Chapter 6

Energy Methods

6

6 Energy Methods

Objectives: It is often convenient to determine displacements/rotations or forces/moments with the aid of energy methods. The pertinent equations are presented in this chapter. For example, it will be shown how the displacement of an arbitrary point of a structure can be calculated using energy methods. These methods also enable us to calculate redundant support reactions of statically indeterminate systems in a simple way. The students will learn how to apply these methods to specific problems.

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6.1 Introduction

In the preceding chapters we have always applied three different types of equations to determine forces/stresses or deformations:

a) The *equilibrium conditions* establish a relation between the applied loads and the internal forces (stress resultants).
b) *Kinematic equations* connect the displacements and the strains.
c) The *constitutive equations* (Hooke's law) connect the stresses and the strains.

Table 6.1 shows the corresponding equations for the three cases tension/compression, bending and torsion. In addition, the bottom row presents the differential equations for the displacements/rotations, assuming constant rigidities, that follow from the three types of equations given in the rows above.

Table 6.1. Basic Equations of Elastostatics

	Tension/ Compression	Bending	Torsion
Equilibrium	$N' = -n$	$M' - V = 0$ $V' = -q$	$M_T' = -m_T$
Kinematics	$\varepsilon = u'$	$\varkappa_B = -\psi'$ $\psi = -w'$	$\varkappa_T = \vartheta'$
Hooke's Law	$N = EA\,\varepsilon$	$M = -EI\,\varkappa_B$	$M_T = GI_T\,\varkappa_T$
	$EA\,u'' = -n$ cf. (1.20b)	$EI\,w^{IV} = q$ cf. (4.34b)	$GI_T\,\vartheta'' = -m_T$ cf. (5.14)

In Volume 1 it was shown how the equilibrium of a rigid body can be investigated with the aid of energy considerations: the principle of virtual work is equivalent to the equilibrium conditions (Volume 1, Section 8.2). Since there are no real displacements in the statics of rigid bodies, we introduced virtual displacements (i.e., imaginary displacements) in order to be able to apply the principle of virtual work. In contrast, the points of elastic structures undergo real displacements. To calculate these displacements it is often advantageous to use energy methods. These methods will be derived in the following sections.

6.2

6.2 Strain Energy and Conservation of Energy

Let us first consider a bar that is subjected to an external tensile force. We assume that the magnitude of this force "slowly" (quasistatically, no dynamic effects due to motion) increases from the initial value zero to its final value F. An arbitrary value between 0 and F is denoted by $\bar{F}$. After the loading process the point of application of the force is displaced by the amount u (Fig. 6.1a). Therefore, the work done by the force is given by

$$U_e = \int_0^u \bar{F}\,\mathrm{d}\bar{u} \qquad (6.1)$$

(the subscript "e" refers to "external" force). If the functional relation between the force $\bar{F}$ and the corresponding displacement $\bar{u}$ is known, the integral in (6.1) can be evaluated. In this chapter it is always assumed that the material behaviour is linearly elastic. Then this relation is given by (1.18) for a bar with length l and constant axial rigidity EA:

$$\bar{u} = \frac{\bar{F}\,l}{EA} \qquad \to \qquad \bar{F} = \frac{EA}{l}\,\bar{u}. \qquad (6.2)$$

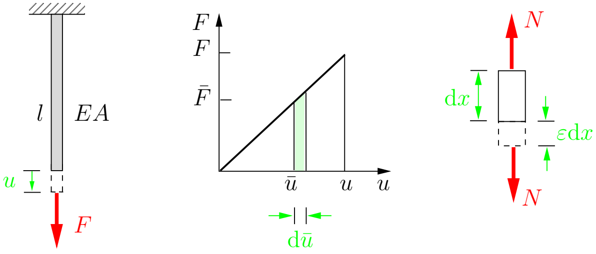


Fig. 6.1

Inserting (6.2) into (6.1) yields

$$U_e = \frac{EA}{l}\frac{u^2}{2} = \frac{1}{2}\frac{F^2\,l}{EA} = \frac{1}{2}\,F\,u. \tag{6.3}$$

The load-displacement diagram is shown in Fig. 6.1b. It illustrates the result (6.3): the integral over the elements of work $\mathrm{d}U_e = \bar{F}\mathrm{d}\bar{u}$ is equal to the area $\frac{1}{2}Fu$ of the triangle.

We will now determine the work U_i that is done by the *internal* forces in the bar. An element of length $\mathrm{d}x$ changes its length by the amount $\varepsilon\,\mathrm{d}x$ under the action of the normal force N (Fig. 6.1c). This force also slowly increases from the initial value of zero to its final value. Since the relation between force and elongation is linear, the work done by the internal forces is (compare to (6.3))

$$\mathrm{d}U_i = \frac{1}{2}\,N\,\varepsilon\,\mathrm{d}x. \tag{6.4}$$

The work done by internal forces (here: the normal force N) is called *internal work* or *strain energy*. The strain energy is stored in the structure just as the potential energy V is stored in a spring (see Volume 1, Equation (8.9)). Note that the strain energy is always positive (also in the case of compression). With the constitutive equation $\varepsilon = N/EA$ (see Table 6.1) we obtain

$$\mathrm{d}U_i = \frac{1}{2}\frac{N^2}{EA}\,\mathrm{d}x = {U_i}^*\,\mathrm{d}x$$

where

$${U_i}^* = \frac{1}{2}\frac{N^2}{EA} \tag{6.5}$$

is the strain energy per unit length. Integration of (6.5) over the length of the bar yields the total strain energy

$$U_i = \int_0^l {U_i}^*\mathrm{d}x = \frac{1}{2}\int_0^l \frac{N^2}{EA}\,\mathrm{d}x. \tag{6.6}$$

In the special case of a constant axial rigidity and a constant normal force $N = F$ we find

$$U_i = \frac{1}{2}\frac{F^2}{EA}\int_0^l \mathrm{d}x = \frac{1}{2}\frac{F^2\,l}{EA}. \tag{6.7}$$

Comparison of (6.7) and (6.3) leads to

$$U_e = U_i\,. \tag{6.8}$$

This equation represents the *principle of conservation of energy*. It was derived for the particular case of a bar under tension or compression; however, it is valid for an arbitrary elastic system (note that only mechanical energy is considered here; energy developed by heat, chemical reactions etc, is disregarded). In words: the work U_e done by the external loads is stored as strain energy U_i in the elastic system. It is regained in the process of unloading: no energy is lost. The principle (6.8) which states that the work done by the external loads is equal to the strain energy is also referred to as *Clapeyron's theorem* (Benoit Paul Emile Clapeyron, 1799–1864).

In order to be able to apply (6.8) to an arbitrary elastic system, we need expressions for the work U_e of the external loads and the strain energy U_i. A force F that is applied to a structure does the work

$$U_e = \frac{1}{2}\,F\,f \tag{6.9a}$$

(compare to (6.3)), where f is the displacement component in the direction of the force at the point where the force is applied (Fig. 6.2a). In the case of an external couple moment M_0, the

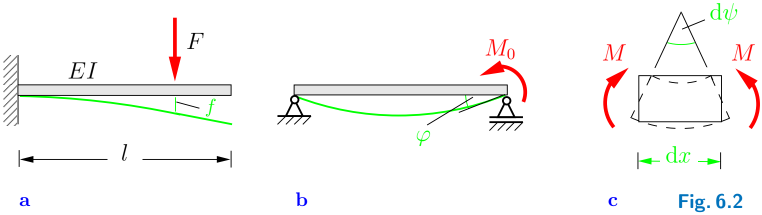


Fig. 6.2

work is given by

$$U_e = \frac{1}{2} M_0 \varphi \tag{6.9b}$$

where φ is the angle of rotation in the direction of M_0 at the point of application of M_0 (Fig. 6.2b).

In contrast to the work of the external loads, the strain energy is obtained using different equations for different types of loading, i.e., for members subjected to a normal force, a bending moment or a torque. We will now derive the corresponding equation for the case of bending. Consider a beam element of length dx. The two end cross sections undergo a relative rotation dψ due to the bending moment M (Fig. 6.2c). The bending moment does the work

$$\mathrm{d}U_i = \frac{1}{2} M \,\mathrm{d}\psi = \frac{1}{2} M \,\psi' \,\mathrm{d}x \tag{6.10}$$

during the rotation. If we insert the constitutive equation $M = EI\psi'$ (see (4.24)) we obtain the strain energy per unit length

$$\mathrm{d}U_i = \frac{1}{2}\frac{M^2}{EI}\,\mathrm{d}x = {U_i}^* \,\mathrm{d}x \qquad \rightarrow \qquad {U_i}^* = \frac{1}{2}\frac{M^2}{EI}. \tag{6.11}$$

Integration over the length l of the beam yields the strain energy

$$U_i = \int_0^l {U_i}^* \,\mathrm{d}x = \frac{1}{2}\int_0^l \frac{M^2}{EI}\,\mathrm{d}x. \tag{6.12}$$

Analogous considerations can be applied to torsion and shear. With the torsional moment (torque) $M_T = GI_T\,\vartheta'$ (see (5.5)) and the shear force $V = GA_S\,\tilde{\gamma}$ (compare to (4.41)) we obtain the strain energies per unit length

$${U_i}^* = \frac{1}{2}\frac{M_T^2}{GI_T} \qquad \text{and} \qquad {U_i}^* = \frac{1}{2}\frac{V^2}{GA_S}. \tag{6.13}$$

Various forms of the strain energies for different types of loading are presented in Table 6.2.

Note that it is assumed that only one load acts on the structure. This load may cause different stress resultants in a member, e.g.,

Table 6.2. Strain Energy $U_i{}^*$ per Unit Length

Tension	Bending	Shear	Torsion
$\frac{1}{2} N \varepsilon$	$\frac{1}{2} M \psi'$	$\frac{1}{2} V \tilde{\gamma}$	$\frac{1}{2} M_T \vartheta'$
$\frac{1}{2} EA \varepsilon^2$	$\frac{1}{2} EI \psi'^2$	$\frac{1}{2} GA_S \tilde{\gamma}^2$	$\frac{1}{2} GI_T \vartheta'^2$
$\frac{1}{2} \frac{N^2}{EA}$	$\frac{1}{2} \frac{M^2}{EI}$	$\frac{1}{2} \frac{V^2}{GA_S}$	$\frac{1}{2} \frac{M_T^2}{GI_T}$

a bending moment M and a shear force V (see Example 6.1) or a bending moment M and a torque M_T (see Example 6.2). In this case we can use the principle of superposition to obtain the strain energy. Hence, the total strain energy of a member which experiences bending, torsion and tension is given by

$$U_i = \frac{1}{2} \int \frac{M^2}{EI} \, \mathrm{d}x + \frac{1}{2} \int \frac{M_T^2}{GI_T} \, \mathrm{d}x + \frac{1}{2} \int \frac{N^2}{EA} \, \mathrm{d}x. \tag{6.14}$$

If the structure is composed of different parts, the total strain energy is the sum of the strain energies of the various parts. Note, however, that the total strain energy due to more than one load is not the sum of the strain energies due to the individual loads acting separately (the strain energy is not a linear function of the loads).

The principle of conservation of energy in the form (6.8) can be applied to statically determinate systems that are subjected to only *one* force or *one* couple moment. It allows only the determination of the displacement at the point and *in* the direction of the external force or the angle of rotation at the point and *in* the direction of the external couple moment. Therefore, its importance in solving practical problems is rather limited. Frequently, structures are subjected to more than only one load and the displacements/rotations have to be determined at arbitrary points of a structure. For these more general cases we have to use an extension of the strain-energy method (6.8) which is also based on the principle of conservation of energy. This method will be derived in Section 6.3.

As an example for the application of (6.8) let us consider the cantilever beam in Fig. 6.2a. The deflection f at the point where the force F is applied follows with (6.9a) and (6.12):

$$U_e = U_i \qquad \to \qquad \frac{1}{2}\,F\,f = \frac{1}{2}\int_0^l \frac{M^2}{EI}\,\mathrm{d}x. \tag{6.15a}$$

Here, M is the bending moment due to the external force F. Similarly, (6.9b) and (6.12) yield the angle of slope at the point of application of the external couple moment M_0 (Fig. 6.2b):

$$U_e = U_i \qquad \to \qquad \frac{1}{2}\,M_0\,\varphi = \frac{1}{2}\int_0^l \frac{M^2}{EI}\,\mathrm{d}x \tag{6.15b}$$

where M is the bending moment caused by M_0.

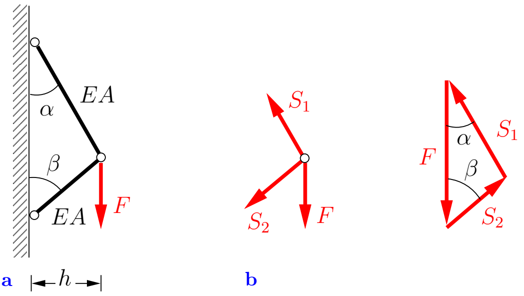
In a truss, the only internal forces (stress resultants) are the normal forces $N_i = S_i$ in the individual members, which are constant. The strain energy in the i-th member is given by $\frac{1}{2}S_i^2\,l_i/E_iA_i$. Consider a truss which consists of n members and which is subjected to only *one* force F. Then the displacement at the point and in the direction of the force follows from

$$U_e = U_i \qquad \to \qquad \frac{1}{2}\,F\,f = \frac{1}{2}\sum_{i=1}^{n}\frac{S_i^2\,l_i}{EA_i}. \tag{6.16}$$

Here, the axial rigidity E_iA_i is denoted by EA_i.

As an example we can calculate the vertical displacement v of the point of application of the force F for the two-bar truss shown in Fig. 6.3a. According to (6.16) this displacement follows from

$$\frac{1}{2}\,F\,v = \frac{1}{2}\left(\frac{S_1^2\,l_1}{EA} + \frac{S_2^2\,l_2}{EA}\right).$$



a **b** **Fig. 6.3**

The forces in the members can be taken from the force triangle in Fig. 6.3b. The law of sines yields

$$S_1 = F\frac{\sin\beta}{\sin(\alpha+\beta)}, \qquad S_2 = -\,F\frac{\sin\alpha}{\sin(\alpha+\beta)}.$$

If we also insert the lengths $l_1 = h/\sin\alpha$ and $l_2 = h/\sin\beta$ of the members we obtain

$$v = \frac{F}{EA}\,\frac{h}{\sin^2(\alpha+\beta)}\left(\frac{\sin^2\beta}{\sin\alpha} + \frac{\sin^2\alpha}{\sin\beta}\right).$$

A graphic-analytical method to determine the displacement of one of the pins of a truss was given in Section 1.5. It can be seen that the strain-energy method avoids the often quite cumbersome geometrical considerations.

We will now show how one can obtain an approximation for the shear correction factor $\varkappa$ of the cross section of a beam with the aid of energy considerations. The shear correction factor was introduced in the constitutive equation (4.25) for the shear force (see also Section 4.6):

$$V = G\,\varkappa\,A(w' + \psi) = GA_S\,(w' + \psi). \tag{6.17}$$

It was assumed that the shear force causes the average shear strain $\tilde{\gamma} = w' + \psi$ in the cross section. The shear area $A_S = \varkappa\,A$ is now obtained by equating the strain energy $U_{i_V}^*$ due to the shear force and the strain energy $U_{i_\tau}^*$ due to the shear stress τ that acts in the cross section. According to Table 6.2 we have

$$U_{i\,V}^{*} = \frac{1}{2}\,V\,\tilde{\gamma} = \frac{1}{2}\frac{V^2}{GA_S}. \tag{6.18}$$

Similarly, the force $\tau\,\mathrm{d}A$ that acts on an element $\mathrm{d}A$ of the cross section leads (with $\tau = G\,\gamma$) to

$$\mathrm{d}\,U_{i\,\tau}^{*} = \frac{1}{2}(\tau\,\mathrm{d}A)\,\gamma = \frac{1}{2}\frac{\tau^2}{G}\,\mathrm{d}A.$$

Integration over the cross section yields the strain energy per unit length

$$U_{i\,\tau}^{*} = \frac{1}{2}\int\frac{\tau^2}{G}\,\mathrm{d}A. \tag{6.19}$$

We now equate (6.18) and (6.19):

$$\frac{1}{2}\frac{V^2}{GA_S} = \frac{1}{2}\int\frac{\tau^2}{G}\,\mathrm{d}A. \tag{6.20}$$

If the distribution of the shear stress τ in the cross section is known, the integral in (6.20) can be evaluated and thus the shear area A_S and the shear correction factor $\varkappa$ can be calculated.

To illustrate the method we consider a rectangular cross section. According to (4.39) the distribution of the shear stress is given by

$$\tau = \frac{3}{2}\frac{V}{A}\left(1 - 4\,\frac{z^2}{h^2}\right)$$

(see Fig. 4.35b). Inserting τ into (6.20) and using $\mathrm{d}A = b\,\mathrm{d}z$ and $A = b\,h$ yields

$$\frac{1}{A_S} = \frac{9}{4}\frac{1}{A^2}\int\limits_{-h/2}^{h/2}\left(1 - 4\,\frac{z^2}{h^2}\right)^2\,b\,\mathrm{d}z = \frac{6}{5}\frac{1}{b\,h}.$$

Hence, we obtain

$$A_S = \frac{5}{6}\,b\,h, \qquad \varkappa = \frac{A_S}{A} = \frac{5}{6} \tag{6.21}$$

for a rectangular cross section. The average shear strain

$$\tilde{\gamma} = w' + \psi = \frac{V}{GA_S} = 1.2\,\frac{V}{GA}$$

is therefore 20% larger than the shear strain that would be caused by a uniform distribution $\tau = V/A$ of the shear stress.

Similar considerations lead to values between 0.8 and 0.9 of the shear correction factor $\varkappa$ for solid cross sections. In the case of an I-beam (see Fig. 4.42), the shear force is essentially supported by the web. Therefore we have to a good approximation

$$A_S \approx A_{\text{web}} = t\,h .$$

For a thin-walled circular cross section Equation (6.20) leads to

$$A_S = \frac{1}{2}A \qquad \text{with} \qquad A = 2\,\pi\,r\,t .$$

It should be noted that the values of the shear correction factor vary considerably depending on the type of the cross section, e.g., solid cross section, thin-walled open or thin-walled closed cross section.

E6.1

Example 6.1 A cantilever beam is subjected to a force F as shown in Fig. 6.4a.

Calculate the deflection f at the free end taking into account the shear deformation.

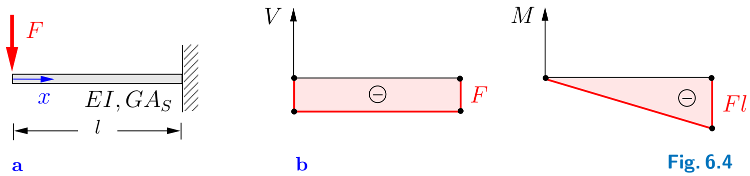


Fig. 6.4

Solution The principle of conservation of energy (6.8) reads

$$\frac{1}{2}\,F\,f = \frac{1}{2}\int \frac{M^2}{EI}\,\mathrm{d}x + \frac{1}{2}\int \frac{V^2}{GA_S}\,\mathrm{d}x$$

where the integrals are taken from Table 6.2. Using the coordinate x as shown in Fig. 6.4a, the stress resultants are given by (Fig. 6.4b)

$$V = -F, \qquad M = -F\,x.$$

Since the rigidities EI and GA_S are constant, we obtain

$$\frac{1}{2}F\,f = \frac{1}{2}\int_0^l \frac{F^2 x^2}{EI}\,\mathrm{d}x + \frac{1}{2}\int_0^l \frac{F^2}{GA_S}\,\mathrm{d}x$$
$$= \frac{1}{2}F^2\frac{l^3}{3\,EI} + \frac{1}{2}F^2\frac{l}{GA_S} \quad\rightarrow\quad \underline{\underline{f = \frac{F\,l^3}{3\,EI} + \frac{F\,l}{GA_S}}}.$$

This problem was already solved in Section 4.6.2 with the aid of the differential equations for the deflection of the Bernoulli beam and the deflection due to shear, respectively. Also, the influence of the shear rigidity was discussed there.

Example 6.2 An angled member carries a load F at the free end (Fig. 6.5a). **E6.2**

Determine the deflection at the point of application of the force.

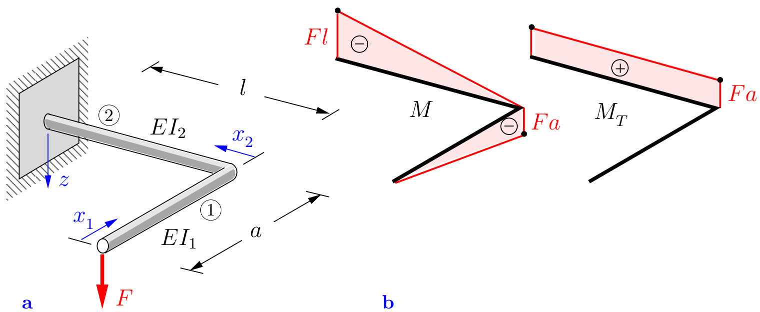


Fig. 6.5

Solution Part ① of the structure is subjected to bending; part ② is subjected to bending and torsion. Therefore, the principle of conservation of energy (6.8) is given by

$$\frac{1}{2}F\,f = \frac{1}{2}\int \frac{M^2}{EI}\,\mathrm{d}x + \frac{1}{2}\int \frac{M_T^2}{GI_T}\,\mathrm{d}x.$$

We use the coordinates x_1 and x_2 as shown in Fig. 6.5a. Then, the stress resultants are the bending moment $M_1 = -F\,x_1$ in beam ① and the bending moment $M_2 = -F\,x_2$ and the torque $M_{T2} = F\,a$ in beam ② (Fig. 6.5b). Thus,

$$\begin{aligned}\frac{1}{2}F\,f &= \frac{1}{2}\int_0^a \frac{F^2\,x_1^2}{EI_1}\,\mathrm{d}x_1 + \frac{1}{2}\int_0^l \frac{F^2\,x_2^2}{EI_2}\,\mathrm{d}x_2 + \frac{1}{2}\int_0^l \frac{F^2\,a^2}{GI_T}\,\mathrm{d}x_2\\ &= \frac{1}{2}\frac{F^2}{EI_1}\frac{a^3}{3} + \frac{1}{2}\frac{F^2}{EI_2}\frac{l^3}{3} + \frac{1}{2}\frac{F^2}{GI_T}a^2\,l\end{aligned}$$

and the deflection is obtained as

$$\underline{\underline{f = F\left\{\frac{a^3}{3\,EI_1} + \frac{l^3}{3\,EI_2} + \frac{a^2\,l}{GI_T}\right\}}}.$$

6.3

6.3 Principle of Virtual Forces and Unit Load Method

The principle of conservation of energy (6.8) enables us to calculate the displacement in the direction of an external force. For example, the *vertical* displacement v of the pin of the two-bar truss in Fig. 6.3 under the action of the *vertical* force F follows from (see (6.16))

$$U_e = U_i \qquad \rightarrow \qquad \frac{1}{2}F\,v = \frac{1}{2}\sum \frac{S_i^2\,l_i}{EA_i}. \tag{6.22}$$

If a *horizontal* force Q is applied to the same truss instead of the vertical force F, the *horizontal* displacement u can be obtained from

$$\frac{1}{2}\,Q\,u = \frac{1}{2}\sum \frac{S_i^2\,l_i}{EA_i}, \tag{6.23}$$

where now S_i are the internal forces in the members due to the force Q. However, we also want to be able to determine the horizontal displacement caused by the vertical force F and the vertical displacement due to the horizontal force Q. In order to achieve this goal we introduce *virtual forces*. These are fictitious forces which are introduced only for the purpose of the calculation. Just as we may determine real forces with the aid of *virtual displacements* (see Volume 1, Section 8.2), we will be able to determine real displacements with the aid of virtual forces.

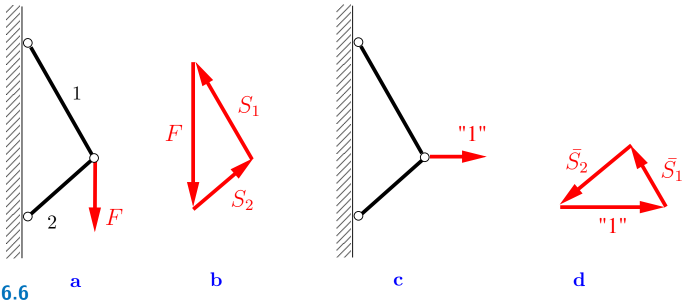


Fig. 6.6

In this section we restrict ourselves to statically determinate systems and illustrate the method with the example of the two-bar truss in Fig. 6.6a. The truss is subjected to the vertical force F as shown; the internal forces in the members are called S_i (Fig. 6.6b). If we want to determine the displacement of the pin in the *horizontal* direction, we first subject the truss only to the virtual force “1“ in the *horizontal* direction (Fig. 6.6c). This force is assumed to be gradually increasing to its final magnitude 1. A force triangle (Fig. 6.6d) yields the corresponding internal forces $\bar{S}_i$. Here and in what follows, forces or kinematical quantities due to a virtual force are always marked by a bar. The horizontal virtual force causes a displacement of the pin; its horizontal component is denoted by $\bar{u}$. During the displacement, the force “1“ does the work

$$U_{e,1} = \frac{1}{2} \cdot 1 \cdot \bar{u}. \qquad (6.24)$$

Subsequently, in addition to the already acting virtual force "1", we apply the vertical force F. Then the corresponding displacement of the pin has the vertical component v and the force F does the work

$$U_{e,2} = \frac{1}{2} F\, v. \qquad (6.25)$$

The horizontal component of the displacement due to F is u. Since the virtual force "1" has been applied before and therefore has the constant magnitude 1 (and since the force "1" and the displacement u have the same direction), it does the work

$$U_{e,3} = 1 \cdot u \qquad (6.26)$$

during the displacement. The total work of the forces during the process described above is given by the sum of the three terms:

$$U_e = \frac{1}{2} \cdot 1 \cdot \bar{u} + \frac{1}{2} F\, v + 1 \cdot u. \qquad (6.27)$$

According to the principle of superposition, the total internal forces in the bars are $\bar{S}_i + S_i$. Thus, the total strain energy in the truss is (compare (6.16))

$$\begin{aligned} U_i &= \frac{1}{2} \sum \frac{(\bar{S}_i + S_i)^2\, l_i}{EA_i} \\ &= \frac{1}{2} \sum \frac{\bar{S}_i^2\, l_i}{EA_i} + \frac{1}{2} \sum \frac{S_i^2\, l_i}{EA_i} + \sum \frac{S_i\, \bar{S}_i\, l_i}{EA_i} \end{aligned} \qquad (6.28)$$

and the principle of conservation of energy (6.8) yields

$$\frac{1}{2} \cdot 1 \cdot \bar{u} + \frac{1}{2} F\, v + 1 \cdot u = \frac{1}{2} \sum \frac{\bar{S}_i^2\, l_i}{EA_i} + \frac{1}{2} \sum \frac{S_i^2\, l_i}{EA_i} + \sum \frac{S_i\, \bar{S}_i\, l_i}{EA_i}.$$

According to (6.22), the second term on the left-hand side is equal to the second term on the right-hand side. Similarly, if we use

(6.23) and set $Q = 1$, we see that the first terms are equal. This leaves the result

$$1 \cdot u = \sum \frac{S_i \, \bar{S}_i \, l_i}{EA_i}. \tag{6.29}$$

Thus, the virtual force "1" in the horizontal direction enables us to determine the real horizontal displacement u due to the vertical force F.

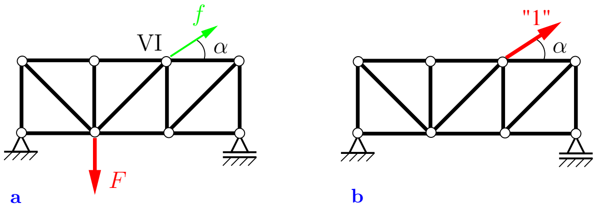


Fig. 6.7 a b

Similar considerations lead to the displacement component in any given direction of an arbitrary pin of a general truss. Assume, for example, that the displacement component f of pin VI of the truss in Fig. 6.7a has to be determined (the direction of f is given by the angle α). In the first step, the forces S_i in the members due to the applied load F have to be calculated. In the second step, the truss is subjected only to the virtual force "1" at pin VI in the direction of f (Fig. 6.7b) and the corresponding forces $\bar{S}_i$ are calculated. Then, (6.29) yields

$$f = \sum \frac{S_i \, \bar{S}_i \, l_i}{EA_i}. \tag{6.30}$$

To obtain (6.30) we have divided (6.29) by the force 1. Thus, the forces $\bar{S}_i$ in (6.30) are due to a dimensionless force 1; hence, they are from now on also dimensionless quantities. Note that they must have the dimension of a force in (6.28) so that S_i and $\bar{S}_i$ can be added.

According to the principle of superposition, Equation (6.30) is valid for a truss subjected to arbitrarily many forces. In general, the quantities S_i are the forces in the members due to the total loading. Equation (6.30) is called the *principle of virtual forces* and its use for the evaluation of displacements is known as the *unit load method*. In summary: if we want to determine the component f of the displacement in a given direction at an arbitrary pin k of the truss, then we have to apply a virtual force "1" in this direction at pin k. The displacement f is obtained from (6.30), where S_i are the forces in the members due to the external forces, $\bar{S}_i$ are the forces in the members due to the virtual force "1" and l_i, EA_i are the lengths and the axial rigidities of the individual members.

In general, we do not know the direction of the displacement of a pin. Therefore we have to apply the method twice: a horizontal force "1" yields the horizontal component of the displacement; a vertical force "1" yields the vertical component. A vector addition of the components leads to the displacement of the pin.

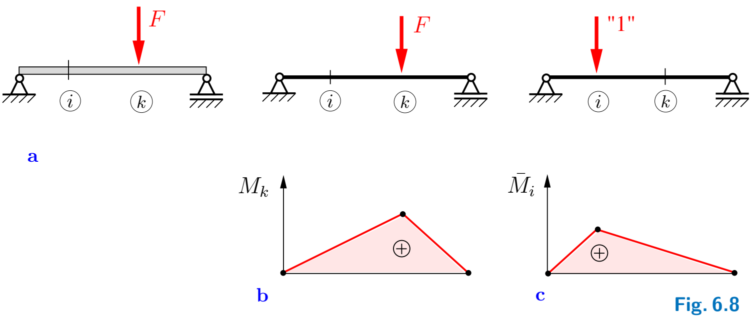


Fig. 6.8

The principle of virtual forces may also be applied to beams, frames, arches, etc. We will now derive the method for the bending of a beam with the aid of an example, namely, the simple beam in Fig. 6.8a subjected to a force at point k. We want to determine the deflection f at point i. For the sake of clarity we will use double subscripts: f_{ik} is the deflection at point i due to the force F at point k. In order to determine this deflection, we first apply a virtual force "1" at point i (Fig. 6.8c). Subsequently, the external force F is applied at point k. Using the same arguments as in the case of a truss, we obtain the total work of these two forces:

$$U_e = \frac{1}{2} \cdot 1 \cdot f_{ii} + \frac{1}{2} F f_{kk} + 1 \cdot f_{ik}. \tag{6.31a}$$

The force “1“ causes the bending moment $\bar{M}_i$, the force F causes the bending moment M_k (Fig. 6.8b, c). Therefore, the total bending moment is given by $\bar{M}_i + M_k$ (here, $\bar{M}_i$ has the dimension of a moment). This yields the strain energy

$$U_i = \frac{1}{2}\int \frac{(\bar{M}_i + M_k)^2}{EI}\,\mathrm{d}x$$

$$= \frac{1}{2}\int \frac{\bar{M}_i^2}{EI}\,\mathrm{d}x + \frac{1}{2}\int \frac{M_k^2}{EI}\,\mathrm{d}x + \int \frac{\bar{M}_i\,M_k}{EI}\,\mathrm{d}x\,. \tag{6.31b}$$

According to (6.15a), the first and the second terms, respectively, in U_e and U_i are equal. Therefore, the principle of conservation of energy (6.8) results in

$$f_{ik} = \int \frac{\bar{M}_i\,M_k}{EI}\,\mathrm{d}x. \tag{6.32}$$

Again, we have divided by the force 1. Thus, the moment $\bar{M}_i$ now has the dimension “length”. Equation (6.32) represents the principle of virtual forces or unit load method for the bending of a beam: the deflection f_{ik} at point i due to a force F at point k is obtained by calculating the bending moment $\bar{M}_i$ due to the dimensionless force “1“ at point i and the bending moment M_k due to the given force F at point k. Equation (6.32) then leads to f_{ik}.

The principle of virtual forces (6.32) is also valid for an arbitrary loading of the beam (several forces, couple moments, line loads). Then M_k is the bending moment due to all the given external loads. In this case, the subscripts i and k are omitted and (6.32) is written in the form

$$f = \int \frac{M\bar{M}}{EI}\,\mathrm{d}x\,. \tag{6.33}$$

Here, M is the bending moment due to the given loads and $\bar{M}$ (dimension: length) is the bending moment due to the dimensionless force “1“ which acts at the point where the deflection is to be determined.

If we want to calculate the angle φ of the slope of the deflection curve at a given point, we apply a dimensionless virtual couple moment “1“ at this point. The angle φ is then obtained from (6.33), where now $\bar{M}$ is the bending moment (dimensionless quantity) due to the virtual couple moment.

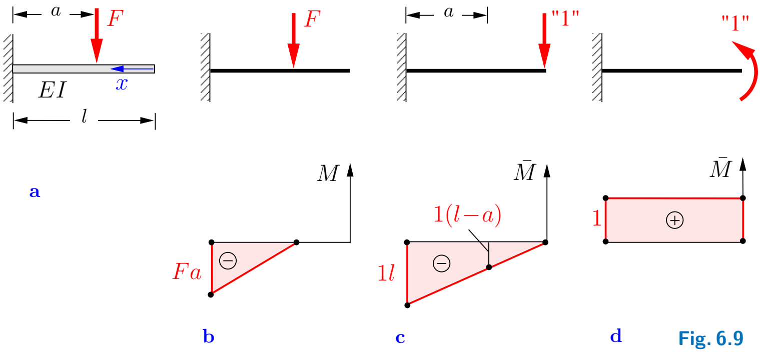


Fig. 6.9

To illustrate the method we consider the cantilever beam subjected to a force F as depicted in Fig. 6.9a. We will calculate the deflection and the angle of slope at the free end. If we use the coordinate x shown in Fig. 6.9a, the bending moment caused by the given force F is (Fig. 6.9b)

$$M = -\,F[x-(l-a)] \qquad \text{for} \qquad x \geq l-a .$$

In order to determine the deflection at the free end we apply the virtual force “1“ at this point. This yields the bending moment (Fig. 6.9c)

$$\bar{M} = -\,1 \cdot x \qquad \text{for} \qquad x \geq 0 .$$

The bending moment $\bar{M}$ has the dimension “length”. The deflection follows from (6.33) (note that the bending moment M is zero in the region $0 \leq x \leq l-a$):

$$f = \int \frac{M\bar{M}}{EI}\,\mathrm{d}x = \frac{1}{EI}\int\limits_{l-a}^{l} (-F)[x-(l-a)](-x)\,\mathrm{d}x$$

$$= \frac{F}{EI}\left[\frac{x^3}{3} - (l-a)\frac{x^2}{2}\right]_{l-a}^{l} = \frac{F\,l^3}{6\,EI}\left[3\left(\frac{a}{l}\right)^2 - \left(\frac{a}{l}\right)^3\right]. \quad (6.34)$$

To determine the angle of slope at the free end we apply the virtual couple moment "1" at this point. The corresponding bending moment is given by $\bar{M} = 1$ (Fig. 6.9d); it is dimensionless. Equation (6.33) yields

$$\varphi = \int \frac{M\bar{M}}{EI}\,\mathrm{d}x = \frac{1}{EI}\int\limits_{l-a}^{l} (-F)[x-(l-a)]\cdot 1\,\mathrm{d}x$$

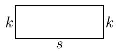
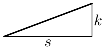
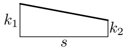
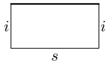
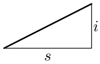
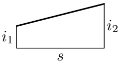
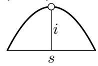
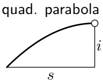
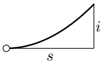
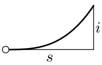
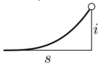
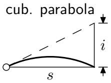
$$= -\,\frac{F}{EI}\left[\frac{x^2}{2} - (l-a)x\right]_{l-a}^{l} = -\,\frac{F}{EI}\frac{a^2}{2}. \quad (6.35)$$

The negative sign indicates that the direction of the actual rotation is opposite to the direction that was chosen for the virtual moment.

In many problems the bending moments are linear, quadratic or cubic functions of x, respectively. Then the integrals in (6.33) can be computed in advance provided that the flexural rigidity is *constant*, and they can be listed in a table. Results of the integrations are presented in Table 6.3. Note that in order to evaluate the integrals, it is irrelevant which of the bending moments is due to the external loads and which one is caused by the virtual load. Therefore, the bar which is used to characterize the quantities due to virtual loads is omitted in Table 6.3. Now the subscripts i and k characterize two bending moments under the integral sign: $\int M_i\,M_k\,\mathrm{d}x$. For example, if M_i is represented by a quadratic parabola and M_k is linear, the notation used in Fig. 6.10 implies

$$M_i = 4\,i\left[\frac{x}{s} - \left(\frac{x}{s}\right)^2\right], \qquad M_k = k\,\frac{x}{s}.$$

Table 6.3. Integrals $\int M_i\, M_k\, \mathrm{d}x$

	M_i	M_k			
					
1		sik	$\frac{1}{2}sik$	$\frac{1}{2}sik$	$\frac{1}{2}si(k_1+k_2)$
2		$\frac{1}{2}sik$	$\frac{1}{3}sik$	$\frac{1}{6}sik$	$\frac{1}{6}si(k_1+2\,k_2)$
3		$\frac{1}{2}s(i_1+ i_2)k$	$\frac{1}{6}s(i_1+ 2\,i_2)k$	$\frac{1}{6}s(2\,i_1+ i_2)k$	$\frac{1}{6}s(2\,i_1\,k_1+ 2\,i_2\,k_2+i_1\,k_2+ i_2\,k_1)$
4	quad. parabola 	$\frac{2}{3}sik$	$\frac{1}{3}sik$	$\frac{1}{3}sik$	$\frac{1}{3}si(k_1+k_2)$
5	quad. parabola 	$\frac{2}{3}sik$	$\frac{5}{12}sik$	$\frac{1}{4}sik$	$\frac{1}{12}si(3\,k_1+5\,k_2)$
6	quad. parabola 	$\frac{1}{3}sik$	$\frac{1}{4}sik$	$\frac{1}{12}sik$	$\frac{1}{12}si(k_1+3\,k_2)$
7	cub. parabola 	$\frac{1}{4}sik$	$\frac{1}{5}sik$	$\frac{1}{20}sik$	$\frac{1}{20}si(k_1+4\,k_2)$
8	cub. parabola 	$\frac{3}{8}sik$	$\frac{11}{40}sik$	$\frac{1}{10}sik$	$\frac{1}{40}si(4\,k_1+11\,k_2)$
9	cub. parabola 	$\frac{1}{4}sik$	$\frac{2}{15}sik$	$\frac{7}{60}sik$	$\frac{1}{60}si(7\,k_1+8\,k_2)$

Quadratic parabola: ∘ indicates maximum
Cubic parabola: ∘ indicates zero value of the triangular load
Trapezium: i_1 and i_2 (k_1 and k_2) may have different algebraic signs

The integral is then obtained as

$$\int_0^s M_i\, M_k\, \mathrm{d}x = \int_0^s 4\, i\left[\frac{x}{s} - \left(\frac{x}{s}\right)^2\right] k\,\frac{x}{s}\,\mathrm{d}x$$

$$= 4\frac{i\,k}{s^2}\left(\frac{s^3}{3} - \frac{s^3}{4}\right) = \frac{1}{3}s\,i\,k.$$

This result can be taken directly from Table 6.3 without the need to integrate. It can be found in the fourth row/second column: $\frac{1}{3}s\,i\,k$.

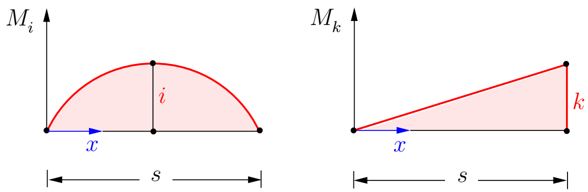


Fig. 6.10

The principle of virtual forces can also be applied to more general types of loading. In the case of a member being subjected to bending, torsion and tension, the deflection is obtained from

$$f = \int \frac{M\bar{M}}{EI}\,\mathrm{d}x + \int \frac{M_T\,\bar{M}_T}{GI_T}\,\mathrm{d}x + \int \frac{N\bar{N}}{EA}\,\mathrm{d}x \qquad (6.36)$$

where M, M_T and N are the bending moment, torque and normal force due to the given load. The virtual force "1" which is applied at the point where the displacement is to be determined leads to the stress resultants $\bar{M}, \bar{M}_T$ and $\bar{N}$. The integrals in (6.36) have to be evaluated for the entire system. If the rigidities GI_T and EA, respectively, are constant in a member, the corresponding integrals

$$\int M_T\,\bar{M}_T\,\mathrm{d}x, \qquad \int N\,\bar{N}\,\mathrm{d}x$$

can also be taken from Table 6.3.

Let us now consider a truss where the i-th member undergoes a change ΔT_i of its temperature. Then, in analogy with (1.17), the change $\alpha_{T_i}\,\Delta T_i\, l_i$ of the length caused by the change of the temperature has to be added to the elongation $\dfrac{S_i\, l_i}{EA_i}$ due to the internal force S_i in (6.30):

$$f = \sum \frac{S_i\,\bar{S}_i\,l_i}{EA_i} + \sum \bar{S}_i\,\alpha_{Ti}\,\Delta T_i\,l_i.$$

Similarly, a moment $M_{\Delta T}$ due to the temperature change (see (4.63)) has to be added to the bending moment M in (6.33) if a beam is subjected to a thermal load (see Section 4.9):

$$f = \int \frac{(M + M_{\Delta T})\,\bar{M}}{EI}\,\mathrm{d}x.$$

E6.3 **Example 6.3** The truss in Fig. 6.11a consists of 17 members (axial rigidity EA).

Determine the vertical displacement f_{V} of pin V.

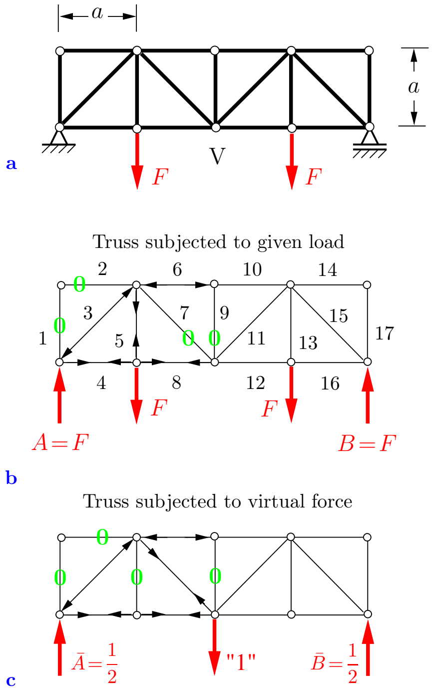


Fig. 6.11

Solution The truss has $j = 10$ joints, $m = 17$ members and $r = 3$ support reactions. Therefore, the necessary isostatic condition $2\,j = m + r$ is satisfied (see Volume 1, Section 6.1).

We determine the displacement with the aid of (6.30). In the first step, the internal forces S_i in the members due to the given loads (Fig. 6.11b) are calculated by the method of joints (Volume 1, Section 6.3.1). The results are presented in a table.

i	S_i	$\bar{S}_i$	l_i	$S_i\,\bar{S}_i\,l_i$
1	0	0	a	0
2	0	0	a	0
3	$-\sqrt{2}F$	$-\sqrt{2}/2$	$\sqrt{2}\,a$	$\sqrt{2}\,F\,a$
4	F	$1/2$	a	$Fa/2$
5	F	0	a	0
6	$-F$	-1	a	$F\,a$
7	0	$\sqrt{2}/2$	$\sqrt{2}\,a$	0
8	F	$1/2$	a	$Fa/2$
9	0	0	a	0
10	$-F$	-1	a	$F\,a$
11	0	$\sqrt{2}/2$	$\sqrt{2}\,a$	0
12	F	$1/2$	a	$F\,a/2$
13	F	0	a	0
14	0	0	a	0
15	$-\sqrt{2}F$	$-\sqrt{2}/2$	$\sqrt{2}\,a$	$\sqrt{2}\,F\,a$
16	F	$1/2$	a	$F\,a/2$
17	0	0	a	0

$$\sum S_i\,\bar{S}_i\,l_i = (4 + 2\sqrt{2})\,F\,a$$

Subsequently, the truss is subjected only to a vertical virtual force "1" at pin V (Fig. 6.11c). The resulting internal forces $\bar{S}_i$ are also recorded in the same table. The products $S_i\,\bar{S}_i\,l_i$ (l_i: length of the member i) are given in the last column of the table. With $EA_i = EA$, the displacement f_V follows from (6.30):

$$\underline{\underline{f_V}} = \sum \frac{S_i\,\bar{S}_i\,l_i}{EA} = \underline{\underline{(4 + 2\sqrt{2})\frac{F\,a}{EA}}}.$$

E6.4

Example 6.4 Determine the horizontal and the vertical components of the displacement of pin B of the truss in Fig. 6.12a. The axial rigidity of the members 1-3 is given by EA; member 4 has the rigidity $2\,EA$.

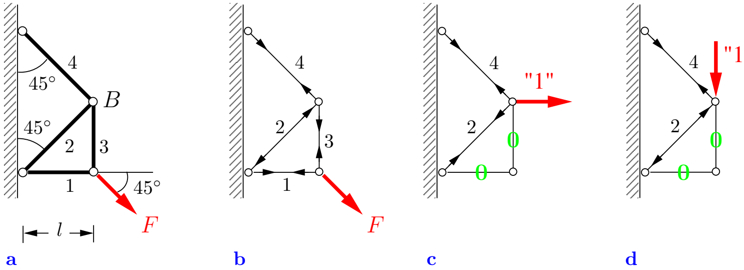


Fig. 6.12

Solution The truss is statically determinate. The internal forces S_i in the members due to the force F (Fig. 6.12b) can be calculated with the aid of the method of joints (Volume 1, Section 6.3.1).

To determine the horizontal displacement of B, we apply a horizontal force "1" at this point (Fig. 6.12c). The corresponding internal forces are denoted by $\bar{S}_{iH}$. Similarly, a vertical force "1" at B (Fig. 6.12d) produces the internal forces $\bar{S}_{iV}$ and leads to the vertical displacement. All the internal forces are given in a table.

i	l_i	S_i	$\bar{S}_{iH}$	$\bar{S}_{iV}$	$S_i\,\bar{S}_{iH}\,l_i$	$S_i\,\bar{S}_{iV}\,l_i$
1	l	$\dfrac{F}{\sqrt{2}}$	0	0	0	0
2	$\sqrt{2}\,l$	$-\dfrac{F}{2}$	$\dfrac{1}{2}\sqrt{2}$	$-\dfrac{1}{2}\sqrt{2}$	$-\dfrac{1}{2}F\,l$	$\dfrac{1}{2}F\,l$
3	l	$\dfrac{F}{\sqrt{2}}$	0	0	0	0
4	$\sqrt{2}\,l$	$\dfrac{F}{2}$	$\dfrac{1}{2}\sqrt{2}$	$\dfrac{1}{2}\sqrt{2}$	$\dfrac{1}{2}F\,l$	$\dfrac{1}{2}F\,l$

The components of the displacement at B follow from (6.30) (note the different axial rigidities of the members):

$$\underline{\underline{f_H}} = \sum \frac{S_i \bar{S}_{iH} l_i}{EA_i} = -\frac{1}{2}\frac{Fl}{EA} + \frac{1}{2}\frac{Fl}{2EA} = \underline{\underline{-\frac{1}{4}\frac{Fl}{EA}}},$$

$$\underline{\underline{f_V}} = \sum \frac{S_i \bar{S}_{iV} l_i}{EA_i} = \frac{1}{2}\frac{Fl}{EA} + \frac{1}{2}\frac{Fl}{2EA} = \underline{\underline{\frac{3}{4}\frac{Fl}{EA}}}.$$

The negative sign of f_H indicates that the horizontal displacement (to the left) is directed in the opposite direction to the force “1“. The vertical displacement is three times as large as the horizontal displacement.

Example 6.5 The frame in Fig. 6.13a (flexural rigidity EI, axial rigidity $EA \to \infty$) is subjected to a constant line load q_0 and a force F. **E6.5**

Calculate the horizontal displacement u_B of the support B.

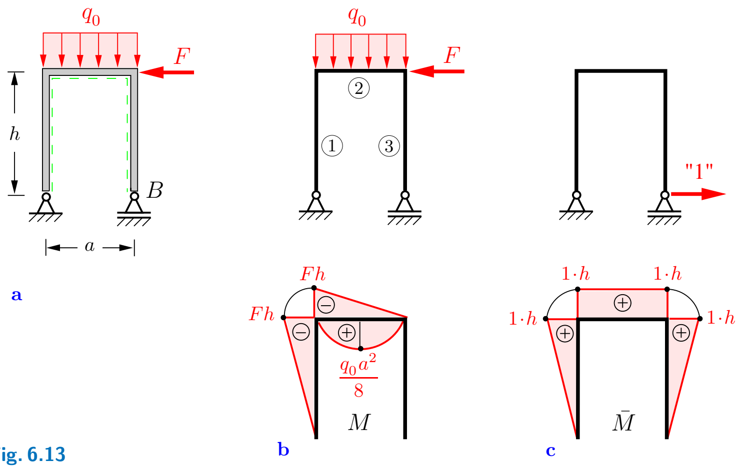


Fig. 6.13

Solution In the first step we determine the bending moment due to the external loads. It is advantageous for the integration that follows to present the moments due to q_0 and F in part ② of the frame in separate graphs (Fig. 6.13b).

Then we subject the frame only to the virtual force “1“ at B and determine the corresponding bending moment (Fig. 6.13c).

The displacement u_B is obtained according to (6.33) by multiplying the bending moments and integrating over the total frame:
Part ① : triangle and triangle

$$\int M\bar{M}\,\mathrm{d}x = \frac{1}{3}h(-F\,h)(1\cdot h) = -\frac{1}{3}F\,h^3$$

Part ② : rectangle and triangle

$$\int M\bar{M}\,\mathrm{d}x = \frac{1}{2}a(-F\,h)(1\cdot h) = -\frac{1}{2}F\,a\,h^2$$

rectangle and quadratic parabola

$$\int M\bar{M}\,\mathrm{d}x = \frac{2}{3}a\frac{q_0\,a^2}{8}(1\cdot h) = \frac{1}{12}q_0\,a^3\,h$$

Part ③ : $\int M\bar{M}\,\mathrm{d}x = 0$ since $M = 0$.

Equation (6.33) yields the displacement:

$$\underline{\underline{EI\,u_B = \frac{1}{12}q_0\,a^3\,h - \frac{1}{6}F\,h^2(2\,h + 3\,a).}}$$

The algebraic signs indicate that the support is displaced to the right due to q_0 and to the left due to F.

E6.6 **Example 6.6** The structure shown in Fig. 6.14a consists of the angled member BCD (flexural rigidity EI) and the two bars 1 and 2 (axial rigidity EA). A couple moment M_0 is applied at point C.

Determine the displacement v_B of the support B and the rotation φ_C at point C.

Solution First we calculate the bending moment M (Fig. 6.14b) and the forces $S_1 = M_0/2\,a$ and $S_2 = -M_0/2\,a$ in the bars due to the external load M_0. In order to find the displacement at B, we apply a virtual force “1“ in the direction of the displacement (Fig. 6.14c) which leads to $\bar{M} = 0$, $\bar{S}_1 = 0$ and $\bar{S}_2 = \sqrt{2}$. Equation (6.36) yields

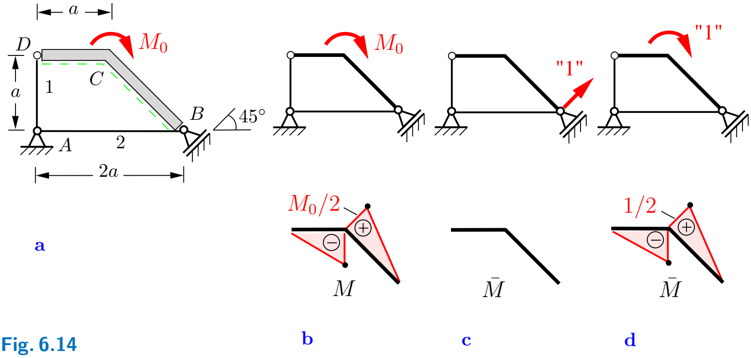


Fig. 6.14

$$\underline{\underline{v_B}} = \frac{S_2\,\bar{S}_2\,l_2}{EA} = -\frac{M_0}{2\,a}\sqrt{2}\,\frac{2\,a}{EA} = \underline{\underline{-\sqrt{2}\,\frac{M_0}{EA}}}.$$

To obtain the rotation at C we apply a virtual couple moment “1“ at this point (Fig. 6.14d). This leads to $\bar{M}$ according to Fig. 6.14d and to $\bar{S}_1 = 1/2\,a$ and $\bar{S}_2 = -1/2\,a$. From

$$\varphi_C = \int \frac{M\,\bar{M}}{EI}\,\mathrm{d}x + \sum \frac{S_i\,\bar{S}_i\,l_i}{EA}$$

(see (6.30) and (6.36)), we obtain the rotation with the aid of Table 6.3:

$$\underline{\underline{\varphi_C}} = \frac{1}{EI}\left[\frac{1}{3}\left(-\frac{M_0}{2}\right)\left(-\frac{1}{2}\right)a + \frac{1}{3}\,\frac{M_0}{2}\,\frac{1}{2}\sqrt{2}\,a\right]$$

$$+\frac{1}{EA}\left[\frac{M_0}{2\,a}\,\frac{1}{2\,a}\,a + \left(-\frac{M_0}{2\,a}\right)\left(-\frac{1}{2\,a}\right)2\,a\right]$$

$$= \underline{\underline{\frac{M_0\,a}{12\,EI}\left[1+\sqrt{2}+9\frac{EI}{EA\,a^2}\right]}}.$$

E6.7 **Example 6.7** Determine the displacement of point C for the frame in Fig. 6.15a.

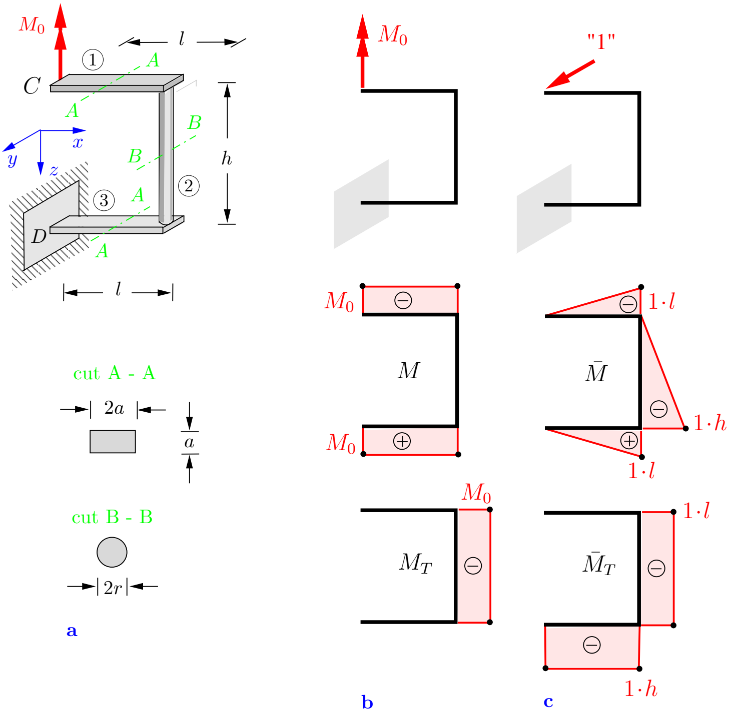


Fig. 6.15

Solution The parts ① and ③ are subjected to bending; part ② is subjected to torsion. One can see by inspection that point C undergoes a displacement v in the y-direction.

In the first step we determine the bending moment M and the torque M_T caused by the external couple moment M_0 (Fig. 6.15b). Note that the algebraic signs for the stress resultants in the individual parts of the system may be chosen arbitrarily. However, the same sign convention has to be applied to the system with the virtual force "1". The stress resultants $\bar{M}$ and $\bar{M}_T$ due to the virtual force in the y-direction are given in Fig. 6.15c. Integration (see (6.36)) leads to

$$v = \int \frac{M\bar{M}}{EI}\,\mathrm{d}x + \int \frac{M_T\bar{M}_T}{GI_T}\,\mathrm{d}x$$

$$= \frac{1}{2}\frac{M_0\,l}{EI_1}\,l + \frac{1}{2}\frac{M_0\,l}{EI_3}\,l + \frac{M_0\,l}{GI_{T2}}\,h.$$

Inserting the moments of inertia

$$I_1 = I_3 = \frac{a\,(2\,a)^3}{12} = \frac{2}{3}a^4, \qquad I_{T2} = \frac{\pi}{2}r^4$$

yields

$$\underline{\underline{v = \frac{3}{2}\frac{M_0\,l^2}{E\,a^4} + \frac{2\,M_0\,l\,h}{G\,\pi\,r^4}}}.$$

Example 6.8 Determine the displacement of point A of the lamp (weight W) in Fig. 6.16a. The weight of the arch (flexural rigidity EI) is negligible. **E6.8**

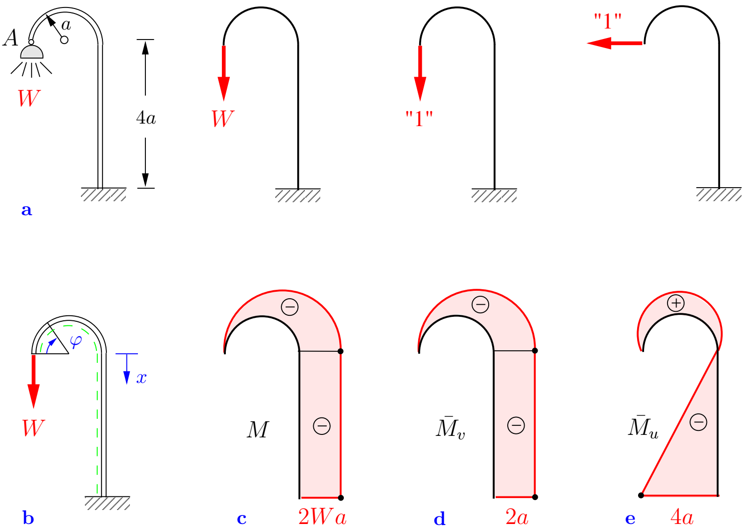


Fig. 6.16

Solution We introduce the dashed line (see Volume 1, Section 7.1) and the coordinates x and φ according to Fig. 6.16b. Then the bending moment M due to the load W is given by (see Fig. 6.16c)

$$M = \begin{cases} -Wa(1-\cos\varphi), & 0 \le \varphi \le \pi, \\ -W2a, & 0 \le x \le 4a. \end{cases}$$

In order to find the vertical displacement v of point A, we apply a vertical virtual force “1“ at A (Fig. 6.16d). We obtain the bending moment $\bar{M}_v$ if we replace W with “1“ in M:

$$\bar{M}_v = \begin{cases} -a(1-\cos\varphi), & 0 \le \varphi \le \pi, \\ -2a, & 0 \le x \le 4a. \end{cases}$$

Integration yields

$$v = \int \frac{M\bar{M}_v}{EI}\,\mathrm{d}s = \frac{1}{EI}\int_0^{\pi} -Wa(1-\cos\varphi)[-a(1-\cos\varphi)]a\,\mathrm{d}\varphi$$

$$+\frac{1}{EI}\int_0^{4a}(-2\,a\,W)(-2\,a)\mathrm{d}x$$

$$= \frac{Wa^3}{EI}\int_0^{\pi}(1-2\,\cos\varphi+\cos^2\,\varphi)\,\mathrm{d}\varphi + \frac{4\,Wa^2}{EI}4\,a$$

$$= \frac{Wa^3}{EI}\left(\frac{3\,\pi}{2}+16\right) \approx 20.7\frac{Wa^3}{EI}.$$

To determine the horizontal displacement u, we apply a horizontal force “1“ at A (Fig. 6.16e). The corresponding bending moment $\bar{M}_u$ is given by

$$\bar{M}_u = \begin{cases} a\sin\varphi, & 0 \le \varphi \le \pi, \\ -x, & 0 \le x \le 4a. \end{cases}$$

Integration leads to

$$u = \int \frac{M\bar{M}_u}{EI}\,\mathrm{d}s = \frac{1}{EI}\int\limits_0^{\pi} -Wa(1-\cos\varphi)a\sin\varphi\,a\,\mathrm{d}\varphi$$

$$+\frac{1}{EI}\int\limits_0^{4a}(-2\,a\,W)(-x)\mathrm{d}x$$

$$= \frac{Wa^3}{EI}(-2+16) = 14\,\frac{Wa^3}{EI}.$$

The total displacement f_A is therefore found to be

$$\underline{\underline{f_A}} = \sqrt{u^2+v^2} \approx \frac{Wa^3}{EI}\sqrt{429+196} = \underline{\underline{25\,\frac{Wa^3}{EI}}}.$$

Note that the vertical load W causes a large horizontal displacement.

6.4 Influence Coefficients and Reciprocal Displacement Theorem

6.4

In Section 6.3 it was shown that the deflection f_{ik} of a beam at an arbitrary point i due to a force F_k at point k can be determined with the aid of the principle of virtual forces (see (6.32)). If the force F_k is the only external load, the deflection of the beam is proportional to this force. Therefore, we can write

$$f_{ik} = \alpha_{ik}\,F_k. \tag{6.37}$$

The proportionality factor α_{ik} is called the *influence coefficient*. It is equal to the deflection at point i due to the force “1“ at point k. As an illustrative example consider the cantilever beam in Fig. 6.9a which is subjected to a force F at point a. The influence coefficient for the deflection at the free end is obtained from (6.34):

$$\alpha_{la} = \frac{f}{F} = \frac{l^3}{6\,EI}\left[3\left(\frac{a}{l}\right)^2 - \left(\frac{a}{l}\right)^3\right].$$

Similarly, the influence coefficient for the deflection at point x for the beam subjected to the couple moment M_0 in Example 4.6 is found to be

$$\alpha_{xl} = \frac{w(x)}{M_0} = \frac{l^2}{6\,EI}\left[\left(\frac{x}{l}\right) - \left(\frac{x}{l}\right)^3\right].$$

Note that the two influence coefficients given here have different dimensions.

If a beam is subjected to n forces F_k, the deflection f at point i is obtained through superposition:

$$f = \sum_k f_{ik} = \alpha_{i1}\,F_1 + \alpha_{i2}\,F_2 + \alpha_{i3}\,F_3 + \ldots + \alpha_{in}\,F_n.$$

Let us now consider a beam that is subjected to two forces as shown in Fig. 6.17a: force F_i acts at point i, force F_k acts at point k. If we first apply F_k and subsequently apply F_i (see Fig. 6.17b), then the total work done by the external forces is given by

$$\begin{aligned} U &= \frac{1}{2}\,f_{kk}\,F_k + \frac{1}{2}\,f_{ii}\,F_i + F_k\,f_{ki} \\ &= \frac{1}{2}\,\alpha_{kk}\,F_k^2 + \frac{1}{2}\,\alpha_{ii}\,F_i^2 + F_k(\alpha_{ki}\,F_i). \end{aligned} \tag{6.38a}$$

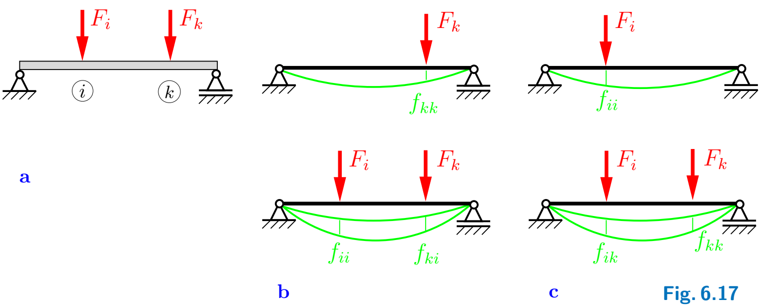


Fig. 6.17

If we first apply F_i and subsequently F_k (Fig. 6.17c), the total work is

$$U = \frac{1}{2} f_{ii} F_i + \frac{1}{2} f_{kk} F_k + F_i f_{ik}$$
$$= \frac{1}{2} \alpha_{ii} F_i^2 + \frac{1}{2} \alpha_{kk} F_k^2 + F_i(\alpha_{ik} F_k). \tag{6.38b}$$

The total strain energy after the two forces have been applied is independent of the sequence of the application. According to the principle of conservation of energy (6.8), the total work of the external forces is also independent of this sequence. We therefore can equate the first lines of (6.38a) and (6.38b) to obtain

$$F_k f_{ki} = F_i f_{ik} \, . \tag{6.39}$$

This is referred to as the *reciprocal work theorem* or *Betti's theorem* (Enrico Betti, 1823–1892 and Lord Rayleigh, 1842–1919). It tells us that the work done by the force F_k during the displacement f_{ki} (which is caused by F_i) is equal to the work done by the force F_i during the displacement f_{ik} (which is caused by F_k). This statement can be generalized to arbitrary elastic systems.

If we equate the second lines of (6.38a) and (6.38b) we obtain the *reciprocal displacement theorem*, also called *Maxwell's reciprocal theorem* (James Clerk Maxwell, 1831–1879):

$$\alpha_{ik} = \alpha_{ki} \, . \tag{6.40}$$

It implies that the deflection α_{ik} at point i due to a force "1" at point k is equal to the deflection α_{ki} at point k due to a force "1" at point i.

Equation (6.40) can also be applied to a system that is subjected to a couple moment. Let us consider, for example, the beam in Fig. 6.18a which is subjected to a force F at point ① and to a

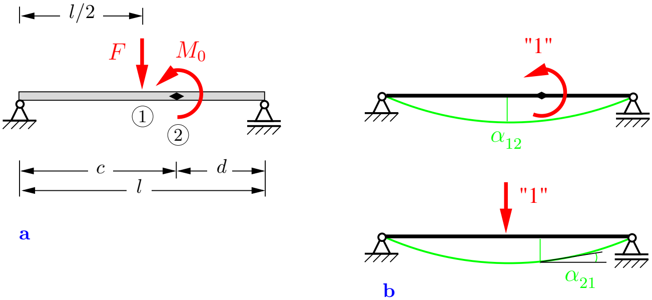


Fig. 6.18

moment M_0 at point ②. This moment causes the deflection

$$f_{12} = \alpha_{12}\, M_0 = -\,\frac{l^2}{6}\left\{\frac{1}{2}\left[3\left(\frac{d}{l}\right)^2 - 1\right] + \frac{1}{8}\right\}\frac{M_0}{EI} \tag{6.41}$$

at point ① (see Table 4.3, Nr. 5, $\xi = 1/2$, $\beta = d/l$).

Now we determine the angle of slope of the deflection curve at point ② which is caused by F. From Table 4.3, Nr. 1, we find the slope at an arbitrary point ξ through differentiation:

$$EI\,w' = \frac{F\,l^2}{6}[\beta(1 - \beta^2 - 3\,\xi^2) + 3\langle\xi - \alpha\rangle^2].$$

In the present example we have to choose $\alpha = \beta = 1/2$ to obtain

$$EI\,w' = \frac{F\,l^2}{6}\left[\frac{1}{2}(1 - \frac{1}{4} - 3\,\xi^2) + 3\langle\xi - \frac{1}{2}\rangle^2\right].$$

The angle has to be taken as positive if its sense of rotation coincides with the sense of rotation of the moment M_0. Thus, $\varphi_{21} = -w'$ at point ②. With $\xi = c/l$ we obtain

$$\varphi_{21} = \alpha_{21}\,F = -\,\frac{l^2}{6\,EI}\left[\frac{1}{2}\left(\frac{3}{4} - 3\left(\frac{c}{l}\right)^2\right) + 3\left(\left(\frac{c}{l}\right) - \frac{1}{2}\right)^2\right]F$$

$$= -\,\frac{l^2}{6\,EI}\left[\frac{3}{2}\left(\frac{c}{l}\right)^2 - 3\left(\frac{c}{l}\right) + \frac{9}{8}\right]F.$$

Using $c = l - d$, this can be written in the form

$$\varphi_{21} = \alpha_{21}\, F = -\, \frac{l^2}{6\, EI} \left\{ \frac{1}{2} \left[3 \left(\frac{d}{l} \right)^2 - 1 \right] + \frac{1}{8} \right\} F. \qquad (6.42)$$

Comparison of (6.41) and (6.42) yields

$$\alpha_{12} = \alpha_{21}.$$

That is, the displacement α_{12} at point ① due to the moment "1" at point ② is equal to the rotation α_{21} at point ② due to the force "1" at point ① (Fig. 6.18b). Note that α_{12} (displacement) and α_{21} (rotation) here have the same dimension.

Maxwell's reciprocal theorem has many useful applications. Note that knowing α_{12} (see (6.41)) in the preceding example, then α_{21} and therefore φ_{21} are also known according to (6.40). Thus, the rather cumbersome calculation to obtain φ_{21} in (6.42) was actually unnecessary.

6.5 Statically Indeterminate Systems

6.5

Statically indeterminate systems were investigated with the aid of the principle of superposition in the Sections 1.4, 1.6 and 4.5.4. In the case of a system which is externally statically indeterminate to the first degree, we removed one of the supports in order to obtain a statically determinate system. In the "0"-system we calculated the displacement $v^{(0)}$ due to the given load at the point where the support was removed. This displacement will now be denoted by α_{10}, i.e., $v^{(0)} = \alpha_{10}$. The new notation is similar to the notation used for the influence coefficients (see Section 6.4). Subsequently, the statically determinate structure was subjected only to the as yet unknown force "X" (the redundant) at the point of the removed support. This system was referred to as the "1"-system. The displacement caused by the force X is $v^{(1)} = X\, \alpha_{11}$, where α_{11} is the displacement caused by the force $X = 1$. The displacement in the given statically indeterminate structure has to be zero due to the actual support:

$$v = v^{(0)} + v^{(1)} = 0. \tag{6.43}$$

This compatibility condition yields the redundant force:

$$\alpha_{10} + X\,\alpha_{11} = 0 \qquad \rightarrow \qquad X = -\,\frac{\alpha_{10}}{\alpha_{11}}. \tag{6.44}$$

Analogous considerations are used if a statically indeterminate beam is made statically determinate by introducing a joint at a point G. In this case we have to apply a moment at G, and the displacement v in (6.43) has to be replaced with the angle φ_G (see Examples 4.11 and 6.12). Similar considerations are valid for a statically indeterminate truss with one redundant bar (internal statical indeterminacy). Then the redundant bar is removed and the displacements in the systems “0“ and “1“ are determined. Compatibility now requires that the change of the distance between the two joints from which the bar was removed is equal to the change of the length of this bar. The force in the statically redundant bar can also be calculated from (6.44), where the coefficients α_{ik} now are the corresponding influence coefficients of the truss (see (6.46)).

In this section we will also apply the principle of superposition, but in contrast to the calculations in the Sections 1.4, 1.6 and 4.5.4, we will now determine the displacements (rotations) with the aid of the principle of virtual forces.

As an illustrative example we consider the beam in Fig. 6.19a; it is statically indeterminate to the first degree. We first remove the support B and replace it with the as yet unknown support reaction X. This leads to the two systems “0“ and “1“ as shown in Fig. 6.19b. In the following derivations we will change the notation: bending moments in a “0“-system are called M_0 from now on and bending moments in a “1“-system are referred to as $\bar{M}_1$ (in Section 4.5.4 they were called $M^{(0)}$ and $M^{(1)} = X\bar{M}_1$). According

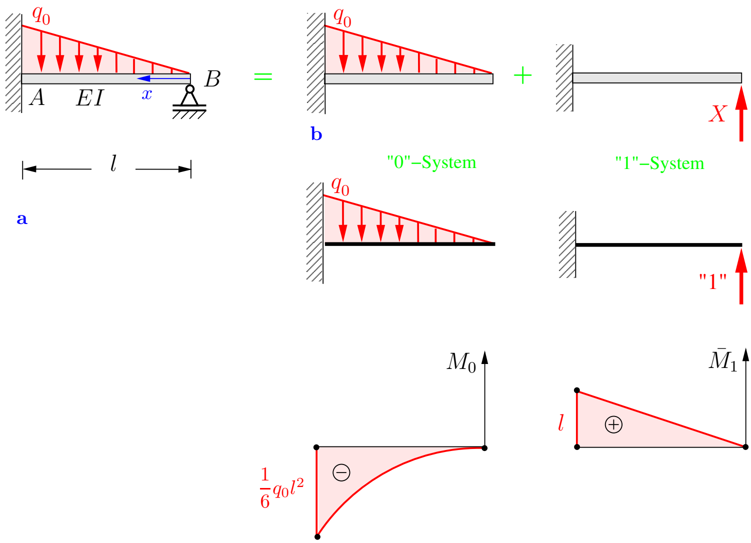


Fig. 6.19

to (6.44) the redundant support reaction X is obtained from

$$X = -\frac{\alpha_{10}}{\alpha_{11}} = -\frac{\displaystyle\int \frac{\bar{M}_1 M_0}{EI}\,\mathrm{d}x}{\displaystyle\int \frac{\bar{M}_1^2}{EI}\,\mathrm{d}x}. \tag{6.45}$$

We use the coordinate x as shown in Fig. 6.19a. Then the bending moments in the systems “0“ and “1“ are given by

$$M_0 = -\frac{1}{2}x\left(q_0\frac{x}{l}\right)\frac{x}{3} = -\frac{q_0}{6\,l}\,x^3, \qquad \bar{M}_1 = x.$$

They lead to

$$\alpha_{10} = \int \frac{\bar{M}_1 M_0}{EI}\,\mathrm{d}x = \frac{1}{EI}\int_0^l x\left(-\frac{q_0}{6\,l}\,x^3\right)\,\mathrm{d}x = -\frac{q_0\,l^4}{30\,EI},$$

$$\alpha_{11} = \int \frac{\bar{M}_1^2}{EI}\,\mathrm{d}x = \frac{1}{EI}\int_0^l x^2\,\mathrm{d}x = \frac{l^3}{3\,EI}.$$

Note that these results could also have been taken from Table 6.3 without performing the integrations. Equation (6.45) yields

$$X = B = -\frac{\alpha_{10}}{\alpha_{11}} = \frac{q_0\, l}{10}.$$

The bending moment in the beam is obtained by superposition:

$$M = M_0 + X\,\bar{M}_1 = -\frac{q_0}{6\,l}\,x^3 + \frac{q_0\,l}{10}\,x.$$

In particular, the moment at the clamped support ($x = l$) is found to be

$$M_A = M(l) = -\frac{q_0\,l^2}{15}.$$

We may also solve this problem using a different “0“-system: the clamped support is now replaced with a joint (Fig. 6.20). Then we have to apply a moment “1“ at this point and the bending moments are given by

$$M_0 = \frac{q_0}{6}\,l\,x - \frac{q_0}{6\,l}\,x^3, \qquad \bar{M}_1 = \frac{x}{l}.$$

Compatibility analogous to (6.43) requires a vanishing slope w'_A at the left-hand side of the beam and leads to the moment at the

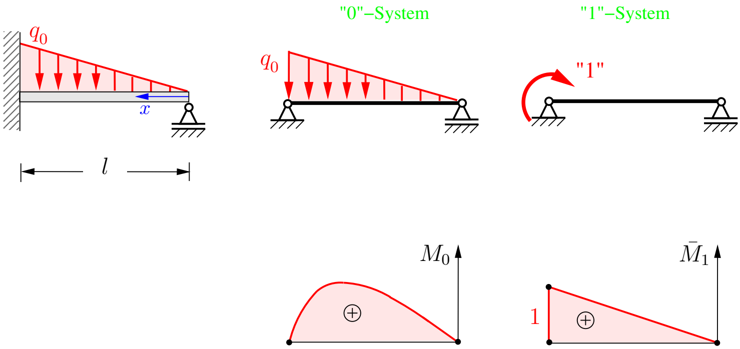


Fig. 6.20

clamping:

$$X = M_A = -\,\frac{\alpha_{10}}{\alpha_{11}} = -\,\frac{\displaystyle\int \frac{\bar{M}_1\,M_0}{EI}\,\mathrm{d}x}{\displaystyle\int \frac{\bar{M}_1^2}{EI}\,\mathrm{d}x} = \frac{\displaystyle\int_0^l \frac{x}{l}\left(\frac{q_0}{6}\,l\,x - \frac{q_0}{6\,l}\,x^3\right)\mathrm{d}x}{\displaystyle\int_0^l \left(\frac{x}{l}\right)^2 \mathrm{d}x}$$

$$= -\,\frac{q_0\,l^2}{15}\,.$$

In the case of a truss which is statically indeterminate to the first degree, we may also use (6.44). The coefficients α_{10} and α_{11} follow according to (6.30):

$$X = -\,\frac{\displaystyle\sum \frac{\bar{S}_i\,S_i^{(0)}\,l_i}{EA_i}}{\displaystyle\sum \frac{\bar{S}_i^2\,l_i}{EA_i}}. \tag{6.46}$$

Here, $S_i^{(0)}$ are the forces in the members of the "0"-system and $\bar{S}_i$ are the forces in the members of the "1"-system.

If there are bending moments, torques and variable normal forces acting in a structure, the redundant follows from (see (6.14))

$$X = -\,\frac{\displaystyle\int \frac{\bar{M}_1\,M_0}{EI}\,\mathrm{d}x + \int \frac{\bar{M}_{T1}\,M_{T0}}{GI_T}\,\mathrm{d}x + \int \frac{\bar{N}_1\,N_0}{EA}\,\mathrm{d}x}{\displaystyle\int \frac{\bar{M}_1^2}{EI}\,\mathrm{d}x + \int \frac{\bar{M}_{T1}^2}{GI_T}\,\mathrm{d}x + \int \frac{\bar{N}_1^2}{EA}\,\mathrm{d}x}. \tag{6.47}$$

After having determined the unknown X, the other support reactions, the stress resultants and the displacements can be calculated.

Finally, we want to indicate the procedure in the case of a system with a statical indeterminacy of degree n. In this case we have to remove n constraints in order to obtain a statically determinate "0"-system. In addition, we have to consider n auxiliary systems to determine the n unknown redundants X_i which can be calculated from n compatibility conditions (see Example 6.12):

$$\begin{aligned} \alpha_{10} + X_1\,\alpha_{11} + \ldots + X_n\,\alpha_{1n} &= 0, \\ \alpha_{20} + X_1\,\alpha_{21} + \ldots + X_n\,\alpha_{2n} &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_{n0} + X_1\,\alpha_{n1} + \ldots + X_n\,\alpha_{nn} &= 0. \end{aligned} \tag{6.48}$$

The displacements in the “0“-system follow from

$$\alpha_{r0} = \int \frac{\bar{M}_r\,M_0}{EI}\,\mathrm{d}x$$

and the displacements in the auxiliary systems are given by

$$\alpha_{ri} = \int \frac{\bar{M}_r\,\bar{M}_i}{EI}\,\mathrm{d}x.$$

Here, M_0 is the bending moment in the “0“-system caused by the given load and $\bar{M}_i$ are the bending moments in the same system due to the virtual forces (moments) acting at the points i ($i = 1, 2, \ldots, n$).

E6.9 **Example 6.9** Determine the forces in the members of the truss in Fig. 6.21a. The axial rigidity EA of the members is given.

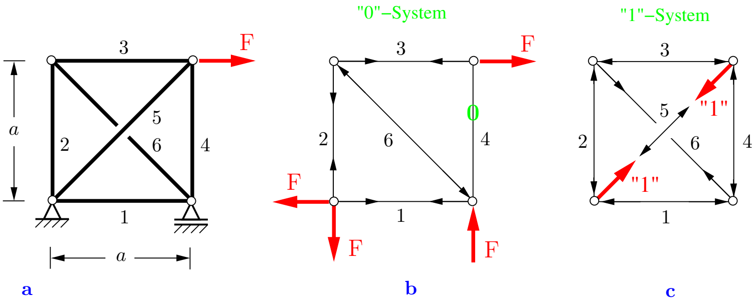


Fig. 6.21

Solution The truss consists of $m = 6$ members and $j = 4$ joints; it has $r = 3$ supports. According to Volume 1, Section 6.1, it is statically indeterminate to the first degree: $m + r - 2j = 1$. We obtain a statically determinate truss if we remove one of the bars.

We choose member 5 which leads to the "0"-system in Fig. 6.21b. Next we subject member 5 to the virtual force "1" ("1"-system, Fig. 6.21c). According to Newton's third law (action equals reaction), forces of the same magnitude act at the joints. The forces in the members of the "0"-system and of the "1"-system, respectively, are calculated and given in the following table.

i	$S_i^{(0)}$	$\bar{S}_i$	l_i	$\bar{S}_i\,S_i^{(0)}\,l_i$	$\bar{S}_i^2\,l_i$	S_i
1	F	$-1/\sqrt{2}$	a	$-F\,a/\sqrt{2}$	$\frac{1}{2}a$	$+\,0.40\,F$
2	F	$-1/\sqrt{2}$	a	$-F\,a/\sqrt{2}$	$\frac{1}{2}a$	$+\,0.40\,F$
3	F	$-1/\sqrt{2}$	a	$-F\,a/\sqrt{2}$	$\frac{1}{2}a$	$+\,0.40\,F$
4	0	$-1/\sqrt{2}$	a	0	$\frac{1}{2}a$	$-\,0.60\,F$
5	0	1	$\sqrt{2}\,a$	0	$\sqrt{2}\,a$	$+\,0.85\,F$
6	$-\sqrt{2}\,F$	1	$\sqrt{2}\,a$	$-2\,a\,F$	$\sqrt{2}\,a$	$-\,0.56\,F$

The unknown force in member 5 is obtained with $\sum \bar{S}_i\,S_i^{(0)}\,l_i = (-2-3/\sqrt{2})\,F\,a$ and $\sum \bar{S}_i^2\,l_i = 2(1+\sqrt{2})a$ from (6.46):

$$X = \underline{\underline{S_5}} = -\frac{\left(-2-\dfrac{3}{\sqrt{2}}\right)F\,a}{2(1+\sqrt{2})\,a} = \frac{3+2\sqrt{2}}{2(2+\sqrt{2})}\,F \approx \underline{\underline{0.85\,F}}.$$

The forces in the other members follow from

$$S_i = S_i^{(0)} + X\,\bar{S}_i.$$

They are given in the last column of the table.

Note that the support reactions can be calculated in advance in the case of a truss which is internally statically indeterminate.

E6.10

Example 6.10 The structure in Fig. 6.22a consists of an angled member (flexural rigidity EI) and two bars (axial rigidity EA). It is subjected to the force F.

Determine the bending moment in the angled member and the forces in the bars.

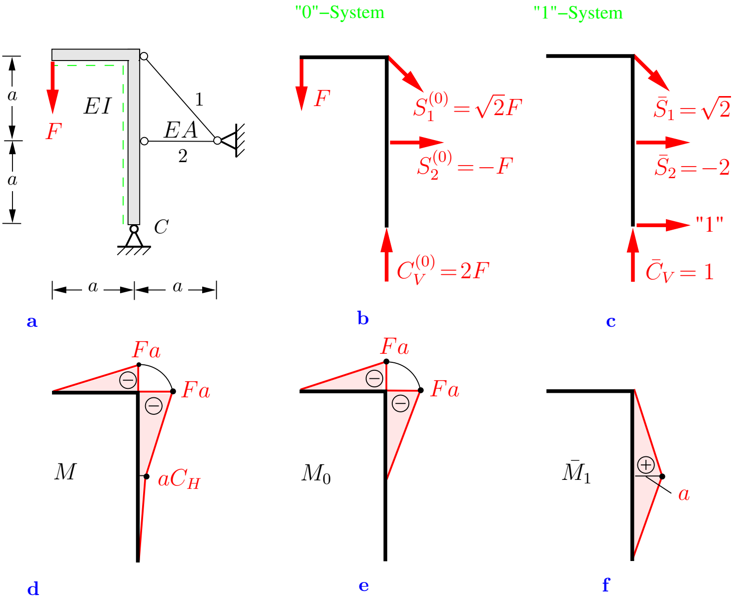


Fig. 6.22

Solution The structure is statically indeterminate to the first degree. To obtain a “0“-system we replace the hinged support at C with a roller support which can move in the horizontal direction. The equilibrium conditions in the “0“-system (Fig. 6.22b) yield

$$C_V^{(0)} = 2\,F, \qquad S_1^{(0)} = \sqrt{2}\,F, \qquad S_2^{(0)} = -\,F.$$

Now we apply a horizontal virtual force “1“ at C (“1“-system, Fig. 6.22c) and obtain

$$\bar{C}_V = 1, \qquad \bar{S}_1 = \sqrt{2}, \qquad \bar{S}_2 = -\,2.$$

The resulting bending moments M_0 and $\bar{M}_1$ are shown in the Figs. 6.22e, f. Since the structure consists of beams and bars, the redundant force C_H follows from (6.47):

$$X = C_H = -\frac{\displaystyle\int \frac{\bar{M}_1\,M_0}{EI}\,\mathrm{d}x + \sum \bar{S}_i \frac{S_i^{(0)}\,l_i}{EA}}{\displaystyle\int \frac{\bar{M}_1^2}{EI}\,\mathrm{d}x + \sum \bar{S}_i^2 \frac{l_i}{EA}}.$$

We use Table 6.3 (triangles with triangles) to calculate the integrals and introduce the parameter $\varkappa = EA\,a^2/EI$ to obtain

$$X = C_H = -\frac{\dfrac{1}{6}a(-F\,a)\dfrac{a}{EI} + \left(\sqrt{2}\sqrt{2}\,F\dfrac{a\sqrt{2}}{EA} + (-2)(-F)\dfrac{a}{EA}\right)}{2\cdot\dfrac{1}{3}\,a\dfrac{a^2}{EI} + \sqrt{2}\sqrt{2}\dfrac{\sqrt{2}\,a}{EA} + (-2)(-2)\dfrac{a}{EA}}$$

$$= \frac{\varkappa - 12(\sqrt{2}+1)}{4\,\varkappa + 12(\sqrt{2}+2)}\,F.$$

The forces in the bars follow from $S_i = S_i^{(0)} + X\,\bar{S}_i$:

$$\underline{\underline{S_1}} = \sqrt{2}\,F + \frac{\varkappa - 12(\sqrt{2}+1)}{4\,\varkappa + 12(\sqrt{2}+2)}\sqrt{2}\,F = \underline{\underline{\frac{5\,\sqrt{2}\,\varkappa + 12(\sqrt{2}+1)}{4\,\varkappa + 12(\sqrt{2}+2)}\,F}},$$

$$\underline{\underline{S_2}} = -\,F + \frac{\varkappa - 12(\sqrt{2}+1)}{4\,\varkappa + 12(\sqrt{2}+2)}(-2)\,F = \underline{\underline{\frac{-\,6\,\varkappa + 12\sqrt{2}}{4\,\varkappa + 12(\sqrt{2}+2)}\,F}}.$$

The bending moment is given by $M = M_0 + X\,\bar{M}_1$ and is displayed in Fig. 6.22d.

Frequently, the stiffness parameter $\varkappa$ is a large number. This is the case, for example, if the beams and the bars are made of the same material and their cross sectional areas are roughly the same. Then, $\varkappa \sim (a/r_g)^2$. Since the length a is much larger than the radius of gyration r_g, we have $\varkappa \gg 1$. In such a case we use $\varkappa \to \infty$ to obtain

$$C_H = \frac{F}{4}, \qquad C_V = \frac{9}{4}\,F, \qquad S_1 = \frac{5}{4}\sqrt{2}\,F, \qquad S_2 = -\,\frac{3}{2}\,F.$$

These are the support reactions and the forces in the bars if the bars are considered to be rigid ($EA \to \infty$).

E6.11 **Example 6.11** The beam (flexural rigidity EI) in Fig. 6.23a is subjected to a moment M_D and a constant line load q_0.

Calculate the moment M_A at the clamped end A.

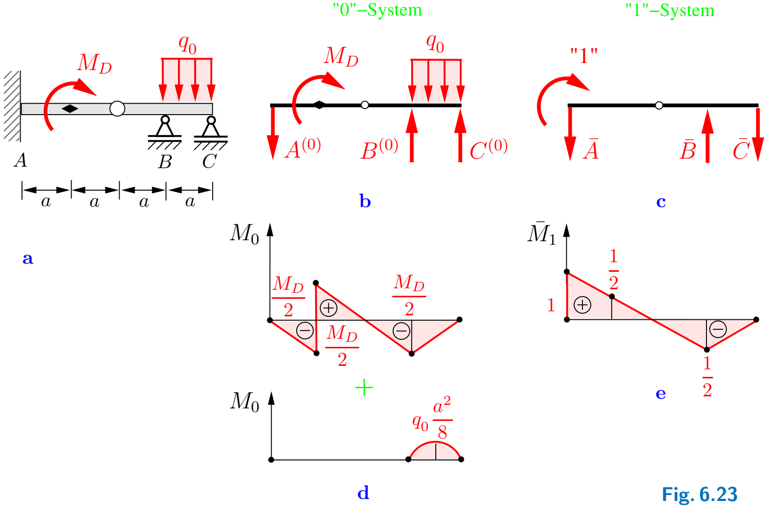


Fig. 6.23

Solution The beam is statically indeterminate to the first degree (see Volume 1, Section 5.3.3). To obtain a statically determinate “0“-system, we remove the clamping and replace it with a hinged support. The equilibrium conditions yield the support reactions (Fig. 6.23b)

$$A^{(0)} = \frac{M_D}{2\,a}, \qquad B^{(0)} = \frac{M_D}{a} + \frac{1}{2} q_0\, a, \qquad C^{(0)} = -\,\frac{M_D}{2\,a} + \frac{1}{2} q_0\, a.$$

The corresponding bending moment M_0 is displayed in Fig. 6.23d, where the moments caused by M_D and q_0 are given in separate graphs.

The moment “1“ in the “1“-system (Fig. 6.23c) causes the support reactions

$$\bar{A} = \frac{1}{2\,a}, \qquad \bar{B} = \frac{1}{a}, \qquad \bar{C} = \frac{1}{2\,a}.$$

The corresponding bending moment $\bar{M}_1$ is shown in Fig. 6.23e.

To determine the unknown moment M_A we apply (6.44): $X = -\alpha_{10}/\alpha_{11}$. The coefficients α_{ik} are calculated with the aid of Table 6.3. This gives

∘ due to q_0 (parabola and triangle):

$$EI\,\alpha_{10q} = \int \bar{M}_1\,M_{0q}\,\mathrm{d}x = \frac{1}{3}\,a\left(-\frac{1}{2}\right)\frac{q_0\,a^2}{8} = -\,\frac{1}{48}\,q_0\,a^3,$$

∘ due to M_D (triangle and trapezium, triangles and triangles):

$$EI\,\alpha_{10M} = \int \bar{M}_1\,M_{0M}\,\mathrm{d}x = \frac{1}{6}\,a\left(-\frac{M_D}{2}\right)\left(1 + 2\cdot\frac{1}{2}\right)$$

$$+\frac{1}{3}\,a\frac{M_D}{2}\frac{1}{2} + \frac{1}{3}\,a\left(-\frac{M_D}{2}\right)\left(-\frac{1}{2}\right)$$

$$+\frac{1}{3}\,a\left(-\frac{M_D}{2}\right)\left(-\frac{1}{2}\right) = \frac{1}{12}\,M_D\,a,$$

∘ due to “1“ (triangles and triangles):

$$EI\,\alpha_{11} = \int \bar{M}_1^2\,\mathrm{d}x = \frac{1}{3}\cdot 1\cdot 1\cdot 2\,a + \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot a$$

$$+\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot a = \frac{5}{6}\,a.$$

With $\alpha_{10} = \alpha_{10q} + \alpha_{10M}$ we obtain

$$X = \underline{\underline{M_A}} = -\,\frac{\alpha_{10}}{\alpha_{11}} = -\,\frac{-\,\dfrac{q_0\,a^3}{48} + \dfrac{1}{12}M_D\,a}{\dfrac{5}{6}\,a} = \underline{\underline{\frac{q_0\,a^2}{40} - \frac{M_D}{10}}}.$$

E6.12

Example 6.12 Determine the support reactions for the frame (flexural rigidity EI) in Fig. 6.24a.

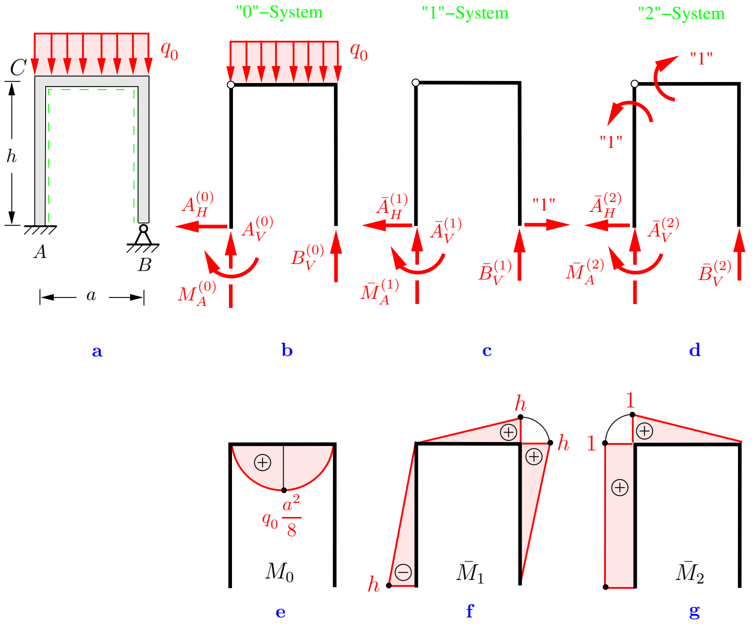


Fig. 6.24

Solution The frame has five support reactions (clamping and hinged support). Therefore it is statically indeterminate to the second degree. In order to obtain a statically determinate "0"-system, we replace the hinged support B with a roller support (which can move in the horizontal direction) and we introduce a hinge at the corner C.

The support reactions in the "0"-system (Fig. 6.24b) are obtained as

$$A_V^{(0)} = B_V^{(0)} = \frac{q_0\, a}{2}, \qquad M_A^{(0)} = 0, \qquad A_H^{(0)} = 0.$$

The corresponding bending moment M_0 is depicted in Fig. 6.24e.

Since we have two redundancies, we need two auxiliary systems. In the "1"-system (Fig. 6.24c) we apply the horizontal force "1" at point B. The support reactions

$$\bar{A}_V^{(1)} = -\,\bar{B}_V^{(1)} = \frac{h}{a}, \qquad \bar{M}_A^{(1)} = -\,h, \qquad \bar{A}_H^{(1)} = 1$$

lead to the bending moment $\bar{M}_1$ which is displayed in Fig. 6.24f. The virtual moment "1" (see Example 4.11) at the corner C in the "2"-system (Fig. 6.24d) causes the support reactions

$$\bar{A}_V^{(2)} = -\,\bar{B}_V^{(2)} = -\,\frac{1}{a}, \quad \bar{M}_A^{(2)} = 1, \quad \bar{A}_H^{(2)} = 0$$

and the bending moment $\bar{M}_2$ (Fig. 6.24g).

The two redundant reactions $X_1 = B_H$ and $X_2 = M_C$ can be determined from two compatibility conditions:

a) the horizontal displacement w_B at B has to be zero,
b) the right angle at C must remain unchanged ($\Delta\, w'_C = 0$).

These conditions are written as (see (6.48))

$$w_B = \alpha_{10} + X_1\,\alpha_{11} + X_2\,\alpha_{12} = 0,$$
$$\Delta w'_C = \alpha_{20} + X_1\,\alpha_{21} + X_2\,\alpha_{22} = 0.$$

We calculate the coefficients α_{ik} with the aid of Table 6.3:

$$EI\,\alpha_{10} = \int \bar{M}_1\,M_0\,\mathrm{d}x = \frac{1}{3}a\,h\,\frac{q_0\,a^2}{8} = \frac{1}{24}q_0\,a^3\,h,$$

$$EI\,\alpha_{20} = \int \bar{M}_2\,M_0\,\mathrm{d}x = \frac{1}{3}a\,\frac{q_0\,a^2}{8} = \frac{1}{24}q_0\,a^3,$$

$$EI\,\alpha_{11} = \int \bar{M}_1^2\,\mathrm{d}x = \frac{1}{3}(h\cdot h^2 + a\,h^2 + h\cdot h^2) = \frac{h^2}{3}(2\,h + a),$$

$$EI\,\alpha_{22} = \int \bar{M}_2^2\,\mathrm{d}x = h + \frac{1}{3}a,$$

$$EI\,\alpha_{12} = \int \bar{M}_1\,\bar{M}_2\,\mathrm{d}x = \frac{1}{2}(-h)h + \frac{1}{6}a\,h$$
$$= \frac{1}{6}h(a - 3\,h) = EI\,\alpha_{21}.$$

Thus, we obtain the system of equations

$$\frac{1}{24} q_0\, a^3\, h + X_1 \frac{h^2}{3}(2\,h + a) + X_2 \frac{1}{6} h(a - 3\,h) = 0,$$

$$\frac{1}{24} q_0\, a^3 + X_1 \frac{1}{6} h(a - 3\,h) + X_2 (h + \frac{1}{3} a) = 0$$

which has the solution

$$X_1 = B_H = -\,\frac{1}{4} q_0\, a^3 \frac{9\,h + a}{15\,h^3 + 26\,a\,h^2 + 3\,h\,a^2},$$

$$X_2 = M_C = -\,\frac{1}{4} q_0\, a^3 \frac{7\,h + a}{15\,h^2 + 26\,a\,h + 3\,a^2}.$$

The superposition of the three systems yields ($A_H^{(0)} = 0$, $M_A^{(0)} = 0$, $A_H^{(2)} = 0$)

$$\underline{\underline{A_V}} = A_V^{(0)} + X_1\,\bar{A}_V^{(1)} + X_2\,\bar{A}_V^{(2)} = \underline{\underline{\frac{15\,h^2 + 25\,a\,h + 3\,a^2}{15\,h^2 + 26\,a\,h + 3\,a^2}\,\frac{q_0\,a}{2}}},$$

$$\underline{\underline{A_H}} = X_1\,\bar{A}_H^{(1)} = -\,\underline{\underline{\frac{1}{4}\,\frac{9\,h + a}{15\,h^3 + 26\,a\,h^2 + 3\,h\,a^2}\,q_0\,a^3}} = \underline{\underline{B_H}},$$

$$\underline{\underline{M_A}} = X_1\,\bar{M}_A^{(1)} + X_2\,\bar{M}_A^{(2)} = \underline{\underline{\frac{1}{4}\,\frac{2\,h}{15\,h^2 + 26\,a\,h + 3\,a^2}\,q_0\,a^3}},$$

$$\underline{\underline{B_V}} = B_V^{(0)} + X_1\,\bar{B}_V^{(1)} + X_2\,\bar{B}_V^{(2)} = \underline{\underline{\frac{15\,h^2 + 27\,a\,h + 3\,a^2}{15\,h^2 + 26\,a\,h + 3\,a^2}\,\frac{q_0\,a}{2}}}.$$

6.6 Supplementary Examples

Detailed solutions to the following examples are given in (**A**) D. Gross et al. *Mechanics of Materials - Formulas and Problems*, Springer 2017 or (**B**) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer 2017.

E6.13

Example 6.13 The truss in Fig. 6.25 consists of five bars (axial rigidity EA) of equal length l.

Determine the forces in the bars that are caused by the external load F.

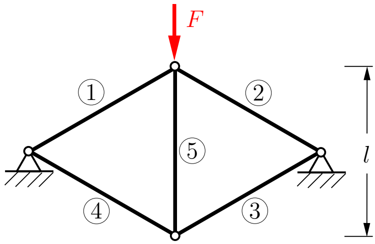


Fig. 6.25

Results: see (**B**) $S_1 = S_2 = -\dfrac{3F}{5}$, $S_3 = S_4 = \dfrac{2F}{5}$, $S_5 = -\dfrac{2F}{5}$.

E6.14

Example 6.14 The structure shown in Fig. 6.26 consists of six elastic bars (axial rigidity EA) of negligible weight and a rigid beam (weight W).

Calculate the support reaction at B due to the weight W.

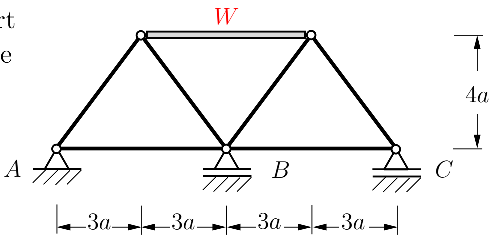


Fig. 6.26

Result: see (**B**) $B = \dfrac{179W}{304}$.

E6.15

Example 6.15 The structure in Fig. 6.27 consists of a beam (axial rigidity $EA \to \infty$, flexural rigidity EI) and two bars (axial rigidity EA). It is subjected to a force F.

Determine the displacement of the point of application of F.

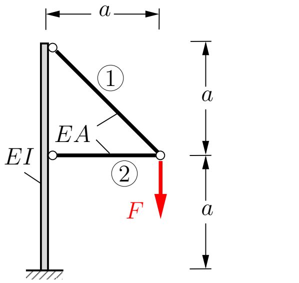


Fig. 6.27

Results: see (**A**) $f_H = \frac{Fa^3}{2EI} - \frac{Fa}{EA}, \; f_V = \frac{4}{3}\frac{Fa^3}{EI} + \frac{(1+2\sqrt{2})Fa}{EA}$.

E6.16

Example 6.16 The truss in Fig. 6.28 is subjected to a force F. The members of the truss have the axial rigidity EA.

Determine the magnitude of F so that the vertical displacement of the point of application of F has the given value f_0.

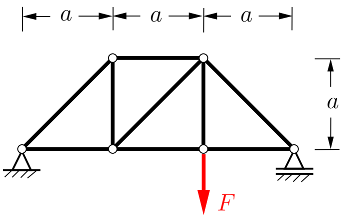


Fig. 6.28

Result: see (**A**) $F = \frac{9EAf_0}{4\left(5+3\sqrt{2}\right)a}$.

E6.17

Example 6.17 Determine the support reaction B of the beam (flexural rigidity EI) and the angle of slope φ_B due to the applied moment M_A (Fig. 6.29).

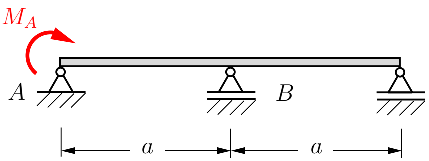


Fig. 6.29

Results: see (**B**) $B = \frac{3M_A}{2a}, \quad \varphi_B = -\frac{M_A a}{12EI}$.

Example 6.18 A circular arch is subjected to a force F as shown in Fig. 6.30. **E6.18**

Determine the displacement of the point of application of the force due to bending.

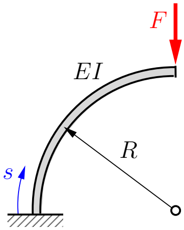


Fig. 6.30

Results: see (**A**) $f_H = \dfrac{FR^3}{4EI}, \quad f_V = \dfrac{\pi FR^3}{4EI}$.

Example 6.19 The two beams (modulus of elasticity E) of the frame shown in Fig. 6.31 have rectangular cross sections with constant width b. The depth h is constant ($h = h_0$) in region AB, whereas in region BC it has a linear taper ($h = h(x)$). A constant line load q_0 acts in region BC. **E6.19**

Calculate the vertical displacement w_C of point C. Neglect axial deformations.

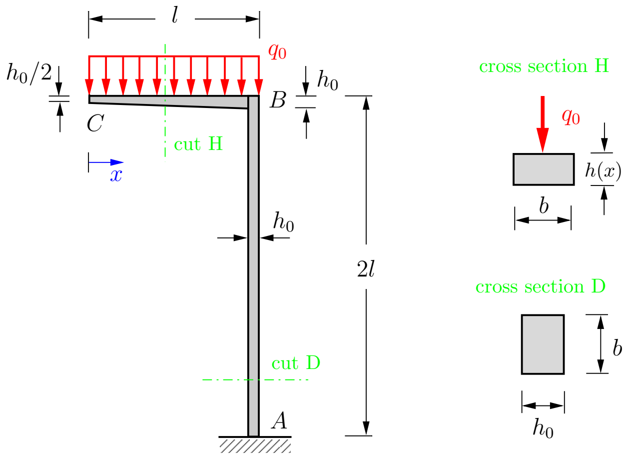


Fig. 6.31

Result: see (**B**) $w_C = 1.2\dfrac{q_0 l^4}{EI_0}$.

E6.20

Example 6.20 A rectangular frame (flexural rigidity EI, axial rigidity $\to \infty$) is subjected to a uniform line load q_0 (Fig. 6.32).

Determine the bending moment in the frame.

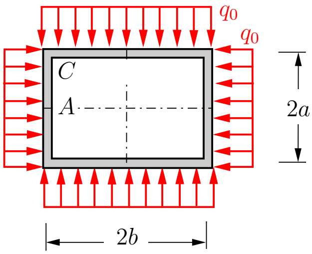


Fig. 6.32

Results: see (**B**) Selected values:

$$M_A = \left(a^2 + 2ab - 2b^2\right) q_0/6, \; M_C = \left(a^2 - ab + b^2\right) q_0/3 .$$

E6.21

Example 6.21 The assembly shown in Fig. 6.33 consists of a frame (flexural rigidity EI, torsional rigidity $GI_T = 3EI/4$, negligible weight) and a wheel (weight $W_1 = W$, radius $r = a/4$). The wheel is attached in a fixed manner to the frame at point C. A rope which is wrapped around the wheel carries a barrel (weight $W_2 = 8W$).

Determine the support reactions and the vertical displacement of point C due to the load.

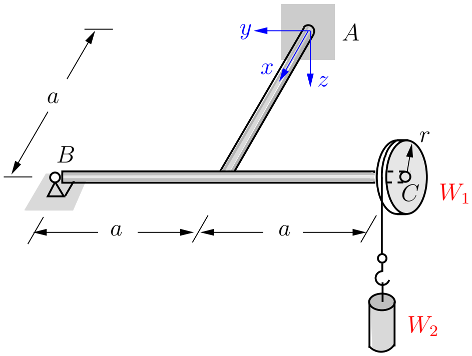


Fig. 6.33

Results: see (**B**)

$$A_z = -13W, \quad B = 4W, \quad M_{Ax} = 5Wa, \quad M_{Ay} = 15Wa,$$

$$w_C = 15\frac{Wa^3}{EI} .$$

Example 6.22 The structure shown in Fig. 6.34 consists of a beam (flexural rigidity EI) and three bars (axial rigidity EA). It is subjected to a force F. **E6.22**

Find the vertical displacement w of the point of application of F.

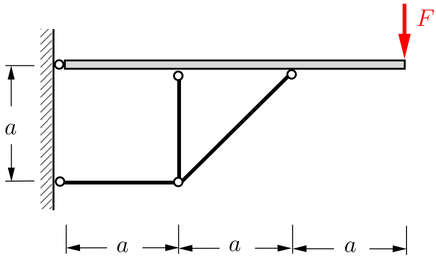


Fig. 6.34

Result: see (**B**) $w = 18\left(1+\sqrt{2}\right)\dfrac{Fa}{EA} + \dfrac{Fa^3}{EI}\,.$

Example 6.23 The structure in Fig. 6.35 consists of a frame (axial rigidity $EA \to \infty$, flexural rigidity EI) and a bar (axial rigidity EA). It is subjected to a uniform line load q_0. **E6.23**

Determine the force S in the bar.

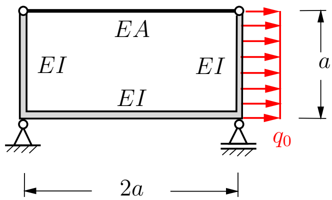


Fig. 6.35

Result: see (**A**) $S = \dfrac{15}{64}\left(1+\dfrac{3EI}{4EAa^2}\right)^{-1} q_0 a\,.$

Example 6.24 A rope S (axial rigidity EA_2) is attached to a cantilever beam (flexural rigidity EI_1) as shown in Fig. 6.36. **E6.24**

Calculate the force S in the rope due to an applied force F at the free end of the cantilever. Disregard axial deformations of the beam.

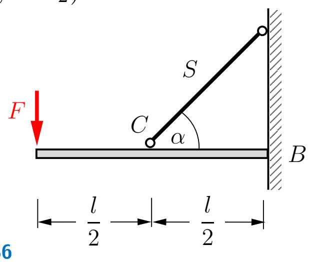


Fig. 6.36

Result: see (**B**) $S = \dfrac{5F\sin\alpha\cos\alpha}{2\sin^2\alpha\cos\alpha + 24\dfrac{(EI)_1}{l^2(EA)_2}}\,.$

E6.25

Example 6.25 The continuous beam in Fig. 6.37 is subjected to a force F at point G.

a) Determine the deflections f_D and f_G at points D and G.
b) Calculate the total deflection f at point G if, in addition to force F, a force $2F$ acts at point D.

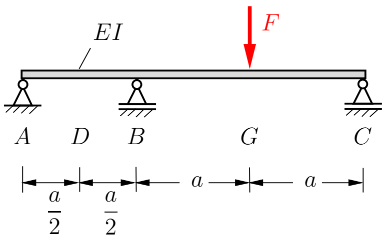


Fig. 6.37

Results: see (**A**) a) $f_D = -\dfrac{1}{64}\dfrac{Fa^3}{EI}$, $f_G = \dfrac{5}{48}\dfrac{Fa^3}{EI}$,

$$\text{b)}\ f = \frac{7}{96}\frac{Fa^3}{EI}\,.$$

E6.26

Example 6.26 Determine the forces S_i in the members (axial rigidity EA) of the truss shown in Fig. 6.38. Calculate the vertical displacement f_F of the point of application of the force F.

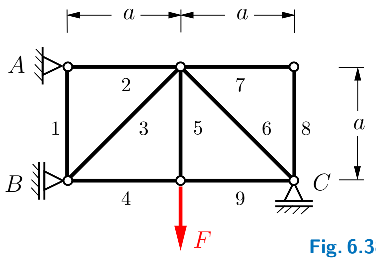


Fig. 6.38

Results: see (**A**) Selected values:

$$S_1 = \frac{4+2\sqrt{2}}{7+4\sqrt{2}}\,F,\ S_2 = \frac{1}{7+4\sqrt{2}}\,F,\ S_3 = -\frac{4+4\sqrt{2}}{7+4\sqrt{2}}\,F,$$

$$f_F = \frac{20+14\sqrt{2}}{7+4\sqrt{2}}\,\frac{Fa}{EA}\,.$$

E6.27

Example 6.27 The beam in Fig. 6.39 (flexural rigidity EI) is pin-supported at its left end and suspended by a rope (axial rigidity EA). It is subjected to a force F.

Determine the vertical displacement f of the point of application of F. Disregard axial deformations of the beam.

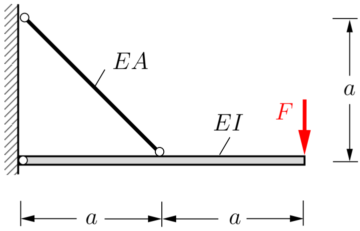


Fig. 6.39

Results: see (**A**) $f = \frac{2}{3}\frac{Fa^3}{EI} + 8\sqrt{2}\frac{Fa}{EA}$.

E6.28

Example 6.28 The truss in in Fig. 6.40 consists of nine members (axial rigidity EA).

Determine the vertical and the horizontal displacements of pin III.

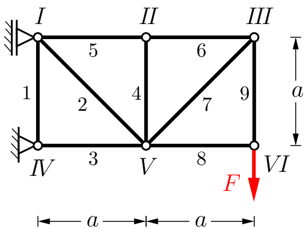


Fig. 6.40

Results: see (**A**) $f_V = (7 + 4\sqrt{2})\frac{Fa}{EA}$, $f_H = 2\frac{Fa}{EA}$.

E6.29

Example 6.29 Calculate the vertical displacement f_B and the angle of slope ψ_B at point B of the frame in Fig. 6.41.

Neglect the axial deformation of the vertical beam.

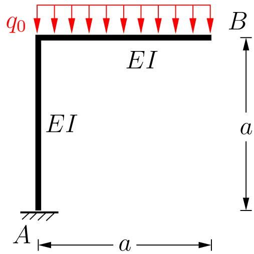


Fig. 6.41

Results: see (**A**) $f_B = \frac{5}{8}\frac{q_0 a^4}{EI}$, $\psi_B = \frac{2}{3}\frac{q_0 a^3}{EI}$.

6.7

6.7 Summary

- Principle of conservation of energy

  $$U_e = U_i\,,$$

  $U_e = \frac{1}{2}\,F f$ work of a force F acting on a linear-elastic bar/beam (analogous relations in the case of an applied moment),

  $U_i = \frac{1}{2}\int \frac{M^2}{EI}\,\mathrm{d}x$ strain energy in bending (analogous relations for torsion, tension/compression).

- Principle of virtual forces (Unit load method)
  - ◇ Statically determinate beam under bending (analogous relations for torsion, tension/compression):

    $$f = \int \frac{M\bar{M}}{EI}\,\mathrm{d}x\,,$$

    M bending moment due to the applied load,
    $\bar{M}$ bending moment due to a virtual force (moment) "1".

    Special case truss: $f = \sum \frac{S_i \bar{S}_i l_i}{EA_i}\,.$
  - ◇ Determination of the redundant X of a beam being statically indeterminate to the first degree:

    $$X = -\frac{\alpha_{10}}{\alpha_{11}}\,, \qquad \alpha_{10} = \int \frac{\bar{M}_1 M_0}{EI}\,\mathrm{d}x\,, \quad \alpha_{11} = \int \frac{\bar{M}_1^2}{EI}\,\mathrm{d}x\,.$$

    M_0 bending moment in the "0"-system,
    $\bar{M}_1$ bending moment in the "1"-system.
    If a system is subjected to bending, torsion and tension/compression, the appropriate terms have to be considered.

- Influence coefficients
  - ◇ α_{ik} displacement at x due to a load "1" at k.
  - ◇ Reciprocal displacement theorem

    $$\alpha_{ik} = \alpha_{ki}\,.$$