Chapter 9

9

Static and Kinetic Friction

9 Static and Kinetic Friction

Objectives: Bodies in contact exert a force on each other. In the case of ideally smooth surfaces, this force acts perpendicularly to the contact plane. If the surfaces are rough, however, there may also be a tangential force component. Students will learn that this tangential component is a reaction force if the bodies adhere, and an active force if the bodies slip. After studying this chapter, students should be able to apply the Coulomb theory of friction to determine the forces in systems with contact.

9.1 Basic Principles

In this textbook so far it has been assumed that all bodies considered have *smooth* surfaces. Then, according to Chapter 2.4, only forces perpendicular to the contact plane can be transferred between two bodies in contact. This is a proper description of the mechanical behavior if the tangential forces occurring in reality due to the *roughness* of the surfaces can be neglected. In this chapter, we will address problems for which this simplification is not valid. First, let us consider the following simple example.

Fig. 9.1a shows a box with weight W resting on a rough horizontal surface. If a sufficiently small horizontal force F is applied to the box, it can be expected to stay at rest. The reason for this behavior is the transfer of a tangential force between the base and the box due to the surfaces' roughness. This tangential force is frequently called *static friction force* H.

Using the notation given in the free-body diagram (Fig. 9.1b), the equilibrium conditions for this system lead to the following relations:

$$\uparrow: \quad N = W\,, \qquad \rightarrow: \quad H = F\,. \tag{9.1}$$

The equilibrium of moments would additionally yield the location of N, which is not needed here.

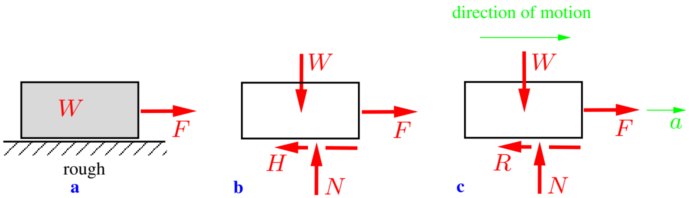


Fig. 9.1

If the force F exceeds a certain limit, the box slips on the base (Fig. 9.1c). Again, a force is transferred between the box and the base due to the roughness of the surfaces. This tangential force is commonly called *kinetic friction force* R. Since it tends to prevent the movement, its orientation is opposed to the direction of the motion. Assuming the acceleration a to be positive to the right, the second equilibrium condition (9.1) is replaced by Newton's second law (cf. Volume 3)

$$\text{mass times acceleration} = \sum \text{forces}\,,$$

i.e., in the current example,

$$m\,a = F - R\,. \tag{9.2}$$

Here, the kinetic friction force R is as yet unknown.

Even though static and kinetic friction are caused by the roughness of the surfaces, their specific nature is fundamentally different. The static friction force H is a *reaction force* that can be determined from the equilibrium conditions for statically determinate systems without requiring additional assumptions. On the other hand, the kinetic friction force R represents an *active force* depending on the surface characteristics of the bodies in contact. In order to keep this fundamental difference in mind, it must be carefully distinguished between *static friction* corresponding to friction in a position of rest and *kinetic friction* related to the movement of bodies in contact. Accordingly, we distinguish precisely between the corresponding *static friction forces* and the *kinetic friction forces*.

Friction forces are altered strongly if other materials are placed between the bodies considered. Every car driver or bicyclist knows the differences between a dry, wet, or even icy road. Lubricants can significantly decrease friction in the case of moveable machine parts. Due to the introductory nature of this chapter, “fluid friction” and related phenomena will not be addressed.

The following investigations are restricted to the case of so-called *dry* friction occurring due to the roughness of any solid body’s surface.

Static and kinetic friction are of great practical relevance. It is static friction that enables motion on solid ground. For instance, wheels of vehicles adhere to the surface of the road. Forces needed for acceleration or deceleration are transferred at the contact areas. If these forces cannot be applied, for example in the case of icy roads, the wheels slide and the desired state of motion cannot be attained.

Screws and nails are able to perform their tasks due to their roughness. This effect is reinforced in the case of screw anchors with the increased asperity of their surfaces.

On the other hand, kinetic friction is often undesirable due to the resulting loss of energy. In the contact areas, mechanical energy is converted to thermal energy, resulting in a temperature increase. While one tries to increase static friction on slippery roads by spreading sand, the kinetic friction of rotating machine parts is reduced by lubricants, as mentioned before. Again, it becomes obvious that static and kinetic friction need to be distinguished carefully.

9.2 Coulomb Theory of Friction

9.2

Let us first consider static friction, using the example in Fig. 9.1b. As long as F is smaller than a certain limit F_0, the box stays at rest and equilibrium yields $H = F$. The tangential force H attains its maximum value $H = H_0$ for $F = F_0$. Charles Augustine de Coulomb (1736–1806) showed in his experiments that this limit force H_0 is in a first approximation proportional to the normal force N:

$$H_0 = \mu_0 \, N \, . \tag{9.3}$$

The proportionality factor μ_0 is commonly referred to as the *coefficient of static friction*. It depends solely on the roughness of surfaces in contact, irrespective of their size. Table 9.1 shows several numerical values for different configurations. Note that coefficients derived from experiments can only be given within certain tolerance limits; the coefficient for "wood on wood", for example, strongly depends on the type of wood and the treatment of the surfaces. It should also be noted that (9.3) relates the tangential force and the normal force only in the limit case when slip is impending; it is *not* an equation for the static friction force H.

A body adheres to its base as long as the *condition of static friction*

$$H \leq H_0 = \mu_0 \, N \tag{9.4a}$$

Table 9.1

	coefficient of static friction μ_0	coefficient of kinetic friction μ
steel on ice	0.03	0.015
steel on steel	$0.15 \ldots 0.5$	$0.1 \ldots 0.4$
steel on Teflon	0.04	0.04
leather on metal	0.4	0.3
wood on wood	0.5	0.3
car tire on snow	$0.7 \ldots 0.9$	$0.5 \ldots 0.8$
ski on snow	$0.1 \ldots 0.3$	$0.04 \ldots 0.2$

is fulfilled. The orientation of the friction force H always opposes the direction of the motion that would occur in the absence of friction. For complex systems, this orientation is often not easily identifiable, and must therefore be assumed arbitrarily. The algebraic sign of the result shows if this assumption was correct (compare e.g., Section 5.1.3). In view of a possible negative algebraic sign of H, we generalise condition (9.4a) as follows:

$$|H| \leq H_0 = \mu_0 \, N \,. \tag{9.4b}$$

The normal force N and the static friction force H can be assembled into a resultant force K (Fig. 9.2a). Its direction is defined by the angle φ, which can be derived from

$$\tan\varphi = \frac{H}{N} \,.$$

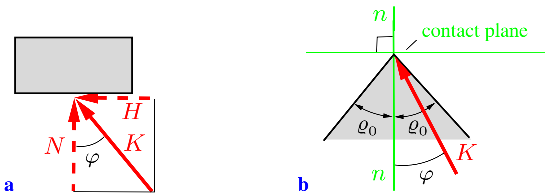


Fig. 9.2

Referring to the limit angle φ_l as ϱ_0 (in the case $H = H_0$) yields

$$\tan\varphi_l = \tan\varrho_0 = \frac{H_0}{N} = \frac{\mu_0 N}{N} = \mu_0 \,.$$

This so-called "angle of static friction" ϱ_0 is related to the coefficient of static friction:

$$\tan\varrho_0 = \mu_0 \,. \tag{9.5}$$

A "static friction wedge" for a plane problem (Fig. 9.2b) is obtained by drawing the static friction angle ϱ_0 on both sides of the normal n. If K is located within this wedge, $H < H_0$ is valid and the body stays at rest.

In three-dimensional space, the static friction angle ϱ_0 can also be interpreted graphically. A body stays at rest if the reaction force K corresponding to an arbitrarily oriented external load is located within the so-called "static friction cone". This cone of revolution around the normal n of the contact plane has an angle of aperture of $2\varrho_0$. If K is located within the static friction cone, $\varphi < \varrho_0$ and consequently $|H| < H_0$ holds (Fig. 9.3).

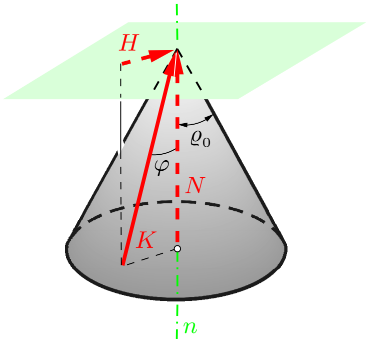


Fig. 9.3

If K lies outside of this cone, equilibrium is no longer possible: the body starts to move. We now will discuss the friction phenomena occurring in this case. Again, Coulomb demonstrated experimentally that the friction force R related to the movement is (in a good approximation)

a) proportional to the normal force N (proportionality factor μ) and
b) oriented in the opposite direction of the velocity vector while being independent of the velocity.

Consequently, the *law of friction* can be stated as follows:

$$R = \mu N . \tag{9.6}$$

The proportionality factor μ is referred to as the *coefficient of kinetic friction*. In general, its value is smaller than the formerly introduced coefficient of static friction μ_0 (compare Table 9.1).

When considering the direction of $\boldsymbol{R}$ by means of a mathematical formula, a unit vector $\boldsymbol{v}/|\boldsymbol{v}|$ oriented in the direction of the velocity vector $\boldsymbol{v}$ must be introduced. Coulomb's friction law then reads

$$\boldsymbol{R} = -\mu N \frac{\boldsymbol{v}}{|\boldsymbol{v}|} ,$$

with the minus sign indicating that the friction force acts in the opposite direction of the velocity vector. In contrast to static friction forces, the sense of direction of kinetic friction forces therefore cannot be assumed arbitrarily.

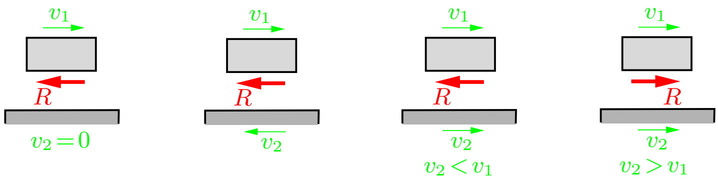


Fig. 9.4

If both the body and the base move (e.g., bulk goods slipping on a band-conveyor) then the direction of the kinetic friction force depends on the relative velocity, i.e., on the difference of the velocities v_1 and v_2 (cf. Volume 3). In Fig. 9.4, the resulting direction of the kinetic friction force exerted on the body is illustrated for different configurations.

In summary, the following three cases have to be distinguished:

a) "Static friction" $H < \mu_0\, N$	The body stays at rest; the corresponding static friction force H can be calculated from the equilibrium conditions.
b) "Limiting friction" $H = \mu_0\, N$	The body is still at rest but on the verge of moving. After a disturbance, the body will be set in motion due to the fact that $\mu < \mu_0$.
c) "Kinetic friction" $R = \mu\, N$	If the body slips, the kinetic friction force R acts as an active force.

When investigating friction phenomena, one has to distinguish between statically determinate and statically indeterminate systems, respectively. In the first case, the reaction forces H and N can be calculated from the equilibrium conditions in a first step. Subsequently, fulfillment of the static friction condition (9.4b) can be checked. In statically indeterminate problems, determining the reaction forces H and N is not possible. In this case, one can only formulate equilibrium conditions as well as static friction conditions at those locations where the bodies adhere. Then a system of algebraic equations and inequalities needs to be solved. However, in this case it is often easier to investigate only the limiting friction case.

Example 9.1 A block with weight W resting on a rough inclined plane (angle of slope α, coefficient of static friction μ_0) is subjected to an external force F (Fig. 9.5a). **E9.1**

Specify the range of F such that the block stays at rest.

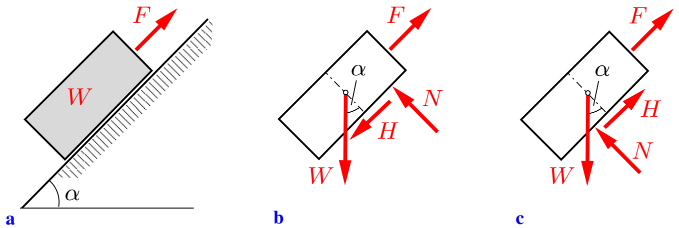


Fig. 9.5

Solution If F is a sufficiently large positive force, the block would move upwards without static friction. The static friction force H then is oriented downwards (Fig. 9.5b). From the equilibrium conditions

$$\nearrow: \quad F - W\sin\alpha - H = 0\,, \qquad \nwarrow: \quad N - W\cos\alpha = 0$$

the static friction force and the normal force can be determined:

$$H = F - W\,\sin\alpha\,, \qquad N = W\,\cos\alpha\,.$$

Inserting these forces into (9.4a) establishes the range of F fulfilling the static friction condition:

$$F - W\sin\alpha \leq \mu_0\,W\cos\alpha \quad \rightarrow \quad F \leq W(\sin\alpha + \mu_0\cos\alpha)\,.$$

Using (9.5), we can reformulate the preceding relation in terms of the static friction angle ϱ_0:

$$F \leq W\,(\sin\alpha + \tan\varrho_0\cos\alpha) = W\,\frac{\sin(\alpha+\varrho_0)}{\cos\varrho_0}\,. \qquad \text{(a)}$$

On the other hand, if F is too small, the block may slip downwards due to its weight. The static friction force preventing this motion is then oriented upwards according to Fig. 9.5c. In this case, from the equilibrium equations

$$\nearrow: \quad F - W\sin\alpha + H = 0\,, \qquad \nwarrow: \quad N - W\cos\alpha = 0$$

and the static friction condition

$$H \leq \mu_0\,N\,,$$

we obtain the following inequality:

$$W\sin\alpha - F \leq \mu_0\,W\cos\alpha\,.$$

This result can be formulated in terms of the static friction angle:

$$F \geq W\,(\sin\alpha - \mu_0\cos\alpha) = W\,\frac{\sin(\alpha-\varrho_0)}{\cos\varrho_0}\,. \qquad \text{(b)}$$

Summarizing the results (a) and (b) yields the following admissible range for the force F:

$$W\,\frac{\sin(\alpha-\varrho_0)}{\cos\varrho_0} \le F \le W\,\frac{\sin(\alpha+\varrho_0)}{\cos\varrho_0}\,. \qquad \text{(c)}$$

Assuming, for example, the case "steel on steel", friction coefficient $\mu_0 = 0.15$ from Table 9.1 yields $\varrho_0 = \arctan 0.15 = 0.149$. Choosing furthermore $\alpha = 10^\circ \mathrel{\widehat{=}} 0.175\,\text{rad}$, we obtain from (c)

$$W\,\frac{\sin(0.175-0.149)}{\cos 0.149} \le F \le W\,\frac{\sin(0.175+0.149)}{\cos 0.149}$$

or

$$0.026\,W \le F \le 0.32\,W\,.$$

In this numerical example, the block stays at rest provided that F is in the range between approximately 3% and 30% of its weight W. If $\alpha < \varrho_0$, F can also take on negative values according to (c).

In the case $\alpha = \varrho_0$, the lower limit of the range of F equals zero. Therefore, the slope of the inclined plane is a direct measure of the coefficient of static friction. A body under the action of only its own weight (i.e., $F = 0$) stays at rest on an inclined plane as long as $\alpha \le \varrho_0$ holds.

Example 9.2 A man with weight Q stands on a ladder as depicted in Fig. 9.6a. E9.2

Determine the maximum position x he can reach on the ladder if a) *only* the floor and b) the floor *and* the wall have rough surfaces. The coefficient of static friction in both cases is μ_0.

Solution a) If the wall surface is smooth, the ladder is subjected only to a normal force N_B at B (Fig. 9.6b). At A, we have a normal force N_A and a static friction force H_A (opposing the movement that would occur without static friction). From the equilibrium conditions

$$\rightarrow:\; N_B - H_A = 0\,, \qquad \uparrow:\; N_A - Q = 0\,, \qquad \overset{\curvearrowleft}{A}:\; x\,Q - h\,N_B = 0$$

the normal force and static friction force at point A can be calculated:

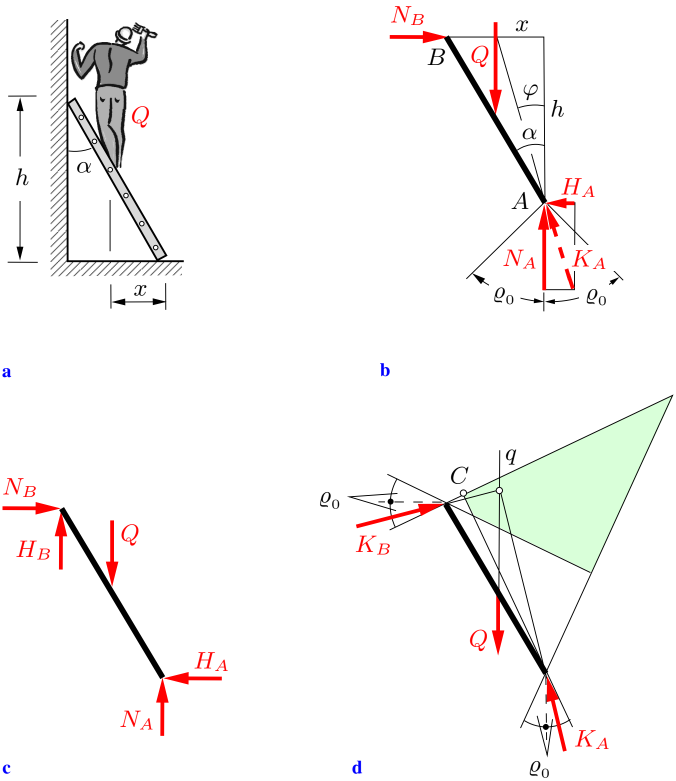


Fig. 9.6

$$H_A = \frac{x}{h} Q \,, \qquad N_A = Q \,.$$

Insertion into the condition of static friction

$$H_A \leq \mu_0 N_A$$

yields the solution

$$\frac{x}{h} Q \leq \mu_0 Q \quad \rightarrow \quad \underline{\underline{x \leq \mu_0 h}} \,.$$

This result can also be obtained in the following way: in equilibrium, the lines of action of the three forces Q, N_B, and K_A (resultant of N_A and H_A) have to intersect at *one* point (Fig. 9.6b). Thus,

$$\tan\varphi = \frac{H_A}{N_A} = \frac{x}{h}\,.$$

The line of action of the reaction force K_A must be located within the static friction cone ($\varphi \le \varrho_0$). Consequently, the ladder stays at rest provided that

$$\frac{x}{h} = \tan\varphi \le \tan\varrho_0 = \mu_0 \quad \rightarrow \quad x \le \mu_0\, h\,.$$

For $\alpha \le \varrho_0$, the stability of the ladder is ensured for all possible values of x, due to $x \le h\tan\alpha$.

b) If the surface of the wall is also *rough*, four unknown reaction forces are exerted on the ladder, according to Fig. 9.6c. However, only three equilibrium equations are available. Hence, these forces cannot be determined unambiguously and the problem is *statically indeterminate*. Nevertheless, one can calculate the admissible range of x from the equilibrium equations

$$\rightarrow: \quad N_B = H_A\,, \qquad \uparrow: \quad N_A + H_B = Q\,,$$

$$\overset{\curvearrowleft}{A}: \quad x\,Q = h\,N_B + (h\tan\alpha)\,H_B$$

and the conditions of static friction

$$H_B \le \mu_0\,N_B\,, \quad H_A \le \mu_0\,N_A\,.$$

Since the solution of the system of equations and inequalities is not straightforward, we prefer the graphical approach illustrated in Fig. 9.6d. For this purpose, the static friction cones are drawn at both points of contact. If the line of action q of the load Q is located within the domain of the overlapping cones marked in green, a multitude of possible reaction forces is conceivable; an example of *one* combination is illustrated in Fig. 9.6d. The ladder starts to slide if q is located to the left of C, since in this case the required static friction force cannot be applied anymore. Obviously, the danger of slipping is decreased or even eliminated in the case of a steeper position of the ladder.

E9.3

Example 9.3 A screw with a flat thread (coefficient of static friction μ_0, pitch h, radius r) is subjected to a vertical load F and a moment M_d as shown in Fig. 9.7a.

State the condition for equilibrium if the normal forces and the static friction forces are uniformly distributed over the screw thread.

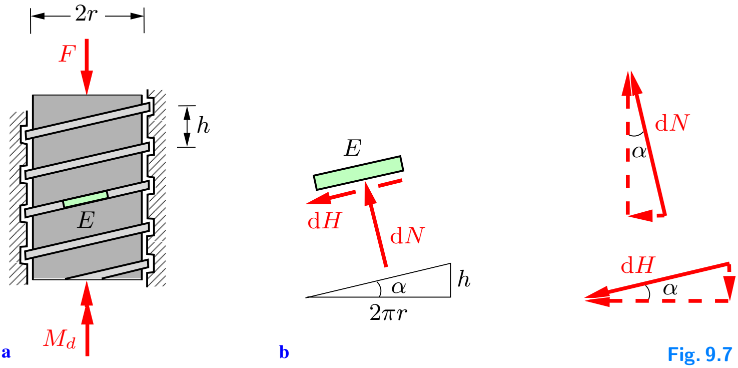


Fig. 9.7

Solution We decompose the normal force dN and the static friction force dH exerted on an element E of the thread into their horizontal and vertical components, as illustrated in Fig. 9.7b. The corresponding angle α can be determined from the pitch h and the unrolled perimeter $2\,\pi\,r$ as $\tan\alpha = h/2\,\pi\,r$. The resultant, i.e., the integral of the vertical components, has to equilibrate the external load F:

$$F = \int \mathrm{d}N\cos\alpha - \int \mathrm{d}H\sin\alpha = \cos\alpha\int \mathrm{d}N - \sin\alpha\int \mathrm{d}H\,. \quad \text{(a)}$$

Furthermore, M_d has to be in equilibrium with the moment resulting from the horizontal components:

$$M_d = \int r\,\mathrm{d}N\sin\alpha + \int r\,\mathrm{d}H\cos\alpha = r\sin\alpha\int \mathrm{d}N + r\cos\alpha\int \mathrm{d}H\,.$$

In combination with (a) we obtain

$$\int \mathrm{d}N = F\cos\alpha + \frac{M_d}{r}\sin\alpha\,, \qquad \int \mathrm{d}H = \frac{M_d}{r}\cos\alpha - F\sin\alpha\,.$$

Introducing the condition of static friction

$$|\mathrm{d}H| \le \mu_0 \, \mathrm{d}N \quad \text{or} \quad \int |\mathrm{d}H| \le \mu_0 \int \mathrm{d}N$$

yields

$$\left| \frac{M_d}{r} \cos\alpha - F \sin\alpha \right| \le \mu_0 \left(F \cos\alpha + \frac{M_d}{r} \sin\alpha \right) .$$

If $M_d/r > F \tan\alpha$, this inequality results in

$$\left| \frac{M_d}{r} - F \tan\alpha \right| = \frac{M_d}{r} - F \tan\alpha \le \mu_0 \left(F + \frac{M_d}{r} \tan\alpha \right)$$

or, using (9.5) and the addition theorem of the tangent function

$$\frac{M_d}{r} \le F \frac{\tan\alpha + \mu_0}{1 - \tan\alpha \, \mu_0} = F \frac{\tan\alpha + \tan\varrho_0}{1 - \tan\alpha \tan\varrho_0} = F \tan(\alpha + \varrho_0) .$$

If, on the other hand, $M_d/r < F \tan\alpha$, we analogously obtain from

$$\left| \frac{M_d}{r} - F \tan\alpha \right| = F \tan\alpha - \frac{M_d}{r} \le \mu_0 \left(F + \frac{M_d}{r} \tan\alpha \right)$$

the condition

$$\frac{M_d}{r} \ge F \tan(\alpha - \varrho_0) .$$

Hence, the screw is in equilibrium provided that

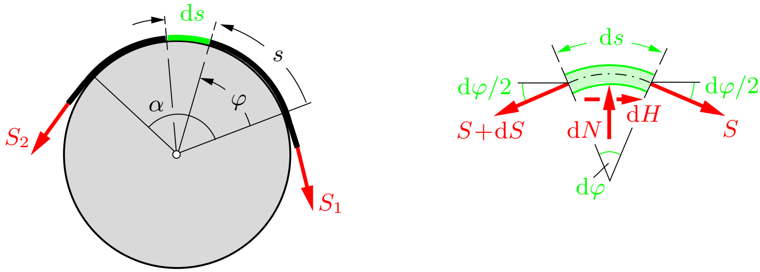
$$\underline{\underline{F \tan(\alpha - \varrho_0) \le \frac{M_d}{r} \le F \tan(\alpha + \varrho_0)}}$$

holds. In particular, if $\alpha \le \varrho_0$ (i.e., $\tan\alpha \le \mu_0$), the load F is supported only by the static friction forces and no additional external moment is required for equilibrium ($M_d = 0$). In this case, the screw is “self-locking”.

9.3 Belt Friction

9.3

If a rope wrapped around a rough post is subjected to a large force on one of its ends, a small force on the other end may be able to



a **b** **Fig. 9.8**

prevent the rope from slipping. In Fig. 9.8a, the rope is wrapped around the post at an angle α. It is assumed that the force S_2 applied to the left end of the rope is larger than the force S_1 exerted on the right end. In order to establish a relation between these forces, we draw the free-body diagram shown in Fig. 9.8b and apply the equilibrium conditions to an element of the rope with length ds. In this context, we take into account that the tension is changing by the infinitesimal force dS along ds. Since $S_2 > S_1$ holds, the rope would slip to the left in a case without friction; i.e. slipping can be prevented only if the static friction force dH is oriented to the right. The equilibrium conditions are

$$\rightarrow: \quad S\cos\frac{\mathrm{d}\varphi}{2} - (S + \mathrm{d}S)\cos\frac{\mathrm{d}\varphi}{2} + \mathrm{d}H = 0\,,$$

$$\uparrow: \quad \mathrm{d}N - S\sin\frac{\mathrm{d}\varphi}{2} - (S + \mathrm{d}S)\sin\frac{\mathrm{d}\varphi}{2} = 0\,.$$

Since dφ is infinitesimally small, we obtain $\cos(\mathrm{d}\varphi/2) \approx 1$ and $\sin(\mathrm{d}\varphi/2) \approx \mathrm{d}\varphi/2$; furthermore, the higher order term $\mathrm{d}S(\mathrm{d}\varphi/2)$ is small and can be neglected in the following. Therefore, the above relations simplify to

$$\mathrm{d}H = \mathrm{d}S\,, \quad \mathrm{d}N = S\,\mathrm{d}\varphi\,. \tag{9.7}$$

Obviously, the three unknowns H, N, and S cannot be determined from these two equations: the system is statically indeterminate. Therefore only the limiting friction case is considered, i.e., when slippage of the rope is impending. In this case (9.3) gives

$$\mathrm{d}H = \mathrm{d}H_0 = \mu_0\,\mathrm{d}N\,.$$

Applying (9.7) yields

$$\mathrm{d}H = \mu_0\, S\, \mathrm{d}\varphi = \mathrm{d}S \quad \rightarrow \quad \mu_0\, \mathrm{d}\varphi = \frac{\mathrm{d}S}{S}\,.$$

Integration over the domain of rope contact produces

$$\mu_0 \int\limits_0^{\alpha} \mathrm{d}\varphi = \int\limits_{S_1}^{S_2} \frac{\mathrm{d}S}{S} \quad \rightarrow \quad \mu_0\, \alpha = \ln \frac{S_2}{S_1}$$

or

$$S_2 = S_1\, \mathrm{e}^{\mu_0 \alpha}\,. \tag{9.8}$$

This formula for belt friction is commonly named after Leonhard Euler (1707–1783) or Johann Albert Eytelwein (1764–1848).

If, in contrast to the initial assumption, $S_1 > S_2$ holds, one simply has to exchange the subscripts to obtain

$$S_1 = S_2\, \mathrm{e}^{\mu_0 \alpha} \qquad \text{or} \qquad S_2 = S_1\, \mathrm{e}^{-\mu_0 \alpha}\,. \tag{9.9}$$

For a given S_1, the system is in equilibrium provided that the value of S_2 remains within the limits given in (9.8) and (9.9):

$$S_1\, \mathrm{e}^{-\mu_0 \alpha} \le S_2 \le S_1\, \mathrm{e}^{\mu_0 \alpha}\,. \tag{9.10}$$

The rope slips to the right if $S_2 < S_1\, \mathrm{e}^{-\mu_0 \alpha}$, whereas it slips to the left for $S_2 > S_1\, \mathrm{e}^{\mu_0 \alpha}$.

The following numerical example gives a sense of the ratio between the two forces. We assume the rope to be wrapped n-times around the post; the coefficient of static friction is given by $\mu_0 = 0.3 \approx 1/\pi$. In this case we obtain

$$\mathrm{e}^{\mu_0 2n\pi} \approx \mathrm{e}^{2n} \approx (7.5)^n \qquad \text{and} \qquad S_1 = \frac{S_2}{\mathrm{e}^{\mu_0 \alpha}} = \frac{S_2}{(7.5)^n}\,.$$

Consequently, the more the rope is wrapped around the support, the smaller is the force S_1 that is required to equilibrate the larger force S_2. This effect is taken advantage of when, for example, mooring a boat.

The Euler-Eytelwein formula can be transferred from static to kinetic belt friction by simply replacing the static coefficient of friction μ_0 with the corresponding kinetic coefficient μ. Kinetic friction may occur if either the rope slips over a fixed drum or the drum rotates while the rope is at rest. The direction of the friction force R is opposite to the direction of the relative velocity (compare Fig. 9.4). When we know in which direction R acts, we also know which of the forces S_1 or S_2 is the larger one. Then

$$\begin{aligned} &\text{for } S_2 > S_1: \quad S_2 = S_1\,\mathrm{e}^{\mu\alpha}\,, \\ &\text{for } S_2 < S_1: \quad S_2 = S_1\,\mathrm{e}^{-\mu\alpha}\,. \end{aligned} \tag{9.11}$$

E9.4

Example 9.4 The cylindrical roller shown in Fig. 9.9a is subjected to a moment M_d. A rough belt (coefficient of static friction μ_0) is wrapped around the roller and connected to a lever.

Determine the minimum value of F such that the roller stays at rest (strap brake).

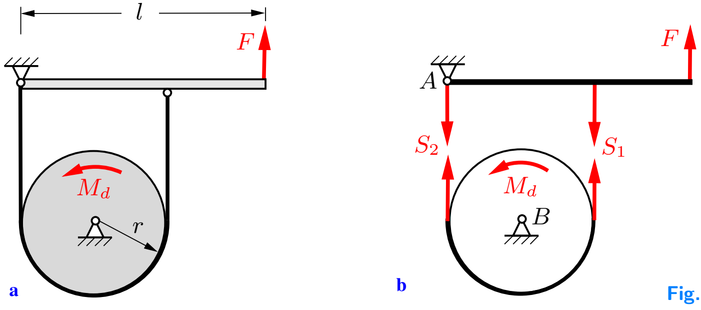


Fig. 9.9

Solution The free-body diagram of the system is given in Fig. 9.9b. The equilibrium of moments for both lever and roller yields:

$$\overset{\curvearrowleft}{A}: \quad l\,F - 2\,r\,S_1 = 0 \quad \rightarrow \quad S_1 = \frac{l}{2\,r}\,F\,,$$

$$\overset{\curvearrowleft}{B}: \quad M_d + (S_1 - S_2)r = 0 \quad \rightarrow \quad S_1 = S_2 - \frac{M_d}{r}\,.$$

Obviously, $S_2 > S_1$ is valid for equilibrium due to the orientation of M_d. Introducing the angle of wrap $\alpha = \pi$, the condition for limiting friction follows from (9.8):

$$S_2 = S_1\, e^{\mu_0 \pi}\,.$$

Hence,

$$S_1 = S_1\, e^{\mu_0 \pi} - \frac{M_d}{r} \quad \to \quad S_1 = \frac{M_d}{r(e^{\mu_0 \pi} - 1)}\,,$$

and we obtain the required force

$$\underline{\underline{F}} = \frac{2\,r}{l}\, S_1 = \underline{\underline{2\,\frac{M_d}{l}\,\frac{1}{e^{\mu_0 \pi} - 1}}}\,.$$

Example 9.5 A block with weight W lies on a rotating drum. It is held by a rope which is fixed at point A (Fig. 9.10a). E9.5

Determine the tension at A if friction acts between the drum and both the block and the rope (each with a coefficient of kinetic friction μ).

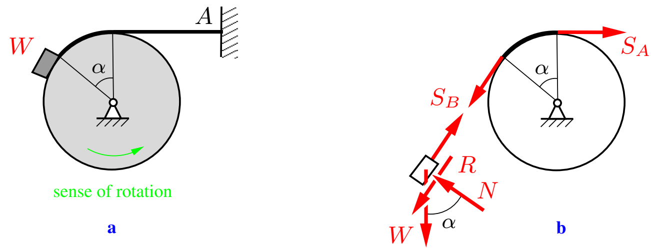


Fig. 9.10

Solution First, the bodies are separated. Due to the movement of the drum, a friction force R is exerted on the block. Its direction is given in the free-body diagram shown in Fig. 9.10b. Furthermore, $S_A > S_B$ holds. From the equilibrium conditions for the block the forces S_B and N are determined as

$$\nearrow: \quad S_B = W \sin\alpha + R\,, \qquad \nwarrow: \quad N = W \cos\alpha\,.$$

Introducing them into the friction laws for the rope (9.11) and for the block (9.6)

$$S_A = S_B\,\mathrm{e}^{\mu\alpha}\,, \qquad R = \mu\,N$$

we obtain

$$\underline{\underline{S_A}} = (W\sin\alpha + R)\,\mathrm{e}^{\mu\alpha} = \underline{\underline{W(\sin\alpha + \mu\cos\alpha)\,\mathrm{e}^{\mu\alpha}}}\,.$$

9.4

9.4 Supplementary Problems

Detailed solutions to most of the following examples are given in (**A**) D. Gross et al. *Formeln und Aufgaben zur Technischen Mechanik 1*, Springer, Berlin 2011 or (**B**) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer, Berlin 2011.

E9.6

Example 9.6 A sphere (weight W_1) and a wedge (weight W_2) are jammed between two vertical walls with rough surfaces (Fig. 9.11). The coefficient of static friction between the sphere and the left wall and between the wedge and the right wall, respectively, is μ_0. The inclined surface O of the wedge is smooth.

Determine the required value of μ_0 in order to keep the system in equilibrium.

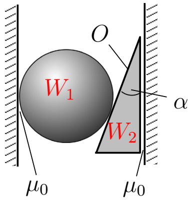


Fig. 9.11

Result: see (**B**) $\mu_0 \geq (1 + W_2/W_1)\tan\alpha$.

E9.7

Example 9.7 The excentric device in Fig. 9.12 is used to exert a large normal force onto the base. The applied force F, desired normal force N, coefficient of static friction μ_0, length l, radius r and angle α are given.

Calculate the required eccentricity e.

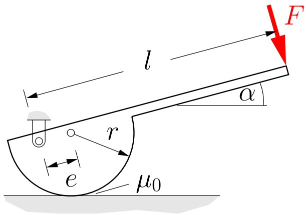


Fig. 9.12

Result: see (**A**) $e > \dfrac{l\,\dfrac{F}{N} - \mu_0\,r}{\cos\alpha - \mu_0\sin\alpha}$.

E9.8

Example 9.8 A horizontal force F is exerted on a vertical lever to prevent a load (weight W) from falling downwards (Fig. 9.13). The drum can rotate without friction about point B; the coefficient of static friction between the drum and the block is μ_0.

Determine the magnitude of the force F needed to prevent the drum from rotating.

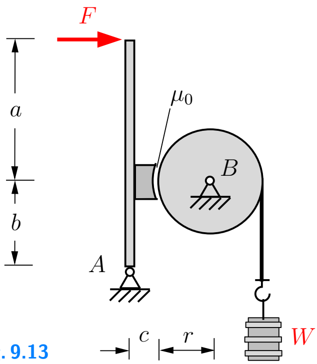


Fig. 9.13

Result: see (**B**) $F \geq \dfrac{b + \mu_0\, c}{\mu_0 (a + b)}\, W\,.$

E9.9

Example 9.9 A wall and a beam (weight $W_2 = W$) keep a roller (weight $W_1 = 3W$) in the position as shown in Fig. 9.14. The beam adheres to the rough base; all the other areas of contact are smooth.

Determine the minimum value of the coefficient of static friction μ_0 between the base and the beam in order to prevent slipping.

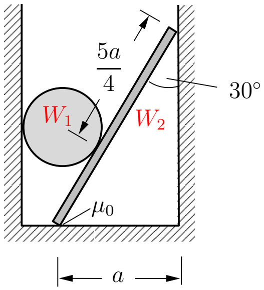


Fig. 9.14

Result: see (**B**) $\mu_0 \geq \sqrt{3}/3\,.$

E9.10

Example 9.10 A block (weight W_2) is clamped between two cylinders (each weight W_1) as shown in Fig. 9.15. All the surfaces are rough (coefficient of static friction μ_0).

Find the maximum value of W_2 in order to prevent slipping.

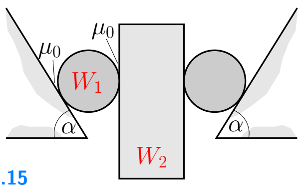


Fig. 9.15

Result: see (**A**) $W_2 < \dfrac{2\mu_0 \sin\alpha}{\cos\alpha - \mu_0 (1 + \sin\alpha)}\, W_1\,.$

E9.11 **Example 9.11** A peg A that can rotate without friction and a fixed peg B are attached to a curved member (weight W) as depicted in Fig. 9.16. The rope supports a load (weight $W_K = W/5$).

Determine the number of coils of the rope around peg B that are needed to prevent slipping. Calculate the angle β in the equilibrium position.

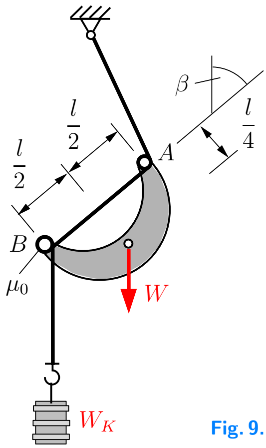


Fig. 9.16

Results: see (**B**) 3 coils are sufficient, $\beta = 19.7°$.

E9.12 **Example 9.12** A block with weight W can move vertically between two smooth walls. It is held by a rope which passes around three fixed rough pegs (coefficient of static friction μ_0) as shown in Fig. 9.17.

Calculate the force F which will ensure that the block remains suspended. Find the forces N_1, N_2 which are exerted from the block onto the walls.

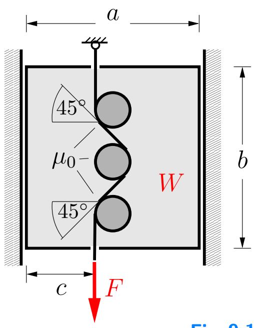


Fig. 9.17

Results: see (**A**) $F > \dfrac{W}{e^{\mu_0 \pi} - 1}$, $N_1 = N_2 = W\dfrac{a - 2c}{2b}$.

Example 9.13 Three cylinders (each radius r, weight W) are arranged as shown in Fig. 9.18. The surfaces of all contact planes are rough (coefficient of static friction μ_0).

Determine the minimum value of μ_0 in order to prevent slipping.

E9.13

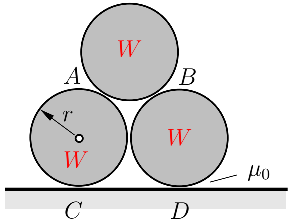


Fig. 9.18

Result: $\mu_0 \geq 0.268$.

Example 9.14 A rotating drum (weight W_1) exerts a normal force and a kinetic friction force on a wedge (Fig. 9.19). The wedge lies on a rough base (coefficient of static friction μ_0).

Find the value of the coefficient of kinetic friction μ between the drum and the wedge that is required to move the wedge to the right.

E9.14

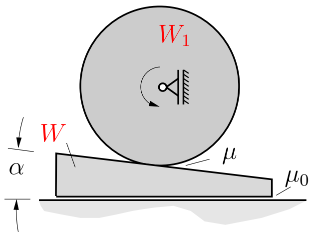


Fig. 9.19

Result: see (**A**) $\mu = \dfrac{\mu_0(1 + W/W_1) + \tan\alpha}{1 - \mu_0(1 + W/W_1)\tan\alpha}$.

E9.15 **Example 9.15** A beam (length $2a$, weight W) rests on support A. The triangle attached to its right end touches a rotating drum (Fig. 9.20). The coefficient of static friction μ_0 at A and the coefficient of kinetic friction μ at B are given.

a) Calculate the maximum allowable value of x in order to prevent slipping at A.

b) Determine the necessary value of μ_0 so that the beam does not slip for arbitrary values of x ($0 \leq x \leq a$).

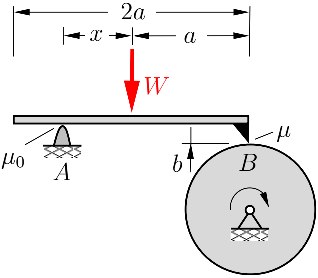


Fig. 9.20

Results: a) $x \leq \dfrac{\mu_0(a+\mu b)}{\mu}$, b) $\mu_0 \geq \dfrac{\mu a}{a+\mu b}$.

E9.16 **Example 9.16** The rotating drum in Fig. 9.21 is encircled by a break band that is tightened by the applied force F. The coefficient of kinetic friction between the drum and the band is μ.

Calculate the magnitude of F that is necessary to induce a given breaking moment M_B if the rotation of the drum is clockwise (c) and if it is counterclockwise (cc).

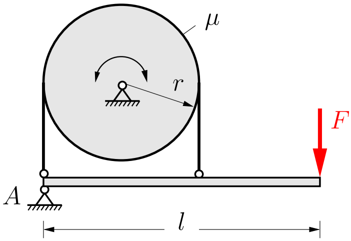


Fig. 9.21

Results: see (**A**) $F_c = \dfrac{2M_B}{l\,(\mathrm{e}^{\mu\pi}-1)}$, $F_{cc} = \dfrac{2M_B\,\mathrm{e}^{\mu\pi}}{l\,(\mathrm{e}^{\mu\pi}-1)}$.

9.5 Summary

- The static friction force H is a reaction force that can be determined directly from the equilibrium conditions in the case of statically determinate systems. Note that the orientation of H can be assumed arbitrarily in the free-body diagram.
- The absolute value of the static friction force H cannot exceed a certain limit friction force H_0. A body adheres to another body if the condition of static friction

  $$|H| \leq H_0 = \mu_0\, N$$

  is fulfilled.
- The kinetic friction force R is an active force given by Coulomb's friction law

  $$R = \mu\, N .$$

  The force R is oriented in the opposite direction of the (relative) velocity vector.
- In the case of static belt friction, the limiting case can be stated with the aid of the Euler-Eytelwein formula:

  $$S_2 = S_1 \mathrm{e}^{\mu_0 \alpha} .$$

- In the case of kinetic belt friction, the forces are also related by means of the Euler-Eytelwein formula if the coefficient of static friction μ_0 is replaced with the kinetic coefficient μ:

  $$S_2 = S_1 \mathrm{e}^{\mu \alpha} .$$