Chapter 8

Work and Potential Energy

8

8 Work and Potential Energy

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Objectives: Students will become familiar with the concepts of *work*, *conservative forces* and *potential energy*. In addition, they will become acquainted with the *principle of virtual work*. After studying this chapter, students should be able to correctly apply this principle in order to determine equilibrium states in nonrigid systems as well as support reactions and internal forces and moments. Finally, it will be shown how to investigate the *stability* of equilibrium states of conservative systems with one degree of freedom.

8.1 Work and Potential Energy

The concept of work involves displacements and therefore belongs to the field of kinetics (see Volume 3) since in the field of statics no movement occurs. However, as will be shown in Section 8.2, problems in the field of statics can also be solved with the aid of work concepts. For this purpose, first the mechanical term "work" is introduced.

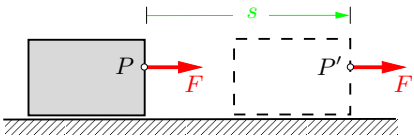


Fig. 8.1

Fig. 8.1 shows a body which is displaced a distance s by a constant force F acting in the direction of the displacement s. Here the *work* of the force F is defined as the product of F and the displacement s of the point of application P:

$$U = F\,s\,.$$

This definition can be generalized with the aid of vector calculus. In Fig. 8.2a the point of application P of a force $\boldsymbol{F}$ moves along an arbitrary path. Let us now consider an infinitesimal displacement $\mathrm{d}\boldsymbol{r}$ from the current position given by vector $\boldsymbol{r}$ to a neighboring position. The infinitesimal work $\mathrm{d}U$ done by the force $\boldsymbol{F}$ is then defined as the scalar product

$$\mathrm{d}U = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}\,. \tag{8.1}$$

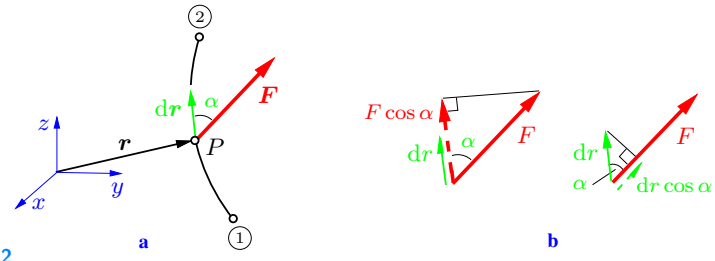


Fig. 8.2

According to (A.19), this product of the vectors $\boldsymbol{F}$ and $\mathrm{d}\boldsymbol{r}$ is given by the scalar value

$$\mathrm{d}U = |\boldsymbol{F}||\mathrm{d}\boldsymbol{r}| \cos\alpha = (F\cos\alpha)\mathrm{d}r = F(\mathrm{d}r\cos\alpha)\,. \tag{8.2}$$

Thus, the work $\mathrm{d}U$ is the displacement $\mathrm{d}r$ multiplied by the force component $F\cos\alpha$ in the direction of the displacement. Alternatively, it may be interpreted as the product of the force F and the component of the displacement $\mathrm{d}r\cos\alpha$ in the direction of the force (Fig. 8.2b). If force and displacement are orthogonal ($\alpha = \pi/2$), then no work will be done: $\mathrm{d}U = 0$.

The work done along an arbitrary path (Fig. 8.2a) from point ① to point ② can be determined from the line-integral

$$U = \int \mathrm{d}U = \int_{①}^{②} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}\,. \tag{8.3}$$

Work has the dimension $[F\,l]$ and is given in the unit named after the physicist James Prescott Joule (1818–1889):

$$1\,\mathrm{J} = 1\,\mathrm{Nm}.$$

The magnitude and direction of the force $\boldsymbol{F}$ in (8.3) may depend on the position vector $\boldsymbol{r}$, i.e., $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r})$. Therefore, the expression "work = force × distance" is only valid when both vectors $\boldsymbol{F}$ and $\mathrm{d}\boldsymbol{r}$ permanently have the same direction ($\alpha = 0$), and the magnitude of $\boldsymbol{F}$ is constant.

As an example, let us consider a system of two bodies with weights W and Q, respectively, connected by an unstretchable cable (Fig. 8.3). If the body on the left side undergoes a downward displacement of length $\mathrm{d}s$, then the body on the right side will be pulled the same length upward along the inclined plane. The work done by weight W is then given by $\mathrm{d}U_W = W\,\mathrm{d}s$, since the directions of the force and the displacement coincide. Only the component $Q\sin\alpha$ in the direction of the displacement is taken into account for the work done by Q, since the component of a force perpendicular to the direction of the displacement does no

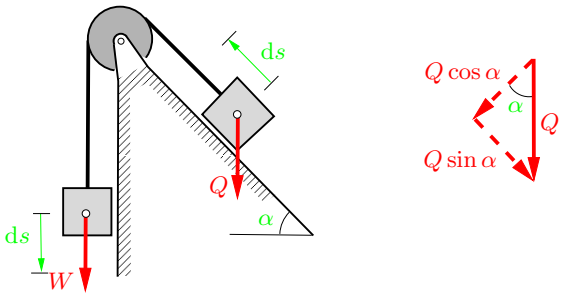


Fig. 8.3

work. As the working component acts in the opposite direction of the displacement, its work is negative: $\mathrm{d}U_Q = -Q \sin\alpha \, \mathrm{d}s$.

As a further example, consider a two-sided lever to which the forces F_1 and F_2 are applied (Fig. 8.4a). During an infinitesimal rotation $\mathrm{d}\varphi$ about the supporting point A, force F_1 does the work (Fig. 8.4b)

$$\mathrm{d}U = F_1 \, \mathrm{d}s_1 = F_1 \, a \, \mathrm{d}\varphi \, .$$

The product of force F_1 and length a of the lever arm is, according to (3.5), the moment M_1 of the force F_1 about A. Therefore, the work can also be expressed by

$$\mathrm{d}U = M_1 \, \mathrm{d}\varphi \, .$$

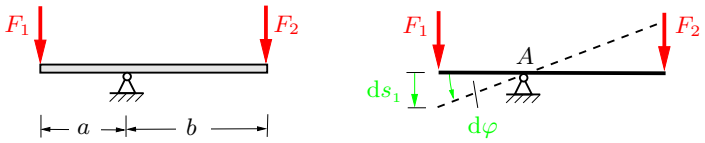


Fig. 8.4

In order to generalize the results obtained so far, we introduce an infinitesimal rotation vector $\mathrm{d}\boldsymbol{\varphi}$, whose direction coincides with the axis of rotation and whose magnitude is the angle $\mathrm{d}\varphi$. Therefore, the work done by a moment vector $\boldsymbol{M}$ during an infinitesimal rotation is given by

$$\mathrm{d}U = \boldsymbol{M} \cdot \mathrm{d}\boldsymbol{\varphi} \, . \tag{8.4}$$

The work done by a moment vector $\boldsymbol{M}$ during a finite rotation of the body on which it acts, follows from integration:

$$U = \int \mathrm{d}U = \int \boldsymbol{M} \cdot \mathrm{d}\boldsymbol{\varphi} \,. \tag{8.5}$$

This equation is analogous to the expression (8.3) for the work of a force. If $\boldsymbol{M}$ and $\mathrm{d}\boldsymbol{\varphi}$ are parallel, then it follows from (8.4) that $\mathrm{d}U = M\mathrm{d}\varphi$. If, in addition, M is constant during a finite rotation φ, then (8.5) yields $U = M\,\varphi$.

Since an angle is dimensionless, moment and work (although they are different physical quantities) have the same dimension $[F\,l]$.

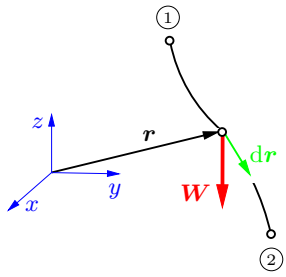


Fig. 8.5

Now a special case of a constant force is considered, namely, the weight W of a body (dead load) in the vicinity of the earth's surface (Fig. 8.5). Let the coordinate z point perpendicularly outward from the earth's surface, then the force vector is given by

$$\boldsymbol{W} = -\,W\,\boldsymbol{e}_z \,.$$

With the change of the position vector

$$\mathrm{d}\boldsymbol{r} = \mathrm{d}x\,\boldsymbol{e}_x + \mathrm{d}y\,\boldsymbol{e}_y + \mathrm{d}z\,\boldsymbol{e}_z$$

and

$$\boldsymbol{e}_z \cdot \boldsymbol{e}_x = \boldsymbol{e}_z \cdot \boldsymbol{e}_y = 0 \,, \quad \boldsymbol{e}_z \cdot \boldsymbol{e}_z = 1 \quad \text{(see (A.22))}$$

we obtain from (8.1)

$$\mathrm{d}U = \boldsymbol{W} \cdot \mathrm{d}\boldsymbol{r} = -W\,\boldsymbol{e}_z \cdot (\mathrm{d}x\,\boldsymbol{e}_x + \mathrm{d}y\,\boldsymbol{e}_y + \mathrm{d}z\,\boldsymbol{e}_z) = -W\,\mathrm{d}z\,.$$

Therefore, according to (8.3), the work done by the weight along the path from ① to ② is given by

$$U = \int \mathrm{d}U = -\int_{z_1}^{z_2} W\,\mathrm{d}z = W\,(z_1 - z_2)\,. \tag{8.6}$$

The work U depends only on the location of the endpoints. Hence, the work done by the weight along arbitrary paths is unchanged as long as the endpoints ① and ② are the same, i.e., the work is *path-independent*.

Forces whose work is path-independent are called *conservative forces* or *potential forces*. These forces, and these forces only, can be derived from a *potential* V, which is defined as

$$V = -U = -\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}\,. \tag{8.7}$$

The quantity V is also referred to as *potential energy*.

As a first example, we consider again the weight W in Fig. 8.6a. With $z_2 = z$, Equation (8.6) yields

$$V(z) = -U = W\,(z - z_1) = W\,z - W\,z_1\,. \tag{8.8a}$$

In Volume 3 it will be shown how a conservative force may be derived from its potential. In our case, the weight W is obtained by calculating the negative derivative of V with respect to the coordinate z:

$$-\frac{\mathrm{d}V}{\mathrm{d}z} = -W\,. \tag{8.8b}$$

The minus sign in front of W indicates that the weight acts in the negative z-direction.

According to (8.8a), the potential depends on the choice of the coordinate system (location of the origin). It is determined up to an arbitrary additive constant (here $W z_1$). However, this constant does not enter into the calculation of the force W, cf. (8.8b). It

also has no influence on the work because the work depends only on the height difference $z - z_1$. Therefore, the location of the coordinate system can be chosen arbitrarily. It is often useful to choose the coordinate system in such a way that the potential is zero at $z = 0$: $V(0) = 0$ (zero level).

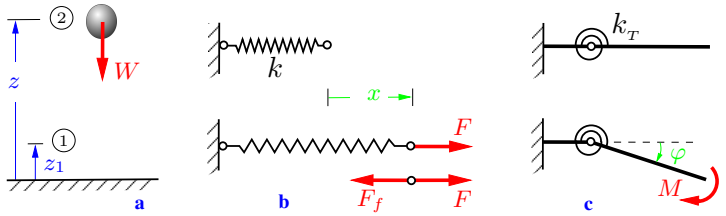


Fig. 8.6

As a further example, the potential of the force in a spring is considered. The spring depicted in Fig. 8.6b is elongated by a length x from its unstretched length through an external force F. It is known from experiments that the linear relation $F = k\,x$ exists between the force F and the spring elongation x, provided the elongation of the spring remains sufficiently small. The spring constant k is a measure of the stiffness of the spring. It has the dimension force divided by length, and therefore the unit $\mathrm{N\,cm^{-1}}$. The spring force F_f (restoring force) is the reaction force associated with F and points in the opposite direction of the elongation. Thus, calculation of the work done by the spring force during elongation yields

$$U = -\int_0^x F_f \,\mathrm{d}\bar{x} = -\int_0^x k\,\bar{x}\,\mathrm{d}\bar{x} = -\frac{1}{2}\,k\,x^2\,.$$

According to (8.7), the potential of the spring force is given by

$$V(x) = \frac{1}{2}\,k\,x^2\,. \tag{8.9}$$

The potential represents the energy stored in the spring due to its elongation. The spring force F_f also follows from the negative

derivative of V with respect to the coordinate x:

$$F_f = -\frac{\mathrm{d}V}{\mathrm{d}x} = -k\,x\,.$$

Consider now a linear torsion spring as depicted in Fig. 8.6c. There is a linear relation $M = k_T\varphi$ between the moment M and the angle of rotation φ. This relation is analogous to $F = k\,x$ in the foregoing case. The torsion spring constant k_T has the dimension $[Fl]$ and therefore the unit N cm or a multiple of it. If in Equation (8.9), k is replaced by k_T and x by φ, the potential of the moment of a torsion spring is obtained:

$$V = \frac{1}{2}\,k_T\varphi^2\,. \tag{8.10}$$

8.2 Principle of Virtual Work

8.2

Up to this point in the chapter we have calculated the work done by a force when the point of application actually moves along a path. However, the concept of work can also be applied to statics, where *no displacements occur*. In this framework, actual displacements must be replaced by *virtual displacements*. Virtual displacements are displacements (or rotations) that are

a) fictitious, i.e., do not exist in reality,
b) infinitesimally small,
c) geometrically (kinematically) admissible, i.e. consistent with the constraints of the system.

In order to distinguish virtual displacements from real displacements $\mathrm{d}\boldsymbol{r}$, we denote virtual displacements as $\delta\boldsymbol{r}$, i.e., with the δ-symbol, taken from the calculus of variations. Accordingly, the *virtual work* done by forces or moments during a virtual displacement is written as

$$\delta U = \boldsymbol{F}\cdot\delta\boldsymbol{r} \qquad \text{(compare (8.1))},$$

$$\delta U = \boldsymbol{M}\cdot\delta\boldsymbol{\varphi} \qquad \text{(compare (8.4))}.$$

We now reconsider the two-sided lever (Fig. 8.7a) and calculate the work done during a virtual displacement. A virtual displace-

ment, i.e., a deflection that is consistent with the constraints of the system, is a rotation with an angle $\delta\varphi$ about the support A (Fig. 8.7b). The total virtual work δU done by the two forces F_1 and F_2 is

$$\delta U = F_1\, a\, \delta\varphi - F_2\, b\, \delta\varphi = (F_1\, a - F_2\, b)\, \delta\varphi\,.$$

The minus sign in the second term takes into account that the force F_2 acts in opposite direction of the virtual deflection $b\,\delta\varphi$. In the equilibrium state, the expression in parentheses disappears due to the equilibrium condition of moments $F_2\, b = F_1\, a$ (Archimedes' law of the lever). Therefore, in this example, the virtual work vanishes in the equilibrium position: $\delta U = 0$. It should be noted that only the *external* forces F_1 and F_2 (cf. Section 1.4) enter into the virtual work, whereas the reaction force in A does not contribute.

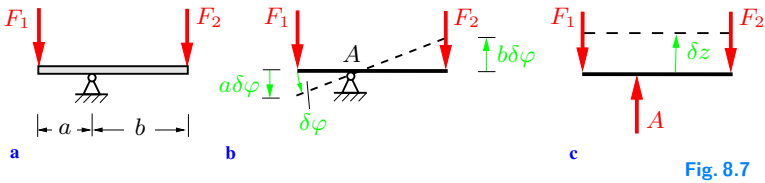


Fig. 8.7

The aforementioned results can be generalized. We postulate as an axiom that for an arbitrary system with an arbitrary number of external forces $\boldsymbol{F}_i^{(e)}$ and external moments $\boldsymbol{M}_i^{(e)}$ the entire virtual work must disappear in an equilibrium state:

$$\delta U = \sum \boldsymbol{F}_i^{(e)} \cdot \delta \boldsymbol{r}_i + \sum \boldsymbol{M}_i^{(e)} \cdot \delta \boldsymbol{\varphi}_i = 0\,. \tag{8.11}$$

Since this equilibrium axiom provides a statement on the work done during virtual displacements, it is called the *principle of virtual work*. It may be expressed as follows:

> A mechanical system is in equilibrium if the virtual work of the external loads (forces and moments) vanishes during an arbitrary virtual displacement.

The axiom of the principle of virtual work $\delta U = 0$ is also often referred to as the *principle of virtual displacements* . In the mechanics of deformable solids, an extended version of this principle has a significant meaning (cf. Volume 2).

The equilibrium conditions can be derived from the principle of virtual work; conversely, the principle of virtual work can be derived from the equilibrium conditions, which also have an axiomatic character. Therefore, the entire field of statics can be based either on the equilibrium conditions or on the principle of virtual work. From a practical point of view, the principle of virtual work offers the great advantage that the number of unknowns in the equations can often be reduced through an appropriate choice of virtual displacements. The drawback is that complicated kinematic conditions may have to be formulated.

The principle of virtual work cannot only be applied to movable systems, but also to systems that are rigidly supported, i.e., immobile. In the case of a rigid system, one or several supports are removed and replaced by the support reactions. These reactions then will be considered to be external loads and consequently taken into account in the principle of virtual work. For example, if we remove in Fig. 8.7b the pin joint at A, it is replaced by the force A (Fig. 8.7c). This force does the virtual work $A\delta z$ during a virtual displacement δz. Hence, the principle of virtual work reads

$$\delta U = A\,\delta z - F_1\,\delta z - F_2\,\delta z = (A - F_1 - F_2)\delta z = 0\,.$$

Since the virtual displacement δz is nonzero, the term in parentheses must vanish, yielding the support force $A = F_1 + F_2$.

8.3 Equilibrium States and Forces in Nonrigid Systems

8.3

In the following we consider systems of rigid bodies that are incompletely constrained and therefore able to move. If the applied forces are prescribed, the corresponding equilibrium configuration can be determined with the aid of the principle of virtual work. On the other hand, if an equilibrium position is prescribed, the

principle of virtual work yields the necessary forces. There are two ways to formulate the principle of virtual work:

a) We can draw the system in an arbitrary configuration and in an adjacent configuration. The magnitudes and directions of the displacements associated with the applied forces can be taken from the drawn figure. The corresponding terms for the virtual work (including the algebraic signs) are inserted into the principle of virtual work.
b) We can choose a coordinate system and describe the coordinates of the point of application of each force in this coordinate system. The virtual displacements are equal to infinitesimal changes of the coordinates. They are obtained formally through differentiation (the δ-symbol can be treated as a differential). For example, if $\boldsymbol{r}$ is a function of a coordinate α, i.e., $\boldsymbol{r} = \boldsymbol{r}(\alpha)$, then $\delta\boldsymbol{r} = (\mathrm{d}\boldsymbol{r}/\mathrm{d}\alpha)\delta\alpha$. The differentiation automatically yields the correct algebraic sign for each term.

Both methods will now be applied to the simple example of a rod (weight negligible) of length l, supported by a pin joint at A (Fig. 8.8a). The rod is loaded by a horizontal force P and a vertical force Q at the free end B. Our aim is to determine the angle α in the equilibrium position. Applying the first method, the rod is considered to be infinitesimally rotated by $\delta\alpha$ from an arbitrary, yet unknown position α (Fig. 8.8b). The point of application of the forces then is displaced by an amount $l\,\delta\alpha\sin\alpha$ upward, and by an amount $l\,\delta\alpha\cos\alpha$ to the left. The principle of virtual work

$$\delta U = P\,l\,\delta\alpha\cos\alpha - Q\,l\,\delta\alpha\sin\alpha = (P\cos\alpha - Q\sin\alpha)\,l\,\delta\alpha = 0$$

yields the equilibrium position ($\delta\alpha \neq 0$!):

$$P\cos\alpha - Q\sin\alpha = 0 \quad \rightarrow \quad \tan\alpha = \frac{P}{Q}\,.$$

In the second, more formal method, we describe the position of point B (point of application of the forces) with the aid of the position vector $\boldsymbol{r}$ that points from the fixed support A to B. In the coordinate system shown in Fig. 8.8c, the coordinates of $\boldsymbol{r}$ are $x = l\sin\alpha$ and $y = l\cos\alpha$. The virtual displacements are obtained

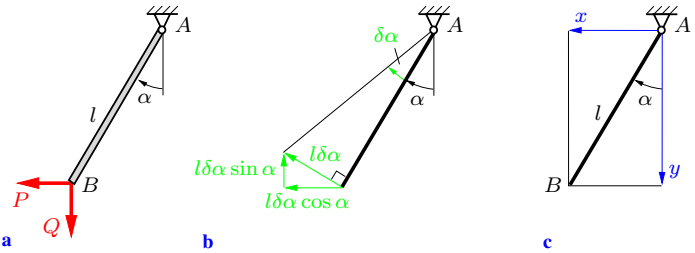


Fig. 8.8

through differentiation:

$$\delta x = \frac{\mathrm{d}x}{\mathrm{d}\alpha}\,\delta\alpha = l\cos\alpha\,\delta\alpha\,, \quad \delta y = \frac{\mathrm{d}y}{\mathrm{d}\alpha}\,\delta\alpha = -\,l\sin\alpha\,\delta\alpha\,.$$

Inserting these relations into the principle of virtual work (both forces point in the directions of the *positive* coordinate axes)

$$\begin{aligned}\delta U &= P\,\delta x + Q\,\delta y = P\,l\cos\alpha\,\delta\alpha - Q\,l\sin\alpha\,\delta\alpha \\ &= (P\cos\alpha - Q\sin\alpha)\,l\,\delta\alpha = 0\end{aligned}$$

leads to the same result as before.

The advantage of the formal method now becomes clear: in the first method the algebraic signs of the virtual displacements have to be determined from observation, whereas the second method automatically gives the correct signs ($\delta\alpha > 0$ yields $\delta y < 0$). The second method is preferable in the case of complicated kinematics (geometry), since it is not always possible to rely on observation. It should be noted that α can also be determined by formulating the moment equilibrium condition $\overset{\curvearrowright}{A}$: $P\,l\cos\alpha - Q\,l\sin\alpha = 0$.

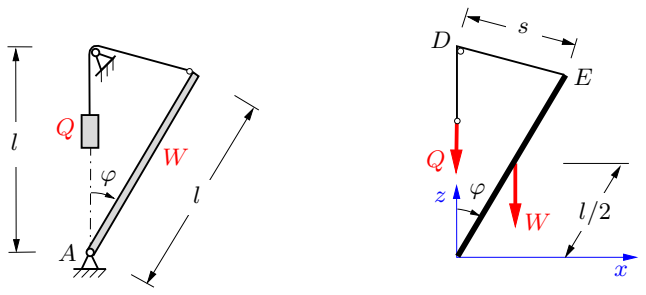
If a system has several independent possibilities of movement (degrees of freedom), the position $\boldsymbol{r}(\alpha, \beta, \ldots)$ of the point of application of a force is given by several independent coordinates $\alpha, \beta, \ldots$. The virtual displacements can then be found analogous to the total differential of a function of several variables:

$$\delta\boldsymbol{r} = \frac{\partial\boldsymbol{r}}{\partial\alpha}\,\delta\alpha + \frac{\partial\boldsymbol{r}}{\partial\beta}\,\delta\beta + \ldots\,. \tag{8.12}$$

E8.1

Example 8.1 A drawbridge of weight W can be raised with the aid of a cable (weight neglected) and a counter-weight Q (Fig. 8.9a).

Determine the positions of equilibrium.



a b

Fig. 8.9

Solution Since the angle φ uniquely describes the position of the system, the system has one degree of freedom.

If the arbitrary position given by φ is changed by a virtual displacement $\delta\varphi$, the points of application of the forces W and Q are displaced. In order to determine the virtual work of W, we have to consider only the change of its elevation, since the weight does no work in a horizontal displacement. With the coordinate system chosen in Fig. 8.9b (origin at the fixed support A), we obtain

$$z_W = \frac{l}{2}\cos\varphi \quad \rightarrow \quad \delta z_W = \frac{\mathrm{d}z_W}{\mathrm{d}\varphi}\,\delta\varphi = -\frac{l}{2}\sin\varphi\,\delta\varphi\,.$$

Determining the virtual displacement of the point of application of Q is more difficult. For this purpose, we introduce the auxiliary coordinate s, i.e., the distance between the fixed point D and point E (Fig. 8.9b):

$$s = 2\,l\sin\frac{\varphi}{2}\,.$$

A virtual displacement $\delta\varphi$ causes a virtual change δs of s, which is equal to the virtual displacement δz_Q of the weight Q (inextensible cable!):

$$\delta z_Q = \delta s = \frac{\mathrm{d}s}{\mathrm{d}\varphi}\,\delta\varphi = l\cos\frac{\varphi}{2}\,\delta\varphi\,.$$

Since both forces point in the direction of the *negative* z-axis, the principle of virtual work yields

$$\delta U = -W\,\delta z_W - Q\,\delta z_Q = \left(W\,\frac{l}{2}\sin\varphi - Q\,l\cos\frac{\varphi}{2}\right)\delta\varphi = 0\,.$$

With $\sin\varphi = 2\sin(\varphi/2)\cos(\varphi/2)$ and $\delta\varphi \neq 0$, we obtain

$$\cos\frac{\varphi}{2}\left(W\sin\frac{\varphi}{2} - Q\right) = 0\,.$$

The first solution $\cos(\varphi/2) = 0$ leads to $\varphi = \pi$. Therefore, the only remaining solution is

$$\underline{\underline{\sin\frac{\varphi}{2} = \frac{Q}{W}}}\,.$$

For φ lying within the technically reasonable region $0 < \varphi < \pi/2$, the inequality $\sin(\varphi/2) < \sin(\pi/4) = \sqrt{2}/2$ must be satisfied. This requirement leads to the following condition for the load Q:

$$\frac{Q}{W} < \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\,.$$

Example 8.2 A car-jack loaded with a weight W is schematically depicted in Fig. 8.10. The height of the screw-thread is h. E8.2

What torque M must be applied to the car-jack in order to keep the jack in equilibrium? The jack-screw, when turned, moves without friction.

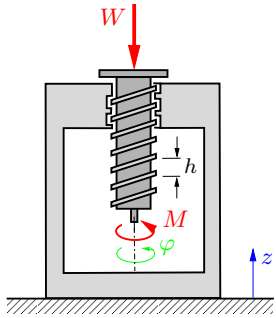


Fig. 8.10

Solution According to the principle of virtual work, the system is in equilibrium when the total work done by force W and torque M vanishes:

$$\delta U = M\delta\varphi - W\,\delta z = 0\,.$$

Here, the directions of M and φ coincide, whereas W and z have opposite directions (Fig. 8.10). The virtual displacements $\delta\varphi$ and δz are not independent. A rotation of $\Delta\varphi = 2\pi$ will raise the jack-screw by the height $\Delta z = h$ of the screw-thread. Therefore, a rotation of $\delta\varphi$ results in $\delta z = (h/2\,\pi)\delta\varphi$. Thus, the principle of virtual work yields

$$\delta U = \left(M - \frac{h}{2\pi}\,W\right)\delta\varphi = 0\,.$$

With $\delta\varphi \neq 0$ we obtain the torque necessary for equilibrium:

$$\underline{\underline{M = \frac{h}{2\,\pi}\,W\,.}}$$

It can be recognized that a large weight W can be raised with a small force K by an externally applied torque $M = l\,K$ if $l \gg h$.

E8.3

Example 8.3 Two pin-jointed rods of weights W_1 and W_2 are displaced from the vertical position by a horizontal force F (Fig. 8.11a).

Determine the equilibrium configuration.

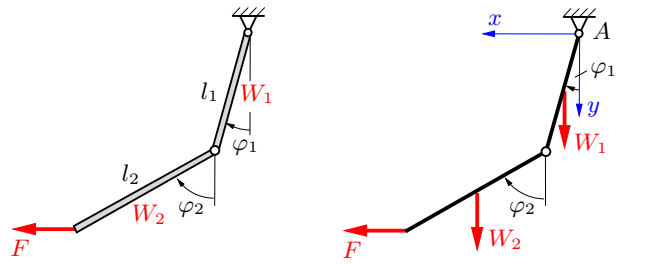


Fig. 8.11

Solution The configuration of the system is uniquely determined by the two angles φ_1 and φ_2. We use a coordinate system that has its origin at the fixed support A (Fig. 8.11b). Then we obtain

the following coordinates of the points of application of the forces:

$$y_1 = \frac{l_1}{2}\cos\varphi_1\,, \quad y_2 = l_1\cos\varphi_1 + \frac{l_2}{2}\cos\varphi_2\,,$$
$$x_F = l_1\sin\varphi_1 + l_2\sin\varphi_2\,.$$

According to (8.12), the virtual displacements are given by

$$\delta y_1 = -\frac{l_1}{2}\sin\varphi_1\,\delta\varphi_1\,, \quad \delta y_2 = -\,l_1\sin\varphi_1\,\delta\varphi_1 - \frac{l_2}{2}\sin\varphi_2\,\delta\varphi_2\,,$$
$$\delta x_F = l_1\cos\varphi_1\,\delta\varphi_1 + l_2\cos\varphi_2\,\delta\varphi_2\,.$$

Therefore, the principle of virtual work leads to

$$\begin{aligned}\delta U &= W_1\,\delta y_1 + W_2\,\delta y_2 + F\,\delta x_F\\ &= W_1\Big(-\frac{l_1}{2}\sin\varphi_1\,\delta\varphi_1\Big) + W_2\Big(-l_1\sin\varphi_1\,\delta\varphi_1 - \frac{l_2}{2}\sin\varphi_2\,\delta\varphi_2\Big)\\ &\quad + F(l_1\cos\varphi_1\,\delta\varphi_1 + l_2\cos\varphi_2\,\delta\varphi_2)\\ &= \Big(F\,l_1\cos\varphi_1 - W_1\,\frac{l_1}{2}\sin\varphi_1 - W_2\,l_1\sin\varphi_1\Big)\delta\varphi_1\\ &\quad + \Big(F\,l_2\cos\varphi_2 - W_2\,\frac{l_2}{2}\sin\varphi_2\Big)\delta\varphi_2 = 0\,.\end{aligned}$$

Since the system has two degrees of freedom, there are two virtual displacements, $\delta\varphi_1$ and $\delta\varphi_2$. They are independent of each other and not equal to zero at the same time. Therefore, the virtual work vanishes only if the expressions in both parentheses are zero:

$$F\,l_1\cos\varphi_1 - W_1\,\frac{l_1}{2}\sin\varphi_1 - W_2\,l_1\sin\varphi_1 = 0$$
$$\rightarrow \quad \underline{\underline{\tan\varphi_1 = \frac{2\,F}{W_1 + 2\,W_2}}}\,,$$
$$F\,l_2\cos\varphi_2 - W_2\,\frac{l_2}{2}\sin\varphi_2 = 0 \quad \rightarrow \quad \underline{\underline{\tan\varphi_2 = \frac{2\,F}{W_2}}}\,.$$

8.4 Reaction Forces and Stress Resultants

8.4

Structures such as beams, frames or trusses are rigidly supported. In order to calculate a reaction force (moment) with the aid of the principle of virtual work, the corresponding support must be

removed and replaced by the support reaction. The point of application of the support reaction is then able to move. Therefore, the reaction force (moment) is considered to be an external force (moment) in the principle of virtual work. Similarly, the force in an internal pin or the stress resultants in beams can be determined by cutting the system appropriately. Due to this imaginary cut, the system becomes movable and the corresponding internal force does virtual work during a virtual displacement. The following examples will explain the procedure.

E8.4

Example 8.4 Determine the force in the pin G of the structure shown in Fig. 8.12a.

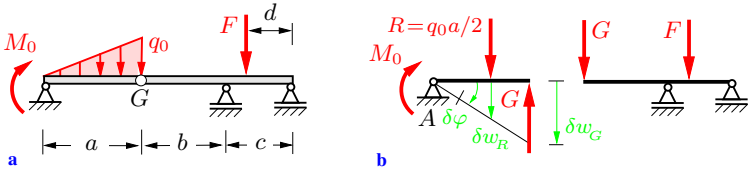


Fig. 8.12

Solution The horizontal component of the force in the pin is zero. In order to determine the vertical component, we divide the structure into two parts by a cut through the pin. The left-hand part is then able to rotate about the point A, whereas the right-hand portion cannot move in the vertical direction (Fig. 8.12b). Furthermore, the triangular load is replaced by its resultant

$$R = q_0\, a/2$$

which is located at a distance $2\,a/3$ from A. With the virtual displacements

$$\delta w_R = \frac{2}{3}\, a\, \delta\varphi\,, \quad \delta w_G = a\, \delta\varphi$$

the principle of virtual work yields

$$\delta U = R\, \delta w_R + M_0\, \delta\varphi - G\, \delta w_G = \frac{q_0\, a}{2}\, \frac{2}{3}\, a\, \delta\varphi + M_0 \delta\varphi - G\, a\, \delta\varphi$$
$$= \left(q_0\, \frac{a^2}{3} + M_0 - G\, a \right) \delta\varphi = 0\,.$$

Since $\delta\varphi \neq 0$, the force in pin G becomes

$$\underline{\underline{G = \frac{q_0\, a}{3} + \frac{M_0}{a}}}\,.$$

The force F acting on the right-hand section has no influence on the force in pin G.

Example 8.5 The structure in Fig. 8.13a consists of three beams that are pin-connected at G_1 and G_2. It is subjected to a concentrated force F and a constant line load $q_0 = F/(3\,a)$. E8.5

Determine the support reactions at A with the aid of the principle of virtual work.

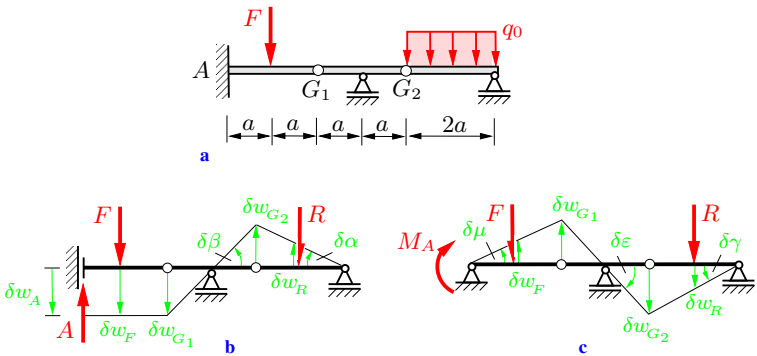


Fig. 8.13

Solution In order to determine the vertical support reaction A, the clamped end of the structure has to be replaced by a parallel motion according to Fig. 8.13b. Then point A can move in the vertical direction and the support reaction A has to be treated as an external load in the principle of virtual work. Since no rotation is possible at A, the moment M_A is a reaction moment and does no work. From Fig. 8.13b we obtain the following relations between the virtual displacements:

$$\delta w_A = \delta w_F = \delta w_{G_1} = a\,\delta\beta\,, \quad \delta w_R = a\,\delta\alpha\,.$$

The angles $\delta\alpha$ and $\delta\beta$ are mutually dependent. Considering the displacement of pin G_2, we find

$$\delta w_{G_2} = 2\,a\,\delta\alpha = a\,\delta\beta \quad \rightarrow \quad \delta\alpha = \tfrac{1}{2}\,\delta\beta\,.$$

From the principle of virtual work

$$\begin{aligned}\delta U &= -A\,\delta w_A + F\,\delta w_F - R\,\delta w_R = -A\,a\,\delta\beta + F\,a\,\delta\beta - \tfrac{2}{3}\,F\,a\,\delta\alpha \\ &= (-A + F - \tfrac{2}{3}\,F\,\tfrac{1}{2})\,a\,\delta\beta = 0\end{aligned}$$

and $\delta\beta \neq 0$, we obtain the support reaction

$$\underline{\underline{A = \frac{2}{3}\,F\,.}}$$

In order to calculate the moment M_A we replace the clamped support by a smooth pin (Fig. 8.13c). Then the left part of the structure can rotate about A and the moment M_A enters into the principle of virtual work as an external moment. Now the point A cannot be vertically displaced. Therefore, force A is a reaction force and does no work. The three angles in Fig. 8.13c are mutually dependent:

$$\begin{aligned}&\delta w_{G_1} = 2\,a\,\delta\mu = a\,\delta\varepsilon\,, \quad \delta w_{G_2} = a\,\delta\varepsilon = 2\,a\,\delta\gamma \\ &\qquad \rightarrow \quad \delta\varepsilon = 2\,\delta\mu \quad \text{and} \quad \delta\gamma = \delta\mu\,.\end{aligned}$$

Considering the algebraic signs (F acts in the opposite direction to δw_F and M_A acts in the opposite direction to $\delta\mu$) the principle of virtual works yields

$$\begin{aligned}\delta U &= -M_A\,\delta\mu - F\,\delta w_F + R\,\delta w_R \\ &= -M_A\,\delta\mu - F\,a\,\delta\mu + \frac{F}{3\,a}\,2\,a\,a\,\delta\gamma \\ &= \Big(-M_A - Fa + \frac{2}{3}\,Fa\Big)\delta\mu = 0 \quad \rightarrow \quad \underline{\underline{M_A = -\frac{1}{3}\,Fa\,.}}\end{aligned}$$

Both calculations show the advantage of the principle of virtual work: the forces G_1 and G_2 in the pins and the remaining support reactions, which appear in the classical calculation (cf. Section 5.3.3), do no work and therefore need not be taken into account when applying the principle of virtual work.

Example 8.6 Determine the force in member 5 of the truss shown in Fig. 8.14a. E8.6

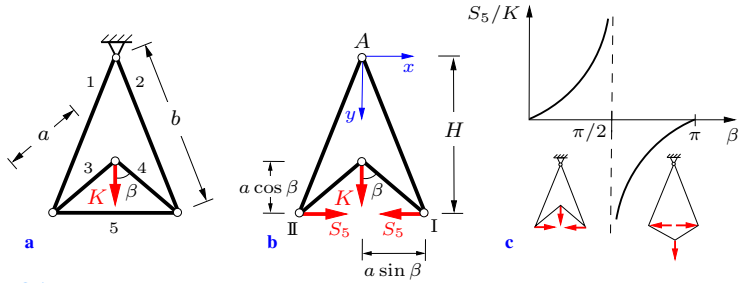


Fig. 8.14

Solution We remove member 5 from the truss. Then the internal forces S_5 at pins I and II act as external forces (Fig. 8.14b). In order to determine the positions of the points of application of the various forces, an x, y-coordinate system is introduced with its origin at the fixed point A. The point of application of force K has the y-coordinate

$$y_K = H - a\cos\beta = \sqrt{b^2 - a^2\sin^2\beta} - a\cos\beta\,.$$

Pin I (point of application of S_5) has the x-coordinate

$$x_{\mathrm{I}} = a\sin\beta\,.$$

A virtual displacement (i.e., a small change of angle β) yields

$$\delta y_K = \frac{\mathrm{d}y_K}{\mathrm{d}\beta}\,\delta\beta = \Big(\frac{-a^2\,2\sin\beta\cos\beta}{2\sqrt{b^2 - a^2\sin^2\beta}} + a\sin\beta\Big)\delta\beta\,,$$

$$\delta x_{\mathrm{I}} = \frac{\mathrm{d}x_{\mathrm{I}}}{\mathrm{d}\beta}\,\delta\beta = a\cos\beta\,\delta\beta\,.$$

Due to the symmetry of the system and the loading, the displacement of pin II (to the right) is equal to the displacement of pin I (to the left). Therefore, the total virtual work performed by the forces is

$$\begin{aligned}\delta U &= K\delta y_K - 2\,S_5\,\delta x_{\mathrm{I}} \\ &= \Big[Ka\sin\beta\Big(1 - \frac{a\cos\beta}{\sqrt{b^2 - a^2\sin^2\beta}}\Big) - 2\,S_5\,a\cos\beta\Big]\delta\beta\,.\end{aligned}$$

Since $\delta U = 0$ and $\delta\beta \neq 0$, we obtain

$$S_5 = \frac{1}{2} K \tan\beta \left(1 - \frac{a\cos\beta}{\sqrt{b^2 - a^2\sin^2\beta}}\right).$$

Fig. 8.14c qualitatively depicts the ratio S_5/K versus β. Because of $b > a$, force S_5 is positive for $0 \leqq \beta < \pi/2$ and negative for $\pi/2 < \beta \leqq \pi$. This result may be verified by inspection.

8.5

8.5 Stability of Equilibrium States

In Section 8.3 the equilibrium states were determined by applying the principle of virtual work $\delta U = 0$. However, experience teaches that there are "different types" of equilibrium states. To characterize these states the concept of *stability* must be introduced. In the following, the discussion is limited to conservative forces and systems with only *one* degree of freedom. In this case the potential V depends only on *one* coordinate. Fig. 8.15 shows two examples where the only applied force is the weight. The first example in Fig. 8.15a shows a sphere (weight W) that lies at the lowest point of a concave surface (equilibrium position). If a small disturbance is imposed, e.g., a small lateral deflection x, the sphere is lifted by an amount Δz and the potential is increased by

$$\Delta V = W\,\Delta z > 0\,.$$

In the second example, a rod supported by a frictionless pin at its upper end is shown in the equilibrium position (vertical position). A small disturbance (small angle φ) raises the center of gravity of the rod (point of application of W) and therefore increases the potential. In both cases, the respective body returns to its equilibrium position when left to itself. Such equilibrium positions are referred to as *stable*.

Now we consider a sphere lying on a horizontal plane and a rod supported at its center of gravity (Fig. 8.15b). The elevation of the points of application of the weights remains unchanged in

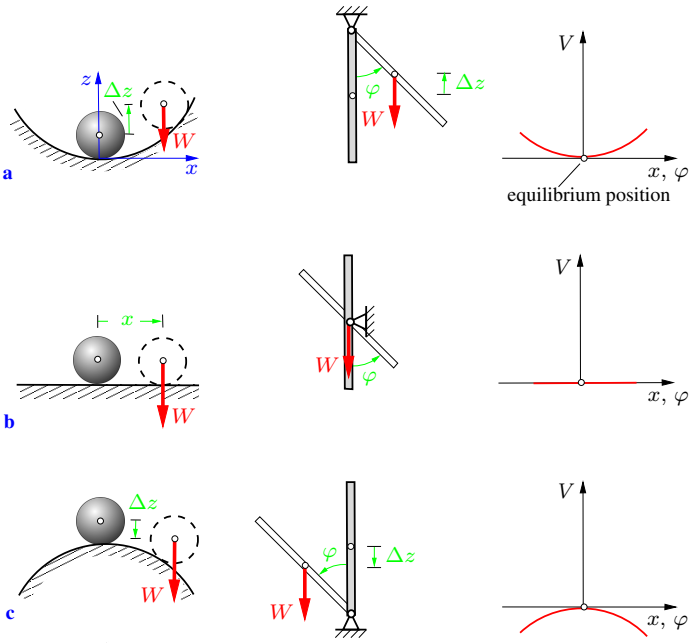


Fig. 8.15

the case of a displacement x or a rotation φ. Therefore, there is no change in the potential:

$$\Delta V = W\,\Delta z = 0\,.$$

The displaced positions are also equilibrium positions. The bodies remain at rest when left to themselves. Such equilibrium states are called *neutral*.

If the sphere is in equilibrium at the highest point of a convex surface or if the rod is supported at its lower end (Fig. 8.15c), then the potential decreases due to a displacement:

$$\Delta V = W\,\Delta z < 0\,.$$

When the sphere or the rod are left to themselves in the displaced

position, they will move further away from the equilibrium state. Such equilibrium states are referred to as *unstable*.

On the right-hand side of Fig. 8.15 the graph of the potential is qualitatively depicted as a function of x (sphere) or as a function of φ (rod) for the three cases. It can be recognized from the diagrams that the potential in each equilibrium state has an extreme value. This extreme value is equivalent to the principle of virtual work that, according to (8.11), requires for equilibrium

$$\delta V = -\,\delta U = 0\,.$$

If the potential depends on one coordinate only, e.g., $V = V(x)$, then

$$\delta V = \frac{\mathrm{d}V}{\mathrm{d}x}\,\delta x = 0\,.$$

Since $\delta x \neq 0$, this equation leads to the following condition for an equilibrium state:

$$\frac{\mathrm{d}V}{\mathrm{d}x} = V' = 0\,. \tag{8.13}$$

The graph of the potential function has a horizontal tangent at the point corresponding to an equilibrium position.

The *criterion of stability* follows from the behavior of the potential curve in the vicinity of the considered equilibrium state. In case a), the potential increases due to a displacement x or a rotation φ; in case c) the potential decreases. Therefore, taking into account the above-mentioned concepts, it can be stated:

$$\Delta V \left\{ \begin{array}{ll} > 0 & \text{stable} \\ \equiv 0 & \text{neutral} \\ < 0 & \text{unstable} \end{array} \right\} \text{ equilibrium}\,. \tag{8.14}$$

Accordingly, the potential has a minimum in a stable equilibrium state and a maximum in an unstable equilibrium state. Consequently, the stability criterion (8.14) can also be formulated in terms of the second derivative of the potential function $V(x)$,

characterizing a minimum and a maximum:

$$\begin{aligned} V''(x_0) > 0 \quad &\rightarrow \quad \text{minimum} \quad \rightarrow \quad \text{stable}\,, \\ V''(x_0) < 0 \quad &\rightarrow \quad \text{maximum} \quad \rightarrow \quad \text{unstable}\,. \end{aligned} \tag{8.15}$$

Here, the position x_0 denotes the equilibrium state.

In the case $V''(x_0) = 0$, further investigations are necessary. Let $V(x_0) = V_0$ be the value of the potential in the equilibrium state x_0. The potential in a neighboring position $x_0 + \delta x$ can be expressed by the Taylor-Series

$$\begin{aligned} V(x_0 + \delta x) = V_0 + V'(x_0)\,\delta x + \tfrac{1}{2}V''(x_0)\,(\delta x)^2 \\ + \tfrac{1}{6}V'''(x_0)\,(\delta x)^3 + \dots\,. \end{aligned}$$

Thus, the change in the potential is

$$\begin{aligned} \Delta V &= V(x_0 + \delta x) - V_0 \\ &= V'(x_0)\,\delta x + \tfrac{1}{2}V''(x_0)\,(\delta x)^2 + \tfrac{1}{6}V'''(x_0)\,(\delta x)^3 + \dots\,. \end{aligned} \tag{8.16}$$

According to (8.13), the first derivative of the potential vanishes in the equilibrium state, i.e. $V'(x_0) = 0$. Therefore, the algebraic sign of the second derivative $V''(x_0)$ of the potential determines whether ΔV is greater or less than zero and hence, whether the equilibrium state is stable or unstable. If $V''(x_0)$ and *all* higher derivatives disappear, then $\Delta V \equiv 0$ according to (8.16). This corresponds to a *neutral* equilibrium state. On the other hand, if $V''(x_0) = 0$ holds and higher derivatives are nonzero, the algebraic sign of the lowest non-vanishing derivative of the potential determines the type of equilibrium.

Example 8.7 Three cogged wheels are supported without friction at their centers. Three mass points (weights W_1, W_2 and W_3) are mounted eccentrically on the wheels as shown in Fig. 8.16a. E8.7

Determine the equilibrium states and investigate their type of stability (i.e., stable or unstable) for $W_1 = W_3 = 2\,W$, $W_2 = W$ and $x = \sqrt{3}\,r$.

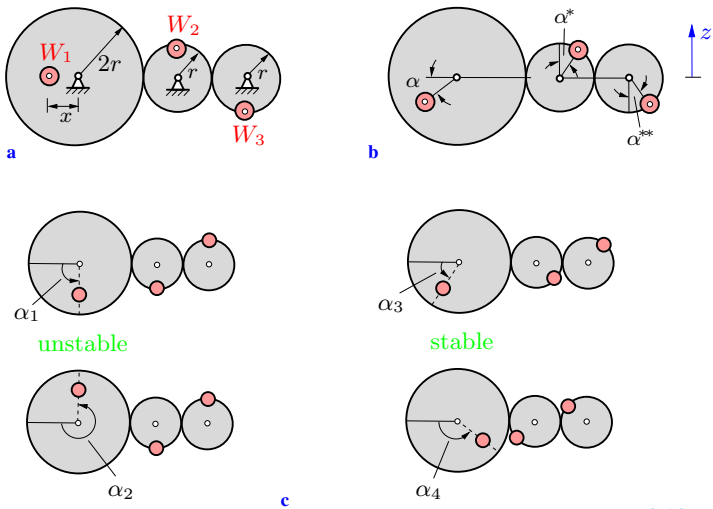


Fig. 8.16

Solution To formulate the potential we introduce the coordinate z as shown in Fig. 8.16b. Let the large wheel be rotated by an arbitrary angle α from the initial position (Fig. 8.16b). This rotation causes the other wheels also to rotate. The arc lengths at the contact points of the wheels must be the same for every wheel. With the given radii, we obtain

$$2\,r\,\alpha = r\,\alpha^{*} = r\,\alpha^{**} \quad \rightarrow \quad \alpha^{*} = \alpha^{**} = 2\,\alpha\,.$$

The coordinates of W_1, W_2 and W_3 are given by

$$z_1 = -\,x\sin\alpha\,, \quad z_2 = r\cos\alpha^{*} = r\cos 2\,\alpha\,,$$

$$z_3 = -\,r\cos\alpha^{**} = -\,r\cos 2\,\alpha\,.$$

Thus, the total potential can be expressed as

$$\begin{aligned} V &= W_1\,z_1 + W_2\,z_2 + W_3\,z_3 \\ &= 2\,W(-\,x\sin\alpha) + W\,r\cos 2\,\alpha + 2\,W(-\,r\cos 2\,\alpha) \\ &= -\,W\,r(2\sqrt{3}\sin\alpha + \cos 2\,\alpha)\,. \end{aligned}$$

According to (8.13), the equilibrium states are determined from

$$V' = \frac{\mathrm{d}V}{\mathrm{d}\alpha} = -\,W\,r(2\sqrt{3}\cos\alpha - 2\sin 2\,\alpha)$$
$$= -\,2\,W\,r\cos\alpha(\sqrt{3} - 2\sin\alpha) = 0\,.$$

A first solution is obtained by setting the first factor to zero:

$$\cos\alpha = 0\,.$$

The corresponding equilibrium positions are

$$\underline{\underline{\alpha_1 = \pi/2}}\,, \quad \underline{\underline{\alpha_2 = 3\,\pi/2}}\,.$$

A second solution follows from the disappearance of the bracketed term:

$$\sqrt{3} - 2\sin\alpha = 0 \quad \rightarrow \quad \sin\alpha = \sqrt{3}/2\,.$$

The corresponding equilibrium positions are

$$\underline{\underline{\alpha_3 = \pi/3}}\,, \quad \underline{\underline{\alpha_4 = 2\,\pi/3}}\,.$$

According to (8.15), the type of equilibrium of the four different positions can be determined with the aid of the second derivative of the potential:

$$V'' = \frac{\mathrm{d}^2V}{\mathrm{d}\alpha^2} = -\,W\,r(-\,2\sqrt{3}\sin\alpha - 4\cos 2\,\alpha)$$
$$= 2\,W\,r(\sqrt{3}\sin\alpha + 2\cos 2\,\alpha)\,.$$

Inserting the solutions α_i yields:

$$V''(\alpha_1) = V''(\tfrac{1}{2}\pi) = 2\,W\,r(\sqrt{3} - 2) < 0 \quad \rightarrow \quad \text{unstable},$$
$$V''(\alpha_2) = V''(\tfrac{3}{2}\pi) = 2\,W\,r(-\sqrt{3} - 2) < 0 \quad \rightarrow \quad \text{unstable},$$
$$V''(\alpha_3) = V''(\tfrac{1}{3}\pi) = 2\,W\,r\left[\sqrt{3}\tfrac{1}{2}\sqrt{3} + 2\left(-\tfrac{1}{2}\right)\right] = W\,r > 0$$
$$\rightarrow \quad \text{stable},$$
$$V''(\alpha_4) = V''(\tfrac{2}{3}\pi) = 2\,W\,r\left[\sqrt{3}\tfrac{1}{2}\sqrt{3} + 2\left(-\tfrac{1}{2}\right)\right] = W\,r > 0$$
$$\rightarrow \quad \text{stable}.$$

Fig. 8.16c shows the four equilibrium configurations.

E8.8

Example 8.8 A weightless rod is subjected to a vertical force F and held on each side by a spring (Fig. 8.17a). The spring constant of each spring is k. An appropriate support keeps the springs in a horizontal position during a rotation of the rod.

Determine the type of stability of the equilibrium positions.

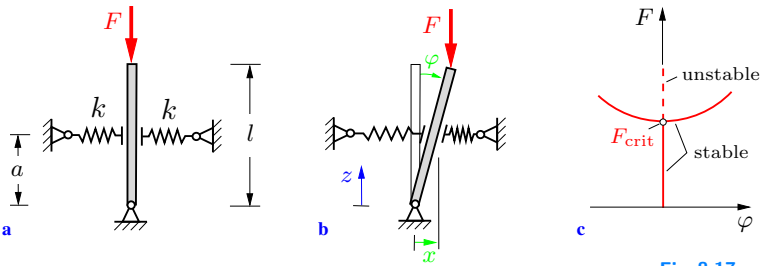


Fig. 8.17

Solution We introduce a coordinate system with its origin at the fixed support (Fig. 8.17b). To determine the potentials of the force F and the spring forces we consider an arbitrary position of the rod. The zero-level of the potential of force F (dead load, cf. Section 8.1) is chosen to be at $z = 0$. Thus, its potential is

$$V_F = F\, z_F \,.$$

According to (8.9) the potential of a spring (spring constant k) due to an elongation x is given by

$$V_f = \tfrac{1}{2}\, k\, x^2 \,.$$

The system has one degree of freedom. Therefore, the total potential depends only on a single coordinate; it is useful to choose the angle φ (Fig. 8.17b). With the geometrical relations $z_F = l\,\cos\varphi$ and $x = a\,\tan\varphi$, the total potential becomes

$$V(\varphi) = F\, l \cos\varphi + 2\,\tfrac{1}{2}\, k\, (a \tan\varphi)^2 \,.$$

We determine the equilibrium positions using condition (8.13):

$$V' = \frac{\mathrm{d}V}{\mathrm{d}\varphi} = -\,F\, l \sin\varphi + 2\, k\, a^2 \frac{\tan\varphi}{\cos^2\varphi} = 0$$

$$\rightarrow \quad \sin\varphi \left(-F\,l + 2\,k\,a^2 \frac{1}{\cos^3\varphi} \right) = 0\,.$$

A first equilibrium position is obtained from

$$\sin\varphi = 0 \quad \rightarrow \quad \underline{\underline{\varphi_1 = 0}}\,.$$

Further solutions follow from the term in parentheses:

$$-F\,l + 2\,k\,a^2 \frac{1}{\cos^3\varphi} = 0 \quad \rightarrow \quad \cos^3\varphi_2 = \frac{2\,k\,a^2}{F\,l}$$

$$\rightarrow \quad \underline{\underline{\varphi_2 = \arccos \sqrt[3]{\frac{2\,k\,a^2}{F\,l}}}}\,. \tag{a}$$

These positions exist only for $Fl > 2k\,a^2$.

In order to investigate the type of stability, the second derivative of the total potential is calculated:

$$V'' = \frac{\mathrm{d}^2 V}{\mathrm{d}\varphi^2} = -F\,l\cos\varphi + 2\,k\,a^2 \, \frac{\cos^2\varphi \, \dfrac{1}{\cos^2\varphi} + \tan\varphi \; 2\cos\varphi \;\; \sin\varphi}{\cos^4\varphi}$$

$$= -F\,l\cos\varphi + 2\,k\,a^2 \, \frac{1 + 2\sin^2\varphi}{\cos^4\varphi}$$

$$= -F\,l\cos\varphi + 2\,k\,a^2 \, \frac{3 - 2\cos^2\varphi}{\cos^4\varphi}\,. \tag{b}$$

Inserting the first solution $\varphi_1 = 0$ yields

$$V''(\varphi_1) = (-F\,l + 2\,k\,a^2) = 2\,k\,a^2 \left(1 - \frac{F\,l}{2\,k\,a^2} \right)\,. \tag{c}$$

The algebraic sign of V'' (and thus the stability of this equilibrium position) depends on the parameters that appear in parentheses. From (c), we obtain

$$V''(\varphi_1) > 0 \quad \text{for} \quad \frac{F\,l}{2\,k\,a^2} < 1 \quad \rightarrow \quad \text{stable position,}$$

$$V''(\varphi_1) < 0 \quad \text{for} \quad \frac{F\,l}{2\,k\,a^2} > 1 \quad \rightarrow \quad \text{unstable position.}$$

The special case

$$\frac{F\,l}{2\,k\,a^2} = 1 \quad \rightarrow \quad F = 2\,\frac{k\,a^2}{l} = F_{\text{crit}} \tag{d}$$

characterizes the "critical load", since at this point of the load-displacement curve in Fig. 8.17c, the stable position $\varphi = 0$ (for $F < F_{\text{crit}}$) changes to an unstable position (for $F > F_{\text{crit}}$).

The second derivative $V''(\varphi_1)$ is equal to zero for $F = F_{\text{crit}}$. To investigate the type of stability at this particular point, it would be necessary to consider higher derivatives of V.

If the angle φ_2 of the second equilibrium position is inserted into (b) and the relation $2\,k\,a^2 = F\,l\cos^3\varphi_2$ is used, we obtain for the second derivative of V

$$\begin{aligned} V''(\varphi_2) &= -\,F\,l\cos\varphi_2\left(1 - \cos^2\varphi_2\,\frac{3 - 2\cos^2\varphi_2}{\cos^4\varphi_2}\right) \\ &= -\,F\,l\cos\varphi_2\left(1 - \frac{3}{\cos^2\varphi_2} + 2\right) \\ &= 3\,F\,l\cos\varphi_2\left(\frac{1}{\cos^2\varphi_2} - 1\right) . \end{aligned}$$

For $0 < \varphi_2 < \pi/2$, we have $\cos\varphi_2 < 1$. Hence, $V''(\varphi_2) > 0$, i.e., the equilibrium position $\varphi = \varphi_2$ is stable. Note, since $\cos\varphi$ is an even function, a second solution $\varphi_2^* = -\varphi_2$ exists. As a consequence, according to (a), $2\,k\,a^2/F\,l < 1$ holds and thus, according to (d), $F > F_{\text{crit}}$. In the special case of $\varphi_2 = 0$ we obtain $V''(\varphi_2) = 0$ and $F = F_{\text{crit}}$.

The result is depicted in Fig. 8.17c: for $F < F_{\text{crit}}$ there exists only one equilibrium position, i.e., $\varphi = 0$; this position is stable. For $F > F_{\text{crit}}$ this position becomes unstable, and two new stable equilibrium positions $\pm\varphi_2$ appear. Hence, three different equilibrium positions exist in this region.

Since a bifurcation of the solution appears for the critical load $F = F_{\text{crit}}$, this particular value in load-displacement diagrams is called the "bifurcation point". The critical load and the bifurcation of a solution play an important role in the analysis of the stability behavior of *elastic* structures (cf. Volume 2).

Example 8.9 The homogeneous body shown in Fig. 8.18a (density ϱ) consists of a half circular cylinder (radius r) and a rectangular block (height h). E8.9

Determine the value of h for which the body is in neutral equilibrium at an arbitrary position.

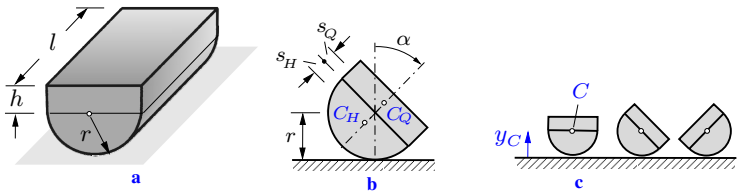


Fig. 8.18

Solution The rectangular block has the weight

$$W_Q = 2\,r\,h\,l\,\varrho\,g\,,$$

and the distance of its center of gravity measured from the separating surfaces of the two parts of the body is

$$s_Q = h/2$$

(Fig. 8.18b). The corresponding values for the half cylinder are

$$W_H = \frac{1}{2}\,\pi\,r^2\,l\,\varrho\,g$$

and

$$s_H = \frac{4}{3}\,\pi\,r$$

(see Table 4.1).

If we choose the surface of the base as the reference level for the potential, we obtain the following expression for an arbitrary position α:

$$V = \varrho\,g\,l\left[\frac{\pi\,r^2}{2}\left(r - \frac{4\,r}{3\,\pi}\cos\alpha\right) + 2\,r\,h\left(r + \frac{h}{2}\cos\alpha\right)\right].$$

The equilibrium positions are determined from

$$V' = \frac{\mathrm{d}V}{\mathrm{d}\alpha} = \varrho\, g\, l \sin\alpha \left[\frac{2}{3}\, r^3 - r h^2\right] = 0\,.$$

A first equilibrium position follows from

$$\sin\alpha = 0 \quad \rightarrow \quad \alpha_1 = 0\,.$$

Therefore, the reference configuration (characterized by $\alpha = 0$) in which the centers of gravity C_Q and C_H lie over each other on a vertical line is an equilibrium position. The derivative V' also vanishes if the term in the brackets becomes zero:

$$\frac{2}{3}\, r^3 - r h^2 = 0 \quad \rightarrow \quad \underline{\underline{h = \sqrt{\frac{2}{3}}\, r}}\,.$$

Equilibrium positions with arbitrary α exist for this specific value of h only.

To apply the stability criterion, we consider the second derivative

$$V'' = \varrho\, g\, l \cos\alpha \left[\frac{2}{3}\, r^3 - r h^2\right]\,.$$

For an arbitrary value of α and $h = \sqrt{2/3}\, r$, the second derivative V'' and all higher derivatives are equal to zero. The equilibrium positions are therefore neutral. The body is then in equilibrium in every arbitrary position, as shown in Fig. 8.18c.

This problem may also be solved without using the potential and its derivatives. An equilibrium position is neutral, if the center of gravity remains at the same elevation during a rotation ($\Delta V \equiv 0$). Therefore, the center of gravity C of the whole body must have the constant elevation r above the base for arbitrary α. Using (4.13), the center of gravity is given by

$$y_C = \frac{\dfrac{\pi\, r^2}{2}\left(r - \dfrac{4\, r}{3\,\pi}\right) + 2\, r h \left(r + \dfrac{h}{2}\right)}{\dfrac{\pi\, r^2}{2} + 2\, r h}\,.$$

After some algebraic manipulations, h follows from the condition $y_C = r$. This value is identical to that calculated above.

Example 8.10 Determine the type of stability of the equilibrium position in Problem 8.1. E8.10

Solution We choose the zero level of the potential of force W at the level of the support A (Fig. 8.9a). The zero level of the potential of force Q is chosen at the height at which the counterweight is located when the drawbridge is in the vertical position ($\varphi = 0$). With $s = 2l\sin(\varphi/2)$, the total potential as a function of the coordinate φ becomes

$$V(\varphi) = \frac{W\,l}{2}\cos\varphi + Q\,s = \frac{W\,l}{2}\cos\varphi + 2Q\,l\sin\frac{\varphi}{2}\,.$$

Using the relation $\sin\varphi = 2\sin(\varphi/2)\cos(\varphi/2)$, the derivatives are obtained as

$$V' = -\frac{W\,l}{2}\sin\varphi + Q\,l\cos\frac{\varphi}{2} = \left(-W\,\sin\frac{\varphi}{2} + Q\right)\,l\cos\frac{\varphi}{2}\,,$$

$$V'' = -\frac{W\,l}{2}\cos\varphi - \frac{Q\,l}{2}\,\sin\frac{\varphi}{2}\,.$$

From $V' = 0$, we again obtain the equilibrium position

$$\sin(\varphi/2) = Q/W.$$

Without further calculations, it can be seen that V'' is negative for $\varphi < \pi/2$. Therefore, the equilibrium position determined in Problem 8.1 is unstable.

8.6 Supplementary Problems 8.6

The detailed solutions to most of the following examples are given in (**A**) D. Gross et al. *Formeln und Aufgaben zur Technischen Mechanik 1*, Springer, Berlin 2011 or (**B**) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer, Berlin 2011.

E8.11

Example 8.11 The mechanism shown in Fig. 8.19 is subjected to a force F and a moment M_0. The weights of the three links may be neglected.

Apply the principle of virtual work to find the equilibrium position $\varphi = \varphi^*$.

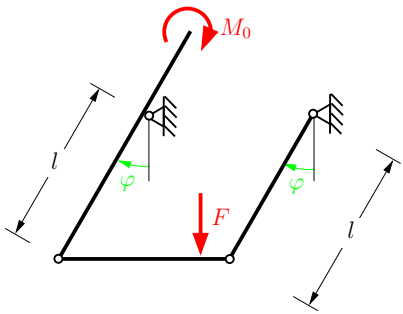


Fig. 8.19

Result: see (**B**) $\varphi^* = \arcsin \dfrac{M_0}{F\,l}$.

E8.12

Example 8.12 A hinged beam is subjected to a line load q_0 and a concentrated force F (Fig. 8.20).

Determine the support reaction B with the aid of the principle of virtual work.

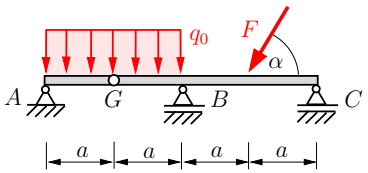


Fig. 8.20

Result: see (**B**) $B = 2\,q_0 a + \frac{1}{2} F \sin\alpha$.

E8.13

Example 8.13 The system in Fig. 8.21 is held by a spring (stiffness k) and a torsion spring (stiffness k_T). The force in the spring and the moment in the torsion spring are zero in the equilibrium position shown in the figure. This equilibrium position is unstable if the applied force F exceeds a critical value F_{crit}.

Find F_{crit}.

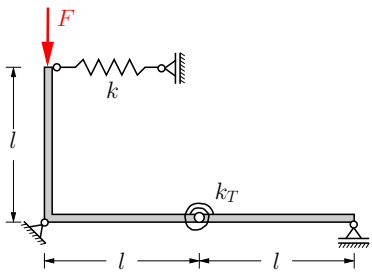


Fig. 8.21

Results: see (**A**) $F_{\text{crit}} = kl + 4k_T/l$.

Example 8.14 A hydraulic ramp is schematically depicted in Fig. 8.22. The two beams (each length l) are pin-connected at their centers M. A car (weight W) stands on the ramp.

Determine the force F which has to be generated in the hydraulic piston and applied to the lever of length a in order to keep the system in equilibrium.

E8.14

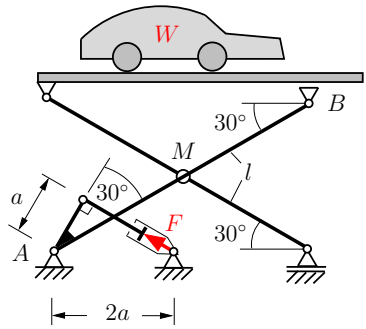


Fig. 8.22

Result: see (**B**) $F = \dfrac{\sqrt{3}\,l}{2\,a}\,W.$

Example 8.15 A wheel (weight W, radius r) rolls on a circular cylinder (radius R) without sliding. It is connected to a wall by a spring (stiffness k). The spring is kept in a horizontal position by the support; the force in the spring is zero in the position shown in Fig. 8.23.

Determine the equilibrium positions and investigate their stability.

E8.15

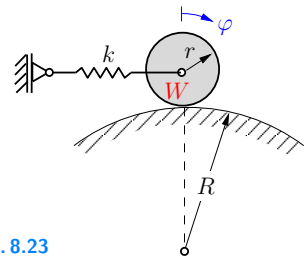


Fig. 8.23

Results: see (**B**) $\varphi_1 = 0$ $(W < k(R+r)$: stable),
$(W \geq k(R+r)$: unstable),

$$\varphi_{2,3} = \pm \arccos \frac{W}{k(R+r)} \quad \text{(unstable)}.$$

E8.16

Example 8.16 The slider crank mechanism shown in Fig. 8.24 consists of the crank $\overline{AC}$ and the connecting rod $\overline{BC}$. Their weights can be neglected in comparison with the force F acting at B.

Determine the moment $M(\alpha)$ which is necessary to keep the system in equilibrium at an arbitrary angle α.

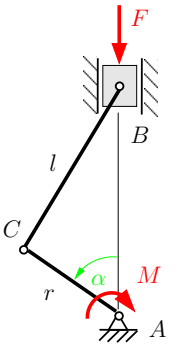


Fig. 8.24

Result: see (**A**) $M = Fr\,\sin\alpha\Big(1 + \dfrac{r\,\cos\alpha}{\sqrt{l^2 - r^2\sin^2\alpha}}\Big)$.

E8.17

Example 8.17 Calculate the force S_1 in member 1 of the structure in Fig. 8.25.

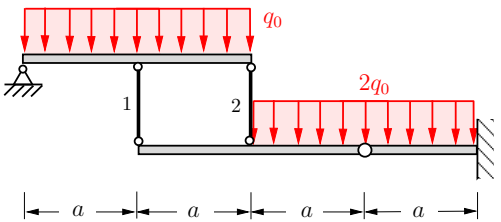


Fig. 8.25

Result: see (**A**) $S_1 = \frac{4}{3}\,q_0 a$.

E8.18

Example 8.18 A concentrated mass m is attached to a circular disk (radius R, mass M) as shown in Fig. 8.26. The disk can roll on an inclined plane (no sliding!).

Determine the positions of equilibrium and investigate their stability.

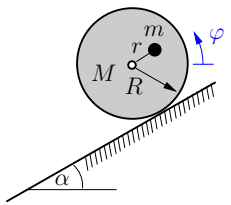


Fig. 8.26

Results: see (**B**) $\varphi_1 = 0 \quad (\kappa = 1,\ \text{unstable})$,

$\varphi_2 = \pm\arccos\kappa \quad (\kappa < 1,\ \text{stable})$,

$\kappa = \cos\varphi = (M+m)R\sin\alpha/(mr)$.

E8.19

Example 8.19 Figure 8.27 shows schematically the door CD (weight W, height $2r$) of a garage. It is supported by a lever BC and a spring AB (stiffness k). The spring is unstretched for $\alpha = \pi$. The distance between the points B and M is denoted by a.

Investigate the stability of the equilibrium configurations for the case $a \ll r$ and $Wr/ka^2 = 3$.

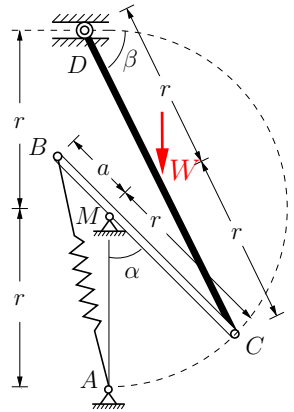


Fig. 8.27

Results: see (**A**) $\alpha_1 = 0$ (unstable), $\alpha_2 = \pi/3$ (stable), $\alpha_3 = \pi$ (unstable).

8.7

8.7 Summary

- The work done by a force $\boldsymbol{F}$ during an infinitesimal displacement $\mathrm{d}\boldsymbol{r}$ of the force application point is $\mathrm{d}U = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$. The work done during a finite displacement is given by

  $$U = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \,.$$

  Remark: The expression "work = force x displacement" is valid if and only if the directions of the force and of the displacement coincide and the force is constant.
- The work U of a conservative force is path-independent. A conservative force can be derived from a potential (potential energy) $V = -U$.
- The potentials of a weight W, a spring force and a torsion-spring moment, respectively, are given by

  $$V = W\,z\,, \qquad V = \frac{1}{2}\,k\,x^2\,, \qquad V = \frac{1}{2}\,k_{_T}\,\varphi^2\,.$$

- Virtual displacements $\delta\boldsymbol{r}$ or rotations $\delta\boldsymbol{\varphi}$ are infinitesimal, imaginary (they do not really exist) and kinematically possible. The virtual work of a force $\boldsymbol{F}$ is given by $\delta U = \boldsymbol{F} \cdot \delta\boldsymbol{r}$, and the virtual work of a moment $\boldsymbol{M}$ by $\delta U = \boldsymbol{M} \cdot \delta\boldsymbol{\varphi}$.
- A mechanical system is in equilibrium if the work done by the external forces disappears during a virtual displacement from this position:

  $$\delta U = 0\,.$$

- In a conservative system, the total potential has an extreme value in an equilibrium position. The equilibrium position x_0 of a system with one degree of freedom is characterized by

  $$V'(x_0) = 0\,.$$

- An equilibrium state is stable (unstable), if its total potential has a minimum (maximum):

  $$V''(x_0) > 0 \quad \rightarrow \text{ stable}\,, \qquad V''(x_0) < 0 \quad \rightarrow \text{ unstable}\,.$$