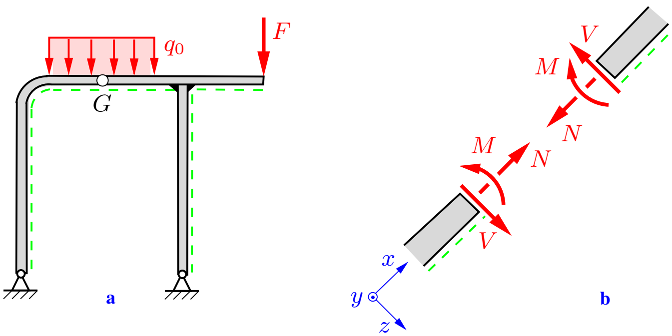


Fig. 7.4

The sign convention for frames and arches may be introduced by drawing a dashed line at one side of each part of the system (Fig. 7.4a). The side with the dashed line can then be interpreted as the "underneath side" of the respective part and the coordinate system can be chosen as the one for a beam: x-axis in the direction of the dashed line, z-axis toward the dashed line ("downward"). Fig. 7.4b shows the stress resultants with their positive directions.

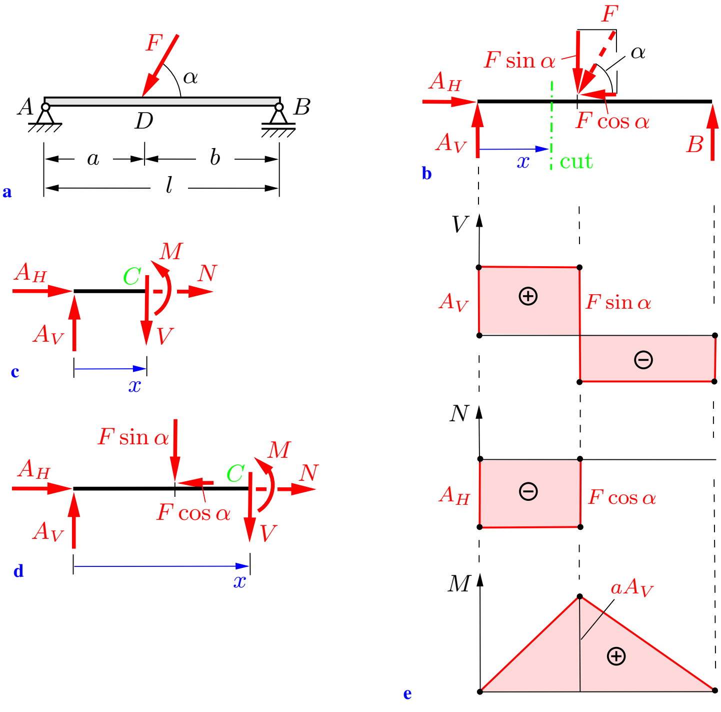


Fig. 7.5

We will now determine the stress resultants for the simply supported beam shown in Fig. 7.5a. First, the support reactions are computed from the equilibrium conditions for the free-body diagram of the beam as a whole (Fig. 7.5b). With $F_V = F \sin\alpha$ and $F_H = F \cos\alpha$ we obtain

$$\begin{aligned}
\rightarrow:&\quad A_H - F_H = 0 \quad\rightarrow\quad A_H = F_H\,,\\
\overset{\curvearrowleft}{A}:&\quad l\,B - a\,F_V = 0 \quad\rightarrow\quad B = \frac{a}{l}\,F_V\,,\\
\overset{\curvearrowleft}{B}:&\quad -\,l\,A_V + b\,F_V = 0 \quad\rightarrow\quad A_V = \frac{b}{l}\,F_V\,.
\end{aligned}$$

In a second step, a coordinate system is chosen and the beam is divided into two parts by a cut at an arbitrary position x between the points A and D ($0 < x < a$, Fig. 7.5b). The free-body diagram of the left-hand segment of the beam is depicted in Fig. 7.5c. We shall always represent the stress resultants at the cut section with their positive directions. If the analysis yields a negative value for a stress resultant, the resultant acts in opposite direction in reality. The equilibrium conditions for this part of the beam yield

$$\begin{aligned}
\rightarrow:&\quad A_H + N = 0 \quad\rightarrow\quad N = -\,A_H = -\,F_H\,,\\
\uparrow:&\quad A_V - V = 0 \quad\rightarrow\quad V = A_V = \frac{b}{l}\,F_V\,,\\
\overset{\curvearrowright}{C}:&\quad x\,A_V - M = 0 \quad\rightarrow\quad M = x\,A_V = x\,\frac{b}{l}\,F_V\,.
\end{aligned}$$

Now we cut the beam at an arbitrary position x between points D and B ($a < x < l$). The free-body diagram of the left-hand part is depicted in Fig. 7.5d. From the equilibrium conditions we obtain

$$\begin{aligned}
\rightarrow:&\quad A_H - F_H + N = 0 \rightarrow N = F_H - A_H = 0\,,\\
\uparrow:&\quad A_V - F_V - V = 0 \rightarrow V = A_V - F_V = \frac{b-l}{l}\,F_V\\
&\qquad\qquad = -\,\frac{a}{l}\,F_V = -\,B\,,\\
\overset{\curvearrowright}{C}:&\quad x\,A_V - (x-a)F_V - M = 0 \rightarrow M = x\,A_V - (x-a)F_V\\
&\qquad\qquad = \left(1 - \frac{x}{l}\right)a\,F_V\,.
\end{aligned}$$

It should be noted that the equilibrium conditions for the corresponding right-hand parts yield the same results for the stress resultants. Usually, the part of the beam with a smaller number of forces will be chosen, since it allows for a simpler calculation of the results.

The stress resultants are functions of the coordinate x; they are shown graphically in Fig. 7.5e. These graphs are called the *shear-force*, *normal-force* and *bending-moment diagrams*, respectively. The shear-force and normal-force diagrams display a jump discontinuity at point D (point of application of the external force F). The jumps have the magnitude of the components F_V and F_H, respectively. The bending-moment diagram shows a slope discontinuity (kink) at D. The maximum bending moment is located at D. It is usually the most important value in the design of a beam (see Volume 2).

This example shows that, in order to determine the stress resultants, the beam may be sectioned at an arbitrary position x between two concentrated loads (external loads or support reactions). The process of sectioning has to be performed for each such span. Since there are discontinuities at the points of application of a concentrated load, these points should not be chosen for a section (compare however Section 7.2.5).

7.2

7.2 Stress Resultants in Straight Beams

Beams are usually subjected to forces perpendicular to their axes. If there are no components of forces (external forces or support reactions) in the direction of the axis of a beam, the normal force vanishes: $N = 0$. In the following subsections, we shall concentrate on such problems.

7.2.1 Beams under Concentrated Loads

In order to determine the stress resultants V and M we choose a coordinate system and imagine the beam cut at an arbitrary position x. The stress resultants are represented with their positive directions in the free-body diagrams; they can be computed from the equilibrium conditions for either portion of the beam. The results of the analysis are usually presented in a shear-force and a bending-moment diagram.

As an alternative to this elementary method, there exists another method to determine the stress resultants. It is based on the differential relationships between the load and the stress resultants and will be presented in the Sections 7.2.2 to 7.2.4.

For the sake of simplicity, we restrict the discussion in the following to beams that are subjected to concentrated loads and to couples. As an example we consider the simply supported beam shown in Fig. 7.6a.

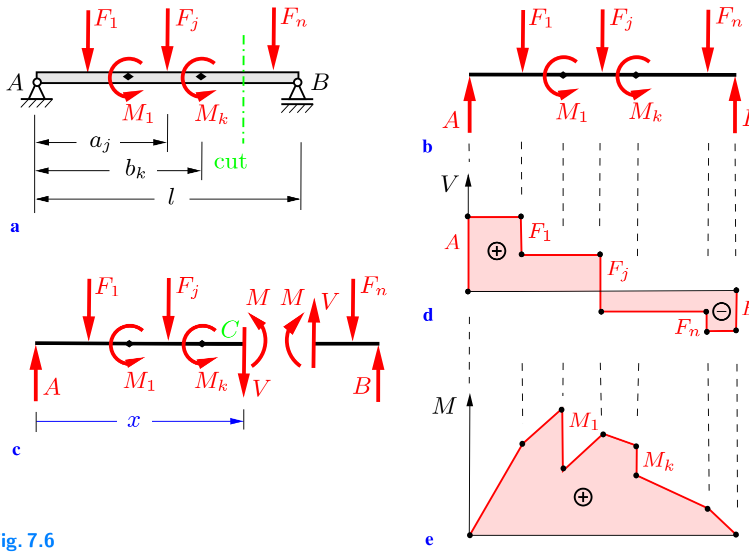


Fig. 7.6

The support reactions are obtained from the equilibrium conditions for the free-body diagram of the beam as a whole (Fig. 7.6b):

$$\overset{\curvearrowleft}{A}: \; l\,B - \sum a_i\,F_i + \sum M_i = 0 \;\rightarrow\; B = \frac{1}{l}\left[\sum a_i\,F_i - \sum M_i\right],$$

$$\overset{\curvearrowleft}{B}: \; -l\,A + \sum (l - a_i)F_i + \sum M_i = 0$$

$$\rightarrow\; A = \frac{1}{l}\left[\sum (l - a_i)F_i + \sum M_i\right].$$

Now, let us imagine the beam cut at an arbitrary position x (Fig. 7.6c). Since the normal force is equal to zero, it is not shown in the free-body diagram. The equilibrium conditions for the left-hand portion of the beam,

$$\uparrow: \quad A - \sum F_i - V = 0\,,$$

$$\overset{\curvearrowleft}{C}: \quad -x\,A + \sum (x - a_i)F_i + \sum M_i + M = 0\,,$$

yield the shear force and the bending moment:

$$V = A - \sum F_i\,, \tag{7.1}$$

$$M = x\,A - \sum (x - a_i)F_i - \sum M_i\,. \tag{7.2}$$

The summations in (7.1) and (7.2) include only the forces F_i and the couples M_i acting at the left-hand portion of the beam.

The stress resultants can also be computed with the equilibrium conditions for the right-hand part of the beam. Usually, the part of the beam that allows for a simpler calculation of the results is chosen.

The shear-force diagram is shown in Fig. 7.6d. According to (7.1), the shear force is piecewise constant. The shear-force diagram has jump discontinuities at the points of application of the concentrated forces F_i. The magnitude of a jump is equal to the magnitude of the respective force.

According to (7.2), the bending moment (Fig. 7.6e) is a piecewise linear function of the coordinate x. The diagram displays slope discontinuities (kinks) at the points of application of the forces F_i and jump discontinuities (magnitudes M_i) at the points of application of the external couples M_i. The supports A and B (hinge and roller support) cannot exert a moment. Therefore, the bending moment is zero at the end-points of a simply supported beam.

A relationship exists between the bending moment and the shear force. If the derivative of (7.2) with respect to x is calculated and (7.1) is applied, we obtain

$$\frac{\mathrm{d}M}{\mathrm{d}x} = A - \sum F_i = V\,. \tag{7.3}$$

The slopes of the straight lines in the bending-moment diagram are thus given by the corresponding values of the shear force.

Example 7.1 The simply supported beam in Fig. 7.7a is subjected to the three forces $F_1 = F$, $F_2 = 2\,F$ and $F_3 = -F$. E7.1

Draw the shear-force and bending-moment diagrams.

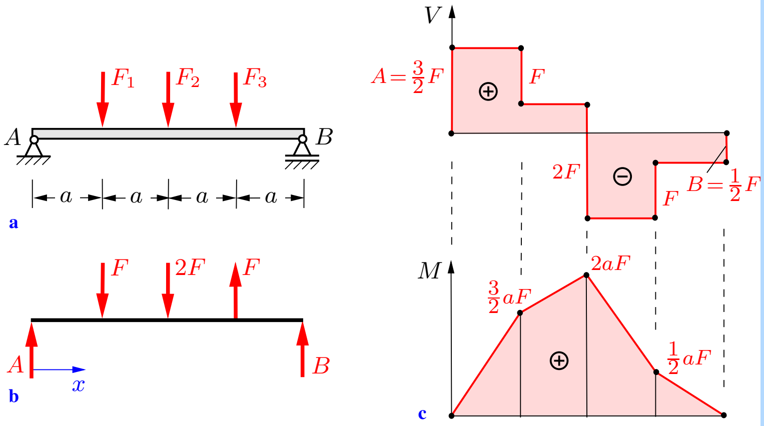


Fig. 7.7

Solution As a first step we draw the free-body diagram of the entire beam (Fig. 7.7b) and compute the support reactions A and B. The equilibrium conditions yield

$$\overset{\curvearrowleft}{A}: \quad -a\,F - 2\,a\,2\,F + 3\,a\,F + 4\,a\,B = 0 \quad \rightarrow \quad B = \frac{1}{2}\,F\,,$$

$$\overset{\curvearrowleft}{B}: \quad -4\,a\,A + 3\,a\,F + 2\,a\,2\,F - a\,F = 0 \quad \rightarrow \quad A = \frac{3}{2}\,F\,.$$

In the next step, we pass imaginary sections at arbitrary positions x in each span between two concentrated loads. The equilibrium of the forces for the left-hand parts of the beam yields the shear force (the corresponding free-body diagrams are not shown in Fig. 7.7):

$$V = A = 3F/2 \qquad \text{for} \quad 0 < x < a\,,$$

$$V = A - F = F/2 \qquad \text{for} \quad a < x < 2\,a\,,$$

$$V = A - F - 2\,F = -\,3F/2 \qquad \text{for} \quad 2\,a < x < 3\,a\,,$$

$$V = A - F - 2\,F + F = -\,F/2 \qquad \text{for} \quad 3\,a < x < 4\,a\,.$$

The bending moment is obtained from the equilibrium of the moments:

$$M = xA = \tfrac{3}{2}\,xF \qquad \text{for} \quad 0 \le x \le a\,,$$

$$M = xA - (x-a)F = (a + \tfrac{1}{2}\,x)F \qquad \text{for} \quad a \le x \le 2a\,,$$

$$M = xA - (x-a)F - (x-2a)2F = (5a - \tfrac{3}{2}\,x)F \qquad \text{for} \quad 2a \le x \le 3a\,,$$

$$M = xA - (x-a)F - (x-2a)2F + (x-3a)F = (2a - \tfrac{1}{2}\,x)F \qquad \text{for} \quad 3a \le x \le 4a\,.$$

It should be noted that it would have been simpler to use the right-hand parts instead of the left-hand parts for $x > 2a$.

The shear-force and bending-moment diagrams are shown in Fig. 7.7c. The bending-moment diagram has a positive (negative) slope in the regions of a positive (negative) shear force.

The values of the stress resultants at the right-hand end of the beam ($x = 4\,a$) may serve as checks:

- the shear force has a jump discontinuity of magnitude B (the diagram should close at $x = 4a$),
- the bending moment is zero (roller support at the end of the beam).

Since the beam in Fig. 7.7a is simply supported, the results for V and M are already given by (7.1) and (7.2).

E7.2

Example 7.2 Determine the shear-force and bending-moment diagrams for the cantilever beam shown in Fig. 7.8a.

Solution First, the support reactions are calculated with the aid of the free-body diagram of the whole beam (Fig. 7.8b). The equilibrium conditions yield

$$\uparrow: \quad A - F = 0 \qquad \rightarrow \qquad A = F\,,$$

$$\stackrel{\curvearrowleft}{A}: \quad -M_A + M_0 - l\,F = 0 \qquad \rightarrow \qquad M_A = M_0 - l\,F = l\,F\,.$$

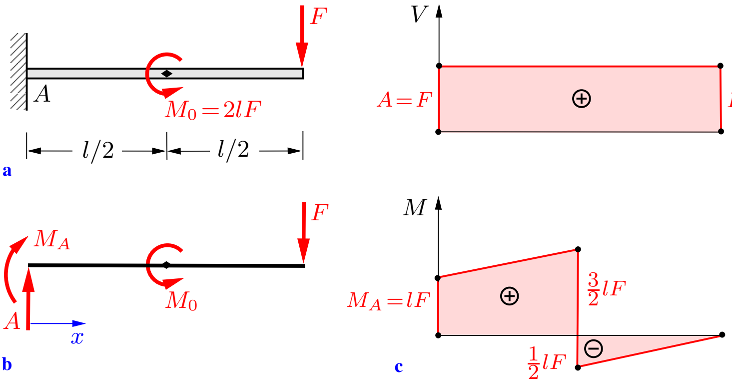


Fig. 7.8

In order to obtain the shear force, we section the beam at an arbitrary position x (since there is no concentrated load acting between both ends of the beam, only one region of x needs to be considered). The shear force follows from the equilibrium condition of the forces in the vertical direction:

$$\underline{\underline{V}} = A = \underline{\underline{F}} \quad \text{for} \quad 0 < x < l\,.$$

Because of the couple M_0 at the center of the beam, two regions of x must be considered when the bending moment is calculated. Accordingly, we pass a cut in the region given by $0 < x < l/2$ and another one in the span $l/2 < x < l$. The equilibrium of the moments yields

$$\underline{\underline{M}} = M_A + x\,A = \underline{\underline{(l+x)F}} \qquad \text{for} \quad 0 < x < \frac{l}{2}\,,$$

$$\underline{\underline{M}} = M_A + x\,A - M_0 = \underline{\underline{(x-l)F}} \quad \text{for} \quad \frac{l}{2} < x \leq l\,.$$

The shear-force and bending-moment diagrams are shown in Fig. 7.8c. The shear force is constant over the entire length of the beam. The bending moment is a linear function of the coordinate x and has a jump of magnitude $M_0 = 2\,l\,F$ at the point of application ($x = l/2$) of the external couple M_0. The two straight lines in the regions $x < l/2$ and $x > l/2$ have the same slope since the shear force has the same value in both regions (see (7.3)).

It should be noted that in this example the support reactions need not be calculated in order to determine the shear force and the bending moment. If we apply the equilibrium conditions to the right-hand portions of the cut beam, the stress resultants are obtained immediately. The support reactions A and M_A can then be found from the diagrams: they are equal to the shear force and the bending moment, respectively, at $x = 0$.

E7.3 **Example 7.3** Draw the diagrams of the stress resultants for the beam shown in Fig. 7.9a.

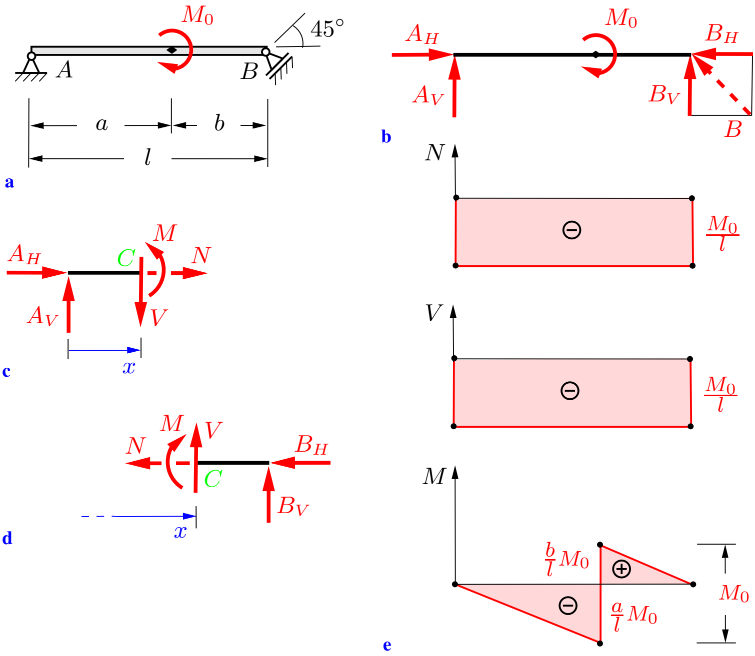


Fig. 7.9

Solution The support reactions follow from the conditions of equilibrium for the whole beam (Fig. 7.9b). With the components $B_H = B_V$ of the reaction force B (action line under 45° against the vertical axis) we obtain

$$\overset{\curvearrowright}{A}: \quad M_0 - l\,B_V = 0 \quad \rightarrow \quad B_V = B_H = \frac{M_0}{l}\,,$$

$$\overset{\curvearrowright}{B}: \quad l\,A_V + M_0 = 0 \quad \rightarrow \quad A_V = -\frac{M_0}{l}\,,$$

$$\rightarrow: \quad A_H - B_H = 0 \quad \rightarrow \quad A_H = \frac{M_0}{l}\,.$$

The forces A and B represent a couple with the moment M_0.

Since there is a discontinuity in the bending moment, two regions of x must be considered to describe it for the entire beam. First, we imagine the beam being cut in the region $x < a$. The equilibrium conditions for the left-hand portion of the beam (Fig. 7.9c) yield

$$\rightarrow: \quad A_H + N = 0 \quad \rightarrow \quad \underline{\underline{N}} = -A_H = \underline{\underline{-\frac{M_0}{l}}}\,,$$

$$\uparrow: \quad A_V - V = 0 \quad \rightarrow \quad \underline{\underline{V}} = A_V = \underline{\underline{-\frac{M_0}{l}}}\,,$$

$$\overset{\curvearrowleft}{C}: \quad -x\,A_V + M = 0 \quad \rightarrow \quad \underline{\underline{M}} = x\,A_V = \underline{\underline{-\frac{x}{l}\,M_0}}\,.$$

In order to obtain the stress resultants to the right of the applied couple, the beam is sectioned at a position $x > a$. Now it is simpler to use the free-body diagram of the right-hand portion of the beam (Fig. 7.9d). Notice the positive directions of the stress resultants (negative face!) in this diagram. The conditions of equilibrium yield

$$\leftarrow: \quad N + B_H = 0 \quad \rightarrow \quad \underline{\underline{N}} = -B_H = \underline{\underline{-\frac{M_0}{l}}}\,,$$

$$\uparrow: \quad V + B_V = 0 \quad \rightarrow \quad \underline{\underline{V}} = -B_V = \underline{\underline{-\frac{M_0}{l}}}\,,$$

$$\overset{\curvearrowright}{C}: \quad M - (l-x)B_V = 0 \quad \rightarrow \quad \underline{\underline{M}} = (l-x)B_V = \underline{\underline{\frac{l-x}{l}\,M_0}}\,.$$

The stress resultants are shown graphically in Fig. 7.9e. The moment diagram has a jump discontinuity at the point of application of the applied couple ($x = a$). The two straight lines in the regions $x < a$ and $x > a$ have the same slope, since the shear force has the same value in both regions (see also (7.3)). The normal force is induced by the support reactions and is constant in the entire beam.

It should be noted that the support reactions are independent of the point of application of the applied couple. The bending-moment diagram, however, depends on this point.

7.2.2 Relationship between Loading and Stress Resultants

A relationship between the shear force V and the bending moment M for beams under concentrated loads has already been given in (7.3). This result is now extended to a beam that is subjected to a load $q(x)$ (force per unit length) that varies continuously over the length of the beam (Fig. 7.10a). Fig. 7.10b shows the free-body diagram of a beam element of infinitesimal length $\mathrm{d}x$. The load q may be considered to be constant over the length $\mathrm{d}x$ since the effect of any change of q disappears in the limit $\mathrm{d}x \to 0$ (compare Section 4.1). The distributed load is replaced by its resultant $\mathrm{d}F = q\,\mathrm{d}x$. The shear force V and the bending moment M act at the location x. They are drawn in their positive directions (negative face!). Proceeding to the location $x + \mathrm{d}x$, the stress resultants have changed by an amount $\mathrm{d}V$ and $\mathrm{d}M$, respectively, to the values $V + \mathrm{d}V$ and $M + \mathrm{d}M$. They are also shown with their positive directions. The conditions of equilibrium yield

$$\uparrow: \quad V - q\,\mathrm{d}x - (V + \mathrm{d}V) = 0 \quad \to \quad q\,\mathrm{d}x + \mathrm{d}V = 0\,, \qquad (7.4)$$

$$\overset{\curvearrowleft}{C}: \quad -M - \mathrm{d}x\,V + \frac{\mathrm{d}x}{2}\,q\,\mathrm{d}x + M + \mathrm{d}M = 0$$

$$\to \quad -V\,\mathrm{d}x + \mathrm{d}M + \frac{1}{2}\,q\,\mathrm{d}x \cdot \mathrm{d}x = 0\,. \qquad (7.5)$$

From (7.4) we obtain

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -\,q\,. \qquad (7.6)$$

Thus, the slope of the shear-force diagram is equal to the negative intensity of the applied loading.

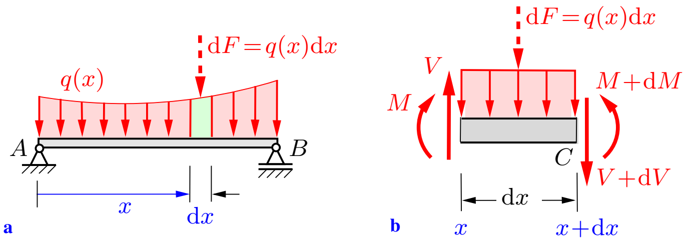


Fig. 7.10

The term in (7.5) containing $\mathrm{d}x \cdot \mathrm{d}x$ is "small of higher order" compared with $\mathrm{d}x$ or $\mathrm{d}M$. Therefore, it vanishes in the limit $\mathrm{d}x \to 0$ and (7.5) reduces to

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V\,. \tag{7.7}$$

The derivative of the bending moment with respect to x is equal to the shear force. This result is already known from the case of beams under concentrated forces (see (7.3)). It should be noted that the algebraic signs in (7.6) and (7.7) result from the sign convention for the stress resultants.

If (7.7) is differentiated and (7.6) introduced, we obtain

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = -\,q\,. \tag{7.8}$$

The differential relations (7.6) and (7.7) may be used, for example, to determine qualitatively the stress resultants and can also serve as checks. For example, if $q = \text{const}$, then the shear force is a linear function of x according to (7.6) and the bending moment is represented by a quadratic parabola according to (7.7). The Table at the end of this section shows the relations between the loading and the stress resultants for several simple examples of q.

The most important value for the design of a beam is usually the magnitude of the maximum bending moment. The corresponding coordinate x for a relative maximum is characterized by the condition of a vanishing shear force (compare (7.7)). It should be noted, however, that the absolute maximum may be located at an end point of the beam or of a span (position of discontinuity).

q	V	M
0	constant	linear
constant	linear	quadratic parabola
linear	quadratic parabola	cubic parabola

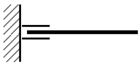
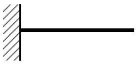
7.2.3 Integration and Boundary Conditions

The relations (7.6) and (7.7) may also be used to quantitatively determine the stress resultants for a given load $q(x)$. If we integrate (7.6) and (7.7), we obtain

$$V = -\int q \, \mathrm{d}x + C_1 \,, \tag{7.9}$$

$$M = \int V \, \mathrm{d}x + C_2 \,. \tag{7.10}$$

The constants of integration C_1 and C_2 can be calculated if the functions (7.9) and (7.10) for the stress resultants are evaluated at positions of x where the values of V or M are known. The corresponding equations are called *boundary conditions*. The following Table shows which stress resultant vanishes at a given support at the end of a beam. Statements $V \neq 0$ and/or $M \neq 0$ cannot be

support		V	M
pin		$\neq 0$	**0**
parallel motion		**0**	$\neq 0$
sliding sleeve		$\neq 0$	$\neq 0$
fixed end		$\neq 0$	$\neq 0$
free end		**0**	**0**

(7.11)

used as boundary conditions.

In contrast to the method of sections (see Section 7.2.1), the support reactions do *not* have to be computed in order to determine the stress resultants V and M: they are automatically obtained through the integration. If, on the other hand, some support reactions are known in advance, they can also be used to determine the constants of integration.

To illustrate the procedure, let us consider the three beams shown in Figs. 7.11a–c. They are subjected to the same load but have different supports. With $q = q_0 = \text{const}$, Equations (7.9) and (7.10) yield

$$V = -\, q_0\, x + C_1\,,$$
$$M = -\,\frac{1}{2}\, q_0\, x^2 + C_1\, x + C_2$$

for each beam.

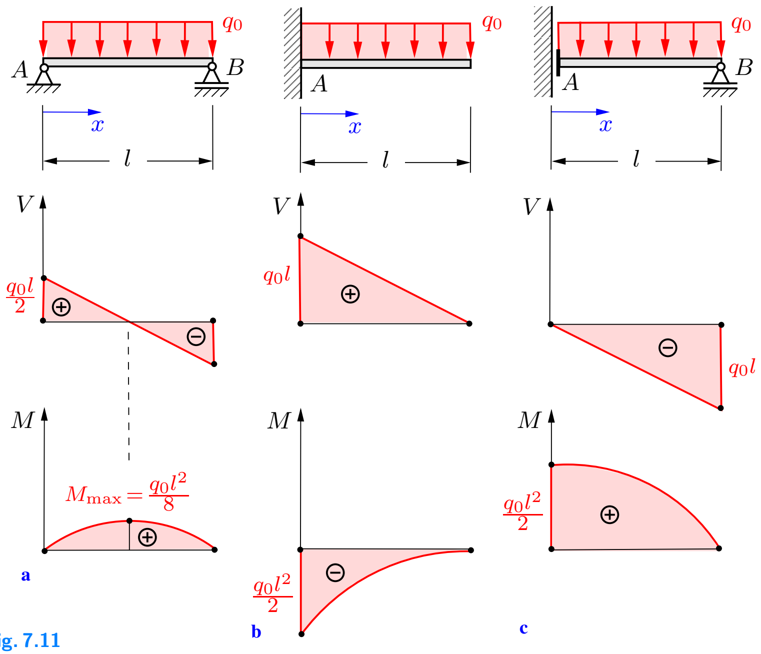


Fig. 7.11

The boundary conditions

$$\text{a) } M(0)=0\,,\quad M(l)=0\,,\qquad \text{b) } V(l)=0\,,\quad M(l)=0\,,\qquad \text{c) } V(0)=0\,,\quad M(l)=0$$

and thus the resulting constants of integration

$$\text{a) } 0=C_2\,,\qquad \text{b) } 0=-\,q_0\,l+C_1\,,\quad \text{c) } 0=C_1\,,$$

$$\text{a), b), c)}\quad 0=-\tfrac{1}{2}\,q_0\,l^2+C_1\,l+C_2$$

$$\rightarrow\quad \begin{cases} C_1=\tfrac{1}{2}\,q_0\,l\,, \\ C_2=0\,, \end{cases}\qquad \begin{cases} C_1=q_0\,l\,, \\ C_2=-\tfrac{1}{2}\,q_0\,l^2\,, \end{cases}\qquad \begin{cases} C_1=0\,, \\ C_2=\tfrac{1}{2}\,q_0\,l^2 \end{cases}$$

are different for each of the cases a) to c). This leads to the following stress resultants (Figs. 7.11a–c):

$$\text{a) } V=\frac{1}{2}\,q_0\,l\left(1-2\,\frac{x}{l}\right)\,,\qquad \text{b) } V=q_0\,l\left(1-\frac{x}{l}\right)\,,$$

$$M=\frac{1}{2}\,q_0\,l^2\,\frac{x}{l}\left(1-\frac{x}{l}\right)\,,\qquad M=-\frac{1}{2}\,q_0\,l^2\left(1-\frac{x}{l}\right)^2\,,$$

$$\text{c) } V=-\,q_0\,x\,,$$

$$M=\frac{1}{2}\,q_0\,l^2\left[1-\left(\frac{x}{l}\right)^2\right]\,.$$

The results are written in such a way that the terms in parentheses are dimensionless. The maximum bending moment $M_{\max}=q_0\,l^2/8$ for the simply supported beam is located at the center of the beam ($x=l/2$: $V=0$).

The support reactions can be taken from the diagrams; they are equal to the values of the stress resultants at the endpoints of the beams:

$$\text{a) } A=V(0)=\tfrac{1}{2}\,q_0\,l\,,\qquad \text{b) } A=V(0)=q_0\,l\,,$$

$$B=-\,V(l)=\tfrac{1}{2}\,q_0\,l\,,\qquad M_A=M(0)=-\,\tfrac{1}{2}\,q_0\,l^2\,,$$

$$\text{c) } M_A=M(0)=\tfrac{1}{2}\,q_0\,l^2\,,$$

$$B=-\,V(l)=q_0\,l\,.$$

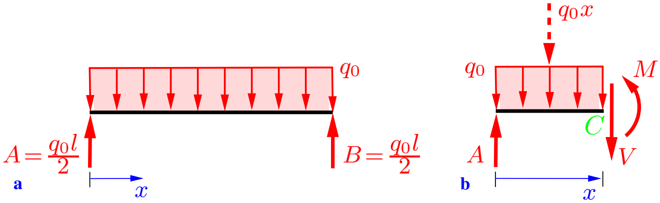


Fig. 7.12

The shear-force and bending-moment diagrams can also be obtained with the method of sections. In order to explain this method in the case of a distributed load, the beam in Fig. 7.11a is reconsidered. In a first step, the support reactions have to be calculated (Fig. 7.12a). Next, the beam is sectioned at an arbitrary position x (Fig. 7.12b). Then we replace the distributed load q_0 by its resultant $q_0 x$ (notice that the distributed load must not be replaced by its resultant for the entire beam, i.e., before the beam is divided by the cut). The equilibrium conditions for the left-hand portion of the beam (Fig. 7.12b) yield the stress resultants

$$\uparrow: \quad A - q_0\, x - V = 0$$

$$\rightarrow \quad V = A - q_0\, x = \frac{1}{2}\, q_0\, l \left(1 - 2\,\frac{x}{l}\right),$$

$$\overset{\curvearrowleft}{C}: \quad -x\,A + \frac{1}{2}\, x\, q_0\, x + M = 0$$

$$\rightarrow \quad M = x\,A - \frac{1}{2}\, q_0\, x^2 = \frac{1}{2}\, q_0\, l^2\, \frac{x}{l}\left(1 - \frac{x}{l}\right).$$

This method is recommended only if the resultant of the distributed load acting at the cut beam and its line of action can easily be given.

Example 7.4 The cantilever beam in Fig. 7.13a is subjected to a triangular line load. **E7.4**

Determine the stress resultants through integration.

Solution In the coordinate system given in Fig. 7.13a, the load is described by the equation $q(x) = q_0(l - x)/l$. Integration leads to

$$V(x) = \frac{q_0}{2l}\,(l - x)^2 + C_1\,, \quad M(x) = -\frac{q_0}{6l}\,(l - x)^3 + C_1\, x + C_2\,,$$

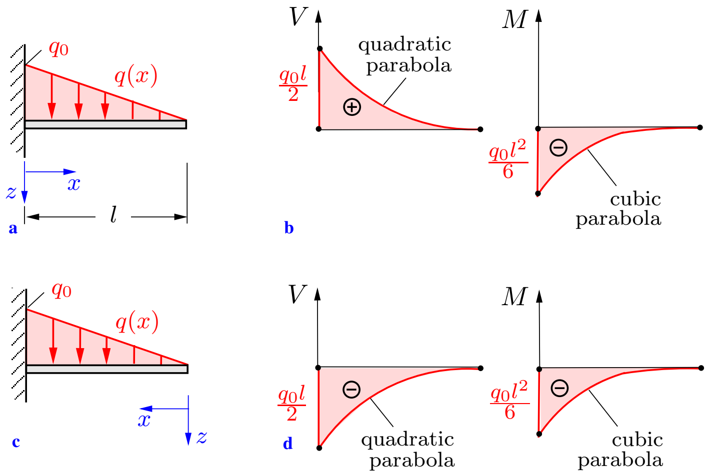


Fig. 7.13

(compare (7.9) and (7.10)). The boundary conditions $V(l) = 0$ and $M(l) = 0$ yield the constants of integration: $C_1 = 0$ and $C_2 = 0$. Hence,

$$\underline{\underline{V(x) = \frac{1}{2}\, q_0\, l \left(1 - \frac{x}{l}\right)^2}}, \quad \underline{\underline{M(x) = -\frac{1}{6}\, q_0\, l^2 \left(1 - \frac{x}{l}\right)^3}},$$

see Fig. 7.13b. The shear-force diagram has a vanishing slope at $x = l$ since the distributed load is zero at the free end of the beam ($V'(l) = -q(l) = 0$). Analogously, the bending-moment diagram has a vanishing slope at $x = l$ since the shear force is zero at the free end ($M'(l) = V(l) = 0$).

In this example, it would have been more convenient to introduce the coordinate system according to Fig. 7.13c. Here, the x-axis points to the left. In this coordinate system, the triangular load is described by the simpler equation $q(x) = q_0\, x/l$, and the integration and boundary conditions $V(0) = 0$ and $M(0) = 0$ yield the stress resultants (Fig. 7.13d)

$$\underline{\underline{V(x) = -\frac{1}{2}\, q_0\, l \left(\frac{x}{l}\right)^2}}, \quad \underline{\underline{M(x) = -\frac{1}{6}\, q_0\, l^2 \left(\frac{x}{l}\right)^3}}.$$

Note that with this choice of the coordinate system the algebraic sign of the shear force is reversed (see Section 7.1).

It should also be noted that the stress resultants can easily be determined with the method of sections.

7.2.4 Matching Conditions

Frequently, the load $q(x)$ is given through different functions of x in different portions of the beam (instead of one function for the entire length of the beam). In this case, the beam must be divided into several regions and the integration of (7.6) and (7.7) must be performed separately in each of these regions.

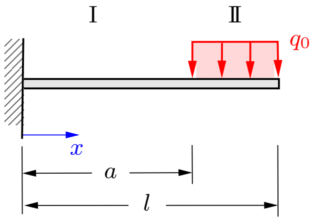


Fig. 7.14

To illustrate the method, the cantilever beam shown in Fig. 7.14 is considered. The load is given by

$$q(x) = \begin{cases} 0 & \text{for } 0 \le x < a\,, \\ q_0 & \text{for } a < x < l\,. \end{cases}$$

Integration in region I ($0 \le x < a$) and region II ($a < x < l$) yields

$$\begin{aligned} \text{I:}\quad q_{\mathrm{I}} &= 0\,, & \text{II:}\quad q_{\mathrm{II}} &= q_0\,, \\ V_{\mathrm{I}} &= C_1\,, & V_{\mathrm{II}} &= -\,q_0\,x + C_3\,, \\ M_{\mathrm{I}} &= C_1\,x + C_2\,, & M_{\mathrm{II}} &= -\,\tfrac{1}{2}\,q_0\,x^2 + C_3\,x + C_4\,. \end{aligned} \tag{7.12}$$

The *two* boundary conditions

$$V_{\mathrm{II}}(l) = 0\,, \quad M_{\mathrm{II}}(l) = 0 \tag{7.13}$$

are not sufficient to determine the *four* constants $C_1 - C_4$ of integration. Therefore, two additional equations must be used. They describe the behaviour of the stress resultants at the point $x = a$ (point of transition from region I to region II). These equations are called *matching conditions*.

The beam is not subjected to a concentrated force or to an external couple at $x = a$. Therefore, there are no jumps in the shear-force or bending-moment diagrams (since $\mathrm{d}V/\mathrm{d}x = -q$ and q has a jump at $x = a$, the shear-force diagram has a jump in the slope). Hence, the matching conditions are

$$V_{\mathrm{I}}(a) = V_{\mathrm{II}}(a)\,, \quad M_{\mathrm{I}}(a) = M_{\mathrm{II}}(a)\,. \tag{7.14}$$

Introducing (7.12) into the boundary conditions (7.13) and the matching conditions (7.14) yields the constants of integration:

$$C_1 = q_0(l-a)\,, \quad C_2 = -\tfrac{1}{2}\,q_0(l^2 - a^2)\,,$$
$$C_3 = q_0\,l\,, \quad C_4 = -\tfrac{1}{2}\,q_0\,l^2\,.$$

As a second example, we consider the beam in Fig. 7.15. It is subjected to a concentrated force F at $x = a$ and an external couple M_0 at $x = b$. Then the shear-force diagram exhibits a jump of magnitude F at $x = a$, whereas the bending-moment diagram is continuous at this point (it has a jump in the slope). The matching conditions for the values of the stress resultants at the transition from region I to region II are therefore

$$V_{\mathrm{II}}(a) = V_{\mathrm{I}}(a) - F\,, \quad M_{\mathrm{II}}(a) = M_{\mathrm{I}}(a)\,. \tag{7.15}$$

The external couple M_0 at $x = b$ causes a jump in the bending-moment diagram; the shear-force diagram is continuous. Hence, the matching conditions at the transition from region II to region

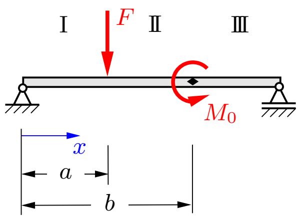
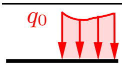
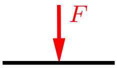
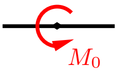


Fig. 7.15

III are given by

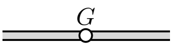
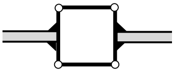
$$V_{\mathrm{III}}(b) = V_{\mathrm{II}}(b)\,, \quad M_{\mathrm{III}}(b) = M_{\mathrm{II}}(b) - M_0\,. \tag{7.16}$$

The following Table shows which loads cause jumps in the stress resultants or in the slopes of the diagrams.

Load	V	M
	kink	–
	jump	kink
	–	jump

If a beam has to be divided into n regions, the integration in each region yields a total of $2n$ constants of integration. They can be determined from $2n - 2$ matching conditions and 2 boundary conditions.

Let us now consider structures composed of several beams that are connected by joints. Since an internal pin cannot exert a moment, the bending moment is zero at the pin: $M = 0$. The shear force is in general not equal to zero at this point: $V \neq 0$. In contrast, at a parallel motion $V = 0$ and $M \neq 0$ are valid. These statements concerning the stress resultants at a connecting member are displayed in the following Table.

connecting member	V	M
	$\neq 0$	**0**
	**0**	$\neq 0$

(7.17)

If an internal pin or a parallel motion exists in a structure, a matching condition is replaced by *one* of the following conditions: bending moment or shear force equal to zero.

A division into regions at a connecting member is not necessary if no concentrated force or external couple acts on the element. Also, no division into regions is required in the case of a distributed load that is described by the same function to the left and to the right of the element.

If a beam must be divided into many regions, a system of equations with many unknowns has to be solved in order to obtain the constants of integration. Therefore, this method is recommended only for beams with very few regions.

E7.5

Example 7.5 A simply supported beam is subjected to a concentrated force and a triangular line load (Fig. 7.16a).

Determine the stress resultants.

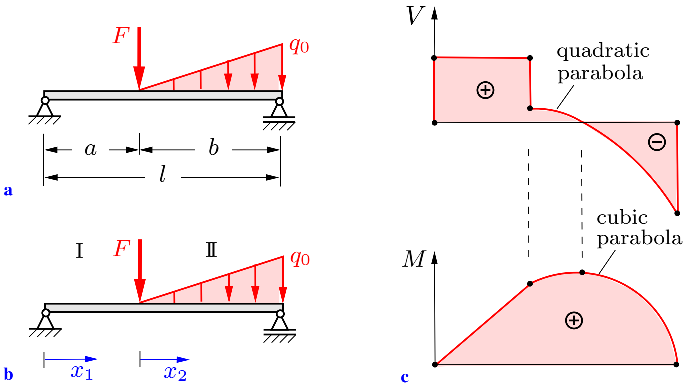


Fig. 7.16

Solution The beam is divided into the two regions I and II according to Fig. 7.16b. We use the coordinate x_1 in the region I and coordinate x_2 in region II (instead of the coordinate x for the entire length of the beam). Integration according to (7.9) and (7.10) in both regions yields

$$\text{I:}\quad q_{\rm I} = 0\,, \qquad\qquad \text{II:}\quad q_{\rm II} = q_0\,\frac{x_2}{b}\,,$$

$$V_{\rm I} = C_1\,, \qquad\qquad V_{\rm II} = -\,q_0\,\frac{x_2^2}{2\,b} + C_3\,,$$

$$M_{\rm I} = C_1\,x_1 + C_2\,, \qquad\qquad M_{\rm II} = -\,q_0\,\frac{x_2^3}{6\,b} + C_3\,x_2 + C_4\,.$$

The boundary conditions and the matching conditions are

$$M_{\mathrm{I}}(x_1 = 0) = 0\,, \quad M_{\mathrm{II}}(x_2 = b) = 0\,,$$
$$V_{\mathrm{II}}(x_2 = 0) = V_{\mathrm{I}}(x_1 = a) - F\,,\ M_{\mathrm{II}}(x_2 = 0) = M_{\mathrm{I}}(x_1 = a)\,.$$

They lead, after some calculation, to the constants of integration:

$$C_1 = \left(\frac{1}{6}\,q_0\,b + F\right)\frac{b}{l}\,, \qquad C_2 = 0\,,$$
$$C_3 = \left(\frac{1}{6}\,q_0\,b - \frac{a}{b}\,F\right)\frac{b}{l}\,, \quad C_4 = \left(\frac{1}{6}\,q_0\,b + F\right)\frac{a\,b}{l}\,.$$

Hence, we obtain the stress resultants (Fig. 7.16c)

$$\underline{\underline{V_{\mathrm{I}}}} = \left(\frac{1}{6}\,q_0\,b + F\right)\frac{b}{l}\,,$$
$$\underline{\underline{V_{\mathrm{II}}}} = -\,q_0\,\frac{x_2^2}{2\,b} + \left(\frac{1}{6}\,q_0\,b - \frac{a}{b}\,F\right)\frac{b}{l}\,,$$
$$\underline{\underline{M_{\mathrm{I}}}} = \left(\frac{1}{6}\,q_0\,b + F\right)\frac{b}{l}\,x_1\,,$$
$$\underline{\underline{M_{\mathrm{II}}}} = -\,q_0\,\frac{x_2^3}{6\,b} + \left(\frac{1}{6}\,q_0 b - \frac{a}{b}\,F\right)\frac{b}{l}\,x_2 + \left(\frac{1}{6}\,q_0\,b + F\right)\frac{a\,b}{l}\,.$$

Example 7.6 Determine the stress resultants for the compound beam in Fig. 7.17a. **E7.6**

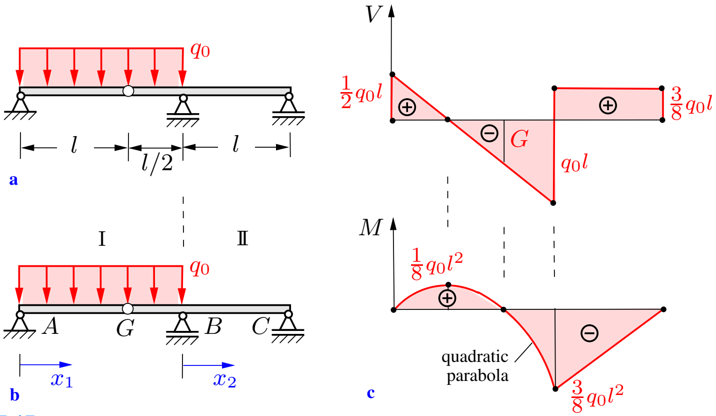


Fig. 7.17

Solution The beam must be divided into two regions at the location of support B (support reaction B and discontinuous load q!). A division into regions at the internal pin G is not required since $q = q_0$ to the left and to the right of the pin. We use the coordinates x_1 and x_2 in regions I and II, respectively (Fig. 7.17b). Integration leads to

$$\text{I:} \quad q_{\text{I}} = q_0 \,, \qquad\qquad \text{II:} \quad q_{\text{II}} = 0,$$

$$V_{\text{I}} = -\, q_0\, x_1 + C_1 \,, \qquad\qquad V_{\text{II}} = C_3 \,,$$

$$M_{\text{I}} = -\, \tfrac{1}{2}\, q_0\, x_1^2 + C_1\, x_1 + C_2 \,, \qquad\qquad M_{\text{II}} = C_3\, x_2 + C_4 \,.$$

The four conditions

$$M_{\text{I}}(x_1 = 0) = 0 \,, \quad M_{\text{II}}(x_2 = l) = 0 \quad \text{(boundary conditions)} \,,$$

$$M_{\text{I}}(x_1 = \tfrac{3}{2} l) = M_{\text{II}}(x_2 = 0) \quad \text{(matching condition)} \,,$$

$$M_{\text{I}}(x_1 = l) = 0 \quad \text{(zero bending moment at internal pin } G\text{)}$$

yield the four constants of integration:

$$C_1 = \tfrac{1}{2}\, q_0\, l \,, \quad C_2 = 0 \,, \quad C_3 = \tfrac{3}{8}\, q_0\, l \,, \quad C_4 = -\, \tfrac{3}{8}\, q_0\, l^2 \,.$$

Thus, we obtain the stress resultants (Fig. 7.17c)

$$\underline{\underline{V_{\text{I}}}} = -\, q_0\, x_1 + \frac{1}{2}\, q_0\, l \,, \qquad \underline{\underline{V_{\text{II}}}} = \frac{3}{8}\, q_0\, l \,,$$

$$\underline{\underline{M_{\text{I}}}} = -\, \frac{1}{2}\, q_0\, x_1^2 + \frac{1}{2}\, q_0\, l\, x_1 \,, \quad \underline{\underline{M_{\text{II}}}} = \frac{3}{8}\, q_0\, l (x_2 - l) \,.$$

As a check, the support reactions are taken from the shear-force diagram:

$$A = V_{\text{I}}(x_1 = 0) = \tfrac{1}{2}\, q_0\, l \,,$$

$$B = V_{\text{II}}(x_2 = 0) - V_{\text{I}}(x_1 = \tfrac{3}{2}\, l) = \tfrac{11}{8}\, q_0\, l \,,$$

$$C = -\, V_{\text{II}}(x_2 = l) = -\, \tfrac{3}{8}\, q_0\, l \,.$$

They are in equilibrium with the resulting external load $3\, q_0\, l/2$.

7.2.5 Pointwise Construction of the Diagrams

To apply the theory to practical problems, it is not always necessary to give the stress resultants as functions of x over the length of the beam. Frequently, it suffices to calculate the stress resultants at several specific points only. The values of the stress resultants at these points are then connected with the curves that are associated with the respective load.

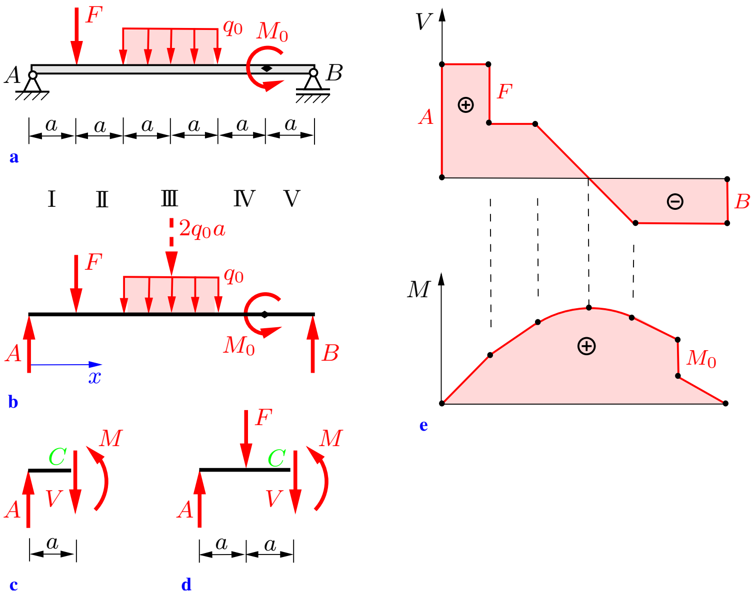


Fig. 7.18

To illustrate the method, the simply supported beam in Fig. 7.18a is considered. As a first step, we compute the support reactions with the conditions of equilibrium for the free-body diagram of the entire beam (Fig. 7.18b):

$$\overset{\curvearrowleft}{B}: \quad -6\,a\,A + 5\,a\,F + 3\,a\,2\,q_0\,a + M_0 = 0$$

$$\rightarrow \quad A = \frac{1}{6}\left(5\,F + 6\,q_0\,a + \frac{M_0}{a}\right),$$

$$\overset{\curvearrowleft}{A}: \quad -a\,F - 3\,a\,2\,q_0\,a + M_0 + 6\,a\,B = 0$$
$$\rightarrow \quad B = \frac{1}{6}\left(F + 6\,q_0\,a - \frac{M_0}{a}\right).$$

The stress resultants exhibit jumps, or jumps in the slopes, at $x = a$, $2\,a$, $4\,a$ and $5\,a$, respectively. At these specific points, the stress resultants are computed using the method of sections. If we cut the beam at $x = a$ immediately to the left of the force F (Fig. 7.18c), we obtain

$$\uparrow: \quad A - V = 0 \quad \rightarrow \quad V(a) = A = \frac{1}{6}\left(5\,F + 6\,q_0\,a + \frac{M_0}{a}\right)$$
$$\text{to the left of } F\,,$$
$$\overset{\curvearrowleft}{C}: \quad -a\,A + M = 0 \quad \rightarrow \quad M(a) = a\,A$$
$$= \frac{1}{6}(5\,a\,F + 6\,q_0\,a^2 + M_0)\,.$$

A cut at $x = 2a$ (Fig. 7.18d) yields

$$\uparrow: \quad A - F - V = 0 \quad \rightarrow \quad V(2\,a) = \frac{1}{6}\left(-F + 6\,q_0\,a + \frac{M_0}{a}\right),$$
$$\overset{\curvearrowleft}{C}: \quad -2\,a\,A + a\,F + M = 0$$
$$\rightarrow \quad M(2\,a) = \frac{1}{3}(2\,a\,F + 6\,q_0\,a^2 + M_0)\,.$$

Similarly, we find

$$V(4\,a) = \frac{1}{6}\left(-F - 6\,q_0\,a + \frac{M_0}{a}\right),$$
$$M(4\,a) = \frac{1}{3}\,(a\,F + 6\,q_0\,a^2 + 2\,M_0)\,,$$
$$V(5\,a) = V(4\,a)\,,$$
$$M(5\,a) = \frac{1}{6}\,(a\,F + 6\,q_0\,a^2 - M_0) \quad \text{to the right of } M_0\,.$$

The distributed load is zero in regions I, II, IV and V. Therefore, the shear force is constant in each of these regions. In region III, the shear force varies linearly since $q = q_0 =$ const. The shear-force diagram has a jump of magnitude F at $x = a$.

Accordingly, the bending moment varies linearly in the regions I, II, IV and V, and it is described by a quadratic parabola in region III (compare the Table in Section 7.2.2). The diagram shows a jump in the slope at $x = a$. Since the shear-force diagram is continuous at $x = 2\,a$ and $x = 4\,a$, the moment diagram has no jumps in the slopes at these points ($V = \mathrm{d}M/\mathrm{d}x$). The external couple M_0 causes a jump in the diagram at $x = 5\,a$.

Fig. 7.18e shows the diagrams of the stress resultants. The maximum bending moment is located at the position of the vanishing shear force.

Example 7.7 Draw the diagrams of the stress resultants for the structure in Fig. 7.19a ($a = 0.5\,\mathrm{m}$, $q_0 = 60\,\mathrm{kN/m}$, $F = 80\,\mathrm{kN}$, $M_0 = 10\,\mathrm{kNm}$). **E7.7**

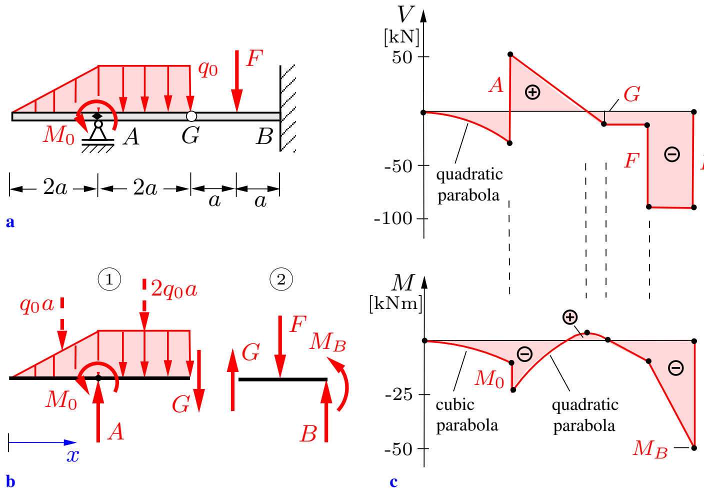


Fig. 7.19

Solution First, we compute the support reactions and the force in the internal pin G (the horizontal components are zero). The equilibrium conditions for the free-body diagrams of the two portions of the structure (Fig. 7.19b) yield

$$① \quad \overset{\curvearrowleft}{A}: \quad \frac{2}{3}\,a\,q_0\,a + M_0 - a\,2\,q_0\,a - 2\,a\,G = 0$$

$$\rightarrow \quad G = -\frac{2}{3}\,q_0\,a + \frac{M_0}{2\,a} = -\,10\,\text{kN}\,,$$

$$\overset{\curvearrowleft}{G}: \quad \frac{8}{3}\,a\,q_0\,a + M_0 - 2\,a\,A + a\,2\,q_0\,a = 0$$

$$\rightarrow \quad A = \frac{7}{3}\,q_0\,a + \frac{M_0}{2\,a} = 80\,\text{kN}\,,$$

$$② \ \overset{\curvearrowleft}{B}: \quad -\,2\,a\,G + a\,F + M_B = 0$$

$$\rightarrow \quad M_B = -\frac{4}{3}\,q_0\,a^2 + M_0 - a\,F = -\,50\,\text{kNm}\,,$$

$$\uparrow: \quad G - F + B = 0$$

$$\rightarrow \quad B = \frac{2}{3}\,q_0\,a - \frac{M_0}{2\,a} + F = 90\,\text{kN}\,.$$

The stress resultants at specific points of the beam are obtained using the method of sections:

$$V(2a) = \begin{cases} -\,q_0\,a = -\,30\,\text{kN} & \text{to the left of } A\,, \\ -\,q_0\,a + A = 50\,\text{kN} & \text{to the right of } A\,, \end{cases}$$

$$V(4a) = -\,q_0\,a + A - 2\,q_0\,a = -\,10\,\text{kN}\,,$$

$$V(5a) = \begin{cases} G = -\,10\,\text{kN} & \text{to the left of } F\,, \\ G - F = -\,90\,\text{kN} & \text{to the right of } F\,, \end{cases}$$

$$V(6a) = -\,B = -\,90\,\text{kN}\,,$$

$$M(2a) = \begin{cases} -\frac{2}{3}\,a\,q_0\,a = -\,10\,\text{kNm} & \text{to the left of } A\,, \\ -\frac{2}{3}\,a\,q_0\,a - M_0 = -\,20\,\text{kNm} & \text{to the right of } A\,, \end{cases}$$

$$M(5a) = a\,G = -\,5\,\text{kNm}\,,$$

$$M(6a) = M_B = -\,50\,\text{kNm}\,.$$

In addition, $V(0) = 0$, $M(0) = 0$ and $M(4\,a) = 0$.

The values of the stress resultants are now connected with the appropriate curves (straight lines or parabolas, Fig. 7.19c). Note that

– at $x = 0$, the quadratic parabola for V (because of $q(0) = 0$) and the cubic parabola for M (because of $V(0) = 0$) have horizontal slopes,
– at $x = 4a$, the slope of the moment diagram has no jump (because of the continuous shear force).

7.3 Stress Resultants in Frames and Arches

7.3

The methods for determining the stress resultants will now be generalized to frames and arches. Note that the differential relations derived in Section 7.2.2 can be applied to straight portions of a frame only; they are *not valid for arches.*

In this section, the discussion is limited to plane (i.e., coplanar) problems and focused on the pointwise construction of the stress-resultants diagrams. According to this method (see Section 7.2.5), the stress resultants are computed at specific points of the structure with the aid of the method of sections. The algebraic signs of the stress resultants are defined using dashed lines (Section 7.1). A frame, in general, experiences also a normal force, even if the external loads act perpendicularly to its members. Therefore, we will always calculate all three stress resultants: bending moment, shear force and normal force.

At the corners of frames where two straight beams are rigidly joined, the equilibrium conditions reveal how the stress resultants

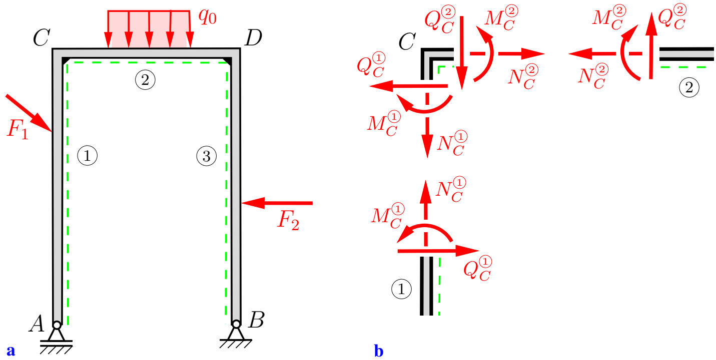


Fig. 7.20

are transferred. As an example, consider the externally unloaded rectangular corner C of the frame shown in Fig. 7.20a. If we free the corner by appropriate cuts, the equilibrium conditions yield (Fig. 7.20b)

$$N_C^{①} = -V_C^{②}\,, \quad V_C^{①} = N_C^{②}\,, \quad M_C^{①} = M_C^{②}\,. \tag{7.18}$$

Whereas the bending moment is transferred unaltered from part ① to part ②, the normal force becomes the shear force and the shear force becomes the normal force. If the beams are joined at a corner under an arbitrary angle, the transferred stress resultants depend on this angle.

E7.8

Example 7.8 Determine the stress resultants for the frame in Fig 7.21a.

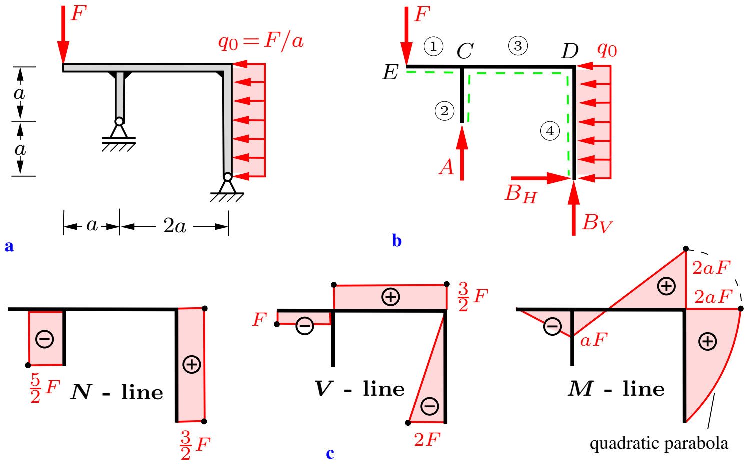


Fig. 7.21

Solution The support reactions are obtained from the equilibrium conditions for the frame as a whole (Fig. 7.21b):

$$A = \frac{5}{2}\,F\,, \quad B_V = -\frac{3}{2}\,F\,, \quad B_H = 2\,F\,.$$

In order to define the algebraic signs of the stress resultants, we introduce the dashed lines according to Fig. 7.21b. The following stress resultants are calculated using the method of sections:

$$N_C^{①} = 0\,, \qquad V_C^{①} = -\,F\,, \qquad M_C^{①} = -\,aF\,,$$

$$N_C^{②} = -\,A = -\,\frac{5}{2}\,F\,, \qquad V_C^{②} = 0\,, \qquad M_C^{②} = 0\,,$$

$$N_D^{③} = 0\,,\ V_D^{③} = -F + A = \frac{3}{2}\,F\,,\ M_D^{③} = -3\,a\,F + 2\,a\,A = 2\,a\,F.$$

The equilibrium conditions at the freed corner D, where parts ③ and ④ are rigidly connected, yield in analogy to (7.18)

$$N_D^{④} = V_D^{③} = \frac{3}{2}\,F\,, \quad V_D^{④} = -\,N_D^{③} = 0\,, \quad M_D^{④} = M_D^{③} = 2\,a\,F\,.$$

With the following values at the ends of the members

$$N_E^{①} = 0\,, \quad V_E^{①} = -\,F\,, \quad M_E^{①} = 0\,,$$
$$N_A^{②} = -\,A = -\,\frac{5}{2}\,F\,, \quad V_A^{②} = 0\,, \quad M_A^{②} = 0\,,$$
$$N_B^{④} = -\,B_V = \frac{3}{2}\,F\,, \quad V_B^{④} = -\,B_H = -\,2\,F\,, \quad M_B^{④} = 0$$

we obtain the stress resultants displayed in Fig. 7.21c.

The equilibrium conditions for the freed bifurcation point C (cuts adjacent to C) may serve as a check. As an example, we consider the stress resultants in the vertical direction showing that the equilibrium condition is fulfilled:

$$V_C^{①} - V_C^{③} - N_C^{②} = 0 \quad \rightarrow \quad -F - \frac{3}{2}\,F + \frac{5}{2}\,F = 0\,.$$

Example 7.9 Determine the stress resultants for the members of the structure in Fig. 7.22a. **E7.9**

Solution First we compute the support reactions. With the components $A_V = A_H$ of the reaction at A, the equilibrium conditions for the entire structure (Fig. 7.22b) yield

$$\overset{\curvearrowright}{B}: \quad 2a\,A_V + 2a\,A_H + 2a\,F = 0 \quad \rightarrow \quad A_V = A_H = -\,\tfrac{1}{2}\,F\,,$$

$$\rightarrow: \qquad A_H + B_H = 0 \quad \rightarrow \quad B_H = \tfrac{1}{2}\,F\,,$$

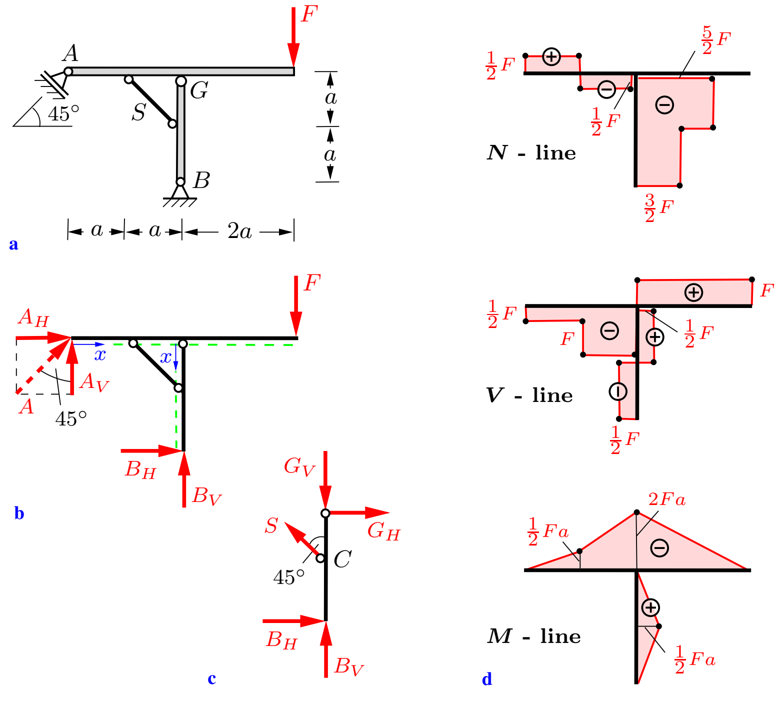


Fig. 7.22

$$\overset{\curvearrowleft}{A}: \quad 2a\,B_V + 2a\,B_H - 4a\,F = 0 \quad \rightarrow \quad B_V = \tfrac{3}{2}\,F\,.$$

Then the force in bar S (= normal force S) and the force in pin G (components G_H and G_V) are calculated using the equilibrium conditions for the vertical beam (Fig. 7.22c):

$$\overset{\curvearrowright}{C}: \quad a\,G_H - a\,B_H = 0 \quad \rightarrow \quad G_H = \frac{F}{2}\,,$$

$$\overset{\curvearrowleft}{G}: \quad 2a\,B_H - a\,\frac{\sqrt{2}}{2}\,S = 0 \quad \rightarrow \quad S = \sqrt{2}\,F\,,$$

$$\uparrow: \quad B_V + \frac{\sqrt{2}}{2}\,S - G_V = 0 \quad \rightarrow \quad G_V = \frac{5}{2}\,F\,.$$

In order to define the algebraic signs of the stress resultants, we introduce the dashed lines according to Fig. 7.22b. The normal-force diagram and the shear-force diagram can be drawn without further computation (note the jumps in the diagrams due to the concentrated forces S, G_H and G_V). The bending-moment diagram can be constructed with the aid of the values at specific points (Fig. 7.22d).

Example 7.10 The circular arch in Fig. 7.23a is subjected to a concentrated force F. **E7.10**

Draw the diagrams of the stress resultants.

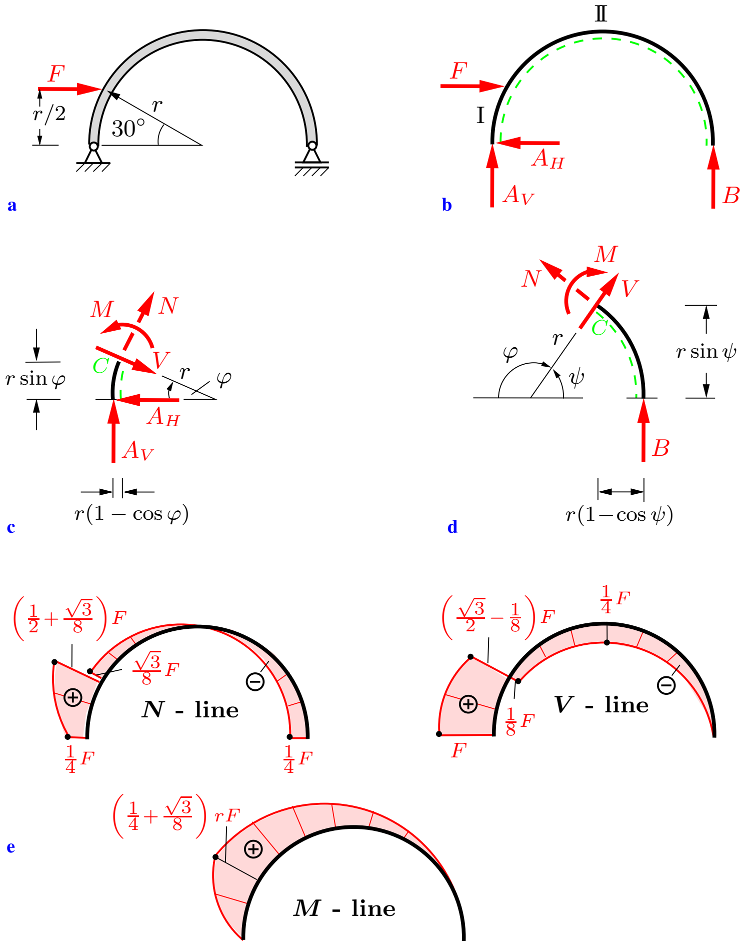


Fig. 7.23

Solution The reactions at the supports follow from the equilibrium conditions for the the entire arch (Fig. 7.23b):

$$B = \frac{1}{4}F\,, \quad A_V = -B = -\frac{1}{4}F\,, \quad A_H = F\,.$$

We introduce the dashed line and cut the arch at an arbitrary position φ in region I $(0 < \varphi < 30°)$. The free-body diagram of the corresponding part of the arch is shown in Fig. 7.23c. The equilibrium conditions lead to the stress resultants:

$$\nearrow: \quad N + A_V \cos\varphi - A_H \sin\varphi = 0$$
$$\to \quad \underline{\underline{N}} = \left(\sin\varphi + \tfrac{1}{4}\cos\varphi\right) F\,,$$
$$\searrow: \quad V - A_V \sin\varphi - A_H \cos\varphi = 0$$
$$\to \quad \underline{\underline{V}} = \left(\cos\varphi - \tfrac{1}{4}\sin\varphi\right) F\,,$$
$$\curvearrowleft C: \quad M - r\sin\varphi\, A_H - r(1-\cos\varphi)A_V = 0$$
$$\to \quad \underline{\underline{M}} = \left(\sin\varphi + \tfrac{1}{4}\cos\varphi - \tfrac{1}{4}\right) r\,F\,.$$

Since the arch is sectioned at an arbitrary position φ, these equations describe the variations of the stress resultants in region I. Similarly, introducing the angle $\psi = \pi - \varphi$, the stress resultants in region II $(30° < \varphi < 180°)$ are obtained from the equilibrium conditions for the free-body diagram in Fig. 7.23d:

$$\nwarrow: \quad N + B\cos\psi = 0$$
$$\to \quad \underline{\underline{N}} = -\tfrac{1}{4}\,F\cos\psi = \tfrac{1}{4}\,F\cos\varphi\,,$$
$$\nearrow: \quad V + B\sin\psi = 0$$
$$\to \quad \underline{\underline{V}} = -\tfrac{1}{4}\,F\sin\psi = -\tfrac{1}{4}\,F\sin\varphi\,,$$
$$\curvearrowleft C: \quad -M + r(1-\cos\psi)B = 0$$
$$\to \quad \underline{\underline{M}} = \tfrac{1}{4}\,(1-\cos\psi)r\,F = \tfrac{1}{4}\,(1+\cos\varphi)r\,F\,.$$

The stress resultants are drawn perpendicularly to the axis of the arch in Fig. 7.23e. The jumps $\Delta N = F/2$ in the normal force and $\Delta V = \sqrt{3}\,F/2$ in the shear force at $\varphi = 30°$ are equal to the components of F tangential and orthogonal to the arch, respectively.

7.4 Stress Resultants in Spatial Structures

The discussion to this point has been limited to plane problems, i.e., plane structures that are subjected to loads acting in the same plane. Now we extend the investigation of the stress resultants to *spatial structures* and to three-dimensional load vectors.

As a simple example, consider the cantilever beam in Fig. 7.24a, which is subjected to the concentrated forces $\boldsymbol{F}_j$ and the external couples $\boldsymbol{M}_j$ acting in arbitrary directions. As in the case of plane problems, we cut the beam at an arbitrary position x (compare Section 7.1). The internal forces acting in the cross-section are replaced by the resultant $\boldsymbol{R}$ (acting at the centroid C of the cross-section) and the corresponding couple $\boldsymbol{M}^{(C)}$. Again, the superscript C is omitted: instead of $\boldsymbol{M}^{(C)}$ we simply write $\boldsymbol{M}$ (Fig. 7.24b). Vectors $\boldsymbol{R}$ and $\boldsymbol{M}$ have, in general, components in all *three* directions of the coordinate system:

$$\boldsymbol{R} = \begin{pmatrix} N \\ V_y \\ V_z \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} M_T \\ M_y \\ M_z \end{pmatrix}. \tag{7.19}$$

The component of the resultant $\boldsymbol{R}$ in the x-direction (*normal* to

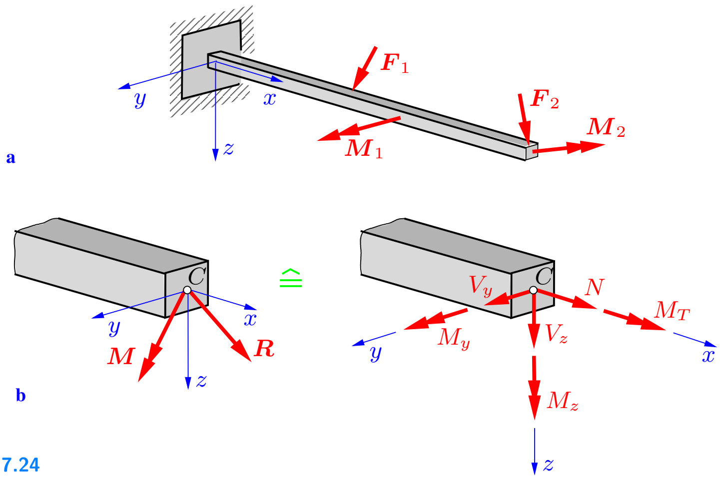


Fig. 7.24

the cross-section) is the normal force N (compare Section 7.1). The components in the y- and z-directions (*perpendicular* to the axis of the beam) are the shear forces V_y and V_z, respectively.

The component M_T of the moment $\boldsymbol{M}$ in the x-direction is called *torque*. In an elastic beam, the torque causes a twist about the longitudinal axis (see Volume 2). The components in the y- and z-directions are the bending moments M_y and M_z, respectively.

The sign convention for the stress resultants coincides with the sign convention for plane problems (see Section 7.1): positive stress resultants at a positive (negative) face point in the positive (negative) directions of the coordinates. Fig. 7.24b shows the stress resultants with their positive directions. In structures composed of several members having different directions, it is helpful to use a different coordinate system for each member.

The stress resultants are determined using the method of sections, i.e., from the equilibrium conditions for a portion of the structure.

E7.11 **Example 7.11** The spatial structure in Fig. 7.25a is subjected to a concentrated force F.

Determine the stress resultants.

Solution We cut the structure at arbitrary positions in the regions ① − ③. If we apply the equilibrium conditions to the cut-off portions of the structure, the stress resultants can be determined without having calculated the support reactions.

To define the algebraic signs of the stress resultants in the three regions, we use three coordinate systems (Fig. 7.25a). First, a cut is passed at an arbitrary position x in region ①. The stress resultants are drawn in the free-body diagram (Fig. 7.25b) with their positive directions (negative face!). The equilibrium conditions (3.34) yield

$$\sum F_{iy} = 0: \quad F - V_y = 0 \quad \rightarrow \quad \underline{\underline{V_y = F}},$$

$$\sum M_{iz} = 0: (c-x)F - M_z = 0 \quad \rightarrow \quad \underline{\underline{M_z = (c-x)F}}.$$

The other stress resultants are zero in region ①.

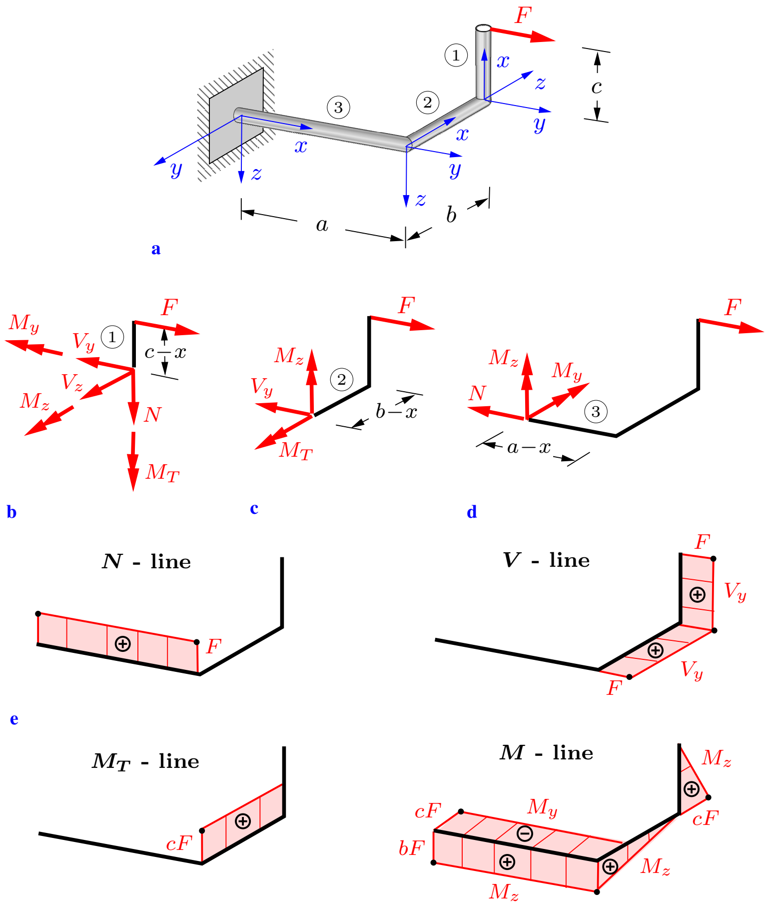


Fig. 7.25

The stress resultants in regions ② and ③ are also obtained using the method of sections. The free-body diagram for region ② (Fig. 7.25c) leads to

$$\sum F_{iy} = 0: \quad F - V_y = 0 \quad \rightarrow \quad \underline{\underline{V_y = F}}\,,$$

$$\sum M_{ix} = 0: \quad c\,F - M_T = 0 \quad \rightarrow \quad \underline{\underline{M_T = c\,F}}\,,$$

$$\sum M_{iz} = 0: \quad (b - x)F - M_z = 0 \quad \rightarrow \quad \underline{\underline{M_z = (b - x)F}}\,,$$

and for region ③ (Fig. 7.25d) we obtain

$$\sum F_{ix} = 0: \qquad F - N = 0 \quad \rightarrow \quad \underline{\underline{N = F}},$$

$$\sum M_{iy} = 0: \; -c\,F - M_y = 0 \quad \rightarrow \quad \underline{\underline{M_y = -c\,F}},$$

$$\sum M_{iz} = 0: \qquad b\,F - M_z = 0 \quad \rightarrow \quad \underline{\underline{M_z = b\,F}}.$$

The stress resultants that are equal to zero are omitted in these free-body diagrams for the sake of clarity of the figures.

The stress resultants are presented in Fig. 7.25e. The support reactions are equal to the values of the stress resultants at the fixed end.

E7.12

Example 7.12 The clamped circular arch in Fig. 7.26a is subjected to a concentrated force F that acts perpendicularly to the plane of the arch.

Determine the stress resultants.

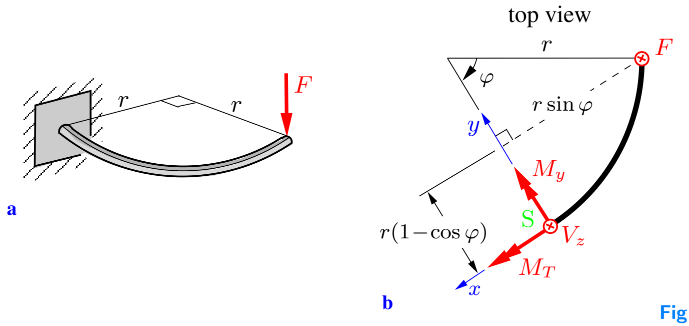


Fig. 7.26

Solution We section the arch at an arbitrary position φ and consider the cut-off portion of the arch (Fig. 7.26b). To define the algebraic signs of the stress resultants, a local x, y, z-coordinate system is used. At the position S given by φ, the shear force V_z, the torque M_T and the bending moment M_y are introduced with their positive directions (positive face). The other stress resultants are zero; they are omitted in the free-body diagram.

The equilibrium conditions yield

$$\sum F_{iz} = 0: \qquad V_z + F = 0 \quad \rightarrow \quad \underline{\underline{V_z = -F}}\,,$$

$$\sum M_{ix}^{(S)} = 0: \quad M_T + F\,r(1-\cos\varphi) = 0$$

$$\rightarrow \quad \underline{\underline{M_T = -F\,r(1-\cos\varphi)}}\,,$$

$$\sum M_{iy}^{(S)} = 0: \qquad M_y + F\,r\sin\varphi = 0$$

$$\rightarrow \quad \underline{\underline{M_y = -F\,r\sin\varphi}}\,.$$

7.5 Supplementary Problems

7.5

Detailed solutions to most of the following examples are given in (**A**) D. Gross et al. *Formeln und Aufgaben zur Technischen Mechanik 1*, Springer, Berlin 2011 or (**B**) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer, Berlin 2011.

E7.13

Example 7.13 A crab on two wheels can move on a beam (weight negligible). Its weight W is linearly distributed as indicated in Fig. 7.27.

Determine the value $\xi = \xi^*$ for which the bending moment attains its maximum value $M_{\max}$. Calculate $M_{\max}$.

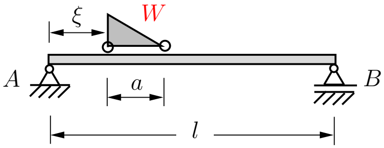


Fig. 7.27

Results: see (**B**) $\quad \xi^* = (3l - a)/6\,, \qquad M_{\max} = \dfrac{1}{36}\left(3 - \dfrac{a}{l}\right)^2 Wl.$

E7.14

Example 7.14 Determine the bending moment for a cantilever subjected to a sinusoidal line load (Fig. 7.28).

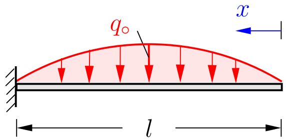


Fig. 7.28

Result: see (**A**) $\quad M(x) = -\dfrac{q_\circ l^2}{\pi}\left(\dfrac{x}{l} - \sin\dfrac{\pi x}{l}\right).$

E7.15 **Example 7.15** The structure in Fig. 7.29 consists of a hinged beam and five bars. It is subjected to two concentrated forces.

Determine the forces in the bars and the bending moment in the beam.

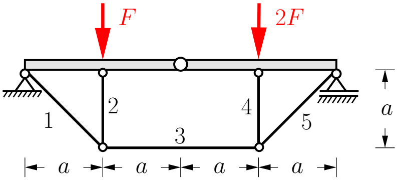


Fig. 7.29

Results: see (**A**) $S_1 = S_5 = \frac{3}{2}\sqrt{2}F\,, \quad S_2 = S_4 = -S_3 = -\frac{3}{2}F\,,$
$M_{max} = aF/4\,.$

E7.16 **Example 7.16** A simply supported beam carries a linearly varying line load as shown in Fig. 7.30.

Calculate the location and the magnitude of the maximum bending moment for $q_1 = 2q_0$.

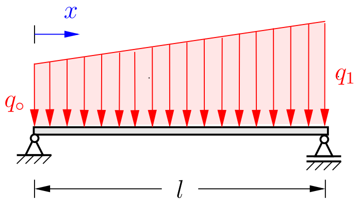


Fig. 7.30

Results: see (**A**) $x^* = 0.53\,l\,, \qquad M_{max} = 0.19\,q_\circ l^2.$

E7.17 **Example 7.17** Draw the shear-force and bending-moment diagrams for the hinged beam shown in Fig. 7.31.

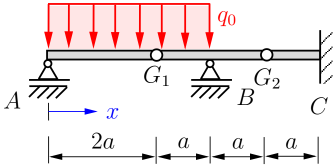


Fig. 7.31

Result: see (**B**) Selected values:

$$V(0) = q_\circ a\,, \quad V(a) = 0\,, \quad V(5a) = 3q_\circ a/2\,,$$
$$M(a) = q_\circ a^2/2\,, \quad M(3a) = -3q_\circ a^2/2\,, \quad M(5a) = 3q_\circ a^2/2\,.$$

Example 7.18 The beam shown in Fig. 7.32 carries a uniformly distributed line load $q_\circ$ and a couple $M_0 = 4q_\circ a^2$.

Draw the shear-force and bending-moment diagrams.

E7.18

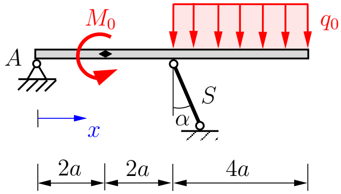


Fig. 7.32

Result: see (**B**) Selected values: $V(a) = -q_\circ a,\ M(4a) = -\,8q_\circ a^2$.

Example 7.19 Determine the distance a of hinge G from the support B (Fig. 7.33) so that the magnitude of the maximum bending moment becomes minimal.

E7.19

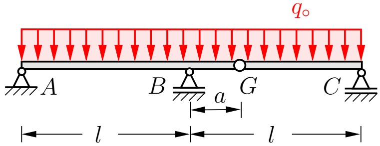


Fig. 7.33

Result: see (**A**) $\quad a = (3 - \sqrt{8})l = 0.172\,l.$

Example 7.20 Determine the stress resultants for the clamped angled member shown in Fig. 7.34.

E7.20

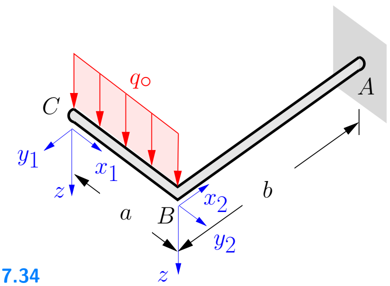


Fig. 7.34

Results: see (**A**)

$$\overline{CB}: \qquad V_z = -q_\circ x_1\,, \quad M_y = -q_\circ x_1^2/2\,,$$

$$\overline{BA}: \qquad V_z = -q_\circ a\,, \quad M_T = q_\circ a^2/2\,, \quad M_y = -q_\circ a x_2\,.$$

E7.21 **Example 7.21** Determine the distance a of the hinge G (Fig. 7.35) so that the magnitude of the maximum bending moment becomes minimal.

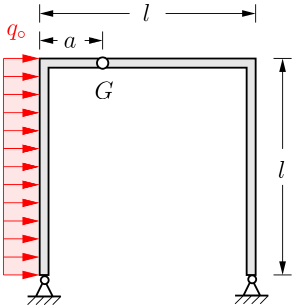


Fig. 7.35

Result: see (**A**) $a = (2\sqrt{3} - 3)l = 0.464\,l$.

E7.22 **Example 7.22** Draw the shear-force and bending-moment diagrams for the frame shown in Fig. 7.36.

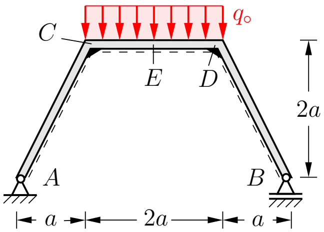


Fig. 7.36

Result: see (**A**) Selected values:

$$V(A) = -V(B) = q_\circ a/\sqrt{5}\,, \quad V(C) = -V(D) = q_\circ a\,,$$
$$M(C) = M(D) = q_\circ a^2\,, \quad M(E) = 3q_\circ a^2/2\,.$$

E7.23 **Example 7.23** The arch shown in Fig. 7.37 carries a constant line load $q_\circ$.

Calculate the maximum values of the normal force and the bending moment.

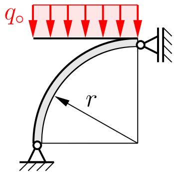


Fig. 7.37

Results: see (**A**) $N_{\max} = 17\,q_\circ r/16\,, \quad M_{\max} = -q_\circ r^2/8.$

Example 7.24 A clamped arch in the form of a quarter-circle (weight negligible) supports a line load $q_\circ$ (Fig. 7.38).

Determine the stress resultants as functions of the coordinate φ.

E7.24

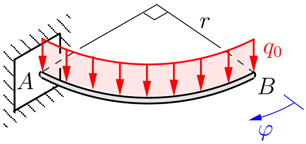


Fig. 7.38

Results: see (**B**) $V_z = -q_\circ r\varphi\,, \quad M_y = -q_\circ r^2(1-\cos\varphi)\,,$

$$M_T = -q_\circ r^2(\varphi - \sin\varphi)\,.$$

Example 7.25 Draw the shear-force and bending-moment diagrams for the hinged beam shown in Fig. 7.39.

E7.25

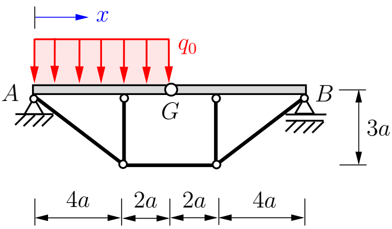


Fig. 7.39

Results: see (**B**) Selected values:

$$V(0) = \tfrac{9}{4}q_0 a\,, \quad V(6a) = -\tfrac{3}{2}q_0 a\,, \quad V(12a) = \tfrac{3}{4}q_0 a\,,$$
$$M(9a/4) = \tfrac{81}{32}q_0 a^2\,, \quad M(4a) = q_0 a^2\,, \quad M(8a) = -3q_0 a^2\,.$$

7.6

7.6 Summary

- In plane problems, the stress resultants in a beam, frame or arch are the normal force N, the shear force V and the bending moment M.
- Sign convention for stress resultants: positive stress resultants at a positive (negative) face point in the positive (negative) directions of the coordinates.
- The stress resultants can be determined using the method of sections:
  - ◇ Pass an imaginary cut through the beam (frame, arch).
  - ◇ Choose a coordinate system.
  - ◇ Draw a free-body diagram of a portion of the structure (stress resultants acting in their positive directions).
  - ◇ Formulate the equilibrium conditions (3 equations in a plane problem, 6 equations in a spatial problem).
  - ◇ Solve the equations.
  - ◇ The system of equations has a unique solution if the structure is (externally and internally) statically determinate.
- The differential relationships

  $$V' = -q\,, \qquad M' = V$$

  are valid for beams and for the straight parts of a frame (not for arches). If the applied load q is known, the stress resultants can be obtained through integration. The constants of integration are determined from boundary conditions or from boundary conditions and matching conditions.
- Frequently, it suffices to compute the stress resultants at several specific points only. The curves between these points are determined by the corresponding loads. Note the relationships between q, V and M.