Chapter 2

Forces with a Common Point of Application

2

2 Forces with a Common Point of Application

Objectives: In this chapter, systems of concentrated forces that have a common point of application are investigated. Such forces are called *concurrent forces*. Note that forces always act on a body; there are no forces without action on a body. In the case of a rigid body, the forces acting on it do not have to have the same point of application; it is sufficient that their lines of action intersect at a common point. Since in this case the force vectors are sliding vectors, they may be applied at any point along their lines of action without changing their effect on the body (principle of transmissibility). If all the forces acting on a body act in a plane, they are called *coplanar forces*.

Students will learn in this chapter how to determine the resultant of a system of concurrent forces and how to resolve force vectors into given directions. They will also learn how to correctly isolate the body under consideration and draw a free-body diagram, in order to be able to formulate the conditions of equilibrium.

2.1 Addition of Forces in a Plane

Consider a body that is subjected to two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, whose lines of action intersect at point A (Fig. 2.1a). It is postulated that the two forces can be replaced by a statically equivalent force $\boldsymbol{R}$. This postulate is an axiom; it is known as the *parallelogram law of forces*. The force $\boldsymbol{R}$ is called the *resultant* of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$. It is the diagonal of the parallelogram for which $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are adjacent sides. The axiom may be expressed in the following way:

> The effect of two nonparallel forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ acting at a point A of a body is the same as the effect of the single force $\boldsymbol{R}$ acting at the same point and obtained as the diagonal of the parallelogram formed by $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$.

The construction of the parallelogram is the geometrical representation of the summation of the vectors (see Appendix A.1):

$$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 \,. \tag{2.1}$$

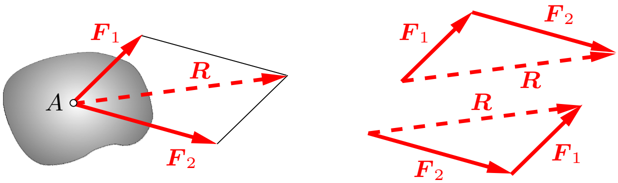


Fig. 2.1 a b

Now consider a system of n forces that all lie in a plane and whose lines of action intersect at point A (Fig. 2.2a). Such a system is called a *coplanar* system of *concurrent forces*. The resultant can be obtained through successive application of the parallelogram law of forces. Mathematically, the summation may be written in the form of the following vector equation:

$$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \ldots + \boldsymbol{F}_n = \sum \boldsymbol{F}_i \,. \tag{2.2}$$

Since the system of forces is reduced to a *single* force, this process is called *reduction*. Note that the forces that act on a rigid body

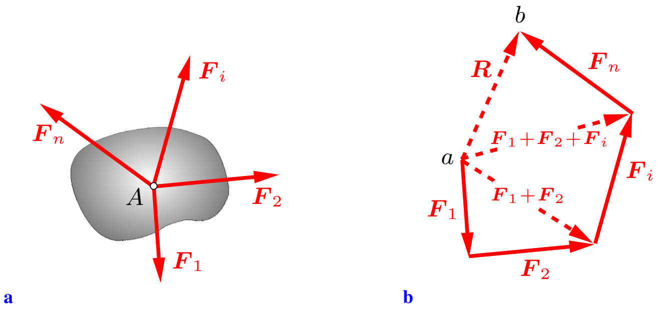


Fig. 2.2

are sliding vectors. Therefore, they do not have to act at point A; only their lines of action have to intersect at this point.

It is not necessary to draw the complete parallelogram to graphically determine the resultant; it is sufficient to draw a *force triangle*, as shown in Fig. 2.1b. This procedure has the disadvantage that the lines of action cannot be seen to intersect at one point. This disadvantage, however, is more than compensated for by the fact that the construction can easily be extended to an arbitrary number of n forces, which are added head-to-tail as shown in Fig. 2.2b. The sequence of the addition is arbitrary; in particular, it is immaterial which vector is chosen to be the first one. The resultant $\boldsymbol{R}$ is the vector that points from the initial point a to the endpoint b of the *force polygon*.

It is appropriate to use a *layout plan* (also called layout diagram) and a *force plan* (also denoted a vector diagram) to solve a problem graphically. The layout plan represents the geometrical specifications of the problem; in general, it has to be drawn to scale (e.g., $1\,\text{cm} \mathrel{\widehat{=}} 1\,\text{m}$). In the case of a system of concurrent forces, it contains only the lines of action of the forces. The force polygon is constructed in a force plan. In the case of a graphical solution, it must be drawn using a scale (e.g., $1\,\text{cm} \mathrel{\widehat{=}} 10\,\text{N}$).

Sometimes problems are solved with just the aid of a sketch of the force plan. The solution is obtained from the force plan, for example, by trigonometry. It is then not necessary to draw the plan to scale. The corresponding method is partly graphical and partly analytical and can be called a “graphic-analytical” method. This procedure is applied, for example, in the Examples 2.1 and 2.4.

Example 2.1 A hook carries two forces F_1 and F_2, which define the angle α (Fig. 2.3a). **E2.1**

Determine the magnitude and direction of the resultant.

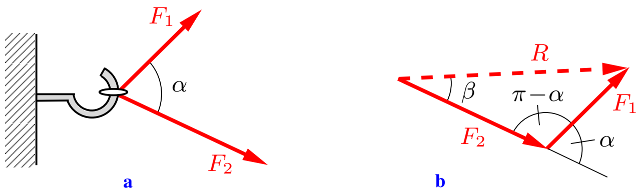


Fig. 2.3

Solution Since the problem will be solved by trigonometry (and since the magnitudes of the forces are not given numerically), a sketch of the force triangle is drawn, but not to scale (Fig. 2.3b).

We assume that the magnitudes of the forces F_1 and F_2 and the angle α are known quantities in this force plan. Then the magnitude of the resultant follows from the law of cosines:

$$R^2 = F_1^2 + F_2^2 - 2\,F_1\,F_2 \cos(\pi - \alpha)$$

or

$$\underline{\underline{R = \sqrt{F_1^2 + F_2^2 + 2\,F_1\,F_2 \cos\alpha}}}\,.$$

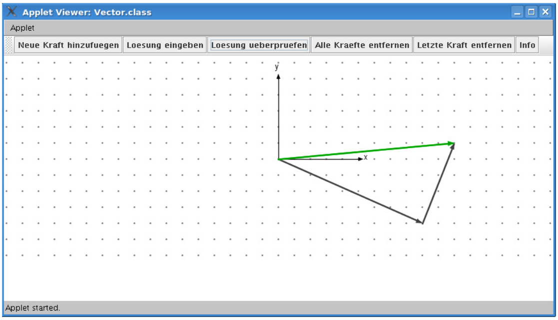
The angle β gives the direction of the resultant R with respect to the force F_2 (Fig. 2.3b). The law of sines yields

$$\frac{\sin\beta}{\sin(\pi - \alpha)} = \frac{F_1}{R}\,.$$

Introducing the result for R and using the trigonometrical relation $\sin(\pi - \alpha) = \sin\alpha$ we obtain

$$\underline{\underline{\sin\beta = \frac{F_1 \sin\alpha}{\sqrt{F_1^2 + F_2^2 + 2\,F_1\,F_2 \cos\alpha}}}}\,.$$

Students may solve this problem and many others concerning the addition of coplanar forces with the aid of the TM-Tool “Resultant of Systems of Coplanar Forces” (see screenshot). This and other TM-Tools can be found at the web address given in the Preface.



E2.2

Example 2.2 An eyebolt is subjected to four forces ($F_1 = 12\,\text{kN}$, $F_2 = 8\,\text{kN}$, $F_3 = 18\,\text{kN}$, $F_4 = 4\,\text{kN}$) that act under given angles ($\alpha_1 = 45°$, $\alpha_2 = 100°$, $\alpha_3 = 205°$, $\alpha_4 = 270°$) with respect to the horizontal (Fig. 2.4a).

Determine the magnitude and direction of the resultant.

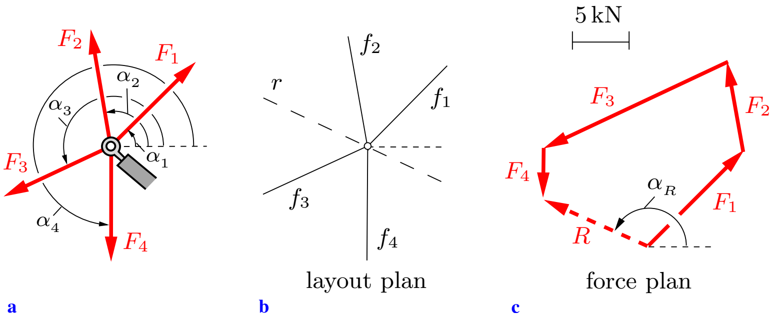


Fig. 2.4

Solution The problem can be solved graphically. First, the layout plan is drawn, showing the lines of action $f_1, \ldots, f_4$ of the forces $F_1, \ldots, F_4$ with their given directions $\alpha_1, \ldots, \alpha_4$ (Fig. 2.4b). Then the force plan is drawn to a chosen scale by adding the given vectors head-to-tail (see Fig. 2.4c and compare Fig. 2.2b). Within the limits of the accuracy of the drawing, the result

$$\underline{\underline{R = 10.5\,\text{kN}}}\,, \quad \underline{\underline{\alpha_R = 155^\circ}}$$

is obtained. Finally, the action line r of the resultant R is drawn into the layout plan.

There are various possible ways to draw the force polygon. Depending on the choice of the first vector and the sequence of the others, different polygons are obtained. They all yield the same resultant R.

2.2 Decomposition of Forces in a Plane, Representation in Cartesian Coordinates

Instead of adding forces to obtain their resultant, it is often desired to replace a force $\boldsymbol{R}$ by two forces that act in the directions of given lines of action f_1 and f_2 (Fig. 2.5a). In this case, the force triangle is constructed by drawing straight lines in the directions of f_1 and f_2 through the initial point and the terminal point of $\boldsymbol{R}$, respectively. Thus, two different force triangles are obtained that unambiguously yield the two unknown force vectors (Fig. 2.5b).

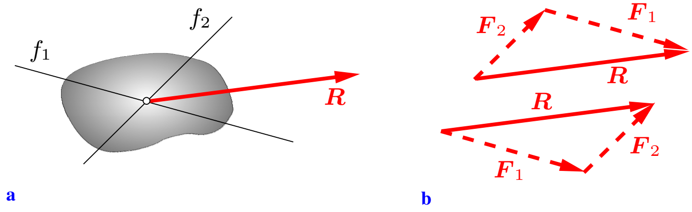


Fig. 2.5

The forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are called the *components* of $\boldsymbol{R}$ in the directions f_1 and f_2, respectively. In coplanar problems, the decomposition of a force into two different directions is unambiguously possible. Note that the resolution into more than two directions cannot be done uniquely: there are an infinite number of ways to resolve the force.

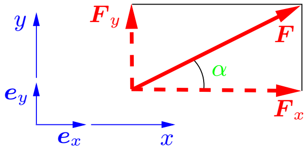


Fig. 2.6

It is usually convenient to resolve forces into two components that are perpendicular to each other. The directions of the components may then be given by the axes x and y of a Cartesian coordinate system (Fig. 2.6). With the unit vectors $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$, the components of $\boldsymbol{F}$ are then written as (compare Appendix A.1)

$$\boldsymbol{F}_x = F_x\,\boldsymbol{e}_x\,, \quad \boldsymbol{F}_y = F_y\,\boldsymbol{e}_y \tag{2.3}$$

and the force $\boldsymbol{F}$ is represented by

$$\boldsymbol{F} = \boldsymbol{F}_x + \boldsymbol{F}_y = F_x\,\boldsymbol{e}_x + F_y\,\boldsymbol{e}_y\,. \tag{2.4}$$

The quantities F_x and F_y are called the *coordinates* of the vector $\boldsymbol{F}$. Note that they are also often called the components of $\boldsymbol{F}$, even though, strictly speaking, the components of $\boldsymbol{F}$ are the vectors $\boldsymbol{F}_x$ and $\boldsymbol{F}_y$. As mentioned in Section 1.2, a vector will often be referred to by writing simply F (instead of $\boldsymbol{F}$) or F_x (instead of $\boldsymbol{F}_x$), especially when this notation cannot lead to confusion (see, for example, Figs. 1.1 and 1.2).

From Fig. 2.6, it can be found that

$$\begin{aligned} F_x &= F\cos\alpha\,, & F_y &= F\sin\alpha\,, \\ F &= \sqrt{F_x^2 + F_y^2}\,, & \tan\alpha &= \frac{F_y}{F_x}\,. \end{aligned} \tag{2.5}$$

In the following, it will be shown that the coordinates of the resultant of a system of concurrent forces can be obtained by simply adding the respective coordinates of the forces. This procedure is demonstrated in Fig. 2.7 with the aid of the example of two forces. The x- and y-components, respectively, of the force $\boldsymbol{F}_i$ are designated with $\boldsymbol{F}_{ix} = F_{ix}\,\boldsymbol{e}_x$ and $\boldsymbol{F}_{iy} = F_{iy}\,\boldsymbol{e}_y$. The resultant then

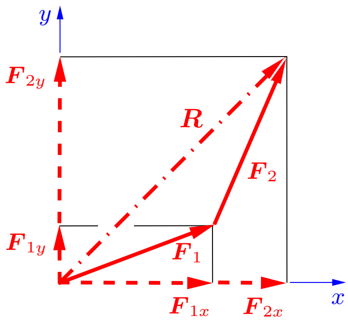


Fig. 2.7

can be written as

$$\boldsymbol{R} = R_x\,\boldsymbol{e}_x + R_y\,\boldsymbol{e}_y = \boldsymbol{F}_1 + \boldsymbol{F}_2 = \boldsymbol{F}_{1x} + \boldsymbol{F}_{1y} + \boldsymbol{F}_{2x} + \boldsymbol{F}_{2y}$$
$$= F_{1x}\,\boldsymbol{e}_x + F_{1y}\,\boldsymbol{e}_y + F_{2x}\,\boldsymbol{e}_x + F_{2y}\,\boldsymbol{e}_y = (F_{1x} + F_{2x})\,\boldsymbol{e}_x + (F_{1y} + F_{2y})\boldsymbol{e}_y\,.$$

Hence, the coordinates of the resultant are obtained as

$$R_x = F_{1x} + F_{2x}\,, \quad R_y = F_{1y} + F_{2y}\,.$$

In the case of a system of n forces, the resultant is given by

$$\boldsymbol{R} = R_x\,\boldsymbol{e}_x + R_y\,\boldsymbol{e}_y = \sum \boldsymbol{F}_i = \sum (F_{ix}\,\boldsymbol{e}_x + F_{iy}\,\boldsymbol{e}_y)$$
$$= \Big(\sum F_{ix}\Big)\,\boldsymbol{e}_x + \Big(\sum F_{iy}\Big)\,\boldsymbol{e}_y \tag{2.6}$$

and the coordinates of the resultant $\boldsymbol{R}$ follow from the summation of the coordinates of the forces:

$$R_x = \sum F_{ix}\,, \quad R_y = \sum F_{iy}\,. \tag{2.7}$$

The magnitude and direction of the resultant are given by (compare (2.5))

$$R = \sqrt{R_x^2 + R_y^2}\,, \quad \tan\alpha_R = \frac{R_y}{R_x}\,. \tag{2.8}$$

In the case of a coplanar force group, the two scalar equations (2.7) are equivalent to the vector equation (2.2).

E2.3

Example 2.3 Solve Example 2.2 with the aid of the representation of the vectors in Cartesian coordinates.

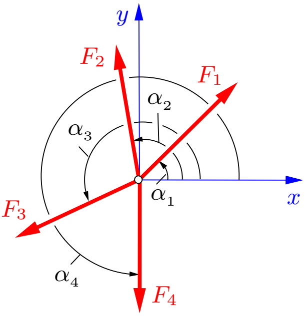


Fig. 2.8

Solution We choose the coordinate system shown in Fig. 2.8, such that the x-axis coincides with the horizontal. The angles are measured from this axis. Then, according to (2.7), the coordinates

$$\begin{aligned} R_x &= F_{1x} + F_{2x} + F_{3x} + F_{4x} \\ &= F_1 \cos\alpha_1 + F_2 \cos\alpha_2 + F_3 \cos\alpha_3 + F_4 \cos\alpha_4 \\ &= 12\cos 45^\circ + 8\cos 100^\circ + 18\cos 205^\circ + 4\cos 270^\circ \\ &= -\,9.22\,\text{kN} \end{aligned}$$

and

$$\begin{aligned} R_y &= F_{1y} + F_{2y} + F_{3y} + F_{4y} \\ &= F_1 \sin\alpha_1 + F_2 \sin\alpha_2 + F_3 \sin\alpha_3 + F_4 \sin\alpha_4 = 4.76\,\text{kN} \end{aligned}$$

are obtained. The magnitude and direction of the resultant follow from (2.8):

$$\underline{\underline{R}} = \sqrt{R_x^2 + R_y^2} = \sqrt{9.22^2 + 4.76^2} = \underline{\underline{10.4\,\text{kN}}}\,,$$

$$\tan\alpha_R = \frac{R_y}{R_x} = -\frac{4.76}{9.22} = -0.52 \quad \rightarrow \quad \underline{\underline{\alpha_R = 152.5^\circ}}\,.$$

2.3 Equilibrium in a Plane

We now investigate the conditions under which a body is in *equilibrium* when subjected to the action of forces. It is known from experience that a body that was originally at rest stays at rest if two forces of equal magnitude are applied that have the same line of action and are oppositely directed (Fig. 2.9). In other words:

> Two forces are in equilibrium if they are oppositely directed on the same line of action and have the same magnitude.

This means that the sum of the two forces, i.e., their resultant, has to be the zero vector:

$$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 = \boldsymbol{0}\,. \tag{2.9}$$

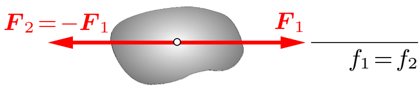


Fig. 2.9

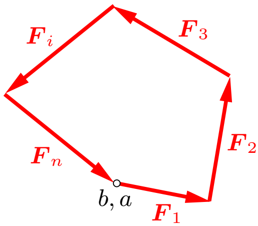


Fig. 2.10

It is also known from Section 2.1 that a system of n concurrent forces $\boldsymbol{F}_i$ can always unambiguously be replaced by its resultant

$$\boldsymbol{R} = \sum \boldsymbol{F}_i\,.$$

Therefore, the *equilibrium condition* (2.9) can immediately be extended to an arbitrary number of forces. A system of concurrent forces is in equilibrium if the resultant is zero:

$$\boldsymbol{R} = \sum \boldsymbol{F}_i = \boldsymbol{0}\,. \tag{2.10}$$

The geometrical interpretation of (2.10) is that of a closed force polygon, i.e., the initial point a and the terminal point b have to coincide (Fig. 2.10).

The resultant force is zero if its components are zero. Therefore, in the case of a coplanar system of forces, the two scalar equilibrium conditions

$$\sum F_{ix} = 0\,, \quad \sum F_{iy} = 0 \tag{2.11}$$

are equivalent to the vector condition (2.10), (compare (2.7)). Thus, a coplanar system of concurrent forces is in equilibrium if the sums of the respective coordinates of the force vectors (here the x- and y-coordinates) vanish.

Consider a problem where the magnitudes and/or the directions of forces need to be determined. Since we have two equilibrium conditions (2.11), only two unknowns can be calculated. Problems that can be solved by applying only the equilibrium conditions are called *statically determinate*. If there are more than two unknowns, the problem is called *statically indeterminate*. Statically indeterminate systems cannot be solved with the aid of the equilibrium conditions alone.

Before the equilibrium conditions for a given problem are written down, a free-body diagram must be constructed. Therefore, the body in consideration must be isolated by imaginary cuts, and all of the forces acting *on* this body (known and unknown forces) must be drawn into the diagram. Only these forces should appear in the equilibrium conditions. Note that the forces exerted *by* the body to the surroundings are not drawn into the free-body diagram.

To solve a given problem analytically, it is generally necessary to introduce a coordinate system. In principle, the directions of the coordinate axes may be chosen arbitrarily. However, an appropriate choice of the axes may save computational work. To apply the equilibrium conditions (2.11), it suffices to determine the coordinates of the forces; in coplanar problems, the force vectors need not be written down explicitly (compare, e.g., Example 2.4).

2.4 Examples of Coplanar Systems of Forces

To be able to apply the above theory to specific problems, a few idealisations of simple structural elements must be introduced. A structural element whose length is large compared to its cross-sectional dimensions and that can sustain only tensile forces in the direction of its axis, is called a *cable* or a *rope* (Fig. 2.11a). Usually the weight of the cable may be neglected in comparison to the force acting in the cable.

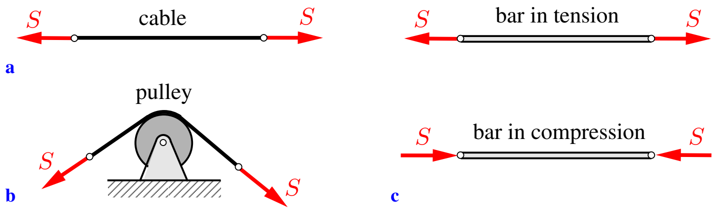


Fig. 2.11

Often a cable is guided over a pulley (Fig. 2.11b). If the bearing friction of the pulley is negligible (ideal pulley), the forces at both ends of the cable are equal in magnitude (see Examples 2.6, 3.3).

A straight structural member with a length much larger than its cross-sectional dimensions that can transfer compressive as well as tensile forces in the direction of its axis is called a *bar* or a *rod* (Fig. 2.11c) (compare Section 5.1.1).

As explained in Section 1.5, the force acting at the point of contact between two bodies can be made visible by separating the bodies (Fig. 2.12a, b). According to Newton's third law (actio = reactio) the contact force K acts with the same magnitude and in an opposite direction on the respective bodies (Fig. 2.12b). It may be resolved into two components, namely, the *normal force* N and the *tangential force* T, respectively. The normal force is perpendicular to the plane of contact, whereas the tangential force lies in this plane. If the two bodies are merely touching each other (i.e., if no connecting elements exist) they can only be pressed against each other (pulling is not possible). Hence, the normal force is oriented towards the interior of the respective body. The tangential force is due to an existing roughness of the surfaces of

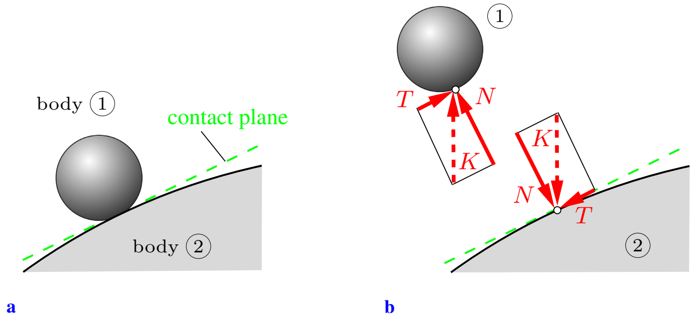


Fig. 2.12

the bodies. In the case of a completely smooth surface of one of the bodies (= idealisation), the tangential force T vanishes. The contact force then coincides with the normal force N.

E2.4 **Example 2.4** Two cables are attached to an eye (Fig. 2.13a). The directions of the forces F_1 and F_2 in the cables are given by the angles α and β.

Determine the magnitude of the force $\boldsymbol{H}$ exerted from the wall onto the eye.

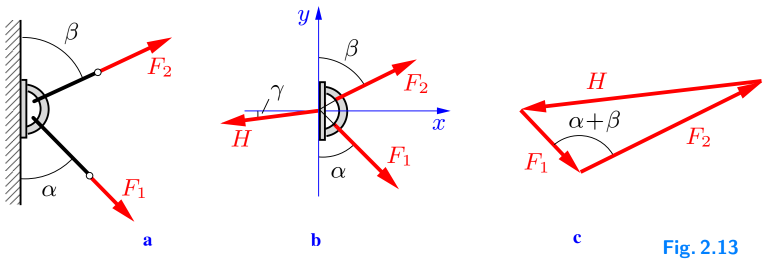


Fig. 2.13

Solution The free-body diagram (Fig. 2.13b) is drawn as the first step. To this end, the eye is separated from the wall by an imaginary cut. Then all of the forces acting on the eye are drawn into the figure: the two given forces F_1 and F_2 and the force H. These three forces are in equilibrium. The free-body diagram contains two unknown quantities, namely, the magnitude of the force H and the angle γ.

The equilibrium conditions are formulated and solved in the second step. We will first present a “graphic-analytical” solution, i.e., a solution that is partly graphical and partly analytical. To this end, the geometrical condition of equilibrium is sketched: the closed force triangle (Fig. 2.13c). Since trigonometry will now be applied to the force plan, it need not be drawn to scale. The law of cosine yields

$$\underline{\underline{H = \sqrt{F_1^2 + F_2^2 - 2F_1F_2\cos(\alpha+\beta)}}}\,.$$

The problem may also be solved analytically by applying the scalar equilibrium conditions (2.11). Then we choose a coordinate system (see Fig. 2.13b), and the coordinates of the force vectors are determined and inserted into (2.11):

$$\begin{aligned}
\sum F_{ix} = 0: &\quad F_1 \sin\alpha + F_2 \sin\beta - H\cos\gamma = 0\\
&\quad \to \quad H\cos\gamma = F_1\sin\alpha + F_2\sin\beta\,,\\
\sum F_{iy} = 0: &\quad -F_1 \cos\alpha + F_2 \cos\beta - H\sin\gamma = 0\\
&\quad \to \quad H\sin\gamma = -F_1\cos\alpha + F_2\cos\beta\,.
\end{aligned}$$

These are *two* equations for the *two* unknowns H and γ. To obtain H, the two equations are squared and added. Using the trigonometrical relation

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

yields

$$H^2 = F_1^2 + F_2^2 - 2\,F_1\,F_2\cos(\alpha+\beta)\,.$$

This result, of course, coincides with the result obtained above.

Example 2.5 A wheel with weight W is held on a *smooth* inclined plane by a cable (Fig. 2.14a). **E2.5**

Determine the force in the cable and the contact force between the plane and the wheel.

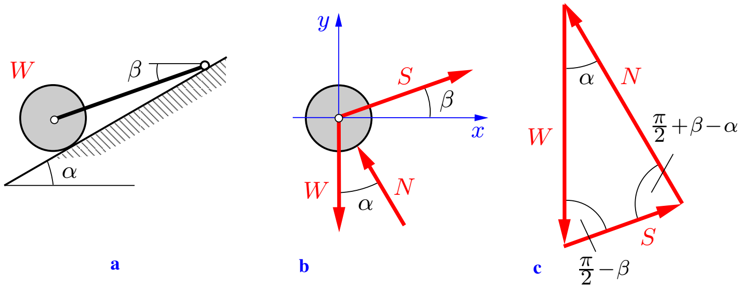


Fig. 2.14

Solution The forces acting at the wheel must satisfy the equilibrium condition (2.10). To make these forces visible, the cable is cut and the wheel is separated from the inclined plane. The free-body diagram (Fig. 2.14b) shows the weight W, the force S in the cable (acting in the direction of the cable) and the contact force N (acting perpendicularly to the inclined plane: smooth surface, $T = 0$!). The three forces S, N and W are concurrent forces; the unknowns are S and N.

First, we solve the problem by graphic-analytical means by sketching (not to scale) the geometrical equilibrium condition, namely, a closed force triangle (Fig. 2.14c). The law of sines yields

$$\underline{\underline{S}} = W\,\frac{\sin\alpha}{\sin(\frac{\pi}{2}+\beta-\alpha)} = \underline{\underline{W\,\frac{\sin\alpha}{\cos(\alpha-\beta)}}}\,,$$

$$\underline{\underline{N}} = W\,\frac{\sin(\frac{\pi}{2}-\beta)}{\sin(\frac{\pi}{2}+\beta-\alpha)} = \underline{\underline{W\,\frac{\cos\beta}{\cos(\alpha-\beta)}}}\,.$$

To solve the problem analytically with the aid of the equilibrium conditions (2.11), we choose a coordinate system (see Fig. 2.14b). Inserting the coordinates of the forces into (2.11) leads to two equations for the two unknowns:

$$\sum F_{ix} = 0 \ : \qquad S\cos\beta - N\sin\alpha = 0\,,$$

$$\sum F_{iy} = 0 \ : \quad S\sin\beta + N\cos\alpha - W = 0\,.$$

Their solution coincides with the solution given above.

Example 2.6 Three boxes (weights W_1, W_2 and W_3) are attached to two cables as shown in Fig. 2.15a. The pulleys are frictionless. **E2.6**

Calculate the angles α_1 and α_2 in the equilibrium configuration.

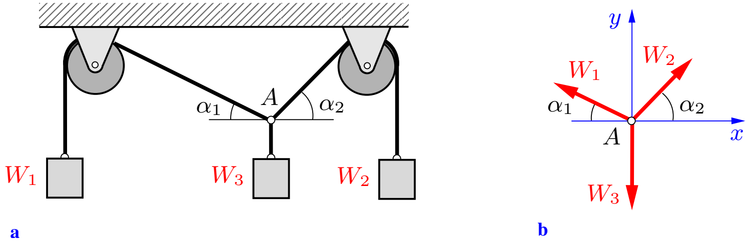


Fig. 2.15

Solution First, point A is isolated by passing imaginary cuts adjacent to this point. The free-body diagram (Fig. 2.15b) shows the forces acting at A; the angles α_1 and α_2 are unknown. Then the coordinate system shown in Fig. 2.15b is chosen and the equilibrium conditions are written down. In plane problems, we shall from now on adopt the following notation: instead of $\sum F_{ix} = 0$ and $\sum F_{iy} = 0$, the symbols $\rightarrow$: and $\uparrow$:, respectively, will be used (sum of all force components in the directions of the arrows equal to zero). Thus,

$$\rightarrow: \quad -W_1 \cos\alpha_1 + W_2 \cos\alpha_2 = 0\,,$$

$$\uparrow: \quad W_1 \sin\alpha_1 + W_2 \sin\alpha_2 - W_3 = 0\,.$$

To compute α_1, the angle α_2 is eliminated by rewriting the equations:

$$W_1 \cos\alpha_1 = W_2 \cos\alpha_2\,,$$

$$W_1 \sin\alpha_1 - W_3 = -W_2 \sin\alpha_2\,.$$

Squaring these equations and then adding them yields

$$\underline{\underline{\sin\alpha_1 = \frac{W_3^2 + W_1^2 - W_2^2}{2\,W_1\,W_3}}}\,.$$

Similarly, we obtain

$$\sin\alpha_2 = \frac{W_3^2 + W_2^2 - W_1^2}{2\,W_2\,W_3}\,.$$

A physically meaningful solution (i.e., an equilibrium configuration) exists only for angles α_1 and α_2 satisfying the conditions $0 < \alpha_1,\ \alpha_2 < \pi/2$. Thus, the weights of the three boxes must be chosen in such a way that both of the numerators are positive and smaller than the denominators.

E2.7

Example 2.7 Two bars 1 and 2 are attached at A and B to a wall by smooth pins. They are pin-connected at C and subjected to a weight W (Fig. 2.16a).

Calculate the forces in the bars.

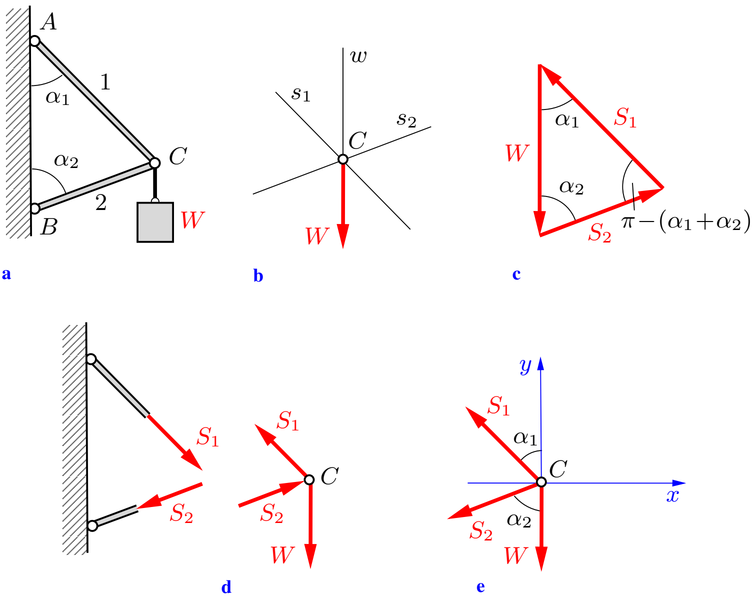


Fig. 2.16

Solution Pin C is isolated by passing cuts through the bars. The forces that act at C are shown in Fig. 2.16d.

First, the graphical solution is indicated. The layout plan is presented in Fig. 2.16b. It contains the given lines of action w, s_1 and s_2 (given by the angles α_1 and α_2) of the forces W, S_1 and S_2, which enables us to draw the closed force triangle (equilibrium condition!) in Fig. 2.16c. To obtain the graphical solution it would be necessary to draw the force plan to scale; in the case of a graphic-analytical solution, no scale is necessary. The law of sines then yields

$$\underline{\underline{S_1 = W\,\frac{\sin\alpha_2}{\sin(\alpha_1+\alpha_2)}}}\,,\qquad \underline{\underline{S_2 = W\,\frac{\sin\alpha_1}{\sin(\alpha_1+\alpha_2)}}}\,.$$

The orientation of the forces S_1 and S_2 can be seen in the force plan. This plan shows the forces that are exerted from the bars onto pin C. The forces exerted from the pin onto the bars have the same magnitude; however, according to Newton's third law they are reversed in direction (Fig. 2.16d). It can be seen that bar 1 is subject to tension and that bar 2 is under compression.

The problem will now be solved analytically with the aid of the equilibrium conditions (2.11). The free-body diagram is shown in Fig. 2.16e. The lines of action of the forces S_1 and S_2 are given. The orientations of the forces along their action lines may, in principle, be chosen arbitrarily in the free-body diagram. It is, however, common practice to assume that the forces in bars are tensile forces, as shown in Fig. 2.16e (see also Sections 5.1.3 and 6.3.1). If the analysis yields a negative value for the force in a bar, this bar is in reality subjected to compression.

The equilibrium conditions in the horizontal and vertical directions

$$\begin{aligned} \rightarrow: &\quad -S_1\sin\alpha_1 - S_2\sin\alpha_2 = 0\,,\\ \uparrow: &\quad S_1\cos\alpha_1 - S_2\cos\alpha_2 - W = 0 \end{aligned}$$

lead to

$$S_1 = W\,\frac{\sin\alpha_2}{\sin(\alpha_1+\alpha_2)}\,,\qquad S_2 = -\,W\,\frac{\sin\alpha_1}{\sin(\alpha_1+\alpha_2)}\,.$$

Since S_2 is negative, the orientation of the vector $\boldsymbol{S}_2$ along its action line is opposite to the orientation chosen in the free-body diagram. Therefore, in reality bar 2 is subjected to compression.

2.5

2.5 Concurrent Systems of Forces in Space

It was shown in Section 2.2 that a force can unambiguously be resolved into two components in a plane. Analogously, a force can be resolved uniquely into three components in space. As indicated

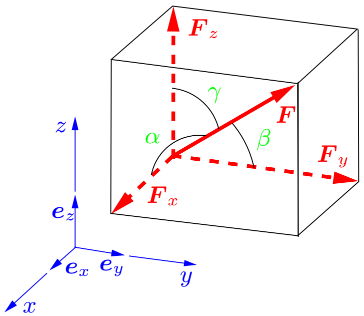


Fig. 2.17

in Section 1.2, a force $\boldsymbol{F}$ may be represented by

$$\boldsymbol{F} = \boldsymbol{F}_x + \boldsymbol{F}_y + \boldsymbol{F}_z = F_x\,\boldsymbol{e}_x + F_y\,\boldsymbol{e}_y + F_z\,\boldsymbol{e}_z \tag{2.12}$$

in a Cartesian coordinate system x, y, z (Fig. 2.17). The magnitude and the direction of $\boldsymbol{F}$ are given by

$$\begin{gathered} F = \sqrt{F_x^2 + F_y^2 + F_z^2}\,, \\ \cos\alpha = \frac{F_x}{F}\,, \quad \cos\beta = \frac{F_y}{F}\,, \quad \cos\gamma = \frac{F_z}{F}\,. \end{gathered} \tag{2.13}$$

The angles α, β and γ are not independent of each other. If the first equation in (2.13) is squared and F_x, F_y and F_z are inserted according to the second equation the following relation is obtained:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1\,. \tag{2.14}$$

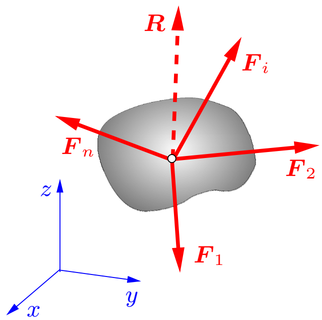


Fig. 2.18

The resultant $\boldsymbol{R}$ of two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ is obtained by constructing the parallelogram of the forces (see Section 2.1) which is expressed mathematically by the vector equation

$$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 \,. \tag{2.15}$$

In the case of a spatial system of n concurrent forces (Fig. 2.18), the resultant is found through a successive application of the parallelogram law of forces in space. As in the case of a system of coplanar forces, the resultant is the sum of the force vectors. Mathematically, this is written as (compare (2.2))

$$\boldsymbol{R} = \sum \boldsymbol{F}_i \,. \tag{2.16}$$

If the forces $\boldsymbol{F}_i$ are represented by their components $\boldsymbol{F}_{ix}$, $\boldsymbol{F}_{iy}$ and $\boldsymbol{F}_{iz}$ according to (2.12), we obtain

$$\begin{aligned}\boldsymbol{R} = R_x\,\boldsymbol{e}_x + R_y\,\boldsymbol{e}_y + R_z\,\boldsymbol{e}_z &= \sum(\boldsymbol{F}_{ix} + \boldsymbol{F}_{iy} + \boldsymbol{F}_{iz})\\ &= \sum(F_{ix}\,\boldsymbol{e}_x + F_{iy}\,\boldsymbol{e}_y + F_{iz}\,\boldsymbol{e}_z)\\ &= \Big(\sum F_{ix}\Big)\,\boldsymbol{e}_x + \Big(\sum F_{iy}\Big)\,\boldsymbol{e}_y + \Big(\sum F_{iz}\Big)\,\boldsymbol{e}_z \,.\end{aligned}$$

The coordinates of the resultant in space are thus given by

$$R_x = \sum F_{ix}\,, \quad R_y = \sum F_{iy}\,, \quad R_z = \sum F_{iz}\,. \tag{2.17}$$

The magnitude and direction of $\boldsymbol{R}$ follow as (compare (2.13))

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}\,, \qquad (2.18)$$
$$\cos\alpha_R = \frac{R_x}{R}\,, \quad \cos\beta_R = \frac{R_y}{R}\,, \quad \cos\gamma_R = \frac{R_z}{R}\,.$$

A spatial system of concurrent forces is in *equilibrium* if the resultant is the zero vector (compare (2.10)):

$$\boldsymbol{R} = \sum \boldsymbol{F}_i = \boldsymbol{0}\,. \qquad (2.19)$$

This vector equation is equivalent to the *three* scalar *equilibrium conditions*

$$\sum F_{ix} = 0\,, \quad \sum F_{iy} = 0\,, \quad \sum F_{iz} = 0\,, \qquad (2.20)$$

which represent a system of three equations for three unknowns.

E2.8 **Example 2.8** A structure consists of two bars 1 and 2 and a rope 3 (weights negligible). It is loaded in A by a box of weight W (Fig. 2.19a).

Determine the forces in the bars and in the rope.

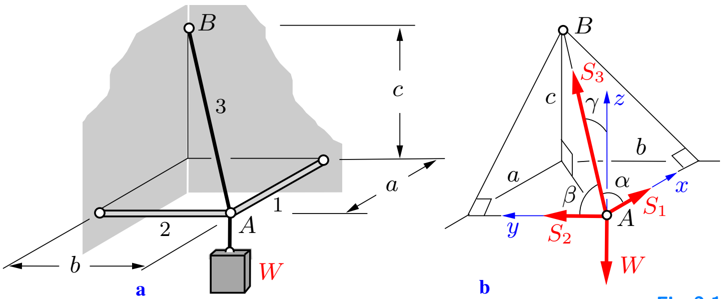


Fig. 2.19

Solution We isolate pin A by passing imaginary sections through the bars and the rope. The internal forces are made visible in the free-body diagram (Fig. 2.19b); they are assumed to be tensile forces. The equilibrium conditions are

$$\begin{aligned}
\sum F_{ix} = 0: &\quad S_1 + S_3 \cos\alpha = 0\,, \\
\sum F_{iy} = 0: &\quad S_2 + S_3 \cos\beta = 0\,, \\
\sum F_{iz} = 0: &\quad S_3 \cos\gamma - W = 0\,.
\end{aligned} \tag{a}$$

With the diagonal $\overline{AB} = \sqrt{a^2+b^2+c^2}$, the angles α, β and γ can be taken from Fig. 2.19b:

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}}\,, \quad \cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}}\,,$$
$$\cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}\,.$$

This yields

$$\underline{\underline{S_3}} = \frac{W}{\cos\gamma} = \underline{\underline{W\,\frac{\sqrt{a^2+b^2+c^2}}{c}}}\,,$$
$$\underline{\underline{S_1}} = -\,S_3\cos\alpha = -\,W\,\frac{\cos\alpha}{\cos\gamma} = \underline{\underline{-\,W\,\frac{a}{c}}}\,,$$
$$\underline{\underline{S_2}} = -\,S_3\cos\beta = -\,W\,\frac{\cos\beta}{\cos\gamma} = \underline{\underline{-\,W\,\frac{b}{c}}}\,.$$

The negative signs for the forces in the bars indicate that the bars are actually in a state of compression; the rope is subjected to tension. This can easily be verified by inspection.

As can be seen, the geometry of this problem is very simple. Therefore, it was possible to apply the equilibrium conditions without resorting to the vector formalism. In the case of a complicated geometry it is, however, recommended that the forces be written down in vector form. This more formal and therefore safer way to solve the problem is now presented.

The force vectors

$$\boldsymbol{S}_1 = S_1\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \boldsymbol{S}_2 = S_2\begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \boldsymbol{W} = W\begin{pmatrix}0\\0\\-1\end{pmatrix}$$

(represented as column vectors, see Appendix A.1) can easily be written down. To obtain the vector $\boldsymbol{S}_3$, we first represent the vector $\boldsymbol{r}_{AB}$, which is directed from A to B:

$$\boldsymbol{r}_{AB} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} .$$

If this vector is divided by its magnitude $r_{AB} = \sqrt{a^2+b^2+c^2}$, the unit vector

$$\boldsymbol{e}_{AB} = \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

in the direction from A to B is obtained. The force vector $\boldsymbol{S}_3$ has the same direction; it is therefore given by

$$\boldsymbol{S}_3 = S_3\, \boldsymbol{e}_{AB} = \frac{S_3}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix} .$$

The equilibrium condition $\sum \boldsymbol{F}_i = \boldsymbol{0}$, i.e.,

$$\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{S}_3 + \boldsymbol{W} = \boldsymbol{0}$$

reads

$$S_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + S_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{S_3}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + W \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

Evaluation yields

$$\begin{aligned}
\sum F_{ix} = 0: &\qquad S_1 + \frac{aS_3}{\sqrt{a^2+b^2+c^2}} = 0 , \\
\sum F_{iy} = 0: &\qquad S_2 + \frac{bS_3}{\sqrt{a^2+b^2+c^2}} = 0 , \\
\sum F_{iz} = 0: &\quad -W + \frac{cS_3}{\sqrt{a^2+b^2+c^2}} = 0 .
\end{aligned}$$

These equations coincide with equations (a).

Note that the quantities S_j are the forces in the bars and in the rope, respectively, which were assumed to be tensile forces. They are *not* the magnitudes of the vectors $\boldsymbol{S}_j$. The determination of the forces leads to $S_1 < 0$ and $S_2 < 0$ (bars in compression), whereas magnitudes of vectors are always non-negative quantities.

Example 2.9 A vertical mast M is supported by two ropes 1 and 2. The force F in rope 3 is given (Fig. 2.20a). E2.9

Determine the forces in ropes 1 and 2 and in the mast.

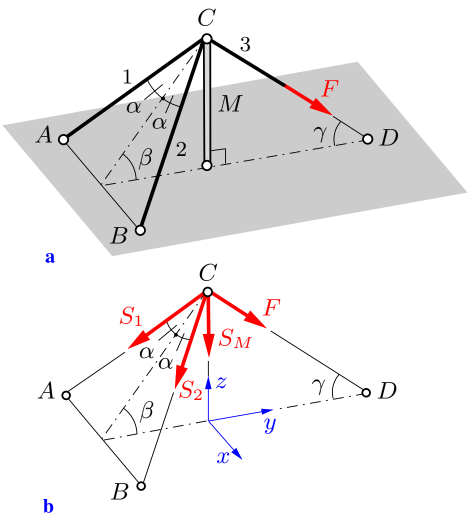


Fig. 2.20

Solution We isolate point C by passing imaginary cuts through the ropes and the mast. The internal forces are assumed to be tensile forces, and are shown in the free-body diagram (Fig. 2.20b). Since the y, z-plane is a plane of symmetry, the forces S_1 and S_2 have to be equal: $S_1 = S_2 = S$ (this may be confirmed by applying the equilibrium condition in the x-direction). The forces S_1 and S_2 are added to obtain their resultant (Fig. 2.20c)

$$S^* = 2\,S\cos\alpha\,.$$

The forces S^*, S_M and F act in the y, z-plane (Fig. 2.20d). The equilibrium conditions

$$\sum F_{iy} = 0 : \quad - S^* \cos\beta + F \cos\gamma = 0 \,,$$
$$\sum F_{iz} = 0 : \quad - S^* \sin\beta - S_M - F \sin\gamma = 0$$

yield, after inserting the relation for S^*,

$$\underline{\underline{S = F \frac{\cos\gamma}{2\cos\alpha\cos\beta}}} \,, \qquad \underline{\underline{S_M = - F \frac{\sin(\beta+\gamma)}{\cos\beta}}} \,.$$

As could be expected, the ropes are subjected to tension ($S > 0$), whereas the mast is under compression ($S_M < 0$).

The special case $\gamma = \pi/2$ is used as a simple check. Then force F acts in the direction of the mast, and $S = 0$ and $S_M = - F$ are obtained with $\cos(\pi/2) = 0$ and $\sin(\beta + \pi/2) = \cos\beta$.

2.6

2.6 Supplementary Problems

Detailed solutions to most of the following examples are given in (**A**) D. Gross et al. *Formeln und Aufgaben zur Technischen Mechanik 1*, Springer, Berlin 2011, or (**B**) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer, Berlin 2011.

E2.10

Example 2.10 A hook is subjected to three forces ($F_1 = 180\,$N, $\alpha_1 = 45°$, $F_2 = 50\,$N, $\alpha_2 = 60°$, $F_3 = 30\,$N) as shown in Fig. 2.21.

Determine the magnitude and direction of the resultant.

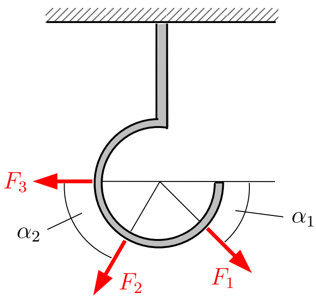
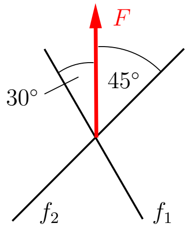


Fig. 2.21

Results: $R = 185\,$N , $\alpha_R = 67°$.

E2.11

Example 2.11 Determine the magnitudes F_1 and F_2 of the components of force $\boldsymbol{F}$ with magnitude $F = 5\,\text{kN}$ in the directions f_1 and f_2 (Fig. 2.22).



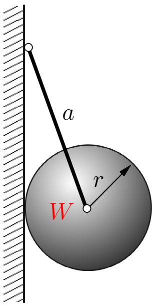
Results: $F_1 = 3.7\,\text{kN}\,, \quad F_2 = 2.6\,\text{kN}\,.$

Fig. 2.22

E2.12

Example 2.12 A smooth sphere (weight $W = 20\,\text{N}$, radius $r = 20\,\text{cm}$) is suspended by a wire (length $a = 60\,\text{cm}$) as shown in Fig. 2.23.

Determine the magnitude of force S in the wire.



Result: see (**A**) $\quad S = 21.2\,\text{N}\,.$

Fig. 2.23

E2.13

Example 2.13 Fig. 2.24 shows a freight elevator. The cable of the winch passes over a smooth pin K. A crate (weight W) is suspended at the end of the cable.

Determine the magnitude of forces S_1 and S_2 in bars 1 and 2.

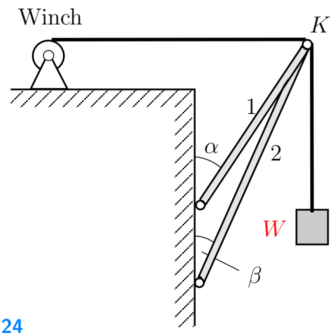


Fig. 2.24

Results: see (**B**) $\quad S_1 = \dfrac{\sin\beta - \cos\beta}{\sin(\alpha - \beta)}\,W\,, \quad S_2 = \dfrac{\cos\alpha - \sin\alpha}{\sin(\alpha - \beta)}\,W\,.$

E2.14 **Example 2.14** A smooth circular cylinder (weight W, radius r) touches an obstacle (height h) as shown in Fig. 2.25.

Find the magnitude of force F necessary to roll the cylinder over the obstacle.

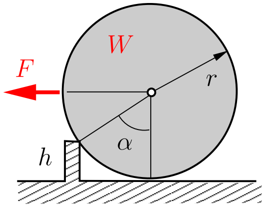


Fig. 2.25

Result: see (**A**) $F = W \tan\alpha\,.$

E2.15 **Example 2.15** A large cylinder (weight $4W$, radius $2r$) lies on top of two small cylinders (each having weight W and radius r) as shown in Fig. 2.26. The small cylinders are connected by a wire S (length $3r$). All surfaces are smooth.

Determine all contact forces and the magnitude of force S in the wire.

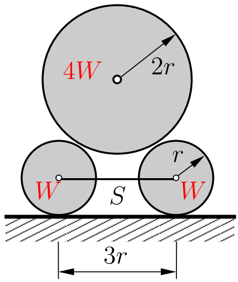


Fig. 2.26

Results: see (**A**) $N_1 = \dfrac{4}{\sqrt{3}}\,W\,, \quad N_2 = 3\,W\,, \quad S = \dfrac{2}{\sqrt{3}}\,W\,.$

E2.16 **Example 2.16** A cable (length l, weight negligible) is attached to two walls at A and B (Fig. 2.27). A cube (weight W) on a frictionless pulley (radius negligible) is suspended by the cable.

Find the distance d of the cube from the left side in the equilibrium position and calculate the force S in the cable.

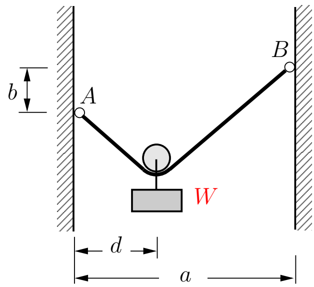


Fig. 2.27

Results: see (**B**) $S = \dfrac{W\,l}{2\sqrt{l^2 - a^2}}\,, \quad d = \dfrac{a}{2}\Big(1 - \dfrac{b}{\sqrt{l^2 - a^2}}\Big)\,.$

Example 2.17 A smooth circular cylinder (weight $W = 500$ N) rests on two fixed supports as shown in Fig. 2.28. It is subjected to a force $F = 200$ N.

a) Calculate the contact forces.

b) Determine the allowable magnitude of F in order to avoid the cylinder from lifting off.

E2.17

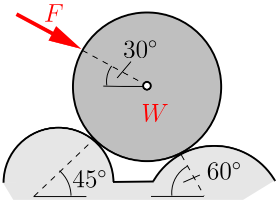


Fig. 2.28

Results: a) $N_l = 155$ N, $N_r = 566$ N, b) $F_{\text{allow}} \leq 500$ N.

Example 2.18 Two cylinders (weights W_1 and W_2) are pin-connected by a bar (weight negligible). They rest on two smooth inclined planes as shown in Fig. 2.29. Given: $W_1 = 200$ N, $W_2 = 300$ N, $\alpha = 60°$.

Calculate the angle φ in the equilibrium position and the corresponding force S in the bar.

E2.18

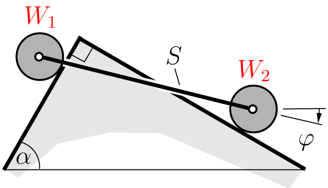


Fig. 2.29

Results: $\varphi = -19°$, $S = 229$ N.

Example 2.19 A spatial system of concurrent forces consists of the three forces

E2.19

$$\boldsymbol{F}_1 = a_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \boldsymbol{F}_2 = a_2 \begin{pmatrix} -9 \\ 6 \\ 9 \end{pmatrix}, \quad \boldsymbol{F}_3 = a_3 \begin{pmatrix} 8 \\ -7 \\ 1 \end{pmatrix}.$$

Their resultant is given by $\boldsymbol{R} = \begin{pmatrix} 30 \\ -28 \\ 44 \end{pmatrix}$ kN.

Determine the unknown constants a_1, a_2, a_3, the magnitudes of the forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ and the angle α between $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$.

Results: (selected values) $a_1 = 2$ kN, $a_2 = 4$ kN, $\alpha = 134.3°$.

E2.20 **Example 2.20** A smooth sphere (weight W, radius R) rests on three points A, B and C. These three points form an equilateral triangle in a horizontal plane. The height of the triangle is $3a = \frac{3}{4}\sqrt{3}\,R$ (see Fig. 2.28). The action line of the applied force F passes through the center of the sphere.

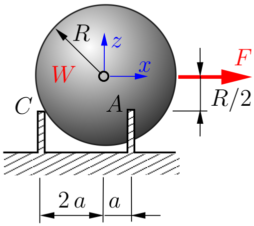


Fig. 2.30

Determine the contact forces at A, B and C. Find the force F required to lift the sphere off at C.

Results: see (**A**) $A = B = \frac{2}{3}\left(W + \frac{1}{\sqrt{3}}F\right), \quad C = \frac{2}{3}\left(W - \frac{2}{\sqrt{3}}F\right),$

$$\boldsymbol{A} = \frac{A}{4}\begin{pmatrix} -\sqrt{3} \\ 3 \\ 2 \end{pmatrix}, \quad \boldsymbol{B} = \frac{B}{4}\begin{pmatrix} -\sqrt{3} \\ -3 \\ 2 \end{pmatrix}, \quad \boldsymbol{C} = \frac{C}{4}\begin{pmatrix} 2\sqrt{3} \\ 0 \\ 2 \end{pmatrix},$$

$$F = \frac{\sqrt{3}}{2}W\,.$$

E2.21 **Example 2.21** The construction shown in Fig. 2.31 consists of three bars that are pin-connected at K. A rope attached to a wall is guided without friction through an eye at K. The free end of the rope is loaded with a crate (weight W).

Calculate the forces in the bars.

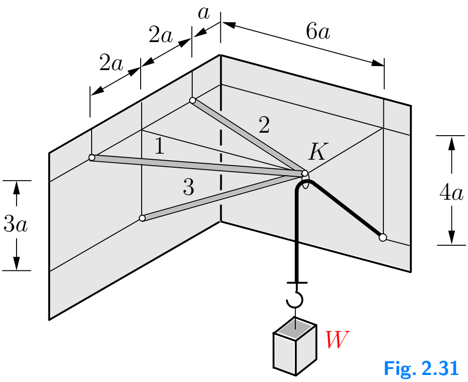


Fig. 2.31

Results: see (**B**) $S_1 = \frac{9}{\sqrt{10}}W\,, \quad S_2 = \frac{3}{\sqrt{10}}W\,, \quad S_3 = -\frac{9}{\sqrt{5}}W\,.$

2.7 Summary

- The lines of action of a system of concurrent forces intersect at a point.
- The resultant of a system of concurrent forces is given by the vector $\boldsymbol{R} = \sum \boldsymbol{F}_i$. In coordinates,

$$R_x = \sum F_{ix}\,, \qquad R_y = \sum F_{iy}\,, \qquad R_z = \sum F_{iz}\,.$$

  In the case of a coplanar system, the z-components vanish. Note: the coordinate system may be chosen arbitrarily; an appropriate choice may save computational work.
- The equilibrium condition for a system of concurrent forces is $\sum \boldsymbol{F}_i = \boldsymbol{0}$. In coordinates,

$$\sum F_{ix} = 0\,, \qquad \sum F_{iy} = 0\,, \qquad \sum F_{iz} = 0\,.$$

  In the case of a coplanar problem, the z-components vanish.
- In order to solve force problems the following steps are usually necessary:
  - ◇ Isolate the body (point).
  - ◇ Sketch the free-body diagram: introduce all of the forces exerted *on* the body; assume the internal forces in bars to be tensile forces.
  - ◇ Choose a coordinate system.
  - ◇ Formulate the equilibrium conditions (3 equations in spatial problems, 2 equations in coplanar problems).
  - ◇ Solve the equilibrium conditions.
- The force acting at the point of contact between two bodies can be made visible by separating the bodies. In the case of smooth surfaces, it is perpendicular to the plane of contact.