Chapter 3

General Systems of Forces, Equilibrium of a Rigid Body

3

3 General Systems of Forces, Equilibrium of a Rigid Body

Objectives: In this chapter general systems of forces are considered, i.e., forces whose lines of action do not intersect at a point. For the analysis, the notion *moment* has to be introduced. Students should learn how coplanar or spatial systems of forces can be reduced and under which conditions they are in equilibrium. They should also learn how to apply the method of sections to obtain a free-body diagram. A correct free-body diagram and an appropriate application of the equilibrium conditions are the key to the solution of a coplanar or a spatial problem.

3.1 General Systems of Forces in a Plane

3.1.1 Couple and Moment of a Couple

In Section 2.1 it was shown that a system of *concurrent forces* can always be reduced to a resultant force. In the following, it

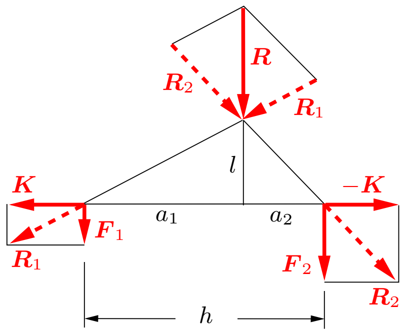

Fig. 3.1

is demonstrated how the resultant $\boldsymbol{R}$ of two *parallel forces* $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ can be found (Fig. 3.1). As a first step, the two forces $\boldsymbol{K}$ and $-\boldsymbol{K}$, which have the same line of action, are added. Since they are in equilibrium they have no effect on a rigid body. Then two parallelograms (here rectangles) are drawn to obtain the forces $\boldsymbol{R}_1 = \boldsymbol{F}_1 + \boldsymbol{K}$ and $\boldsymbol{R}_2 = \boldsymbol{F}_2 + (-\boldsymbol{K})$. These forces, which are statically equivalent to the given system of parallel forces, represent a system of concurrent forces. They may be moved along their respective lines of action (sliding vectors!) to their point of intersection. There another parallelogram of forces is constructed that yields the resultant

$$\boldsymbol{R} = \boldsymbol{R}_1 + \boldsymbol{R}_2 = \boldsymbol{F}_1 + \boldsymbol{F}_2 \,. \tag{3.1}$$

This graphical solution yields the magnitude R of the resultant as well as the location of its action line. From Fig. 3.1 it can be found that

$$\begin{aligned} &R = F_1 + F_2 \,, \\ &h = a_1 + a_2 \,, \quad \frac{a_1}{l} = \frac{K}{F_1} \,, \quad \frac{a_2}{l} = \frac{K}{F_2} \,. \end{aligned} \tag{3.2}$$

Thus, the magnitude of the resultant $\boldsymbol{R}$ of the parallel forces is simply the algebraic sum of the magnitudes of the forces. Equation (3.2) also yields the *principle of the lever* by Archimedes

$$a_1 F_1 = a_2 F_2 \tag{3.3}$$

and the distances

$$a_1 = \frac{F_2}{F_1 + F_2} h = \frac{F_2}{R} h, \quad a_2 = \frac{F_1}{F_1 + F_2} h = \frac{F_1}{R} h. \tag{3.4}$$

Hence, the method described above always gives the resultant force and the location of its action line unless the denominator in (3.4) is zero. In this case the two forces are said to form a *couple*.

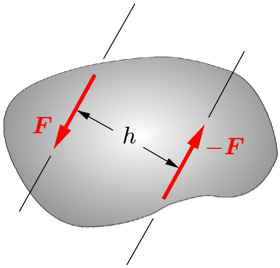

Fig. 3.2

A *couple* consists of two forces having equal magnitude, parallel action lines and opposite directions (Fig. 3.2). In this case the method to find the resultant fails to work. With $F_2 = -F_1$ the equations (3.2) and (3.4) lead to $R = 0$ and a_1, $a_2 \to \pm\infty$. Hence, a couple can *not* be reduced to a resultant single force.

Although the resultant force of a couple is zero the couple has an effect on the body on which it acts: it tends to rotate the body. Fig. 3.3 shows three examples of couples: a) a wheel which is to be turned, b) a screw driver acting on the head of a screw and c) a "clamped" beam whose free end is twisted. As can be seen, a couple has a *sense of rotation*: either clockwise or counterclockwise. Similar to the notion of a "concentrated" force the couple is an idealization which replaces the action of the area forces.

We now investigate the quantities which define a couple and its properties. The effect of a couple on a rigid body is unambiguously determined by its *moment*. The moment incorporates two

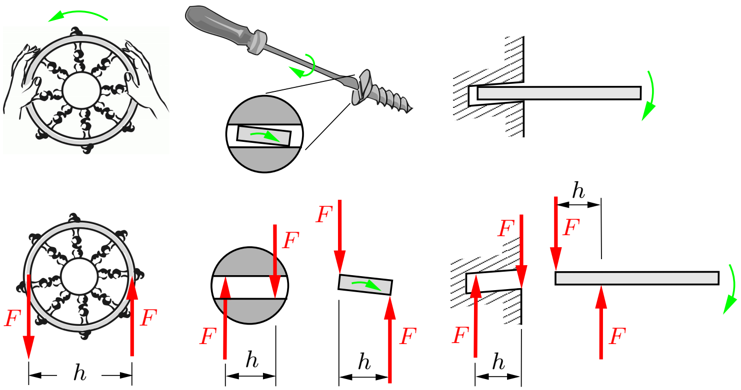

Fig. 3.3 a b c

quantities: first, its magnitude M which is given by the product of the perpendicular distance h of the action lines (Fig. 3.2) and the magnitude F of the forces

$$M = h\,F\,, \tag{3.5}$$

and, secondly, its sense of rotation. In the figures, the sense of rotation is represented by a curved arrow ($\curvearrowleft$ or $\curvearrowright$). The quantities *magnitude* M and *sense of rotation* $\curvearrowleft$ indicate that a couple moment is a vector in three-dimensional space. The moment has the dimension length times force $[l\,F]$, and is expressed, for example, in the unit Nm. In order to avoid confusion with the unit mN $\widehat{=}$ Milli-Newton, the sequence of the units of length and force is exchanged: Nm $\widehat{=}$ Newton-Meter.

Fig. 3.4 shows that a couple with a given moment can be produced by arbitrarily many different pairs of forces. If the forces K and $-K$ are added to the given couple (forces F, perpendicular distance h) a statically equivalent couple is obtained (forces F', perpendicular distance h'). The couple moment, i.e., the sense of rotation and the magnitude of the moment

$$M = h'\,F' = (h\sin\alpha)\left(\frac{F}{\sin\alpha}\right) = h\,F$$

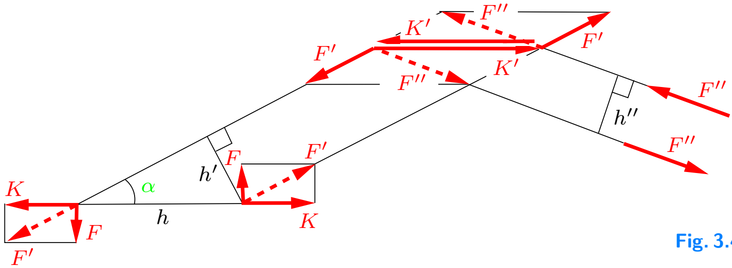

Fig. 3.4

remain unchanged. By successive application of this procedure, a couple may be moved arbitrarily in the plane without any change of its moment. Hence, in contrast to a force, a couple is *not* bound to a line of action. Therefore, it may be applied at arbitrary points of a rigid body: it always has the same turning effect.

A couple is uniquely described by its moment. Hence, the two forces F and $-F$ in the following are replaced by the couple moment. In particular, in the figures we shall replace the two forces by a curved arrow, i.e., $\curvearrowleft\ M$, as shown in Fig. 3.5. This notation incorporates the magnitude M of the couple moment and the sense of rotation (curved arrow); it is analogous to the notation $\nearrow F$ (arrow and magnitude of the force).

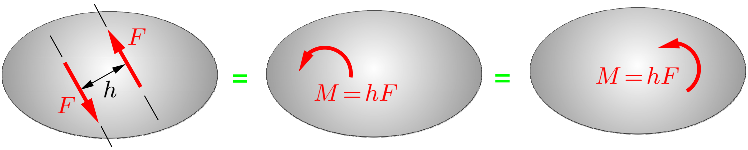

Fig. 3.5

The law of action and reaction (Section 1.5) states that every force has a counteracting force of the same magnitude but opposite direction. By analogy, every couple moment has a counteracting couple moment of equal magnitude but with an opposite sense of rotation. For example, the screwdriver in Fig. 3.3b exerts a moment $M = h\,F$ on the screw which acts clockwise, whereas the screw exerts a moment of equal magnitude in the counterclockwise direction on the screwdriver.

If several couples act on a rigid body they may be appropriately moved and rotated and then added to yield a resultant moment

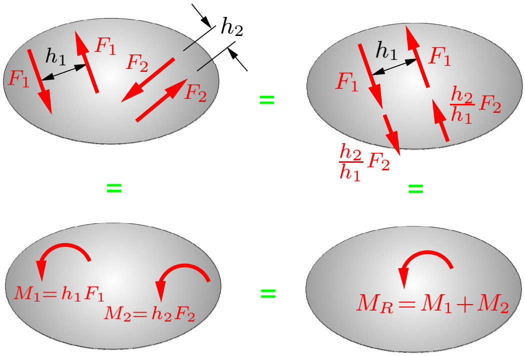

Fig. 3.6

M_R (Fig. 3.6). The couple moments are added algebraically taking into account their algebraic signs (given by their respective senses of rotation):

$$M_R = \sum M_i \,. \tag{3.6}$$

If the sum of the moments is zero, the resultant couple moment, and therefore the tendency to rotate the body, vanish. Thus, the equilibrium condition for a system of couple moments is

$$M_R = \sum M_i = 0 \,. \tag{3.7}$$

3.1.2 Moment of a Force

A force acting on a rigid body is a sliding vector: it may be moved along its line of action without changing the effect on the body. With the aid of the notion of the couple moment, we now will investigate how a force may be moved to a parallel line of action. Consider in Fig. 3.7 a force F whose line of action f is assumed to be moved to the line f', which is parallel to f and passes through point 0. The perpendicular distance of the two lines is given by h. As a first step, the forces F and $-F$ are introduced on the line f'. These two forces are in equilibrium. One of the forces and the originally given force (action line f) represent a couple. The couple moment is given by its magnitude $M^{(0)} = h\,F$ and the sense of rotation. The system consisting of force F with action line f' and couple moment $M^{(0)} = h\,F$ is statically equivalent to force F with action line f. The quantity $M^{(0)} = hF$ is called the *moment of the force* F about (with respect to) point 0. The superscript (0) indicates the *reference point*. The perpendicular distance of point 0 from the action line f is called the *lever arm* of force F with respect to 0. The *sense of rotation* of the moment is given by the sense of rotation of force F about 0.

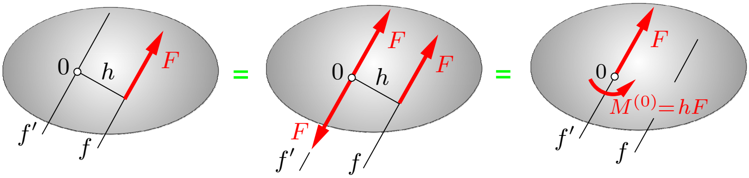

Fig. 3.7

It should be noted that a couple moment is independent of the point of reference, whereas the magnitude and sense of rotation of the moment of a force depend on this point.

Often it is advantageous to replace a force $\boldsymbol{F}$ by its Cartesian components $\boldsymbol{F}_x = F_x\,\boldsymbol{e}_x$ and $\boldsymbol{F}_y = F_y\,\boldsymbol{e}_y$ (Fig. 3.8). Adopting the commonly used sign convention that a moment is *positive* if it tends to rotate the body *counterclockwise* when viewed from above ($\curvearrowleft$), the moment of the force F about point 0 in Fig. 3.8 is given by $M^{(0)} = h\,F$. Using the relations

$$h = x\sin\alpha - y\cos\alpha$$

and

$$\sin\alpha = F_y/F\,, \quad \cos\alpha = F_x/F$$

the moment can also be represented as

$$M^{(0)} = hF = \left(x\,\frac{F_y}{F} - y\,\frac{F_x}{F}\right)F = x\,F_y - y\,F_x\,. \tag{3.8}$$

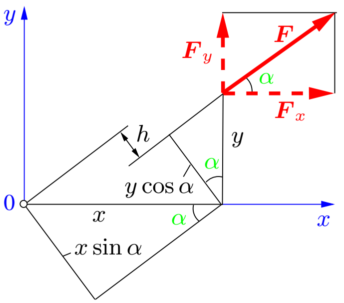

Fig. 3.8

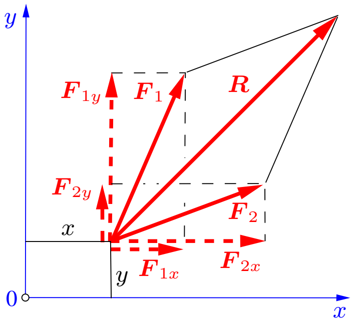

Fig. 3.9

Hence, the moment is equal to the sum of the moments of the force components about 0. Note the senses of rotation of the respective components: they determine the algebraic signs in the summation.

Consider now two forces F_1 and F_2 and their resultant R (Fig. 3.9). The moments of the two forces with respect to point 0 are

$$M_1^{(0)} = x\,F_{1y} - y\,F_{1x}\,, \quad M_2^{(0)} = x\,F_{2y} - y\,F_{2x}\,,$$

and their sum is given by

$$M_1^{(0)} + M_2^{(0)} = x\,(F_{1y} + F_{2y}) - y\,(F_{1x} + F_{2x}) = x\,R_y - y\,R_x = M_R^{(0)}\,.$$

Therefore, it is immaterial whether the forces are added first and then the moment is determined or if the sum of the individual moments is calculated. This property holds for an arbitrary number of forces:

The sum of the moments of single forces is equal to the moment of their resultant.

3.1.3 Resultant of Systems of Coplanar Forces

Consider a rigid body that is subjected to a general system of coplanar forces (Fig. 3.10). To investigate how this system can be reduced to a simpler system, a reference point A is chosen and the action lines of the forces are moved without changing their

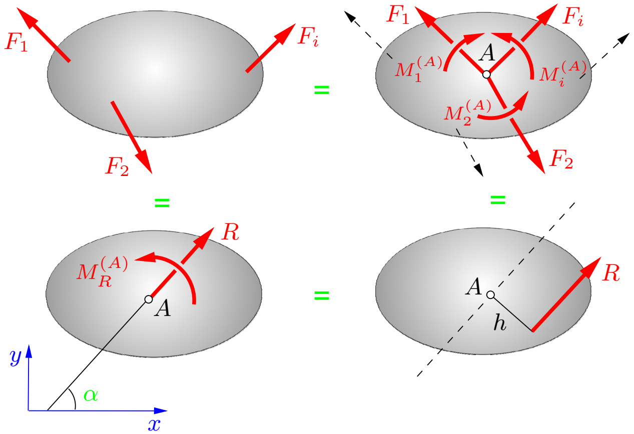

Fig. 3.10

directions until they pass through A. To avoid changing the effect of the forces on the body, the respective moments of the forces about A must be introduced. Hence, the given general system of forces is replaced by a system of concurrent forces and a system of moments. These two systems can be reduced to a resultant force R with the components R_x and R_y and a resultant moment $M_R^{(A)}$. According to (2.7) and (3.6), they are given by

$$R_x = \sum F_{ix}\,, \quad R_y = \sum F_{iy}\,, \quad M_R^{(A)} = \sum M_i^{(A)}\,. \qquad (3.9)$$

The magnitude and direction of the resultant force can be calculated from

$$R = \sqrt{R_x^2 + R_y^2}\,, \quad \tan\alpha = \frac{R_y}{R_x}\,. \qquad (3.10)$$

The system of the resultant R (action line through A) and the moment $M_R^{(A)}$ may be further simplified. It is equivalent to the single force R alone if the action line is moved appropriately. The perpendicular distance h (Fig. 3.10) must be chosen in such a way that the moment $M_R^{(A)}$ equals hR, i.e., $hR = M_R^{(A)}$, which yields

$$h = \frac{M_R^{(A)}}{R}\,. \qquad (3.11)$$

If $M_R^{(A)} = 0$ and $R \neq 0$, Equation (3.11) gives $h = 0$. In this case the action line of the resultant of the general system of forces passes through A. On the other hand, if $R = 0$ and $M_R^{(A)} \neq 0$, a further reduction is not possible: the system of forces is reduced to only a moment (i.e., a couple), which is independent of the choice of the reference point.

Equations (3.9) to (3.11) can be used to calculate the magnitude and direction of the resultant as well as the location of its action line.

Example 3.1 A disc is subjected to four forces as shown in Fig. 3.11a. The forces have the given magnitudes F or $2\,F$, respectively. **E3.1**

Determine the magnitude and direction of the resultant and the location of its line of action.

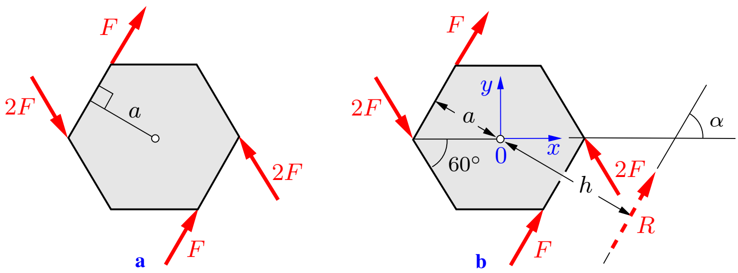

Fig. 3.11

Solution We choose a coordinate system x, y (Fig. 3.11b), and its origin 0 is taken as the point of reference. According to the sign convention, positive moments tend to rotate the disk counterclockwise (↶). Thus, from (3.9) we obtain

$$R_x = \sum F_{ix} = 2\,F\cos 60^\circ + F\cos 60^\circ + F\cos 60^\circ - 2\,F\cos 60^\circ = F\,,$$

$$R_y = \sum F_{iy} = -\,2\,F\sin 60^\circ + F\sin 60^\circ + F\sin 60^\circ + 2\,F\sin 60^\circ = \sqrt{3}\,F\,,$$

$$M_R^{(0)} = \sum M_i^{(0)} = 2\,a\,F + a\,F + 2\,a\,F - a\,F = 4\,a\,F\,,$$

which yield (see (3.10))

$$\underline{\underline{R}} = \sqrt{R_x^2 + R_y^2} = \underline{\underline{2\,F}}\,, \quad \tan\alpha = \frac{R_y}{R_x} = \sqrt{3} \quad \rightarrow \quad \underline{\underline{\alpha = 60^\circ}}\,.$$

The perpendicular distance of the resultant from point 0 follows from (3.11):

$$\underline{\underline{h}} = \frac{M_R^{(0)}}{R} = \frac{4\,a\,F}{2\,F} = \underline{\underline{2\,a}}\,.$$

3.1.4 Equilibrium Conditions

As shown in Section 3.1.3 a general system of coplanar forces can be reduced to a system of concurrent forces and a system of moments with respect to an arbitrary reference point A. The system of moments consists of the moments of the forces and of possible couple moments. The systems will be subjected to the conditions of equilibrium (2.11) and (3.7), respectively. Hence, a rigid body under the action of a general system of coplanar forces is in equilibrium if the following equilibrium conditions are satisfied:

$$\sum F_{ix} = 0\,, \quad \sum F_{iy} = 0\,, \quad \sum M_i^{(A)} = 0\,. \qquad (3.12)$$

The number of equilibrium conditions (three) equals the number of the possible motions (three) of a body in a coplanar problem: translations in the x- and y-directions, respectively, and a rotation about an axis that is perpendicular to the x,y-plane. The body is said to have three *degrees of freedom*.

It is now shown that the point of reference in the moment equation of (3.12) can be chosen arbitrarily. In order to do this, we formulate the moment equation with respect to point A (see Fig. 3.12):

$$\begin{aligned}\sum M_i^{(A)} &= \sum\{(x_i - x_A)F_{iy} - (y_i - y_A)F_{ix}\} \\ &= \sum(x_iF_{iy} - y_iF_{ix}) - x_A\sum F_{iy} + y_A\sum F_{ix} \qquad (3.13) \\ &= \sum M_i^{(B)} - x_A\sum F_{iy} + y_A\sum F_{ix}\,.\end{aligned}$$

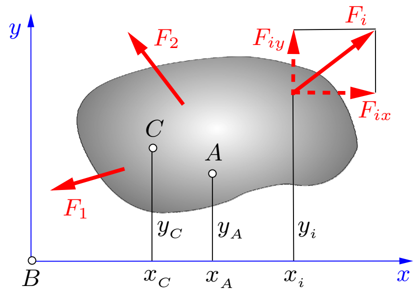

Fig. 3.12

If the equilibrium conditions (3.12) are satisfied, Equation (3.13) immediately yields $\sum M_i^{(B)} = 0$. On the other hand, if $\sum F_{ix} = 0$, $\sum F_{iy} = 0$ and $\sum M_i^{(B)} = 0$, then $\sum M_i^{(A)} = 0$ also has to be satisfied. Therefore it is immaterial which point is chosen as the point of reference.

Instead of using *two* force conditions and *one* moment condition, *one* force condition and *two* moment conditions may be applied. Introducing the conditions

$$\sum F_{ix} = 0\,, \quad \sum M_i^{(A)} = 0\,, \quad \sum M_i^{(B)} = 0 \tag{3.14}$$

into (3.13), $\sum F_{iy} = 0$ is also satisfied if $x_A \neq 0$. Hence, the equilibrium conditions (3.14) are equivalent to the conditions (3.12) if the two points A and B are not lying on a straight line (here the y-axis) that is perpendicular to the direction of the force equilibrium (here the x-direction). Similarly, the conditions

$$\sum F_{iy} = 0\,, \quad \sum M_i^{(A)} = 0\,, \quad \sum M_i^{(B)} = 0 \tag{3.15}$$

also lead to $\sum F_{ix} = 0$ if $y_A \neq 0$.

Three points A, B and C may also be chosen and only moment equations of equilibrium used, as follows:

$$\sum M_i^{(A)} = 0\,, \quad \sum M_i^{(B)} = 0\,, \quad \sum M_i^{(C)} = 0\,. \tag{3.16}$$

These equations are equivalent to (3.12) if the points A, B and C are not lying on a straight line. In order to prove this statement we use (3.13) and the corresponding relation for an arbitrary point C:

$$\begin{aligned} \sum M_i^{(A)} &= \sum M_i^{(B)} - x_A \sum F_{iy} + y_A \sum F_{ix}\,, \\ \sum M_i^{(C)} &= \sum M_i^{(B)} - x_C \sum F_{iy} + y_C \sum F_{ix}\,. \end{aligned} \tag{3.17}$$

Introducing (3.16) yields

$$-x_A \sum F_{iy} + y_A \sum F_{ix} = 0\,, \quad -x_C \sum F_{iy} + y_C \sum F_{ix} = 0\,,$$

and eliminating $\sum F_{iy}$ and $\sum F_{ix}$, respectively, leads to

$$\Big(-x_C\,\frac{y_A}{x_A} + y_C\Big) \sum F_{ix} = 0\,, \quad \Big(-x_C + \frac{x_A}{y_A}\,y_C\Big) \sum F_{iy} = 0\,.$$

Therefore, $\sum F_{ix} = 0$ and $\sum F_{iy} = 0$ is ensured if the terms in the parentheses are nonzero, i.e., if $y_A/x_A \neq y_C/x_C$. This means that the points A and C, respectively, must not lie on the same straight line passing through the origin B of the coordinate system.

In principle, it is irrelevant whether one applies the equilibrium conditions (3.12), (3.14) or (3.16) to solve a given problem. In practice, however, it may be advantageous to use one form or the other.

To apply a moment equilibrium condition (e.g., $\sum M_i^{(A)} = 0$), it is necessary to specify a reference point and a positive sense of rotation (e.g., counterclockwise). In the following, the symbol $\overset{\curvearrowleft}{A}:$ is used to signify that the sum of all moments about point A must be equal to zero and that moments in the direction of the curved arrow are taken to be positive. This notation is analogous to the notation for the equilibrium of forces (e.g. $\rightarrow$:).

Consider again a general system of coplanar forces. According to (3.12) and to the results of Section 3.1.3, one can always reduce the system to one of the following four cases:

1. *Resultant* does not pass through the reference point A (Fig. 3.13a):

$$\boldsymbol{R} \neq \boldsymbol{0}\,, \quad M^{(A)} \neq 0\,.$$

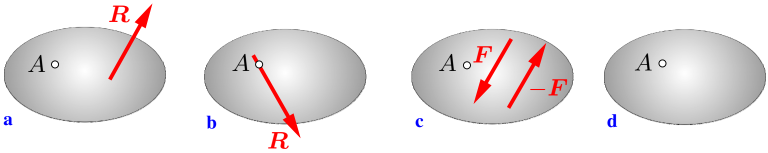

Fig. 3.13

2. *Resultant* passes through the reference point A (Fig. 3.13b):

$$\boldsymbol{R} \neq \mathbf{0}\,, \quad M^{(A)} = 0\,.$$

3. *Couple* (independent of the reference point A)(Fig. 3.13c):

$$\boldsymbol{R} = \mathbf{0}\,, \quad M^{(A)} = M \neq 0\,.$$

4. *Equilibrium* (Fig. 3.13d):

$$\boldsymbol{R} = \mathbf{0}\,, \quad M^{(A)} = 0\,.$$

E3.2

Example 3.2 The beam shown in Fig. 3.14a can rotate about its support (see Chapter 5). It is loaded by two forces F_1 and F_2. Its weight may be neglected.

Determine the location of the support so that the beam is in equilibrium. Find the force A exerted on the beam from the support.

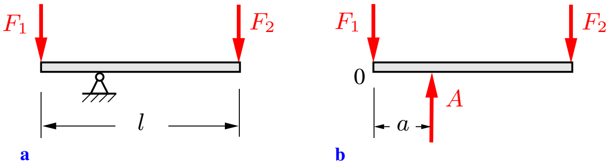

Fig. 3.14

Solution The required distance of the support from point 0 is denoted by a (Fig. 3.14b). The beam is isolated in the free-body diagram and the force A (= support reaction, see Chapter 5) is introduced. Since the forces F_1 and F_2 act in the vertical direction (the horizontal components are zero), force A also has to be vertical. This follows from the equilibrium condition in the horizontal direction.

The reference point for the equilibrium condition of moments may be chosen arbitrarily. It is, however, practical to choose a point on the action line of one of the forces. Then the lever arm of this force is zero and the force does not appear in the moment equation. If point 0 is chosen, the equilibrium conditions are (the force equilibrium in the horizontal direction is identically satisfied)

$$\uparrow: \quad A - F_1 - F_2 = 0\,, \qquad \overset{\curvearrowleft}{0}: \quad a\,A - l\,F_2 = 0\,.$$

This yields

$$\underline{\underline{A = F_1 + F_2}}\,, \qquad \underline{\underline{a = \frac{F_2}{F_1 + F_2}\,l}}\,.$$

As a check, point A is chosen as the reference point. Then the moment equilibrium is given by

$$\overset{\curvearrowleft}{A}: \quad a\,F_1 - (l - a)\,F_2 = 0\,,$$

which leads to the same result as found above.

E3.3 **Example 3.3** A cable is guided over an ideal pulley and subjected to forces S_1 and S_2, which act under the given angles α and β (Fig. 3.15a). The two forces are in equilibrium.

If force S_1 is given, determine the required force S_2 and the force exerted at 0 from the support on the pulley.

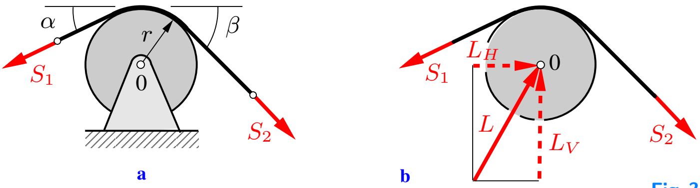

Fig. 3.15

Solution In order to solve the problem, the pulley is isolated (Fig. 3.15b). The solution of the first part of the problem is found with the aid of the equilibrium of moments about point 0. The force L acting at 0 on the pulley has no lever arm, and in an ideal pulley there exists no moment induced by friction. Therefore, the moment equilibrium condition yields

$$\overset{\curvearrowleft}{0}: \quad r\,S_1 - r\,S_2 = 0 \quad \rightarrow \quad \underline{\underline{S_2 = S_1}}\,.$$

This result is already known from Section 2.4, Fig. 2.11b.

Since the direction of the force L is unknown, it is resolved into its components, L_H and L_V, in the horizontal and the vertical directions, respectively. The equilibrium conditions

$$\uparrow: \quad L_V - S_1 \sin\alpha - S_2 \sin\beta = 0\,,$$
$$\rightarrow: \quad L_H - S_1 \cos\alpha + S_2 \cos\beta = 0$$

and $S_2 = S_1$ lead to the result

$$\underline{\underline{L_V = S_1(\sin\alpha + \sin\beta)}}\,, \quad \underline{\underline{L_H = S_1(\cos\alpha - \cos\beta)}}\,.$$

In the special case of $\alpha = \beta$, we get $L_V = 2\,S_1 \sin\alpha$, $L_H = 0$. If $\alpha = \beta = \pi/2$, then $L_V = 2\,S_1$.

Example 3.4 A homogeneous beam (length $4\,a$, weight W) is suspended at C by a rope. The beam touches the smooth vertical walls at A and B (Fig. 3.16a). **E3.4**

Find the force in the rope and the contact forces at A and B.

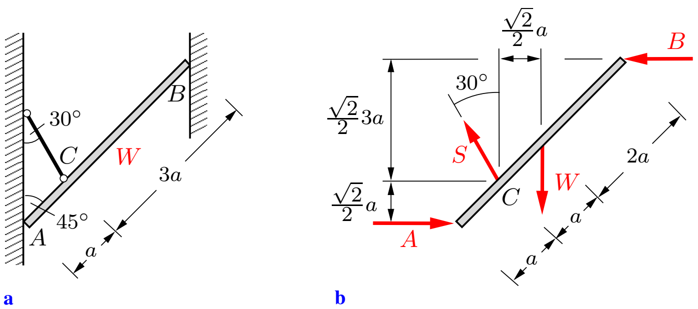

Fig. 3.16

Solution We isolate the beam by cutting the rope and removing the two walls. The free-body diagram (Fig. 3.16b) shows the contact forces A and B acting perpendicularly to the planes of contact (smooth walls), the internal force S in the rope and the weight W (acting at the center of the beam, compare Chapter 4). To formulate the sum of the moments, point C is chosen as reference point; the lever arms of the forces follow from simple geometrical relations. The equilibrium conditions then are given by

$$\uparrow: \quad S\cos 30^\circ - W = 0\,,$$

$$\rightarrow: \quad A - B - S\sin 30^\circ = 0\,,$$

$$\stackrel{\curvearrowleft}{C}: \quad \frac{\sqrt{2}}{2}\,a\,A - \frac{\sqrt{2}}{2}\,a\,W + \frac{\sqrt{2}}{2}\,3\,a\,B = 0\,.$$

With $\cos 30^\circ = \sqrt{3}/2$, $\sin 30^\circ = 1/2$ the three unknown forces are obtained as

$$\underline{\underline{S = \frac{2\sqrt{3}}{3}\,W}}\,, \quad \underline{\underline{A = \frac{1+\sqrt{3}}{4}\,W}}\,, \quad \underline{\underline{B = \frac{3-\sqrt{3}}{12}\,W}}\,.$$

E3.5

Example 3.5 A beam (length $l = \sqrt{2}\,r$, weight negligible) lies inside a smooth spherical shell with radius r (Fig. 3.17a).

If a weight W is attached to the beam, determine the distance x from the left end point of the beam required to keep the beam in equilibrium with the angle $\alpha = 15^\circ$. Calculate the contact forces at A and B.

Solution The free-body diagram (Fig. 3.17b) shows the forces acting on the isolated beam. The contact forces are orthogonal to the respective contact planes (smooth surfaces). Therefore, they are directed towards the center 0 of the sphere. The isosceles triangle $0\,A\,B$ displays a right angle at 0 because of the given lengths r and $l = \sqrt{2}\,r$. Forces A and B are inclined with angles 60° and 30°, respectively. If forces A and B are resolved into their components perpendicular and parallel to the beam, the following equilibrium conditions are obtained:

$$\uparrow: \quad A\sin 60^\circ + B\sin 30^\circ - W = 0\,,$$

$$\rightarrow: \quad A\cos 60^\circ - B\cos 30^\circ = 0\,,$$

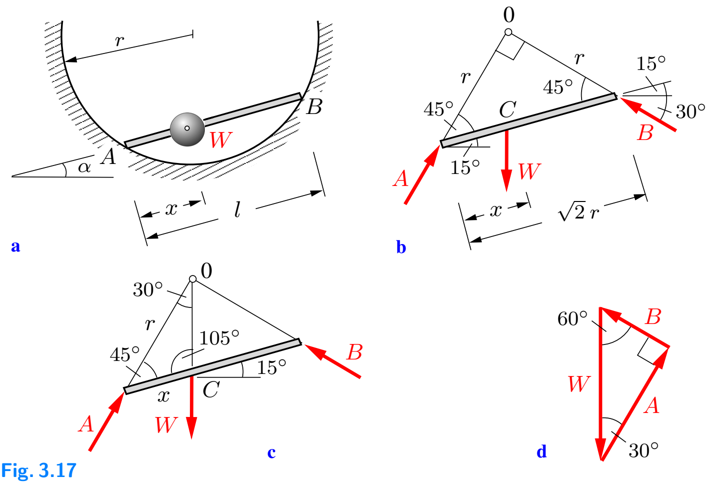

Fig. 3.17

$$\overset{\curvearrowleft}{C}: \quad -x(A\sin 45^\circ) + (l-x)(B\sin 45^\circ) = 0\,.$$

These are three equations for the three unknowns A, B and x. The force equations lead to the contact forces (note: $\sin 30^\circ = \cos 60^\circ = 1/2$, $\sin 60^\circ = \cos 30^\circ = \sqrt{3}/2$):

$$\underline{\underline{A = \frac{\sqrt{3}}{2}\,W}}\,, \qquad \underline{\underline{B = \frac{1}{2}\,W}}\,.$$

If these results and $\sin 45^\circ = \sqrt{2}/2$ are introduced into the moment equation, the required distance is obtained:

$$\underline{\underline{x}} = l\,\frac{B}{A+B} = \underline{\underline{\frac{l}{\sqrt{3}+1}}}\,.$$

The problem may also be solved graphic-analytically. If three forces are in equilibrium, they must be concurrent forces. Since the action lines of A and B intersect at 0, the action line of W also must pass through this point (Fig. 3.17c). The law of sines is now applied to the triangle $0\,A\,C$. With $\sin 105^\circ = \sin(45^\circ + 60^\circ) = \sin 45^\circ \cos 60^\circ + \cos 45^\circ \sin 60^\circ = (\sqrt{2}/4)(1+\sqrt{3})$ and $r = l/\sqrt{2}$ it leads again to

$$\underline{\underline{x}} = r\,\frac{\sin 30°}{\sin 105°} = \frac{l}{\sqrt{2}}\,\frac{1/2}{\frac{\sqrt{2}}{4}(1+\sqrt{3})} = \underline{\underline{\frac{l}{1+\sqrt{3}}}}\,.$$

The contact forces A and B are determined from a sketch of the force plan (not to scale, see Fig. 3.17d):

$$\underline{\underline{A}} = W\cos 30° = \underline{\underline{\frac{\sqrt{3}}{2}\,W}}\,, \quad \underline{\underline{B}} = W\sin 30° = \underline{\underline{\frac{W}{2}}}\,.$$

E3.6 **Example 3.6** A lever (length l) that is subjected to a vertical force F (Fig. 3.18a) exerts a contact force on a circular cylinder (radius r, weight W). The weight of the lever may be neglected. All surfaces are smooth.

Determine the contact force between the cylinder and the floor if the height h of the step is equal to the radius r of the cylinder.

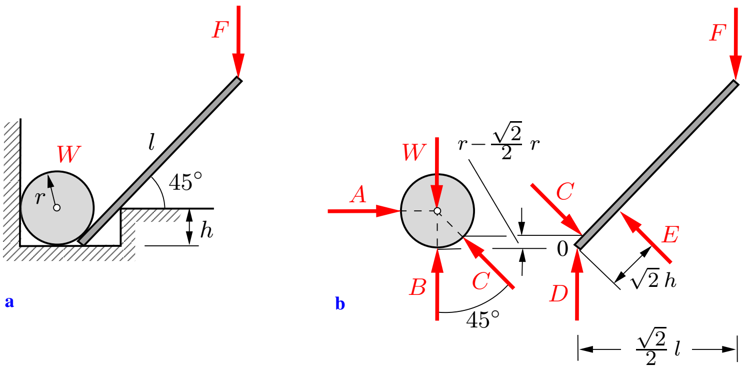

Fig. 3.18

Solution The cylinder and the lever are isolated and the contact forces A to E, which are perpendicular to the planes of contact at the respective points of contact, are introduced (Fig. 3.18b). Note that the floor and the lever represent the planes of contact at D and E, respectively. The equilibrium conditions for the lever are given by

$$\rightarrow: \quad \frac{\sqrt{2}}{2}\,C - \frac{\sqrt{2}}{2}\,E = 0\,,$$

$$\uparrow: \quad D - \frac{\sqrt{2}}{2} C + \frac{\sqrt{2}}{2} E - F = 0\,,$$
$$\overset{\curvearrowleft}{0}: \quad \sqrt{2}\, r \Big(1 - \frac{\sqrt{2}}{2}\Big) C - \sqrt{2}\, h\, E + \frac{\sqrt{2}}{2}\, l\, F = 0$$

and equilibrium at the cylinder (concurrent forces) requires

$$\rightarrow: \quad A - \frac{\sqrt{2}}{2} C = 0\,,$$
$$\uparrow: \quad B + \frac{\sqrt{2}}{2} C - W = 0\,.$$

These are five equations for the five unknown forces A to E. With $h = r$, they yield the contact force at point B:

$$\underline{\underline{B = W - \frac{l}{2\,r} F}}\,.$$

In the case of $F = (2\,r/l)W$, contact force B vanishes. For larger values of F there is no equilibrium: the cylinder will be lifted.

3.2 General Systems of Forces in Space

3.2

3.2.1 The Moment Vector

In order to investigate general systems of forces in space, the *moment vector* is now introduced. To this end, the coplanar problem that was already treated in Section 3.1.2 (see Fig. 3.8) is reconsidered in Fig. 3.19. The force $\boldsymbol{F}$, which acts in the x, y-plane, has a moment $M^{(0)}$ about point 0. With $\boldsymbol{F}_x = F_x\, \boldsymbol{e}_x$, etc., it is given by

$$M_z^{(0)} = h\, F = x\, F_y - y\, F_x \qquad (3.18)$$

(compare (3.8)). The algebraic sign (positive sense of rotation) is chosen as in Section 3.1.2. The subscript z indicates that $M_z^{(0)}$ exerts a moment about the z-axis.

The two quantities (magnitude and sense of rotation) by which a couple is defined in coplanar problems may be expressed mathematically by the moment vector

$$\boldsymbol{M}_z^{(0)} = M_z^{(0)} \boldsymbol{e}_z \, . \tag{3.19}$$

The vector $\boldsymbol{M}_z^{(0)}$ points in the direction of the z-axis. It incorporates the magnitude $M_z^{(0)}$ and the positive sense of rotation. The positive sense of rotation is determined by the *right-hand rule* (*corkscrew rule*): if we look in the direction of the positive z-axis, a positive moment tends to rotate the body clockwise.

In order to distinguish between force vectors and moment vectors in the figures, a moment vector is represented with a double head, as shown in Fig. 3.19. Note: force vectors and moment vectors have different dimensions; therefore they can never be added.

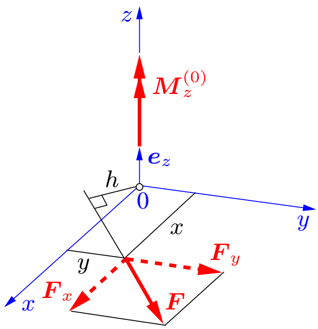

Fig. 3.19

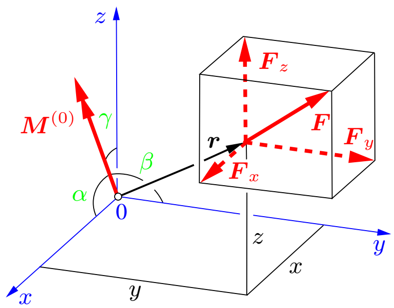

Fig. 3.20

In coplanar problems, the body can only be rotated about the z-axis. Therefore, the moment vector has only the component $M_z^{(0)}$. In spatial systems, there are three possibilities of rotation (about the three axes x, y and z). Hence, the moment vector has the three components $M_x^{(0)}$, $M_y^{(0)}$ and $M_z^{(0)}$:

$$\boldsymbol{M}^{(0)} = M_x^{(0)} \, \boldsymbol{e}_x + M_y^{(0)} \, \boldsymbol{e}_y + M_z^{(0)} \, \boldsymbol{e}_z \, . \tag{3.20}$$

Fig. 3.20 shows that the components, i.e., the moments about the coordinate axes, are obtained as follows:

$$M_x^{(0)} = yF_z - zF_y,\ M_y^{(0)} = zF_x - xF_z,\ M_z^{(0)} = xF_y - yF_x. \quad (3.21)$$

The magnitude and direction of the moment vector are given by

$$|\boldsymbol{M}^{(0)}| = M^{(0)} = \sqrt{[M_x^{(0)}]^2 + [M_y^{(0)}]^2 + [M_z^{(0)}]^2}\,,$$
$$\cos\alpha = \frac{M_x^{(0)}}{M^{(0)}}\,, \quad \cos\beta = \frac{M_y^{(0)}}{M^{(0)}}\,, \quad \cos\gamma = \frac{M_z^{(0)}}{M^{(0)}}\,. \quad (3.22)$$

Formally, the moment vector $\boldsymbol{M}^{(0)}$ may be represented by the *vector product*

$$\boldsymbol{M}^{(0)} = \boldsymbol{r} \times \boldsymbol{F}\,, \quad (3.23)$$

where vector $\boldsymbol{r}$ is the *position vector* pointing from the reference point 0 to the point of application of force $\boldsymbol{F}$, i.e., to an arbitrary point on the action line of $\boldsymbol{F}$. With

$$\boldsymbol{r} = x\,\boldsymbol{e}_x + y\,\boldsymbol{e}_y + z\,\boldsymbol{e}_z\,, \quad \boldsymbol{F} = F_x\,\boldsymbol{e}_x + F_y\,\boldsymbol{e}_y + F_z\,\boldsymbol{e}_z$$

and the following cross-products (compare Appendix A.1),

$$\begin{aligned}
\boldsymbol{e}_x \times \boldsymbol{e}_x &= \boldsymbol{0}\,, & \boldsymbol{e}_x \times \boldsymbol{e}_y &= \boldsymbol{e}_z\,, & \boldsymbol{e}_x \times \boldsymbol{e}_z &= -\boldsymbol{e}_y\,,\\
\boldsymbol{e}_y \times \boldsymbol{e}_x &= -\boldsymbol{e}_z\,, & \boldsymbol{e}_y \times \boldsymbol{e}_y &= \boldsymbol{0}\,, & \boldsymbol{e}_y \times \boldsymbol{e}_z &= \boldsymbol{e}_x\,,\\
\boldsymbol{e}_z \times \boldsymbol{e}_x &= \boldsymbol{e}_y\,, & \boldsymbol{e}_z \times \boldsymbol{e}_y &= -\boldsymbol{e}_x\,, & \boldsymbol{e}_z \times \boldsymbol{e}_z &= \boldsymbol{0}
\end{aligned}$$

equation (3.23) yields

$$\begin{aligned}
\boldsymbol{M}^{(0)} &= (x\,\boldsymbol{e}_x + y\,\boldsymbol{e}_y + z\,\boldsymbol{e}_z) \times (F_x\,\boldsymbol{e}_x + F_y\,\boldsymbol{e}_y + F_z\,\boldsymbol{e}_z)\\
&= (y\,F_z - z\,F_y)\,\boldsymbol{e}_x + (z\,F_x - x\,F_z)\,\boldsymbol{e}_y + (x\,F_y - y\,F_x)\,\boldsymbol{e}_z\\
&= M_x^{(0)}\,\boldsymbol{e}_x + M_y^{(0)}\,\boldsymbol{e}_y + M_z^{(0)}\,\boldsymbol{e}_z\,. \quad (3.24)
\end{aligned}$$

According to the properties of a cross-product, the moment vector $\boldsymbol{M}^{(0)}$ is perpendicular to the plane determined by $\boldsymbol{r}$ and $\boldsymbol{F}$ (Fig. 3.21). Its magnitude is numerically equal to the area of the parallelogram formed by $\boldsymbol{r}$ and $\boldsymbol{F}$:

$$M^{(0)} = r\,F\sin\varphi = h\,F\,. \quad (3.25)$$

Hence, the moment is equal to the product of the lever arm h and the force F.

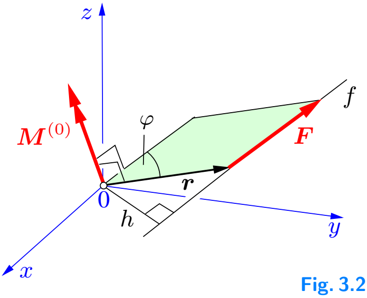

Fig. 3.21

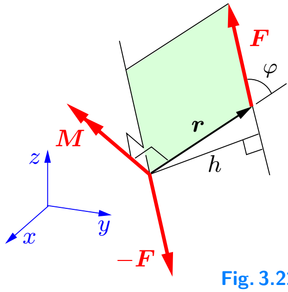

Fig. 3.22

The moment of a couple in space (Fig. 3.22) may be represented by the same formalism. Here,

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} . \tag{3.26}$$

The vector $\boldsymbol{r}$ points from an arbitrary point on the action line of $-\boldsymbol{F}$ to an arbitrary point on the action line of $\boldsymbol{F}$. As before, the moment vector $\boldsymbol{M}$ is orthogonal to the plane determined by $\boldsymbol{r}$ and $\boldsymbol{F}$. Its sense of rotation follows from the rule of the right-handed screw and its magnitude is numerically equal to the area of the parallelogram formed by $\boldsymbol{r}$ and $\boldsymbol{F}$ (lever arm times force):

$$M = h\,F . \tag{3.27}$$

The properties of *couple moments* and of *moments of a force* in space correspond to their properties in coplanar problems. In a plane, couple moments may be moved without changing the effect on a rigid body. In space, the vector of a couple moment can be moved parallel to its line of action and along this line without changing the effect. Whereas the force vector is bound to its action line (sliding vector) the vector of a couple moment is a *free vector*.

If a body in space is subjected to several couple moments $\boldsymbol{M}_i$, the resultant moment $\boldsymbol{M}_R$ is obtained as the vector sum

$$\boldsymbol{M}_R = \sum \boldsymbol{M}_i , \tag{3.28}$$

which reads in components

$$M_{Rx} = \sum M_{ix}\,, \quad M_{Ry} = \sum M_{iy}\,, \quad M_{Rz} = \sum M_{iz}\,. \qquad (3.29)$$

If the sum of the moments is zero, the resulting moment $\boldsymbol{M}_R$ and hence the rotational effect on the body vanishes. Then the moment equilibrium condition

$$\boldsymbol{M}_R = \sum \boldsymbol{M}_i = \boldsymbol{0} \qquad (3.30)$$

is satisfied. In components,

$$\sum M_{ix} = 0\,, \quad \sum M_{iy} = 0\,, \quad \sum M_{iz} = 0\,. \qquad (3.31)$$

Example 3.7 A rope passes over an ideal pulley as shown in Fig. 3.23a. It carries a crate with weight W and is held at point C. The radius of the pulley may be neglected. **E3.7**

Determine the resultant moment of the forces in the rope about point A.

Solution The internal forces S_1 and S_2 in the rope are made visible by cuts through the rope. Their action on point B of the structure is shown in Fig. 3.23b. Since the bearing friction of the pulley is negligible (ideal pulley) both forces are equal, and equilibrium at the crate yields $S_1 = S_2 = W$.

To represent the moments of the forces, a coordinate system is introduced. Moment $\boldsymbol{M}_1^{(A)}$ of force $\boldsymbol{S}_1$ about A (represented as a column vector) has a component only in the x-direction:

$$\boldsymbol{M}_1^{(A)} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} aW\,. \qquad \text{(a)}$$

Moment $\boldsymbol{M}_2^{(A)}$ of force $\boldsymbol{S}_2$ may be obtained with the aid of the

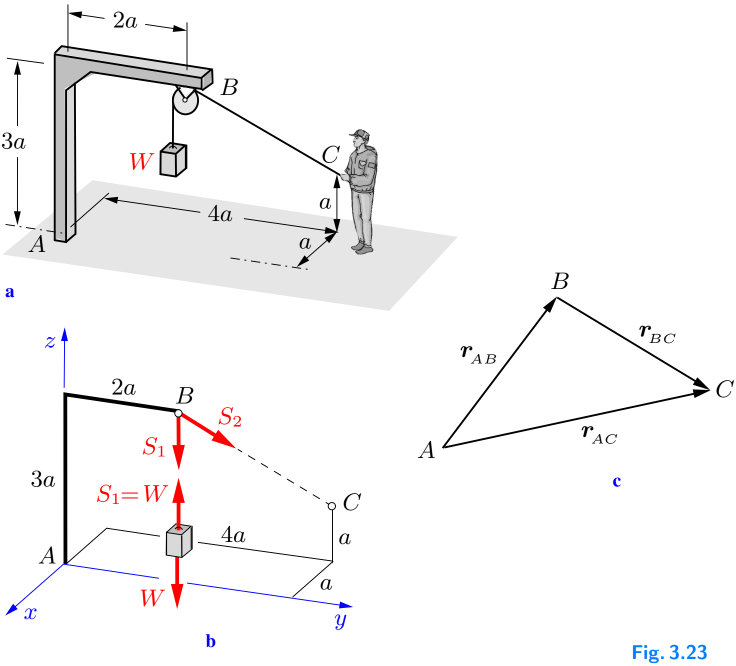

Fig. 3.23

cross-product (see (3.23))

$$\boldsymbol{M}_2^{(A)} = \boldsymbol{r}_{AB} \times \boldsymbol{S}_2 \,. \tag{b}$$

The vector $\boldsymbol{r}_{AB}$ from reference point A to the point of application B of force $\boldsymbol{S}_2$ is given by

$$\boldsymbol{r}_{AB} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} a \,. \tag{c}$$

The force $\boldsymbol{S}_2$ may be represented by its magnitude $S_2 = W$ and the unit vector $\boldsymbol{e}_2$, which points in the direction of $\boldsymbol{S}_2$, i.e., $\boldsymbol{S}_2 = S_2\,\boldsymbol{e}_2$. To obtain vector $\boldsymbol{e}_2$, we first give vector $\boldsymbol{r}_{BC}$, which points from B to C (see Fig. 3.23c):

$$\boldsymbol{r}_{BC} = \boldsymbol{r}_{AC} - \boldsymbol{r}_{AB} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} a - \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} a = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} a \,.$$

If this vector is divided by its magnitude $|\boldsymbol{r}_{BC}|$, the unit vector $\boldsymbol{e}_2$ is obtained:

$$\boldsymbol{e}_2 = \frac{\boldsymbol{r}_{BC}}{|\boldsymbol{r}_{BC}|} = \frac{1}{a\sqrt{1+4+4}} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} a = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} .$$

Therefore,

$$\boldsymbol{S}_2 = S_2\, \boldsymbol{e}_2 = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} W . \tag{d}$$

With (c) and (d) the vector product (b) yields (compare Appendix (A.31))

$$\begin{aligned} \boldsymbol{M}_2^{(A)} &= \boldsymbol{r}_{AB} \times \boldsymbol{S}_2 = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ 0 & 2a & 3a \\ -W/3 & 2W/3 & -2W/3 \end{vmatrix} \\ &= (-4/3 - 2)aW\, \boldsymbol{e}_x - aW\, \boldsymbol{e}_y + (2/3)aW\, \boldsymbol{e}_z . \end{aligned}$$

This vector may be written as a column vector:

$$\boldsymbol{M}_2^{(A)} = \frac{1}{3} \begin{pmatrix} -10 \\ -3 \\ 2 \end{pmatrix} aW . \tag{e}$$

The resultant moment $\boldsymbol{M}_R^{(A)}$ of the forces $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ about point A is the sum of the moments (a) and (e), (compare (3.32)):

$$\underline{\underline{\boldsymbol{M}_R^{(A)}}} = \boldsymbol{M}_1^{(A)} + \boldsymbol{M}_2^{(A)} = \underline{\underline{\frac{1}{3} \begin{pmatrix} -16 \\ -3 \\ 2 \end{pmatrix} aW}} .$$

3.2.2 Equilibrium Conditions

Consider a general system of forces in space (Fig. 3.24). This system can be reduced to a statically equivalent system that consists of a resultant force and a resultant moment. Similar to the procedure used in a coplanar problem (compare Section 3.1.3), an

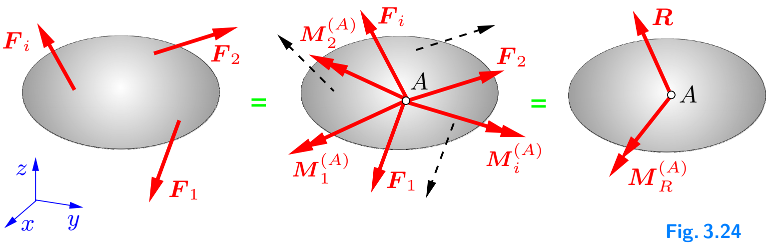

Fig. 3.24

arbitrary reference point A is chosen in space. The forces $\boldsymbol{F}_i$ are then moved to parallel lines of action that pass through this point. Since the effect of the forces on the body must not be changed, the corresponding moments $\boldsymbol{M}_i^{(A)}$ of the forces have to be introduced. Now the system of concurrent forces and the system of moments may be represented by the resultant force $\boldsymbol{R}$ and the resultant moment $\boldsymbol{M}_R^{(A)}$, respectively:

$$\boldsymbol{R} = \sum \boldsymbol{F}_i\,, \quad \boldsymbol{M}_R^{(A)} = \sum \boldsymbol{M}_i^{(A)}\,. \tag{3.32}$$

The resultant force $\boldsymbol{R}$ is independent of the choice of point A; the resultant moment $\boldsymbol{M}_R^{(A)}$, however, depends on this choice. Hence, there are many possible ways to reduce a given general system of forces to a resultant force and a resultant moment.

A general system of forces is in equilibrium if the resultant force $\boldsymbol{R}$ and the resultant moment $\boldsymbol{M}_R^{(A)}$ about an arbitrary point A vanish:

$$\sum \boldsymbol{F}_i = \boldsymbol{0}\,, \quad \sum \boldsymbol{M}_i^{(A)} = \boldsymbol{0}\,. \tag{3.33}$$

In components,

$$\begin{aligned} \sum F_{ix} &= 0\,, & \sum M_{ix}^{(A)} &= 0\,,\\ \sum F_{iy} &= 0\,, & \sum M_{iy}^{(A)} &= 0\,,\\ \sum F_{iz} &= 0\,, & \sum M_{iz}^{(A)} &= 0\,. \end{aligned} \tag{3.34}$$

The fact that there are six scalar equilibrium conditions corresponds to the six degrees of freedom of a rigid body in space: translations in the x-, y- and z-directions and rotations about the corresponding coordinate axes. It can be shown that the reference point A may be chosen arbitrarily, as in a coplanar problem.

Consider now the special case of a system of parallel forces. Let, for example, all the forces act in the z-direction. Then $F_{ix} = 0$ and $F_{iy} = 0$, and the equilibrium conditions (3.34) reduce to

$$\sum F_{iz} = 0\,, \quad \sum M_{ix}^{(A)} = 0\,, \quad \sum M_{iy}^{(A)} = 0\,. \tag{3.35}$$

In this case, the equilibrium conditions in the x- and y-directions for the forces and the moment equation about the axis through A which is parallel to the z-axis are identically satisfied.

Example 3.8 A rectangular block (lengths a, b and c) is subjected to six forces, F_1 to F_6 (Fig. 3.25a). **E3.8**

Calculate the resultant $\boldsymbol{R}$, the resultant moments $\boldsymbol{M}_R^{(A)}$ and $\boldsymbol{M}_R^{(B)}$ with respect to points A and B and their magnitudes. Assume $F_1 = F_2 = F$, $F_3 = F_4 = 2\,F$, $F_5 = F_6 = 3\,F$, $b = a$, $c = 2\,a$.

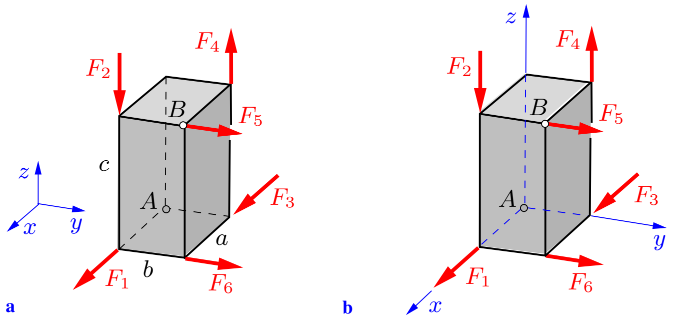

Fig. 3.25

Solution The components of the resultant force vector are obtained as the sum of the given force components:

$$R_x = F_1 + F_3 = 3F, \quad R_y = F_5 + F_6 = 6F, \quad R_z = -F_2 + F_4 = F.$$

Hence, $\boldsymbol{R}$ can be written as the column vector (see Appendix A.1)

$$\underline{\underline{\boldsymbol{R} = \begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} F}}\,, \quad \underline{\underline{R}} = \sqrt{3^2+6^2+1^2}\,F = \underline{\underline{\sqrt{46}\,F}}\,.$$

To determine the resultant moment about point A, the coordinate system is chosen such that its origin is at A (Fig. 3.25b). The components of the moment then are obtained as

$$M_{Rx}^{(A)} = \sum M_{ix}^{(A)} = b\,F_4 - c\,F_5 = -\,4\,a\,F\,,$$

$$M_{Ry}^{(A)} = \sum M_{iy}^{(A)} = a\,F_2 = a\,F\,,$$

$$M_{Rz}^{(A)} = \sum M_{iz}^{(A)} = a\,F_5 + a\,F_6 - b\,F_3 = 4\,a\,F\,,$$

and vector $\boldsymbol{M}_R^{(A)}$ can be written as

$$\underline{\underline{\boldsymbol{M}_R^{(A)} = \begin{pmatrix} M_{Rx}^{(A)} \\ M_{Ry}^{(A)} \\ M_{Rz}^{(A)} \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 4 \end{pmatrix} a\,F}}\,,$$

$$\underline{\underline{M_R^{(A)}}} = \sqrt{4^2+1^2+4^2}\,a\,F = \underline{\underline{\sqrt{33}\,a\,F}}\,.$$

Similarly, for point B

$$M_{Rx}^{(B)} = b\,F_2 + c\,F_6 = 7\,a\,F\,,$$

$$M_{Ry}^{(B)} = -\,c\,F_1 - c\,F_3 + a\,F_4 = -\,4\,a\,F\,,$$

$$M_{Rz}^{(B)} = b\,F_1 = a\,F$$

and

$$\underline{\underline{\boldsymbol{M}_R^{(B)} = \begin{pmatrix} 7 \\ -4 \\ 1 \end{pmatrix} a\,F}}\,, \quad \underline{\underline{M_R^{(B)}}} = \sqrt{7^2+4^2+1^2}\,a\,F = \underline{\underline{\sqrt{66}\,a\,F}}$$

are obtained. The resultant moments about A and B, respectively, are different!

Example 3.9 A homogeneous plate with weight W is supported by six bars and loaded by a force F (Fig. 3.26a). E3.9

Calculate the forces in the bars.

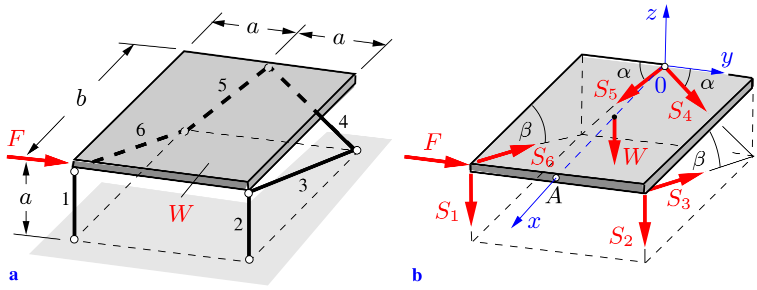

Fig. 3.26

Solution First, the free-body diagram is sketched (Fig. 3.26b). It displays the weight W (acting at the center of the plate, compare Chapter 4), the load F and the internal forces S_1 to S_6 in the bars (these are assumed to be tensile forces). In addition, the auxiliary angles α and β are introduced. Choosing the coordinate system such that as many moments as possible about its origin are zero, the following equilibrium conditions are obtained:

$$
\begin{aligned}
\sum F_{ix} = 0 :& \quad -S_3\cos\beta - S_6\cos\beta = 0\,,\\
\sum F_{iy} = 0 :& \quad S_4\cos\alpha - S_5\cos\alpha + F = 0\,,\\
\sum F_{iz} = 0 :& \quad -S_1 - S_2 - S_3\sin\beta - S_6\sin\beta - S_4\sin\alpha\\
& \quad -S_5\sin\alpha - W = 0\,,\\
\sum M_{ix}^{(0)} = 0 :& \quad a\,S_1 - a\,S_2 + a\,S_6\sin\beta - a\,S_3\sin\beta = 0\,,\\
\sum M_{iy}^{(0)} = 0 :& \quad \frac{b}{2}W + b\,S_1 + b\,S_2 + b\,S_6\sin\beta + b\,S_3\sin\beta = 0\,,\\
\sum M_{iz}^{(0)} = 0 :& \quad b\,F + a\,S_3\cos\beta - a\,S_6\cos\beta = 0\,.
\end{aligned}
$$

Using the trigonometrical relations

$$\cos\alpha = \sin\alpha = \frac{a}{\sqrt{2\,a^2}} = \frac{\sqrt{2}}{2}\,,$$

$$\cos\beta = \frac{b}{\sqrt{a^2+b^2}}\,, \qquad \sin\beta = \frac{a}{\sqrt{a^2+b^2}}$$

the first and the sixth equilibrium condition lead to

$$\underline{\underline{S_3 = -\,S_6 = -\frac{\sqrt{a^2+b^2}}{2\,a}\,F\,.}}$$

Then the fourth and fifth equations yield

$$\underline{\underline{S_1 = -\frac{W}{4} - \frac{F}{2}}}\,, \qquad \underline{\underline{S_2 = -\frac{W}{4} + \frac{F}{2}}}\,.$$

Finally, from the second and third equations

$$\underline{\underline{S_4 = -\frac{1}{\sqrt{2}}\left(\frac{W}{2} + F\right)}}\,, \qquad \underline{\underline{S_5 = -\frac{1}{\sqrt{2}}\left(\frac{W}{2} - F\right)}}$$

are obtained. As a check, it is verified that the equilibrium of moments is satisfied if an axis is chosen that is parallel to the y-axis and passes through point A:

$$\begin{aligned}\sum M_{iy}^{(A)} &= -\frac{b}{2}\,W - b\,S_4 \sin\alpha - b\,S_5 \sin\alpha \\ &= -b\left[\frac{W}{2} - \frac{1}{\sqrt{2}}\left(\frac{W}{2}+F\right)\frac{\sqrt{2}}{2} - \frac{1}{\sqrt{2}}\left(\frac{W}{2}-F\right)\frac{\sqrt{2}}{2}\right] = 0.\end{aligned}$$

E3.10

Example 3.10 An angled member is in equilibrium under the action of four forces (Fig. 3.27a). The forces are perpendicular to the plane determined by the member; the weight of the member is negligible.

If the force F is given, calculate the forces A, B and C.

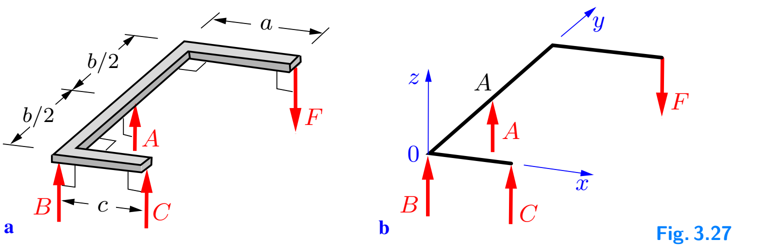

Fig. 3.27

Solution We draw the free-body diagram and introduce a coordinate system (Fig. 3.27b). If the origin 0 is chosen as the reference point, the equilibrium conditions (3.35) are

$$\sum F_{iz} = 0 : \quad A + B + C - F = 0\,,$$
$$\sum M_{ix}^{(0)} = 0 : \quad \frac{b}{2}\,A - b\,F = 0\,,$$
$$\sum M_{iy}^{(0)} = 0 : \quad -c\,C + a\,F = 0\,.$$

They have the solution

$$\underline{\underline{A = 2\,F}}\,, \quad \underline{\underline{C = \frac{a}{c}\,F}}\,, \quad \underline{\underline{B = -\left(1 + \frac{a}{c}\right)F}}\,.$$

As a check, point A is chosen as the reference point instead of point 0. Then the moment equation

$$\sum M_{ix}^{(A)} = 0 : \quad -\frac{b}{2}\,B - \frac{b}{2}\,C - \frac{b}{2}\,F = 0$$

is used instead of the second equation above, which leads to the same solution.

3.3 Supplementary Problems

3.3

Detailed solutions to most of the following examples are given in (**A**) D. Gross et al. *Formeln und Aufgaben zur Technischen Mechanik 1*, Springer, Berlin 2011 or (**B**) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer, Berlin 2011.

Example 3.11 A uniform pole (length l, weight W) leans against a corner as shown in Fig. 3.28. A rope S prevents the pole from sliding. All surfaces are smooth. E3.11

Determine the force S in the rope.

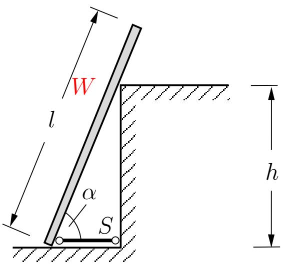

Fig. 3.28

Result: see (**B**) $S = \frac{Wl}{2h}\sin^2\alpha\cos\alpha\,.$

E3.12 **Example 3.12** A uniform beam (length l, weight W) is inserted into an opening (Fig. 3.29). The surfaces are smooth.

Calculate the magnitude of the force F required to hold the beam in equilibrium. Is the result valid for an arbitrary ratio a/l?

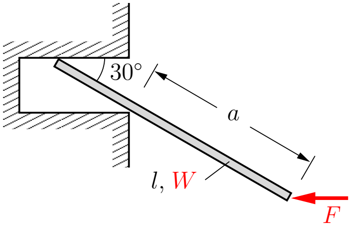

Fig. 3.29

Result: see (**B**) $F = \frac{\sqrt{3}}{6-8a/l}W\,.$

The result is valid only if the contact forces between the beam and the surfaces of the opening are positive, which leads to the requirement $3/8 < a/l < 3/4$.

E3.13 **Example 3.13** Two smooth rollers (each having weight W and radius r) are connected by a rope (length a) as shown in Fig. 3.30. A lever (length l) subjected to a vertical force F exerts contact forces on the rollers.

Determine the contact forces between the rollers and the horizontal plane.

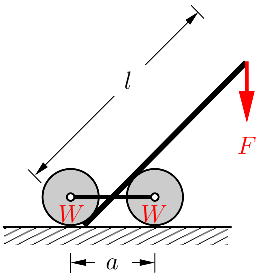

Fig. 3.30

Results: see (**A**)

$$N_1 = W - F\frac{l}{a}\sqrt{1-4(\frac{r}{a})^2}\,, \qquad N_2 = W + F\frac{l}{a}\sqrt{1-4(\frac{r}{a})^2}$$

where N_1 and N_2 are the contact forces acting on the left and on the right roller, respectively.

E3.14

Example 3.14 Two smooth spheres (each having weight W and radius r) rest in a thin-walled circular cylinder (weight Q, radius $R = 4r/3$) as shown in Fig. 3.31.

Find the magnitude of Q required to prevent the cylinder from falling over.

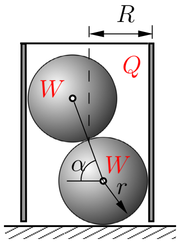

Fig. 3.31

Result: see (**A**) $Q > W/2$.

E3.15

Example 3.15 A rigid body is subjected to three forces: $\boldsymbol{F}_1 = F(-2,3,1)^T$, $\boldsymbol{F}_2 = F(7,1,-4)^T$, $\boldsymbol{F}_3 = F(3,-1,-3)^T$. Their points of application are given by the position vectors $\boldsymbol{r}_1 = a(4,3,2)^T$, $\boldsymbol{r}_2 = a(3,2,4)^T$, $\boldsymbol{r}_3 = a(3,5,0)^T$.

Determine the resultant force $\boldsymbol{R}$ and the resultant moment $\boldsymbol{M}_R^{(A)}$ with respect to point A given by $\boldsymbol{r}_A = a(3,2,1)^T$.

Results: $\boldsymbol{R} = F(8,3,-6)^T$, $\boldsymbol{M}_R^{(A)} = aF(-15,15,-4)^T$.

E3.16

Example 3.16 A plate in the form of a rectangular triangle (weight negligible) is supported by six bars. It is subjected to the forces F and Q (Fig. 3.32).

Calculate the forces in the bars.

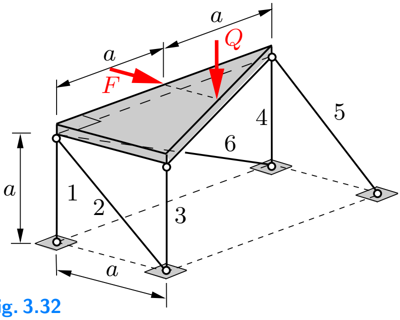

Fig. 3.32

Results: see (**A**) $S_1 = F/2, \quad S_2 = S_5 = -\sqrt{2}\,F/2,$
$S_3 = -Q/2, \quad S_4 = (F-Q)/2, \quad S_6 = 0.$

E3.17

Example 3.17 A homogeneous rectangular plate (weight W) is supported by six bars. The plate is subjected to a vertical load F (Fig. 3.33).

Calculate the forces in the bars.

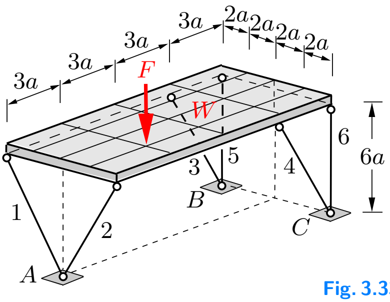

Fig. 3.33

Results: see (**B**) $S_1 = S_2 = -\sqrt{13}\,(2W + 3F)/24,$

$$S_3 = S_4 = 0, \quad S_5 = -(2W - F)/8, \quad S_6 = -(2W + 3F)/8.$$

E3.18

Example 3.18 A rectangular plate of negligible weight is suspended by three vertical wires as shown in Fig. 3.34.

a) Assume that the plate is subjected to a concentrated vertical force Q. Determine the location of the point of application of Q so that the forces in the wires are equal.

b) Calculate the forces in the wires if the plate is subjected to a vertical constant area load p.

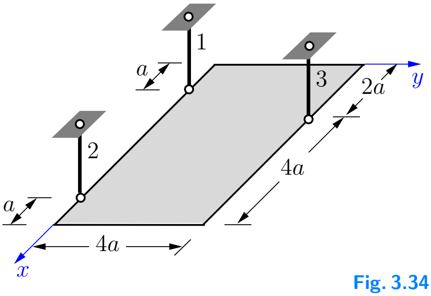

Fig. 3.34

Results: see (**A**) a) $x_Q = 8\,a/3\,, \quad y_Q = 4\,a/3.$

b) $S_1 = 3pa^2\,, \quad S_2 = 9pa^2\,, \quad S_3 = 12pa^2.$

Example 3.19 The circular arch in Fig. 3.35 is subjected to a uniform tangential line load q_0.

Determine the resultant force $\boldsymbol{R}$ and the resultant moment $\boldsymbol{M}_R^{(A)}$ with respect to the center A of the circle. If the load is reduced to a single force alone, find the corresponding line of action.

E3.19

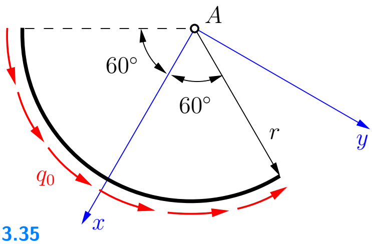

Fig. 3.35

Results: see (**B**) $R_x = 0\,, \quad R_y = \sqrt{3}\,q_0 r\,, \quad M_R^{(A)} = 2\pi q_0 r^2/3.$

$x_R = 1.21\,r\,, \quad y_R$ arbitrary.

Example 3.20 A sphere (weight W_S) is held between a beam (weight W_B) and a wall as shown in Fig. 3.36. The surface of the sphere is smooth. The beam is supported by a hinge at A and a rope at B.

Calculate the force S in the rope.

E3.20

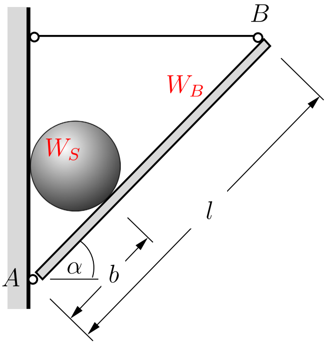

Fig. 3.36

Result: $$S = \frac{W_B}{2}\cot\alpha + \frac{b\,W_S}{l\,\sin\alpha\cos\alpha}\,.$$

3.4

3.4 Summary

- A couple consists of two forces having equal magnitudes, parallel action lines and opposite directions.
  The effect of a couple is uniquely given by its moment $\boldsymbol{M}$. The couple moment is determined by its magnitude $M = h\,F$ and its sense of rotation.
  A couple moment is not bound to an action line: it can be applied at arbitrary points of a rigid body without changing its effect on the body.
- The moment of a force $\boldsymbol{F}$ with respect to a point A is defined as $\boldsymbol{M}^{(A)} = \boldsymbol{r} \times \boldsymbol{F}$ where $\boldsymbol{r}$ is the vector pointing from point A to an arbitrary point on the action line of $\boldsymbol{F}$. The moment $\boldsymbol{M}^{(A)}$ has the magnitude $M^{(A)} = h\,F$ (h = lever arm) and a sense of rotation.
  In the case of a coplanar system of forces, the moment vector has only one component $M^{(A)} = h\,F$ (perpendicular to the plane) and a sense of rotation about A.
- A general system of forces can be reduced to a resultant force $\boldsymbol{R}$ and a resultant moment $\boldsymbol{M}_R^{(A)}$ with respect to an arbitrary point A.
- A general system of forces is in equilibrium if the resultant force $\boldsymbol{R}$ and the resultant moment $\boldsymbol{M}_R^{(A)}$ vanish:

  $$\sum \boldsymbol{F}_i = \boldsymbol{0}\,, \qquad \sum \boldsymbol{M}_i^{(A)} = \boldsymbol{0}\,.$$

  These equations represent three force conditions and three moment conditions in spatial problems.
  In the case of a coplanar system of forces, the equilibrium conditions reduce to

  $$\sum F_{ix} = 0\,, \qquad \sum F_{iy} = 0\,, \qquad \sum M_i^{(A)} = 0\,.$$