



Chapter 5

Support Reactions

5

5 Support Reactions

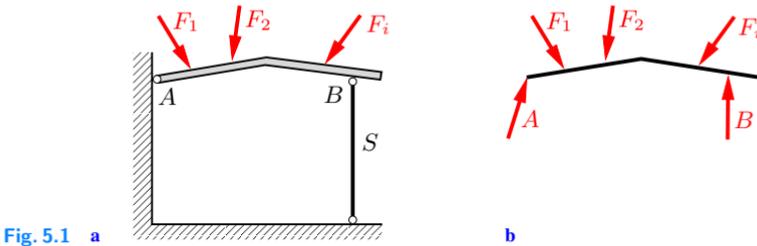
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——— **Objectives:** In this chapter, the most common kinds of supports of simple structures and the different connecting elements of multi-part structures are introduced. We will discuss their characteristic features and how they can be classified, so that the students will be able to decide whether or not a structure is statically and kinematically determinate. Students will also learn from this chapter how the forces and couple moments appearing at the supports and the connecting elements of a loaded structure can be determined. Here, the most important steps are the sketch of the free-body diagram and the correct application of the equilibrium conditions.

5.1 Plane Structures

5.1.1 Supports

Structures can be classified according to their geometrical shape and the loads acting on them. A straight slender structural element (cross-sectional dimensions much smaller than its length) that is loaded solely in the axial direction (tension or compression) is called a *bar* or a *rod* (see Section 2.4). If the same geometrical object is subjected to a load perpendicular to its axis, it is called a *beam*. A curved beam is usually designated as an *arch*. Structures consisting of inclined, rigidly joined beams are called *frames*. A plane structure with a thickness much smaller than its characteristic in-plane length is called a *disk* if it is solely loaded in its plane, e.g., by in-plane forces. If the same geometrical structure is loaded perpendicularly to its midplane it is called a *plate*. If such a structure is curved it is a *shell*.



Structures are connected to their surroundings by *supports* whose main purpose is to fix the structure in space in a specific position. Secondly, supports transmit forces. As a simple example, consider the “roof” in Fig. 5.1a, loaded by external forces F_i , joined at A to a vertical wall by a pin, and supported at B by the strut S. Forces are transmitted to the wall and the ground via the supports A and B. According to the law of action and reaction (actio = reactio) the same forces are exerted in opposite directions from the wall and the ground onto the roof. These forces from the environment onto the structure are reaction forces (cf. Section 1.4), and are termed *support reactions*. They become visible in

the free-body diagram (Fig. 5.1b), where they are generally denoted by the same symbols as the supports, i.e. by A and B in this example.

The following discussion is limited to single-part structures located and loaded in a plane. A free body with no restraints has three degrees of freedom, i.e., it can be independently displaced in three different ways: by two translations in different directions and by one rotation about an axis perpendicular to the plane (cf. Section 3.1.4). Supports (restraints) reduce the feasible displacements: each support reaction imposes a constraint. Let r be the number of support reactions. Then the number f of degrees of freedom of a body in a plane is given by

$$f = 3 - r \quad (5.1)$$

(for exceptions see Section 5.1.2).

We now will consider different types of supports and classify them by the number of support reactions involved.

Supports that can transmit only one single reaction ($r = 1$). Examples of this type of support are the roller support, the simple support and the support by a strut, cf. Fig. 5.2a–c. In this case, the direction of the reaction force is known (here vertical), whereas its magnitude is unknown.

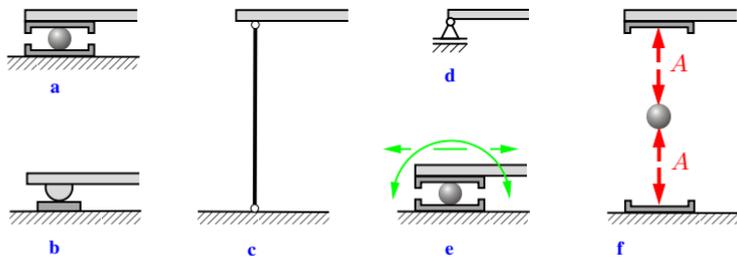


Fig. 5.2

Figure 5.2f shows the free-body diagram for the roller support. If the contact areas are assumed as frictionless, all contact forces can be considered to act perpendicular to the respective contact surfaces. With this assumption the action line of the reaction force A is determined. Figure 5.2e indicates the displacements that

are unconstrained by the support: a horizontal translation and a rotation. A vertical translation is excluded through the support's restraint. If the support reaction A changes its sign, i.e., if it is reversed in the direction along the action line, a lift-off must be prevented by an appropriate support construction. From now on a simple support will be depicted by the symbol shown in Fig. 5.2d.

Supports that can transmit two reactions ($r = 2$). Examples of this type of support are the hinged support and the support by two struts (Fig. 5.3a, b), which are depicted symbolically in Fig. 5.3c, d.

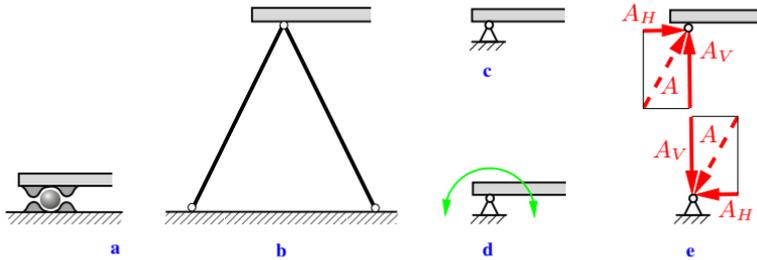


Fig. 5.3

As shown in Fig. 5.3d, the hinged support allows a rotation but not a displacement in any direction. Accordingly, it can transmit a reaction force A of arbitrary magnitude and arbitrary direction that can be resolved into its horizontal and vertical components A_H and A_V (Fig. 5.3e).

Additional variants of supports transmitting two reactions are the parallel motion and the sliding sleeve (Fig. 5.4a, b). Their free-body diagrams (Fig. 5.4c, d) show that in both cases one force and

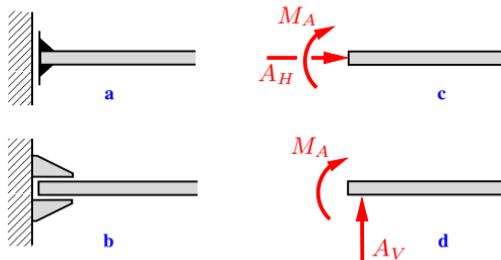


Fig. 5.4

one couple moment can be transmitted. A displacement in one single direction is possible in each case; a displacement in any other direction or a rotation are not possible.

The rotational degree of freedom disappears if a support by two struts is complemented by an additional, somewhat shifted, third strut (Fig. 5.5a): the structure becomes immobile. In addition to the two force components, the support can now also transmit a couple moment, i.e., in total *three reactions*: $r = 3$.

The same situation appears in the case of a clamped support (fixed support) according to Fig. 5.5b which symbolically is depicted in Fig. 5.5c. The free-body diagram in Fig. 5.5d shows that the clamped support can transmit a reaction force A of arbitrary magnitude and direction (or A_H and A_V) and a couple moment M_A .

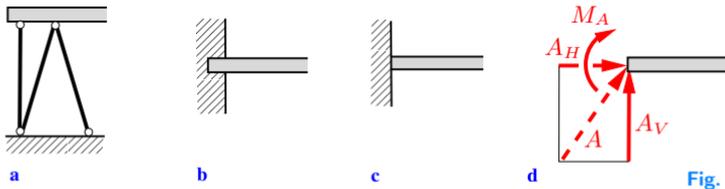


Fig. 5.5

5.1.2 Statical Determinacy

A structure is called *statically determinate* if the support reactions can be calculated from the *three* equilibrium conditions (3.12). Since the number of unknowns must coincide with the number of equations, *three* unknown reactions (forces or couple moments) must exist at the supports: $r = 3$. It will be explained later that this necessary condition may not be sufficient for the determination of the support reactions.

The beam in Fig. 5.6a is supported by the hinged support A and the simple support B . Accordingly, the three unknown support reactions A_H , A_V and B exist. Therefore, with $r = 3$ it follows from (5.1) that the beam is immobile: $f = 3 - r = 0$; it is statically determinate.

The support reactions of the clamped beam in Fig. 5.6b consist of the two force components A_H and A_V and the couple moment M_A . Figure 5.6c shows a disk supported by the three struts A , B and C , each transmitting one reaction. In both cases, with $r = 3$ and $f = 0$, the support is statically determinate.

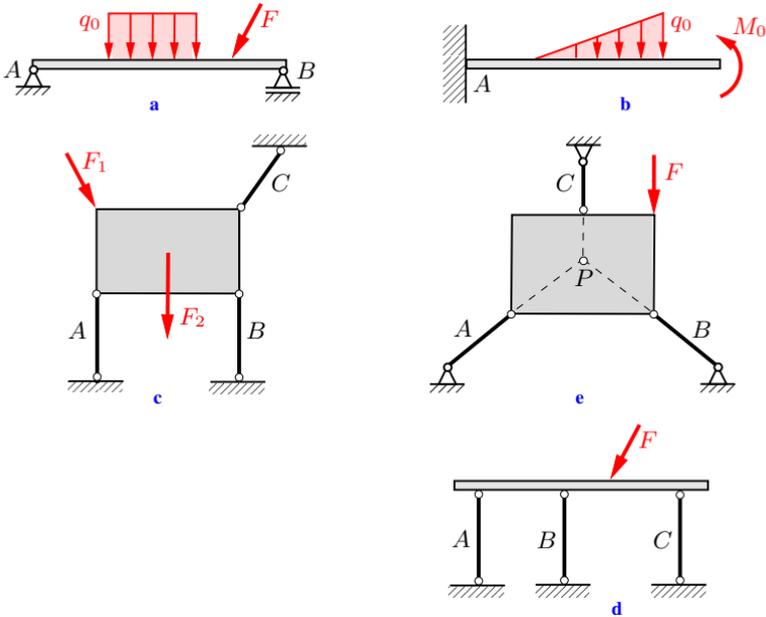


Fig. 5.6

In contrast, Fig. 5.6d shows a beam that is supported by three *parallel* struts A , B and C . Here too, the number of unknown support reactions coincides with the number of equilibrium conditions: the necessary condition for statical determinacy is satisfied. However, the reaction forces *cannot* be calculated from the equilibrium conditions. Here $r = 3$ does not imply $f = 0$ (exceptional case!): the beam can be displaced in a horizontal direction. Such exceptional cases must be excluded. A structure that may undergo finite or infinitesimal displacements is called *kinematically indeterminate* (cf. also Sections 5.3.4 and 6.1).

The disk in Fig. 5.6e is also kinematically indeterminate. Since the action lines of the reaction forces intersect at point P , the

supports allow an infinitesimal rotation about this point. It can be seen immediately that the supports in Figs. 5.6d and e are not statically determinate. In the case of the beam, the equilibrium condition for the horizontal force components cannot be fulfilled ($\sum F_{iH} \neq 0$), whereas for the disk, the equilibrium of the moments with respect to P cannot be satisfied ($\sum M_i^{(P)} \neq 0$).

In the case of a plane problem, a structure is supported statically and kinematically determinate if it is immobile and exactly three support reactions appear. These may be

- a) three forces which are not all parallel and not central,
- b) two nonparallel forces and one moment.

It must be emphasized that the statical determinacy of a structure is solely dependent on the supports and not on the loading.

If additional supports are attached to a statically determinate structure, more than three support reactions exist, which can no longer be determined solely from the three equilibrium conditions. Such a structure is called *statically indeterminate*.

For example, if the clamped beam in Fig. 5.6b is additionally supported by the simple support B (see Fig. 5.7a), the number of unknown reactions increases from three to four. In this case, one redundant reaction (force or couple moment) is present. The beam is therefore statically indeterminate with *one degree of static indeterminacy*.



Fig. 5.7

Generally, a structure is statically indeterminate with a degree x of statical indeterminacy if the number of unknown support reactions exceeds the number of available equilibrium conditions by x . Consequently, for the beam in Fig. 5.7b, the degree of statical indeterminacy is equal to two, since $r = 2 + 3 \cdot 1 = 5$.

The support reactions of statically indeterminate structures can only be determined if they are not considered to be rigid but if

their deformations are taken into account. The relevant methods will be discussed in Volume 2, *Mechanics of Materials*.

5.1.3 Determination of the Support Reactions

In order to determine the support reactions, the method of sections is applied (cf. Section 1.4): the body is freed from its supports and their action on the body is replaced by the unknown reactions.

As an example, consider the beam in Fig. 5.8a, which is supported by the strut A and the two simple supports B and C . The reaction forces become visible in the free-body diagram (Fig. 5.8b). Their sense of direction along the prescribed action lines can be chosen arbitrarily. However, for the strut the sign convention for rods is applied (see Sections 2.4 and 6.3.1) and it is assumed to be subject to tension. The assumptions are correct if the analysis yields positive values for the reaction forces, whereas a reaction force is oppositely directed in the case of a negative sign.

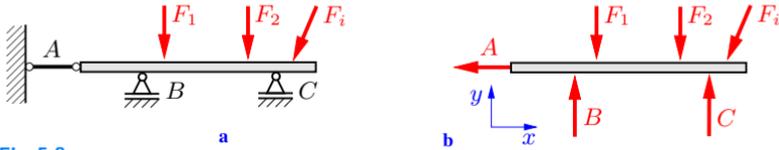


Fig. 5.8

All of the forces acting on the isolated body (i.e., active forces and reaction forces) must fulfill the equilibrium conditions (3.12):

$$\sum F_{ix} = 0, \quad \sum F_{iy} = 0, \quad \sum M_i^{(P)} = 0. \quad (5.2)$$

Here, P is a reference point that may be chosen arbitrarily. The support reactions can be calculated from (5.2).

Example 5.1 The beam shown in Fig. 5.9a is loaded by the force F which acts under an angle α .

Determine the reaction forces at the supports A and B .

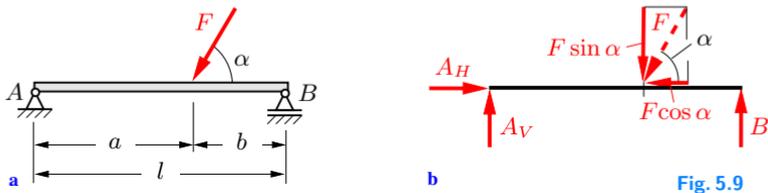


Fig. 5.9

Solution The beam is rigidly supported; the support A transmits two reactions and support B one reaction. In total, the three unknown reaction forces A_H , A_V and B exist: therefore, the beam is statically determinate. We free the beam from its supports and make the reaction forces visible in the free-body diagram (Fig. 5.9b) where we choose their senses of direction along the action lines freely. Hence, the equilibrium conditions are given by

$$\uparrow: A_V - F \sin \alpha + B = 0, \quad (a)$$

$$\rightarrow: A_H - F \cos \alpha = 0 \quad \rightarrow \quad \underline{\underline{A_H = F \cos \alpha}},$$

$$\curvearrowleft_A: -a F \sin \alpha + l B = 0 \quad \rightarrow \quad \underline{\underline{B = \frac{a}{l} F \sin \alpha}}. \quad (b)$$

Introducing B and the geometric relation $a + b = l$ into (a) yields

$$\underline{\underline{A_V}} = F \sin \alpha - B = \left(1 - \frac{a}{l}\right) F \sin \alpha = \underline{\underline{\frac{b}{l} F \sin \alpha}}.$$

As a check, the equilibrium condition for the couple moments about another reference point is applied:

$$\curvearrowleft_B: -l A_V + b F \sin \alpha = 0 \quad \rightarrow \quad A_V = \frac{b}{l} F \sin \alpha.$$

This equation, in contrast to Equation (a), directly yields the reaction force A_V . Thus, application of the equilibrium conditions (3.14) instead of (3.12) would have been advantageous in this case.

E5.2 Example 5.2 The clamped beam shown in Fig. 5.10a is loaded by the two forces F_1 and F_2 .

Determine the reactions at the support.

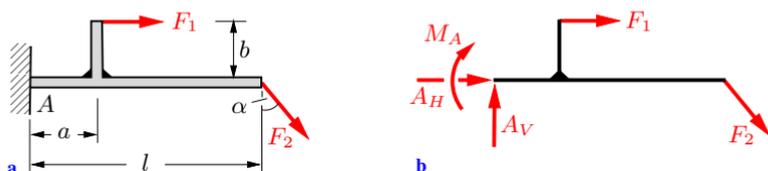


Fig. 5.10

Solution The fixed support A transmits three reactions: two force components A_H , A_V and the moment M_A . They are made visible in the free-body diagram (Fig. 5.10b) where their senses of direction have been chosen arbitrarily. Thus, the equilibrium conditions (5.2) yield

$$\begin{aligned} \uparrow: \quad A_V - F_2 \cos \alpha &= 0 & \rightarrow \quad \underline{\underline{A_V = F_2 \cos \alpha}}, \\ \rightarrow: \quad A_H + F_1 + F_2 \sin \alpha &= 0 & \rightarrow \quad \underline{\underline{A_H = -(F_1 + F_2 \sin \alpha)}}, \\ \curvearrowleft: \quad M_A + b F_1 + l F_2 \cos \alpha &= 0 & \rightarrow \quad \underline{\underline{M_A = -(b F_1 + l F_2 \cos \alpha)}}. \end{aligned}$$

The negative signs of A_H and M_A indicate that these reactions in reality are directed oppositely to the directions chosen in the free-body diagram.

5.2 Spatial Structures

A body that can move freely in space has six degrees of freedom: three translations in x -, y - and z -direction and three rotations about the three axes. Supports constrain the possible displacements. As in the plane case, the different types of support are classified by the number of transferable support reactions.

The strut in Fig. 5.11a can transfer only *one* force in the direction of its axis. Therefore, $r = 1$ for spatial as well as for plane structures. In contrast, the hinged support in Fig. 5.11b transfers *three* force components in space (in x -, y - and z -direction), i.e., $r = 3$. The fixed support or clamping (Fig. 5.11c) transfers six reactions in space ($r = 6$): the force components in the three

coordinate directions as well as the moments about the three axes. The sliding sleeve in Fig. 5.11d can transfer two moments and two force components, provided that the beam with a circular cross-section can rotate freely about its axis; for this type of support, $r = 4$ is valid. When the support and the beam have rectangular cross-sections, which makes a rotation impossible, moments about all three axes can be transferred. This leads to $r = 5$.

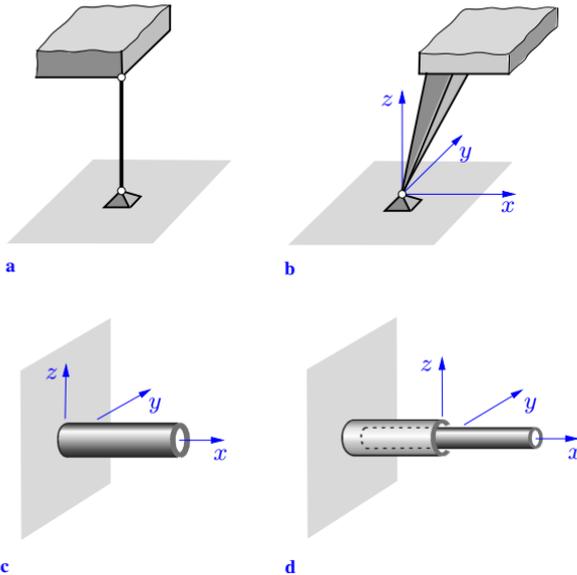


Fig. 5.11

A spatial structure is statically determinate when it is immobile and the support reactions can be calculated from the *six* equilibrium conditions (3.34), see also Section 5.3.4. Thus, in total *six* reactions must exist at the supports. As in the case of plane structures, these reactions are calculated by applying the method of sections.

E5.3 **Example 5.3** The rectangular lever which is clamped at A (Fig. 5.12a) is loaded by the line load q_0 , two forces F_1 , F_2 and the moment M_0 .

Determine the support reactions.

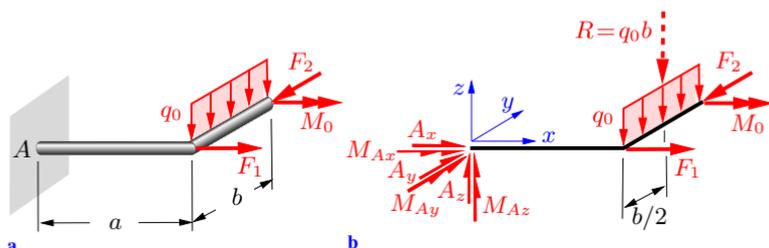


Fig. 5.12

Solution We free the lever from the fixed support and make the reactions visible in the free-body diagram. According to the clamped support, the three force components A_x , A_y , A_z and the three moment components M_{Ax} , M_{Ay} , M_{Az} exist (Fig. 5.12b). Their directions are chosen in such a way that they coincide with the positive coordinate directions. The line load can be replaced by its resultant $R = q_0 b$. The equilibrium conditions (3.34) then yield

$$\begin{aligned} \sum F_{ix} = 0: \quad A_x + F_1 &= 0 && \rightarrow \underline{\underline{A_x = -F_1}}, \\ \sum F_{iy} = 0: \quad A_y - F_2 &= 0 && \rightarrow \underline{\underline{A_y = F_2}}, \\ \sum F_{iz} = 0: \quad A_z - q_0 b &= 0 && \rightarrow \underline{\underline{A_z = q_0 b}}, \\ \sum M_{ix}^{(A)} = 0: \quad M_{Ax} + M_0 - \frac{b}{2}(q_0 b) &= 0 && \rightarrow \underline{\underline{M_{Ax} = \frac{q_0 b^2}{2} - M_0}}, \\ \sum M_{iy}^{(A)} = 0: \quad M_{Ay} + a(q_0 b) &= 0 && \rightarrow \underline{\underline{M_{Ay} = -q_0 a b}}, \\ \sum M_{iz}^{(A)} = 0: \quad M_{Az} - a F_2 &= 0 && \rightarrow \underline{\underline{M_{Az} = a F_2}}. \end{aligned}$$

Example 5.4 A spatial frame is supported at A , B and C (Fig. 5.13a). It is loaded by the line load q_0 , the forces F_1 , F_2 and the moment M_0 .

Determine the support reactions.

Solution The hinged support A transfers the three force components A_x , A_y , A_z (Fig. 5.13b). At the support B , forces B_x and B_z act in the directions of the struts, and at the simple (moveable) support, force C acts perpendicularly to the horizontal plane, i.e.,

E5.4

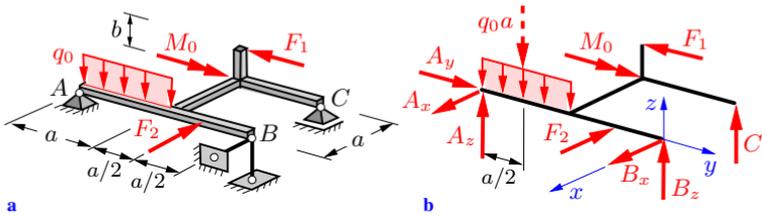


Fig. 5.13

in the direction of the z -axis. Hence, the equilibrium equations for the forces are

$$\sum F_{ix} = 0: A_x + B_x - F_2 = 0, \quad (\text{a})$$

$$\sum F_{iy} = 0: A_y - F_1 = 0 \quad \rightarrow \underline{\underline{A_y = F_1}},$$

$$\sum F_{iz} = 0: A_z + B_z + C - q_0 a = 0. \quad (\text{b})$$

To formulate the equilibrium of moments we choose axes through the reference point B:

$$\sum M_{ix}^{(B)} = 0: -2a A_z + \frac{3}{2}a(q_0 a) + b F_1 = 0 \rightarrow \underline{\underline{A_z = \frac{3}{4}q_0 a + \frac{b}{2a}F_1}},$$

$$\sum M_{iy}^{(B)} = 0: a C + M_0 = 0 \quad \rightarrow \underline{\underline{C = -\frac{1}{a}M_0}},$$

$$\sum M_{iz}^{(B)} = 0: 2a A_x + a F_1 - \frac{a}{2}F_2 = 0 \quad \rightarrow \underline{\underline{A_x = -\frac{1}{2}F_1 + \frac{1}{4}F_2}}.$$

With the results for A_x , A_z and C , (a) and (b) yield

$$\underline{\underline{B_x}} = -A_x + F_2 = \underline{\underline{\frac{1}{2}F_1 + \frac{3}{4}F_2}},$$

$$\underline{\underline{B_z}} = q_0 a - A_z - C = \underline{\underline{\frac{1}{4}q_0 a - \frac{b}{2a}F_1 + \frac{1}{a}M_0}}.$$

5.3

5.3 Multi-Part Structures

5.3.1 Statical Determinacy

Structures often consist not only of one single part but of a number of rigid bodies that are appropriately connected. The connecting members transfer forces and moments, respectively, which can

be made visible by passing cuts through the connections. In the following the discussion is restricted to *plane* structures.

The connecting member between two rigid bodies ① and ② of a structure can be, for example, a strut S , a hinge G or a parallel motion P (Fig. 5.14a–c). The *strut* transfers only one single force S in its axial direction. In this case, the number v of joint reactions is $v = 1$. In contrast, a *hinge* can transfer a force in an arbitrary direction, i.e. the force components G_H and G_V . Since the hinge is assumed to be frictionless, it offers no resistance to a rotation: it cannot transfer a moment. Therefore, the number of joint reactions in this case is $v = 2$. The *parallel motion* prevents a relative rotation and a relative displacement in the horizontal direction of the connected bodies; however, it allows a vertical displacement. Therefore, only a horizontal force N and a moment M can be transferred: again $v = 2$. According to the principle *actio = reactio*, the joint reactions act in opposite directions on the two bodies.

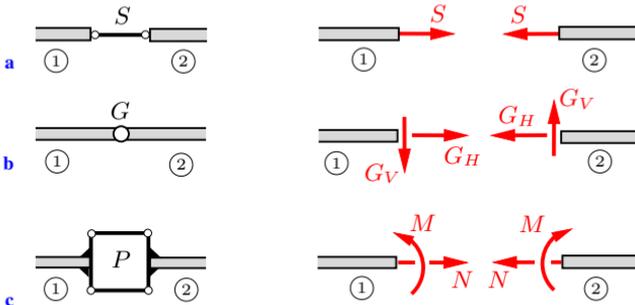


Fig. 5.14 c

In order to determine the support reactions and the forces and moments transferred by the connecting members the method of sections is applied: we free the different bodies of the structure by removing all of the joints and supports and replace them by the joint and support reactions.

Three equilibrium conditions can be formulated for each body of the structure. Therefore, there are in total $3n$ equations if the structure consists of n bodies. Let r be the number of support reactions and v be the number of transferred joint reactions. We

call the multi-body structure statically determinate if the r support reactions and the v joint reactions can be calculated from the $3n$ equilibrium conditions. The necessary condition for static determinacy is that the number of equations and the number of unknowns are equal:

$$r + v = 3n. \quad (5.3)$$

Moreover, if the structure is rigid, this condition is sufficient for static determinacy. Condition (5.3) also includes the special case of a statically determinate single body where $n = 1$, $v = 0$ and $r = 3$ (cf. Section 5.1.2).

As examples, let us consider the multi-part structures depicted in Fig. 5.15. The structure shown in Fig. 5.15a consists of $n = 2$ beams ① and ②, connected by the hinge G , and it is supported by the fixed support A and strut B . Hinge G transfers $v = 2$ force components, and at the fixed support and the strut, $r = 3 + 1 = 4$ support reactions exist. Hence, since $4 + 2 = 3 \cdot 2$, the necessary condition (5.3) for static determinacy is fulfilled. The structure in Fig. 5.15b consists of three beams ① - ③ and the disk ④; i.e., $n = 4$. The four hinges $G_1 - G_4$ transfer $v = 4 \cdot 2 = 8$ joint reactions. At the support A , two reactions exist and each of the

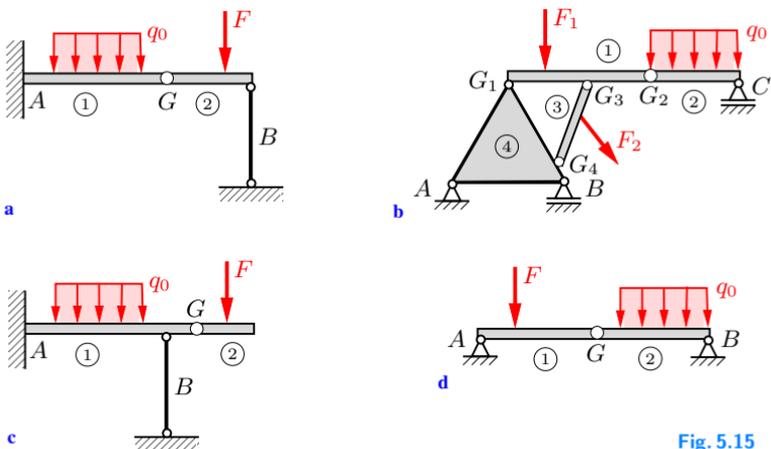


Fig. 5.15

supports B and C transfers one reaction; this results in a total of $r = 2 + 1 + 1 = 4$. Introducing these numbers into (5.3) shows that the necessary condition for statical determinacy is again satisfied: $4 + 8 = 3 \cdot 4$. Since both structures are rigid, they are statically determinate.

If the strut in Fig. 5.15a is attached to beam ① instead of beam ② as shown in Fig. 5.15c, the necessary condition for statical determinacy is still fulfilled. However, this structure is kinematically indeterminate (beam ② is moveable) and therefore useless. The structure in Fig. 5.15d is also kinematically indeterminate. Even though hinge G cannot undergo finite displacements, it can still be displaced infinitesimally upwards or downwards.

Example 5.5 The structure shown in Fig. 5.16a consists of the beam ① and the angled part ②, which are connected by the hinge G . The angled part is clamped at A and the beam is supported at B . The system is loaded by the force F .

Determine the support and joint reactions.

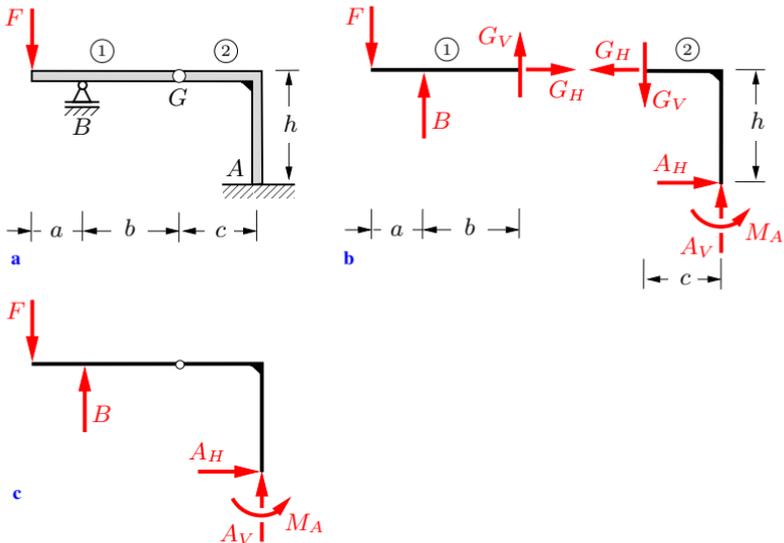


Fig. 5.16

Solution Since $r = 3 + 1 = 4$, $v = 2$ and $n = 2$, condition (5.3) is fulfilled: $4 + 2 = 3 \cdot 2$. Furthermore, since the structure is immobile, it is statically determinate.

We separate the bodies ① and ②, remove the supports and draw the free-body diagram (Fig. 5.16b). The directions of G_H and G_V can be chosen freely for one of the two bodies. Their directions for the second body are determined through the principle *actio = reactio*. The equilibrium conditions for body ① yield

$$\rightarrow: \underline{\underline{G_H = 0}},$$

$$\curvearrowright G: (a+b)F - bB = 0 \quad \rightarrow \quad \underline{\underline{B = \frac{a+b}{b} F}},$$

$$\curvearrowright B: aF + bG_V = 0 \quad \rightarrow \quad \underline{\underline{G_V = -\frac{a}{b} F}}.$$

From the equilibrium conditions for body ② in conjunction with the results for G_H and G_V , we obtain

$$\uparrow: -G_V + A_V = 0 \quad \rightarrow \quad \underline{\underline{A_V = G_V = -\frac{a}{b} F}},$$

$$\rightarrow: -G_H + A_H = 0 \quad \rightarrow \quad \underline{\underline{A_H = G_H = 0}},$$

$$\curvearrowright A: M_A + hG_H + cG_V = 0 \quad \rightarrow \quad \underline{\underline{M_A = -hG_H - cG_V = \frac{ac}{b} F}}.$$

The negative signs of G_V and A_V indicate that their directions in reality are opposite to those assumed in the free-body diagram.

As a check, the equilibrium conditions are applied to the complete system (Fig. 5.16c), where the hinge G is assumed to be frozen:

$$\uparrow: -F + B + A_V = 0 \quad \rightarrow \quad -F + \frac{a+b}{b} F - \frac{a}{b} F = 0,$$

$$\rightarrow: A_H = 0,$$

$$\begin{aligned} \curvearrowright B: \quad aF + M_A + hA_H + (b+c)A_V &= 0 \\ &\rightarrow \quad aF + \frac{ac}{b} F - (b+c)\frac{a}{b} F = 0. \end{aligned}$$

Example 5.6 The symmetrical sawbuck in Fig. 5.17a consists of two beams connected at hinge C and fixed by the rope S . It is loaded with a frictionless cylinder of weight W .

Determine the support reactions at A and B , the force S in the rope and the joint reaction in C . The weight of the sawbuck can be neglected.

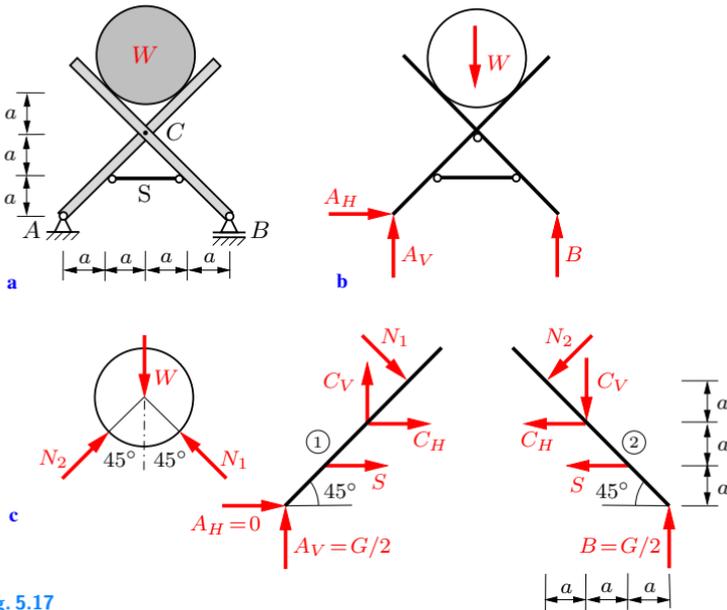


Fig. 5.17

Solution Since only three support reactions are present (see Fig. 5.17b), they can be determined by applying the equilibrium conditions to the complete system:

$$\begin{aligned}
 \rightarrow: \quad & \underline{\underline{A_H = 0}}, \\
 \curvearrowleft A: \quad & -2aW + 4aB = 0 \quad \rightarrow \quad \underline{\underline{B = W/2}}, \\
 \curvearrowleft B: \quad & -4aA_V + 2aW = 0 \quad \rightarrow \quad \underline{\underline{A_V = W/2}}.
 \end{aligned} \tag{a}$$

In order to determine the forces in the rope and the hinge, we separate the structure into its two parts ($n = 2$). In the hinge C and rope S , in total $v = 2 + 1 = 3$ forces are transferred (Fig. 5.17c). With $r = 3$, the necessary condition (5.3) for statical determinacy is fulfilled: $3 + 3 = 3 \cdot 2$.

Since the surface of the cylinder is assumed to be frictionless, the contact forces N_1 and N_2 between the beams and the cylinder act in directions normal to the respective contact planes. Therefore, with $\sin 45^\circ = \sqrt{2}/2$, the equilibrium conditions for the cylinder read

$$\begin{aligned} \rightarrow: \quad \frac{\sqrt{2}}{2} N_2 - \frac{\sqrt{2}}{2} N_1 &= 0 & \rightarrow N_1 &= N_2, \\ \uparrow: \quad -W + \frac{\sqrt{2}}{2} N_2 + \frac{\sqrt{2}}{2} N_1 &= 0 & \rightarrow N_1 = N_2 &= \frac{\sqrt{2}}{2} W. \end{aligned} \quad (\text{b})$$

From the equilibrium conditions for the beam ② we obtain with (a) and (b):

$$\begin{aligned} \curvearrowright C: \quad \sqrt{2} a N_2 - a S + 2 a B &= 0 & \rightarrow \underline{S} &= 2 B + \sqrt{2} N_2 = \underline{\underline{2W}}, \\ \uparrow: \quad -\frac{\sqrt{2}}{2} N_2 - C_V + B &= 0 & \rightarrow \underline{C_V} &= B - \frac{\sqrt{2}}{2} N_2 = \underline{0}, \\ \rightarrow: \quad -\frac{\sqrt{2}}{2} N_2 - C_H - S &= 0 & \rightarrow \underline{C_H} &= -\frac{\sqrt{2}}{2} N_2 - S = \underline{\underline{-\frac{5}{2}W}}. \end{aligned}$$

The same result is obtained when the equilibrium conditions are applied to beam ①. By symmetry considerations, it can be concluded from Fig. 5.17c with no calculation that $N_1 = N_2$ and $C_V = 0$.

5.3.2 Three-Hinged Arch

The arch shown in Fig. 5.18a is statically determinate because it is immobile and in total three support reactions exist at A and B . In a real structure, the arch AB is not rigid but deforms under applied loads. If B is a roller support, this may lead to a large deformation that cannot be tolerated.

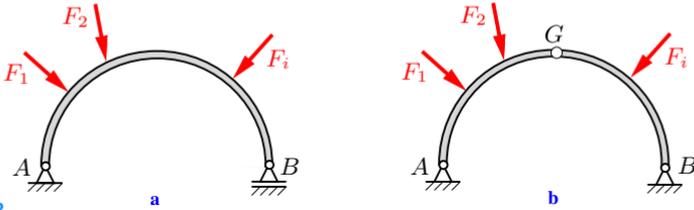


Fig. 5.18

Such a displacement is prevented if A and B are designed as hinged supports. As a consequence, the static determinacy of the structure gets lost. However, static determinacy can be re-established if an additional hinge G is introduced at an arbitrary location (Fig. 5.18b). Such a structure is called a *three-hinged arch*. It consists of $n = 2$ bodies connected by the hinge G , which transfers $v = 2$ joint reactions. Since the supports A and B transfer $r = 2 + 2 = 4$ support reactions, the condition for static determinacy (5.3) is fulfilled: $4 + 2 = 3 \cdot 2$. Therefore, taking the immobility of the structure into account, the three-hinged arch is statically determinate.

The two bodies of a three-hinged arch need not necessarily be arch shaped. An arbitrary structure consisting of two bodies connected by a hinge and supported by two hinged supports (in total *three* hinges) is also called a three-hinged arch from now on. Two examples are shown in Fig. 5.19: a) a frame and b) a truss consisting of two single trusses (compare Chapter 6).

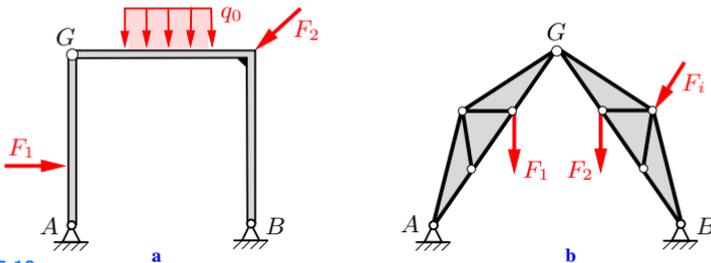


Fig. 5.19

In order to calculate the forces at the supports and the hinge, we isolate the two bodies ① and ② (cf. Fig. 5.20a,b) and apply the equilibrium conditions to each body. From the $2 \cdot 3 = 6$ equations,

the unknowns A_H , A_V , B_H , B_V , G_H and G_V can be calculated. As a check, the equilibrium conditions can be applied to the complete system where the hinge is regarded as being frozen.

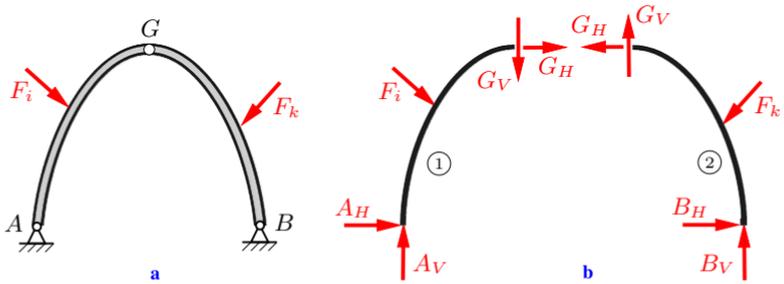


Fig. 5.20

E5.7 **Example 5.7** The structure shown in Fig. 5.21a consists of two beams, joined by the hinge G and supported in A and B by hinged supports. The system is loaded by the forces $F_1 = F$ and $F_2 = 2F$. Determine the forces at the supports and the hinge.

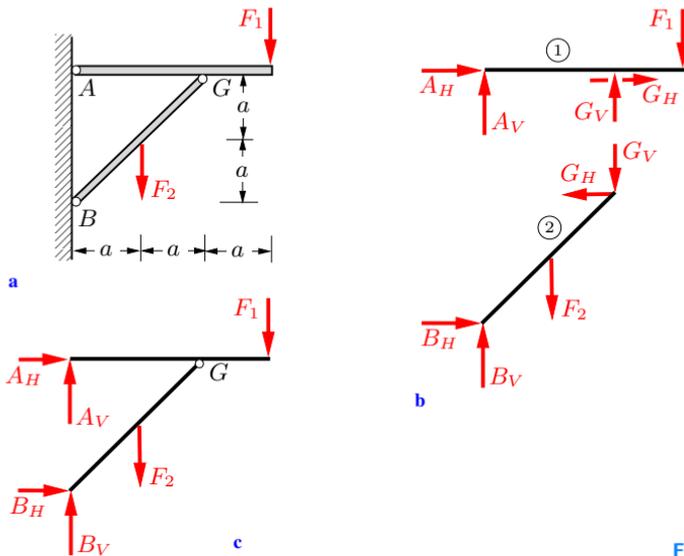


Fig. 5.21

Solution The structure is a three-hinged arch. In order to calculate the unknown forces we separate the bodies ① and ② and

draw the free-body diagram (Fig. 5.21b). The equilibrium conditions for beam ① read

$$\begin{aligned}\widehat{A}: \quad 2a G_V - 3a F_1 = 0 &\rightarrow \underline{\underline{G_V}} = \frac{3}{2} F_1 = \underline{\underline{\frac{3}{2} F}}, \\ \widehat{G}: \quad -2a A_V - a F_1 = 0 &\rightarrow \underline{\underline{A_V}} = -\frac{1}{2} F_1 = \underline{\underline{-\frac{1}{2} F}}, \\ \rightarrow: \quad A_H + G_H = 0.\end{aligned}$$

For beam ② we obtain

$$\begin{aligned}\widehat{B}: \quad -a F_2 - 2a G_V + 2a G_H = 0, \\ \widehat{G}: \quad 2a B_H - 2a B_V + a F_2 = 0, \\ \rightarrow: \quad B_H - G_H = 0.\end{aligned}$$

Solving the system of equations yields

$$\begin{aligned}\underline{\underline{G_H}} = \frac{1}{2} F_2 + G_V = \underline{\underline{\frac{5}{2} F}}, \quad \underline{\underline{B_H}} = G_H = \underline{\underline{\frac{5}{2} F}}, \\ \underline{\underline{B_V}} = \frac{1}{2} F_2 + B_H = \underline{\underline{\frac{7}{2} F}}, \quad \underline{\underline{A_H}} = -G_H = \underline{\underline{-\frac{5}{2} F}}.\end{aligned}$$

As a check we use the force equilibrium for the complete (frozen) system according to Fig. 5.21c:

$$\begin{aligned}\uparrow: \quad A_V + B_V - F_1 - F_2 = 0 &\rightarrow -\frac{1}{2} F + \frac{7}{2} F - F - 2F = 0, \\ \rightarrow: \quad A_H + B_H = 0 &\rightarrow -\frac{5}{2} F + \frac{5}{2} F = 0.\end{aligned}$$

5.3.3 Hinged Beam

Structures with a wide span width are necessarily often supported by more than two supports. As an example, consider the beam shown in Fig. 5.22a. Since $r = 5$, the system is statically indeterminate with two degrees of statical indeterminacy (see Section 5.1.2). Therefore, the calculation of the support reactions solely from the equilibrium conditions is impossible.

A statically determinate multi-body structure can be obtained

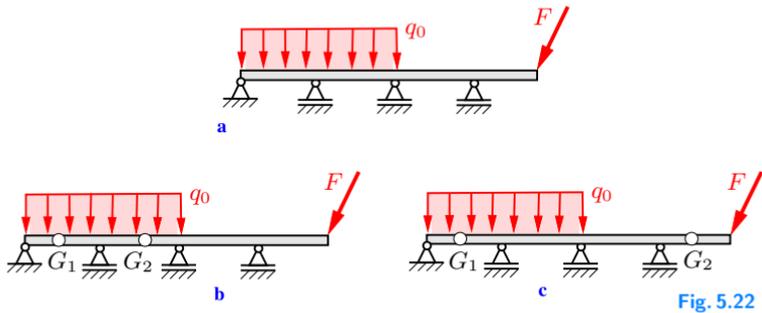


Fig. 5.22

if the continuous beam is divided into several parts by introducing an appropriate number of hinges. Such a structure is called a *hinged beam*.

If the number of these hinges is g , the continuous beam is divided into $n = g + 1$ parts. Since each hinge transfers two force components, the number of joint reactions is $v = 2g$. Therefore, according to (5.3) the necessary condition for static determinacy takes the form

$$r + v = 3n \quad \rightarrow \quad r + 2g = 3(g + 1). \quad (5.4)$$

Thus, the necessary number of hinges is given by

$$g = r - 3. \quad (5.5)$$

The beam in Fig. 5.22a has $r = 5$ support reactions. Therefore, according to (5.5) two hinges are necessary: $g = 5 - 3 = 2$. There are various possibilities for arranging the hinges; the support and joint reactions depend on their positions. One possibility is depicted in Fig. 5.22b. In contrast, Fig. 5.22c shows a hinge arrangement leading to a movable, i.e., kinematically indeterminate structure that is statically useless.

To determine the support and joint reactions, we first divide the hinged beam into its parts and subsequently apply the equilibrium conditions to each body.

E5.8 **Example 5.8** The hinged beam shown in Fig. 5.23a is loaded by the single force F and the line load q_0 .

Determine the support and hinge forces.

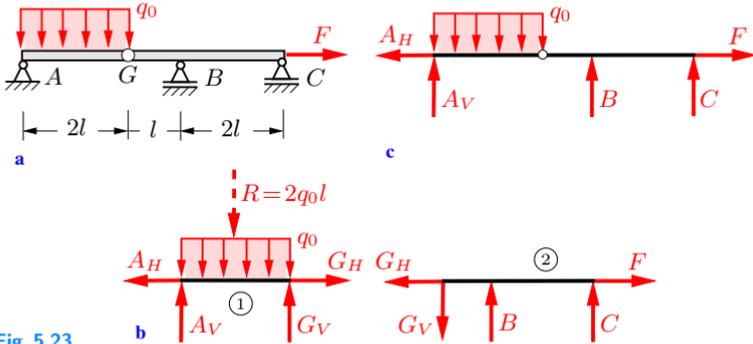


Fig. 5.23

Solution The system is statically determinate. We separate the two bodies and draw the free-body diagram (Fig. 5.23b). The line load can be replaced by its statically equivalent resultant force $R = 2q_0l$ acting in the middle of beam ①.

It is often advantageous to use moment equations with respect to the hinges and the supports. The unknowns can then be calculated successively from equations with only one unknown. Here, the equilibrium conditions for beam ① read

$$\begin{aligned} \hat{A}: \quad -lR + 2lG_V &= 0 \quad \rightarrow \quad \underline{\underline{G_V}} = \frac{1}{2}R = \underline{\underline{q_0l}}, \\ \hat{G}: \quad -2lA_V + lR &= 0 \quad \rightarrow \quad \underline{\underline{A_V}} = \frac{1}{2}R = \underline{\underline{q_0l}}, \\ \rightarrow: \quad -A_H + G_H &= 0 \end{aligned}$$

and those for beam ② are

$$\begin{aligned} \hat{B}: \quad lG_V + 2lC &= 0, \\ \hat{C}: \quad 3lG_V - 2lB &= 0, \\ \rightarrow: \quad -G_H + F &= 0 \quad \rightarrow \quad \underline{\underline{G_H}} = \underline{\underline{F}}. \end{aligned}$$

The solution of the remaining system of equations is

$$\begin{aligned} \underline{\underline{A_H}} = G_H = \underline{\underline{F}}, \quad \underline{\underline{B}} &= \frac{3}{2}G_V = \underline{\underline{\frac{3}{2}q_0l}}, \\ \underline{\underline{C}} &= -\frac{1}{2}G_V = \underline{\underline{-\frac{1}{2}q_0l}}. \end{aligned}$$

As a check, the force equilibrium conditions for the complete system are used (Fig. 5.23c):

$$\begin{aligned} \rightarrow: \quad & -A_H + F = 0 \quad \rightarrow \quad -F + F = 0, \\ \uparrow: \quad & A_V - 2q_0 l + B + C = 0 \\ & \rightarrow \quad q_0 l - 2q_0 l + \frac{3}{2}q_0 l - \frac{1}{2}q_0 l = 0. \end{aligned}$$

5.3.4 Kinematical Determinacy

In this section, statical and kinematical determinacy or indeterminacy is discussed in more detail than in Section 5.3.1. Again, we will restrict the discussion to plane multi-part structures.

The number f of degrees of freedom of an n body system without any joints is given by $3n$ (3 degrees of freedom for each body). This number is reduced by the number of restraints r through supports and the number of restraints v through joints:

$$f = 3n - (r + v). \quad (5.6)$$

Each restraint r and v , respectively, is associated with one support or joint reaction. Furthermore, the number of available equilibrium conditions is given by $3n$ (three equations for each body).

For $f > 0$ the system is movable. In contrast, for $f < 0$ the number $r + v$ of support and joint reactions exceeds the number $3n$ of equilibrium conditions by x . In this case, the system is statically indeterminate where the degree x of statical indeterminacy is given by

$$x = -f = r + v - 3n. \quad (5.7)$$

Even though it is impossible to determine *all* support and joint reactions of statically indeterminate systems solely from the equilibrium conditions, it may be possible to calculate some reactions in certain cases. For example, let us consider the system in Fig. 5.24a, where on account of $n = 2$, $r = 5$ and $v = 2$, the degree of statical indeterminacy is equal to one. Nevertheless, for beam \overline{GC} , the force components in the hinge G and the force in the simple support C can be calculated for a given loading from

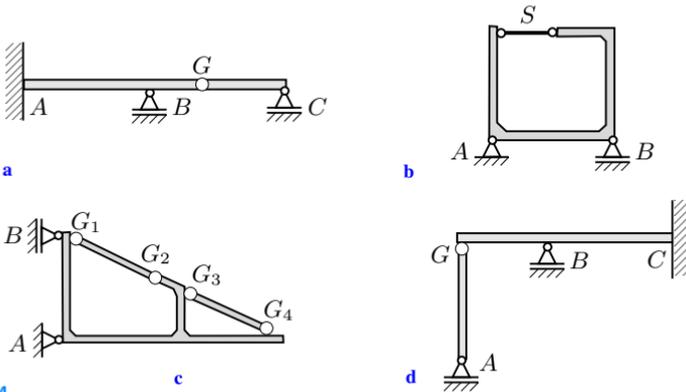


Fig. 5.24

the three equilibrium conditions. Two other examples of systems with one or two degrees of statical indeterminacy, respectively, are depicted in Figs. 5.24b and c. In both cases the support reactions can be calculated from the equilibrium conditions applied to the complete structures: the systems are *externally statically determinate*. However, the joint reactions (force in the strut, forces in the hinges) between the parts of the systems cannot be calculated. Such systems are called *internally statically indeterminate*.

Statically indeterminate systems in certain cases can undergo finite or infinitesimal displacements, i.e., they are then also kinematically indeterminate. As an example, the system shown in Fig. 5.24d is one-degree statically indeterminate because $n = 2$, $r = 5$ and $v = 2$. However, it can be seen that the system is not immovable since the vertical beam can rotate infinitesimally about G . It is evident that such a structure is not able to carry arbitrary loads.

Finally, for $f = 3n - (r + v) = 0$ the necessary condition for statical determinacy is fulfilled (cf. (5.3)). Then all support and joint reactions can be calculated except in the exceptional case of a movable system.

Now we will answer the question of how we can determine whether or not a multi-part structure is movable. Let us first consider only plane systems fulfilling the necessary condition for statical determinacy ($f = 0$). Whether the system is movable or not can

always be formally decided by rewriting the equilibrium conditions in the form of a linear system of equations (cf. Appendix A.2):

$$\mathbf{A} \mathbf{x} = \mathbf{b}. \quad (5.8)$$

Here, $\mathbf{b} = (b_1, \dots, b_{3n})^T$ is given by the prescribed loading, $\mathbf{x} = (x_1, \dots, x_{3n})^T$ represents the unknown support and joint reactions, and the matrix \mathbf{A} is given through the coefficients found through writing down the equilibrium conditions. The system of equations has a unique solution if the determinant of matrix \mathbf{A} is nonzero:

$$\det \mathbf{A} \neq 0. \quad (5.9)$$

The multi-part structure with $f = 0$ is then not only statically but also kinematically determinate. This condition is very general; it also holds for an arbitrary spatial system.

E5.9 **Example 5.9** Formulate the equilibrium conditions for the beam shown in Fig. 5.25a ($0 \leq \alpha \leq \pi$) in the form $\mathbf{A} \mathbf{x} = \mathbf{b}$. Calculate the determinant of the matrix \mathbf{A} . Is the system statically useful for every value of angle α ?

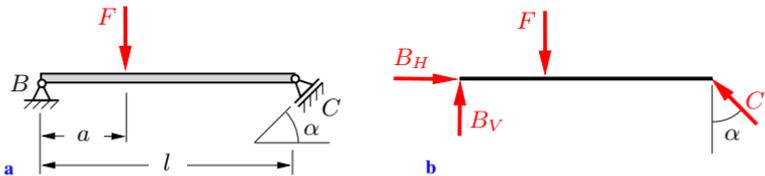


Fig. 5.25

Solution The equilibrium conditions (cf. Fig. 5.25b)

$$\rightarrow: \quad B_H - C \sin \alpha = 0,$$

$$\uparrow: \quad B_V + C \cos \alpha - F = 0,$$

$$\widehat{C}: \quad lB_V - (l - a)F = 0$$

can be written in matrix representation as

$$\begin{pmatrix} 1 & 0 & -\sin \alpha \\ 0 & 1 & \cos \alpha \\ 0 & l & 0 \end{pmatrix} \begin{pmatrix} B_H \\ B_V \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ F \\ (l-a)F \end{pmatrix} \quad \text{i.e., } \mathbf{A} \mathbf{x} = \mathbf{b}.$$

The determinant of \mathbf{A} is evaluated through cofactor expansion along the first column:

$$\underline{\underline{\det \mathbf{A}}} = \begin{vmatrix} 1 & 0 & -\sin \alpha \\ 0 & 1 & \cos \alpha \\ 0 & l & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & \cos \alpha \\ l & 0 \end{vmatrix} = \underline{\underline{-l \cos \alpha}}.$$

Consequently, we find that

$$\det \mathbf{A} \begin{cases} \neq 0 & \text{for } \alpha \neq \pi/2, \\ = 0 & \text{for } \alpha = \pi/2. \end{cases}$$

Thus, the beam is supported kinematically determinate (immobile) for $\alpha \neq \pi/2$ and kinematically indeterminate solely for $\alpha = \pi/2$. In the latter case, the simple support C can move in the vertical direction. Then the beam may rotate infinitesimally about the hinged support B and is therefore statically useless.

It should be mentioned that even though the beam is formally useful for angles α near $\pi/2$, such a construction should be avoided from the technical point of view because the forces in the supports become very large in this case.

5.4 Supplementary Problems

Detailed solutions to most of the following examples are given in (A) D. Gross et al. *Formeln und Aufgaben zur Technischen Mechanik 1*, Springer, Berlin 2011 or (B) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer, Berlin 2011.

E5.10 **Example 5.10** The beam in Fig. 5.26 is supported by three struts and subjected to a triangular line load.

Determine the forces in the struts.

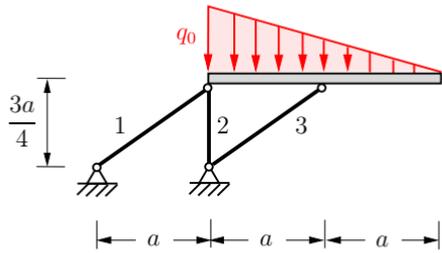


Fig. 5.26

Results: see (B) $S_1 = 10 q_0 a / 9$, $S_2 = -q_0 a$, $S_3 = -10 q_0 a / 9$.

E5.11 **Example 5.11** The structure shown in Fig. 5.27 consists of a beam and three bars. It carries a concentrated force F .

Determine the support reaction at A and the forces in the bars.

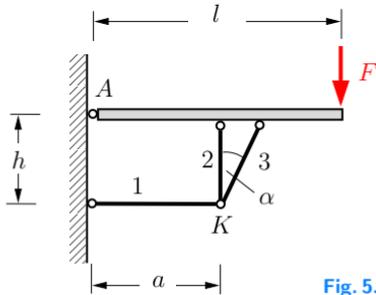


Fig. 5.27

Results: see (B) $A_V = F$, $A_H = -lF/h$,

$$S_1 = -lF/h, \quad S_2 = lF/(h \tan \alpha), \quad S_3 = -lF/(h \sin \alpha).$$

E5.12 **Example 5.12** The simply supported beam (length $a = 1$ m) shown in Fig. 5.28 is subjected to the three concentrated forces $F_1 = 4$ kN, $F_2 = 2$ kN, $F_3 = 3$ kN, the line load $q_0 = 5$ kN/m and the moment $M_0 = 4$ kNm.

Calculate the support reactions.

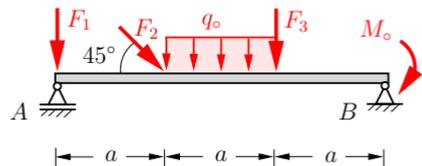


Fig. 5.28

Results: see (A) $A = 7.11$ kN, $B_H = 1.41$ kN, $B_V = 6.30$ kN.

Example 5.13 Find the support reactions for the hinged beam shown in Fig. 5.29.

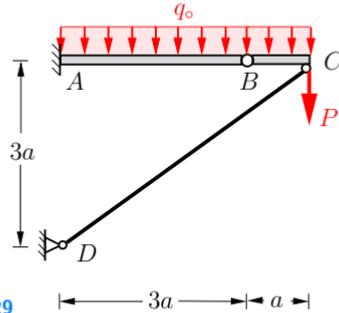


Fig. 5.29

Results: see (A) $A_V = 7q_0a/2$, $A_H = -4P/3 - 2q_0a/3$,
 $D = 5P/3 + 5q_0a/6$, $M_A = -3q_0a^2$.

Example 5.14 The hinged beam in Fig. 5.30 carries a concentrated force and a triangular line load.

Determine the support reactions and the force in the hinge.

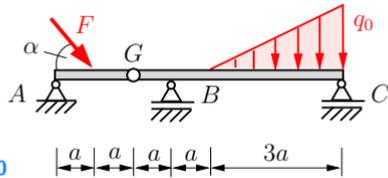


Fig. 5.30

Results: see (B) $B = (3q_0a + 5F \sin \alpha)/8$, $C = (9q_0a - F \sin \alpha)/8$,
 $A_V = F \sin \alpha/2$, $A_H = F \cos \alpha$, $G_V = F \sin \alpha/2$, $G_H = 0$.

Example 5.15 Determine the support reactions for the structure shown in Fig. 5.31. The pulley is frictionless.

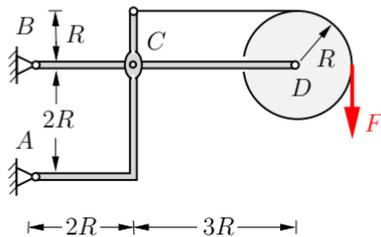


Fig. 5.31

Results: see (A) $A_H = 3F$, $A_V = 5F/2$,
 $B_H = -3F$, $B_V = -3F/2$.

E5.13

E5.14

E5.15

E5.16 **Example 5.16** A homogeneous beam (weight W) hangs on a crane (Fig. 5.32).

Determine the support reactions at A and B and the force at hinge C .

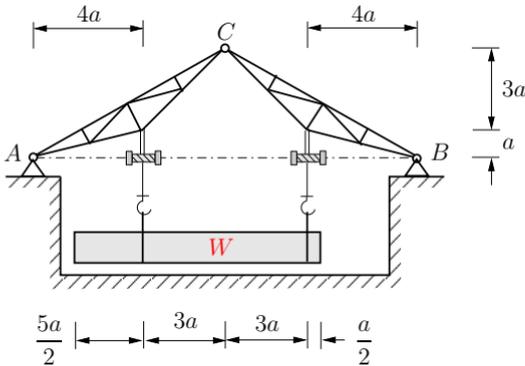


Fig. 5.32

Results: see (B) $A_V = 4W/7$, $A_H = W/2$, $B_V = 3W/7$,
 $B_H = -W/2$, $C_V = 2W/21$, $C_H = -W/2$.

E5.17 **Example 5.17** A mast (weight W_1) has a hinged support (ball-and-socket connection) at A .

In addition it is supported by two struts. Its upper end carries a weight W_2 (Fig. 5.33).

Determine the reaction force at A and the forces in the struts.

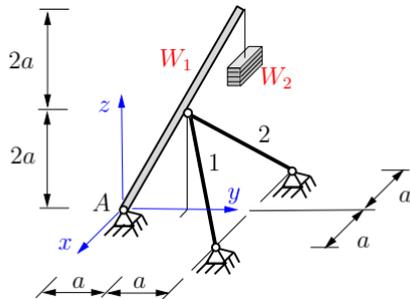


Fig. 5.33

Results: see (B) $A_x = 0$, $A_y = (W_1 + 2W_2)/4$, $A_z = W_1/2$,
 $S_1 = S_2 = -\sqrt{6}(W_1 + 2W_2)/8$.

Example 5.18 Determine the support reactions for the frame shown in Fig. 5.34.

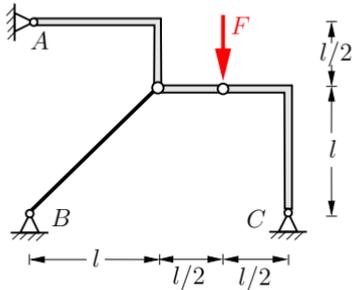


Fig. 5.34

Results: see (A) $A_H = F/3$, $A_V = -F/6$,
 $B = \sqrt{2}F/6$, $C_H = F/2$, $C_V = F$.

Example 5.19 Calculate the support reactions for the spatial structure in Fig. 5.35.

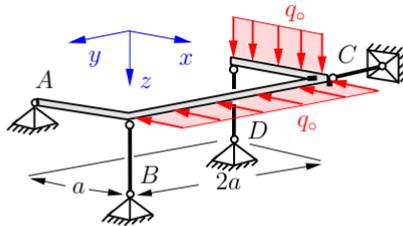


Fig. 5.35

Results: see (A) $A_x = -A_y = 2q_0a$, $A_z = q_0a/2$,
 $B_z = -q_0a/2$, $C_y = 2q_0a$, $D_z = -q_0a$.

E5.18

E5.19

5.5 Summary

- Supports and connecting elements, respectively, are classified according to the number of transferred reaction forces and couple moments. A simple support transfers one reaction, a hinged support two reactions etc. Analogously, a hinge as joining element transfers two reactions etc.
- A structure is statically determinate if all support and joint reactions can be calculated from the equilibrium conditions. This is the case if the number of unknown support and joint reactions is equal to the number of equilibrium conditions and the structure is immobile.
- A structure is kinematically determinate if it is immobile. A structure that can undergo finite or infinitesimal displacements is kinematically indeterminate.
- To calculate the support and joint reactions usually the following steps are necessary:
 - ◊ Removal of the supports from the structure and separation of the individual bodies.
 - ◊ Sketch of the free-body diagrams; all acting forces and couple moments as well as all reaction forces and couple moments must be drawn.
 - ◊ Formulation of the equilibrium conditions. In the plane case there are 3 equations for each body, e.g.

$$\sum F_{ix} = 0, \quad \sum F_{iy} = 0, \quad \sum M_i^{(A)} = 0,$$

where A is an arbitrary (appropriately chosen) reference point. In the spatial case there are 6 equilibrium conditions for each body.

- ◊ Calculation of the unknowns by resolution of the system of equations. Note: the number of equilibrium conditions and the number of unknowns must be equal!
- ◊ The system of equations has a unique solution if the determinant of the coefficient matrix is nonzero. Then the structure is statically and kinematically determinate.