Chapter 2

Stress

2

2 Stress

Objectives: In Chapter 1 the notion of stress in a bar has been introduced. We will now generalize the concept of stress to make it applicable to arbitrary structures. For this purpose the *stress tensor* is introduced. Subsequently we will discuss in detail the *plane stress state* that appears in thin sheets or plates under in-plane loading. This state is fully determined by stress components in two sections perpendicular to each other. We will see that the normal stress and the shear stress take on extreme values for specific directions of the section.

The students will learn how to analyse the plane stress state and how to determine the stresses in different sections.

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2.1 Stress Vector and Stress Tensor

So far, stresses have been calculated only in bars. To be able to determine stresses also in other structures we must generalize the concept of stress. For this purpose let us consider a body which is loaded arbitrarily, e.g. by single forces $\boldsymbol{F}_i$ and area forces $\boldsymbol{p}$ (Fig. 2.1a). The external load generates internal forces. In an imaginary section $s-s$ through the body the internal area forces (stresses) are distributed over the entire area A. In contrast to the bar where these stresses are constant over the cross section (see Section 1.1) they now generally vary throughout the section.

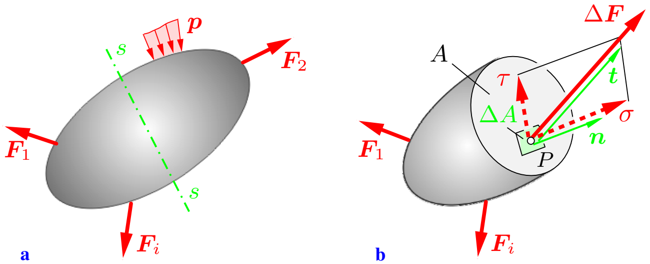

Fig. 2.1 **a** **b**

Since the stress is no longer the same everywhere in the section, it must be defined at an arbitrary point P of the cross section (Fig. 2.1b). The area element ΔA containing P is subjected to the resultant internal force $\Delta\boldsymbol{F}$ (note: according to the law of action and reaction the same force acts in the opposite cross section with opposite direction). The average stress in the area element is defined as the ratio $\Delta\boldsymbol{F}/\Delta A$ (force per area). We assume that the ratio $\Delta\boldsymbol{F}/\Delta A$ in the limit $\Delta A \to 0$ tends to a finite value:

$$\boldsymbol{t} = \lim_{\Delta A \to 0} \frac{\Delta \boldsymbol{F}}{\Delta A} = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}A}\,. \tag{2.1}$$

This limit value is called *stress vector* $\boldsymbol{t}$.

The stress vector can be decomposed into a component normal to the cross section at point P and a component tangential to the cross section. We call the normal component *normal stress* σ and the tangential component *shear stress* τ.

In general, the stress vector $\boldsymbol{t}$ depends on the location of point P in the section area A. The stress distribution in the section is known when the stress vector $\boldsymbol{t}$ is known for all points of A. However, the *stress state* at a point P of the section is not yet sufficiently determined by $\boldsymbol{t}$ for the following reason. If we choose sections through P having *different directions*, different forces will act in the sections because of the different orientation of the area elements. Therefore, the stresses also depend on the orientation of the section which is characterized by the normal vector $\boldsymbol{n}$ (cf. stresses (1.3) in a bar for different directions of the section).

It can be shown that the stress state at point P is uniquely determined by three stress vectors for three sections through P, perpendicular to each other. It is useful to choose the directions of a Cartesian coordinate system for the respective orientations. The three sections can most easily be visualized if we imagine them to be the surfaces of a volume element with edge lengths dx, dy and dz at point P (Fig. 2.2a). A stress vector acts on each of its six surfaces. It can be decomposed into its components perpendicular to the section (= normal stress) and tangential to the section (= shear stress). The shear stress subsequently can be further decomposed into its components according to the coordinate directions. To characterize the components double subscripts are used: the first subscript indicates the orientation of the section by the direction of its normal vector whereas the second subscript indicates the direction of the stress component. For example, τ_{yx} is a shear stress acting in a section whose normal points in y-direction; the stress itself points in x-direction (Fig. 2.2a).

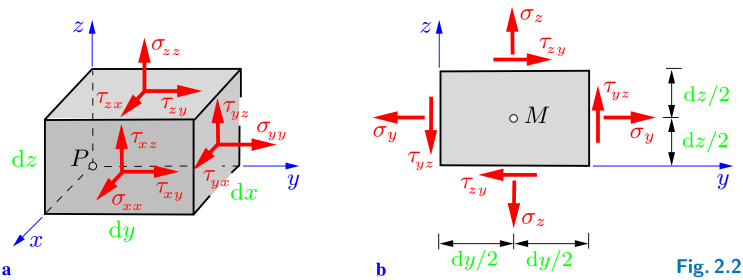

Fig. 2.2

The notation can be simplified for the normal stresses. In this case the directions of the normal to the section and of the stress component coincide. Thus, both subscripts are always equal and one of them can be omitted without losing information:

$$\sigma_{xx} = \sigma_x\,, \qquad \sigma_{yy} = \sigma_y\,, \qquad \sigma_{zz} = \sigma_z\,.$$

From now on we will adopt this shorter notation.

Using the introduced notation, the stress vector, for example in the section with the normal vector pointing in y-direction, can be written as

$$\boldsymbol{t} = \tau_{yx}\,\boldsymbol{e}_x + \sigma_y\,\boldsymbol{e}_y + \tau_{yz}\,\boldsymbol{e}_z\,. \tag{2.2}$$

The *sign convention* for the stresses is the same as for the stress resultants (cf. Volume 1, Section 7.1):

Positive stresses at a *positive (negative)* face point in *positive (negative)* directions of the coordinates.

Accordingly, positive (negative) normal stresses cause tension (compression) in the volume element. Figure 2.2a shows positive stresses acting on the positive faces.

By means of the decomposition of the three stress vectors into their components we have obtained three normal stresses $(\sigma_x, \sigma_y, \sigma_z)$ and six shear stresses $(\tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz}, \tau_{zx}, \tau_{zy})$. However, not all shear stresses are independent of each other. This can be shown by formulating the equilibrium condition for the moments about an axis parallel to the x-axis through the center of the volume element (cf. Fig. 2.2b). Since equilibrium statements are valid for forces, the stresses must be multiplied by the associated area elements:

$$\overset{\curvearrowleft}{C}: \quad 2\frac{\mathrm{d}y}{2}(\tau_{yz}\,\mathrm{d}x\;\mathrm{d}z) - 2\frac{\mathrm{d}z}{2}(\tau_{zy}\,\mathrm{d}x\;\mathrm{d}y) = 0 \quad \rightarrow \quad \tau_{yz} = \tau_{zy}\,.$$

Two further relations are obtained from the moment equilibrium about the other axes:

$$\tau_{xy} = \tau_{yx}\,, \qquad \tau_{xz} = \tau_{zx}\,, \qquad \tau_{yz} = \tau_{zy}\,. \tag{2.3}$$

In words:

The shear stresses with the same subscripts in two orthogonal sections (e.g. τ_{xy} and τ_{yx}) are equal.

They are sometimes called *complementary shear stresses*. Since they have the same algebraic sign they are directed either towards or away from the common edge of the cubic volume element (cf. Fig. 2.2). As a result of (2.3) there exist only six independent stress components.

The components of the three stress vectors can be arranged in a matrix:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}. \tag{2.4}$$

The main diagonal contains the normal stresses; the remaining elements are the shear stresses. The matrix (2.4) is *symmetric* because of (2.3).

The quantity $\boldsymbol{\sigma}$ is called *stress tensor* (the concept *tensor* will be explained in Section 2.2.1). The elements of (2.4) are the components of the stress tensor. The *stress state* at a material point is uniquely defined by the stress vectors for three sections, orthogonal to each other, and consequently by the stress tensor (2.4).

2.2

2.2 Plane Stress

We will now examine the state of stress in a *disk*. This plane structural element has a thickness t much smaller than its in-plane dimensions and it is loaded solely *in* its plane by in-plane forces (Fig. 2.3). The upper and the lower face of the disk are load-free. Since no external forces in the z-direction exist, we can assume with sufficient accuracy that also no stresses will appear in this direction:

$$\tau_{xz} = \tau_{yz} = \sigma_z = 0\,.$$

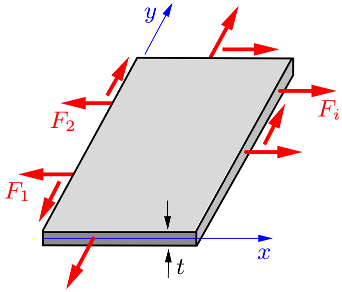

Fig. 2.3

Because of the small thickness we furthermore can assume that the stresses σ_x, σ_y and $\tau_{xy} = \tau_{yx}$ are constant across the thickness of the disk. Such a stress distribution is called a *state of plane stress*. In this case, the third row and the third column of the matrix (2.4) vanish and we get

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}\,.$$

In general, the stresses depend on the location, i.e. on the coordinates x and y. In the special case when the stresses are independent of the location, the stress state is called *homogeneous*.

2.2.1 Coordinate Transformation

Up to now only stresses in sections parallel to the coordinate axes have been considered. Now we will show how from these stresses, the stresses in an arbitrary section perpendicular to the disk can be determined. For this purpose we consider an infinitesimal wedge-shaped element of thickness t cut out from the disk (Fig. 2.4). The directions of the sections are characterized by the x, y-coordinate system and the angle φ. We introduce a ξ, η-system which is rotated with respect to the x, y-system by the angle φ and whose ξ-axis is normal to the inclined section. Here φ is counted positive *counterclockwise*.

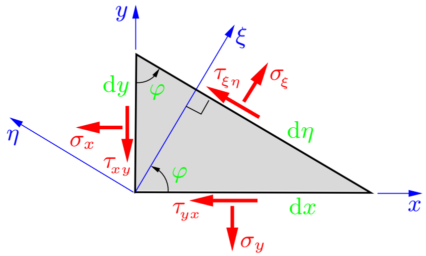

Fig. 2.4

According to the coordinate directions, the stresses in the inclined section are denoted as σ_ξ and $\tau_{\xi\eta}$. The corresponding cross section is given by $\mathrm{d}A = \mathrm{d}\eta\, t$. The other two cross sections perpendicular to the y- and x-axis, respectively, are $\mathrm{d}A \sin\varphi$ and $\mathrm{d}A\cos\varphi$. The equilibrium conditions for the forces in ξ- and in η-direction are

$$\begin{aligned}\nearrow: \quad & \sigma_\xi\,\mathrm{d}A - (\sigma_x\,\mathrm{d}A\cos\varphi)\cos\varphi - (\tau_{xy}\,\mathrm{d}A\cos\varphi)\sin\varphi \\ & \qquad - (\sigma_y\,\mathrm{d}A\sin\varphi)\sin\varphi - (\tau_{yx}\,\mathrm{d}A\sin\varphi)\cos\varphi = 0\,,\end{aligned}$$

$$\begin{aligned}\nwarrow: \quad & \tau_{\xi\eta}\,\mathrm{d}A + (\sigma_x\,\mathrm{d}A\cos\varphi)\sin\varphi - (\tau_{xy}\,\mathrm{d}A\cos\varphi)\cos\varphi \\ & \qquad - (\sigma_y\,\mathrm{d}A\sin\varphi)\cos\varphi + (\tau_{yx}\,\mathrm{d}A\sin\varphi)\sin\varphi = 0\,.\end{aligned}$$

Taking into account $\tau_{yx} = \tau_{xy}$, we get

$$\begin{aligned}\sigma_\xi &= \sigma_x\cos^2\varphi + \sigma_y\sin^2\varphi + 2\,\tau_{xy}\sin\varphi\cos\varphi\,, \\ \tau_{\xi\eta} &= -(\sigma_x - \sigma_y)\sin\varphi\cos\varphi + \tau_{xy}(\cos^2\varphi - \sin^2\varphi)\,.\end{aligned} \tag{2.5a}$$

Additionally, we will now determine the normal stress σ_η which acts in a section with the normal pointing in η-direction. The cutting angle of this section is given by $\varphi + \pi/2$. Therefore, σ_η is obtained by replacing in the first equation of (2.5a) the normal stress σ_ξ by σ_η and the angle φ by $\varphi + \pi/2$. Recalling that $\cos(\varphi + \pi/2) = -\sin\varphi$ and $\sin(\varphi + \pi/2) = \cos\varphi$, we obtain

$$\sigma_\eta = \sigma_x\sin^2\varphi + \sigma_y\cos^2\varphi - 2\,\tau_{xy}\cos\varphi\sin\varphi\,. \tag{2.5b}$$

Usually, the Equations (2.5a, b) are written in a different form. Using the standard trigonometric relations

$$\cos^2\varphi = \frac{1}{2}(1+\cos 2\varphi)\,, \qquad 2\sin\varphi\cos\varphi = \sin 2\varphi\,,$$
$$\sin^2\varphi = \frac{1}{2}(1-\cos 2\varphi)\,, \qquad \cos^2\varphi - \sin^2\varphi = \cos 2\varphi$$

we get

$$\begin{aligned}
\sigma_\xi &= \frac{1}{2}(\sigma_x+\sigma_y) + \frac{1}{2}(\sigma_x-\sigma_y)\cos 2\varphi + \tau_{xy}\sin 2\varphi\,,\\
\sigma_\eta &= \frac{1}{2}(\sigma_x+\sigma_y) - \frac{1}{2}(\sigma_x-\sigma_y)\cos 2\varphi - \tau_{xy}\sin 2\varphi\,,\\
\tau_{\xi\eta} &= \qquad\qquad\quad - \frac{1}{2}(\sigma_x-\sigma_y)\sin 2\varphi + \tau_{xy}\cos 2\varphi\,.
\end{aligned} \tag{2.6}$$

The stresses σ_x, σ_y and τ_{xy} are the components of the stress tensor in the x, y-system. From these stresses, using (2.6), the components σ_ξ, σ_η and $\tau_{\xi\eta}$ in the ξ, η-system can be determined. Equations (2.6) are called *transformation relations* for the components of the stress tensor. Fig. 2.5 shows the stresses in the x, y-system and in the ξ, η-system at the corresponding elements. Note that the stresses in either of the coordinate systems represent one and the same state of stress at a given point of the disk.

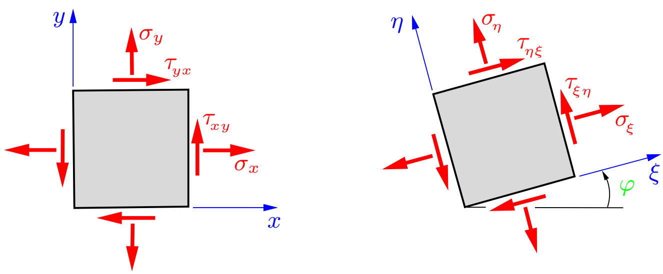

Fig. 2.5

A quantity whose components have *two* coordinate subscripts and which are transformed by a certain rule from one coordinate system to a rotated coordinate system is called a *second rank ten-*

sor. For the stress tensor the rule for the transition from the x, y-system to the ξ, η-system is given by the transformation relations (2.6). We will become familiar with other 2nd rank tensors in Sections 3.1 and 4.2. It should be mentioned that the components of vectors also fulfill specific transformation relations. Because vector components have only *one* subscript, vectors are sometimes called 1st rank tensors.

When adding the first two equations in (2.6) we obtain

$$\sigma_\xi + \sigma_\eta = \sigma_x + \sigma_y \,. \tag{2.7}$$

Thus, the sum of the normal stresses has the same value in each coordinate system. For this reason, the sum $\sigma_x + \sigma_y$ is called an *invariant* of the stress tensor. It can also be verified by simple algebraic manipulation that the determinant $\sigma_x \sigma_y - \tau_{xy}^2$ of the matrix of the stress tensor is a further invariant, that is $\sigma_x \sigma_y - \tau_{xy}^2 = \sigma_\xi \sigma_\eta - \tau_{\xi\eta}^2$.

We finally consider the special case of equal normal stresses ($\sigma_x = \sigma_y$) and vanishing shear stresses ($\tau_{xy} = 0$) in the x, y-system. Equation (2.6) then yields

$$\sigma_\xi = \sigma_\eta = \sigma_x = \sigma_y \,, \qquad \tau_{\xi\eta} = 0 \,.$$

Accordingly, the normal stresses for *all* directions of the sections are the same (i.e. they are independent of φ) whereas the shear stresses always vanish. Such a state of stress is called *hydrostatic* because it corresponds to the pressure in a fluid at rest where the normal stress is the same in all directions.

It should be noted that a disk also can be sectioned in such a way that the normal does not lie in the plane of the disk (slanted section). This case is not discussed here; the reader is referred to the literature.

2.2.2 Principal Stresses

According to (2.6) the stresses σ_ξ, σ_η and $\tau_{\xi\eta}$ depend on the direction of the section, i.e. on the angle φ. We now determine the angle for which these stresses have maximum and minimum values and we calculate these extreme values.

The normal stresses reach extreme values when $\mathrm{d}\sigma_\xi/\mathrm{d}\varphi = 0$ and when $\mathrm{d}\sigma_\eta/\mathrm{d}\varphi = 0$, respectively. Both conditions lead to

$$-(\sigma_x - \sigma_y)\sin 2\varphi + 2\,\tau_{xy}\cos 2\varphi = 0.$$

Hence, the angle $\varphi = \varphi^*$ that leads to a maximum or a minimum is given by

$$\tan 2\varphi^* = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y}\,. \tag{2.8}$$

The tangent function is π-periodic, that is, it satisfies $\tan 2\varphi^* = \tan 2(\varphi^* + \pi/2)$. Therefore, there exist two directions of the sections, φ^* and $\varphi^* + \pi/2$, perpendicular to each other, for which (2.8) is fulfilled. These directions of the sections are called *principal directions*.

The normal stresses which correspond to the principal directions are determined by introducing the condition (2.8) for φ^* into Equation (2.6) for σ_ξ or σ_η, respectively. Here, the following trigonometric relations are used:

$$\begin{aligned}\cos 2\varphi^* &= \frac{1}{\sqrt{1+\tan^2 2\varphi^*}} = \frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau_{xy}^2}}\,,\\ \sin 2\varphi^* &= \frac{\tan 2\varphi^*}{\sqrt{1+\tan^2 2\varphi^*}} = \frac{2\,\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau_{xy}^2}}\,.\end{aligned} \tag{2.9}$$

Using the notations σ_1 and σ_2 for the extreme values of the stresses we obtain

$$\sigma_{1,2} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{\frac{1}{2}(\sigma_x - \sigma_y)^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau_{xy}^2}} \pm \frac{2\,\tau_{xy}^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau_{xy}^2}}$$

or

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}\,, \tag{2.10}$$

respectively. The two normal stresses σ_1 and σ_2 are called *principal stresses*. Typically, they are numbered such that $\sigma_1 > \sigma_2$ (positive sign of the square root for σ_1).

Equation (2.8) provides two values for the angles φ^* and $\varphi^* + \pi/2$. These two angles can be assigned to the stresses σ_1 and σ_2, for example, by introducing one of them into the first equation of (2.6). Doing so, the associated normal stress, either σ_1 or σ_2, is obtained.

If the angles φ^* or $\varphi^* + \pi/2$, respectively, are introduced into the third equation of (2.6), we find $\tau_{\xi\eta} = 0$. Thus, the shear stresses vanish in sections where the normal stresses take on their extreme (principal) values σ_1 and σ_2. Inversely, when the shear stress in a section is zero, the normal stress in this section is a principal stress.

A coordinate system with its axes pointing in the principal directions is called *principal coordinate system*. We denote the axes by 1 and 2: the 1-axis points in the direction of σ_1 (first principal direction), the 2-axis in σ_2-direction (second principal direction). In Figs. 2.6a and b the stresses at an element in the x, y-system and in the principal coordinate system are displayed.

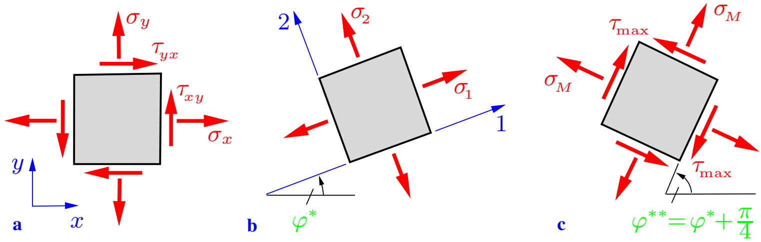

Fig. 2.6

We will now determine the extreme values of the shear stresses and the associated directions of the sections. From the condition

$$\frac{d\tau_{\xi\eta}}{d\varphi} = 0 \qquad \rightarrow \qquad -(\sigma_x - \sigma_y)\cos 2\,\varphi - 2\,\tau_{xy}\sin 2\,\varphi = 0$$

the angle $\varphi = \varphi^{**}$ for an extreme value is obtained:

$$\tan 2\,\varphi^{**} = -\frac{\sigma_x - \sigma_y}{2\,\tau_{xy}}\,. \tag{2.11}$$

Again this equation defines the two perpendicular angles φ^{**} and $\varphi^{**}+\pi/2$ where the shear stress reaches maximum or minimum values. By comparing (2.11) with (2.8) it can be seen that $\tan 2\,\varphi^{**} = -1/\tan 2\,\varphi^{*}$. Recalling of the trigonometric identity $\tan(\alpha+\pi/2) = -1/\tan\alpha$ this implies that the directions $2\varphi^{**}$ and $2\,\varphi^{*}$ are perpendicular to each other. As a consequence, the direction φ^{**} of the extreme shear stress is rotated by 45° with respect to the direction φ^{*} of the extreme normal stress.

The extreme shear stresses are obtained by introducing (2.11) into (2.6) and using (2.9) to give

$$\tau_{\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}\,. \tag{2.12a}$$

Since they differ only in the sign (i.e. in the sense of direction) both stresses are commonly called *maximum shear stresses*. Using the principal stresses (2.10) the maximum shear stress $\tau_{\max}$ can also be written as

$$\tau_{\max} = \pm\frac{1}{2}(\sigma_1 - \sigma_2)\,. \tag{2.12b}$$

The sense of direction of the maximum shear stress can be found by choosing the rotation angle of the ξ,η-system to be φ^{**}. Introducing φ^{**} into the third equation of (2.6) the shear stress $\tau_{\xi\eta} = \tau_{\max}$ is obtained including its correct sign.

Introducing φ^{**} into the first or second equation of (2.6) leads to a normal stress in the sections where the shear stress is maximum. We denote this stress as σ_M; it is given by

$$\sigma_M = \frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(\sigma_1 + \sigma_2)\,. \tag{2.13}$$

Therefore, the normal stresses generally do *not* vanish in the sections with extreme shear stresses. Fig. 2.6c shows the stresses in the respective sections.

E2.1

Example 2.1 The homogeneous state of plane stress in a metal sheet is given by $\sigma_x = -64$ MPa, $\sigma_y = 32$ MPa and $\tau_{xy} = -20$ MPa. Fig. 2.7a shows the stresses and their directions as they act in the sheet.

Determine

a) the stresses in a section which is inclined at an angle of 60° to the x-axis,
b) the principal stresses and principal directions,
c) the maximum shear stress and the associated directions of the sections.

Display the stresses at an element for each case.

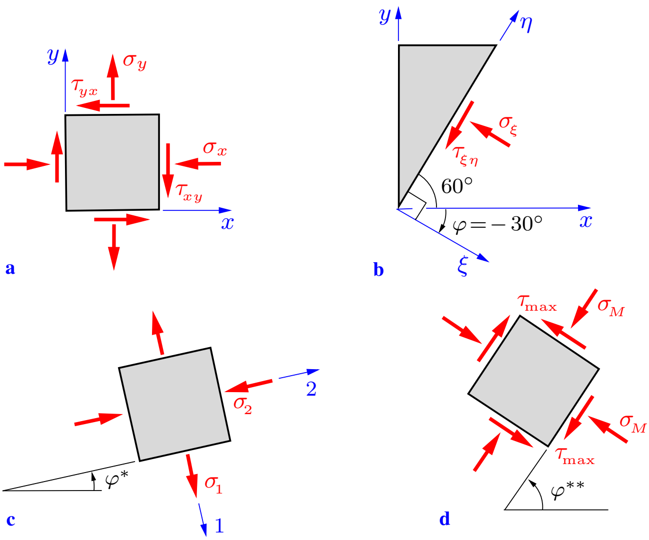

Fig. 2.7

Solution a) We cut the sheet in the prescribed direction. To characterize the section, a ξ,η-system is introduced whose ξ-axis is normal to the section (Fig. 2.7b, compare Fig. 2.5). Since it emanates from the x, y-system by a *clockwise* rotation of 30°, the rotation angle is negative: $\varphi = -30°$. Thus, from (2.6) we obtain the stresses

$$\underline{\underline{\sigma_\xi}} = \tfrac{1}{2}(-64+32) + \tfrac{1}{2}(-64-32)\cos(-60°) - 20\sin(-60°) = \underline{\underline{-22.7\ \text{MPa}}}\,,$$

$$\underline{\underline{\tau_{\xi\eta}}} = -\tfrac{1}{2}(-64-32)\sin(-60°) - 20\cos(-60°) = \underline{\underline{-51.6\ \text{MPa}}}\,.$$

Both stresses are negative. They are directed as shown in Fig. 2.7b.
b) The principal stresses are calculated by applying (2.10):

$$\sigma_{1,2} = \frac{-64+32}{2} \pm \sqrt{\left(\frac{-64-32}{2}\right)^2 + (-20)^2}$$

$$\rightarrow \quad \underline{\underline{\sigma_1 = 36\ \text{MPa}}}\,, \qquad \underline{\underline{\sigma_2 = -68\ \text{MPa}}}\,. \qquad \text{(a)}$$

One of the associated principal directions follows from (2.8):

$$\tan 2\varphi^* = \frac{2(-20)}{-64-32} = 0.417 \quad \rightarrow \quad \underline{\underline{\varphi^* = 11.3°}}\,.$$

To decide which principal stress is associated with this principal direction, we introduce the angle φ^* into the first equation of (2.6) and obtain

$$\sigma_\xi(\varphi^*) = \tfrac{1}{2}(-64+32) + \tfrac{1}{2}(-64-32)\cos(22.6°) - 20\sin(22.6°) = -68\ \text{MPa} = \sigma_2\,.$$

Accordingly, the principal stress σ_2 is associated with the angle φ^*. The principal stress σ_1 acts in a section perpendicular to it (Fig. 2.7c).
c) The maximum shear stresses are determined with (a) from (2.12b):

$$\underline{\underline{\tau_{\max}}} = \pm\frac{1}{2}(36+68) = \underline{\underline{\pm 52\ \text{MPa}}}\,.$$

The associated directions of the sections are rotated by 45° with respect to the principal directions. Hence, we get

$$\underline{\underline{\varphi^{**} = 56.3^\circ\,.}}$$

The direction of $\tau_{\max}$ follows after inserting φ^{**} into (2.6) from the positive sign of $\tau_{\xi\eta}(\varphi^{**})$. The associated normal stresses are given according to (2.13) by

$$\sigma_M = \frac{1}{2}(-64 + 32) = -16 \text{ MPa}\,.$$

In Fig. 2.7d the stresses are displayed with their true directions.

2.2.3 Mohr's Circle

Using the transformation relations (2.6), the stresses σ_ξ, σ_η and $\tau_{\xi\eta}$ for a ξ,η-system can be calculated from the stresses σ_x, σ_y and τ_{xy}. These relations also allow a simple and useful geometric representation. For this purpose, in a first step, the relations (2.6) for σ_ξ and $\tau_{\xi\eta}$ are rewritten:

$$\begin{aligned} \sigma_\xi - \frac{1}{2}(\sigma_x + \sigma_y) &= \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\,\varphi + \tau_{xy}\sin 2\,\varphi\,, \\ \tau_{\xi\eta} &= -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\,\varphi + \tau_{xy}\cos 2\,\varphi\,. \end{aligned} \tag{2.14}$$

By squaring and adding, the angle φ can be eliminated:

$$\left[\sigma_\xi - \frac{1}{2}(\sigma_x + \sigma_y)\right]^2 + \tau_{\xi\eta}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\,. \tag{2.15}$$

If we use in (2.14) the corresponding equation for σ_η instead of the equation for σ_ξ, we find that in (2.15) σ_ξ will be replaced by σ_η. In what follows we therefore omit the subscripts ξ and η.

For given stresses σ_x, σ_y and τ_{xy} the right-hand side of (2.15) is a fixed value which we abbreviate with r^2:

$$r^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\,. \tag{2.16}$$

With $\sigma_M = \frac{1}{2}(\sigma_x + \sigma_y)$ and (2.16) Equation (2.15) then takes the form

$$(\sigma - \sigma_M)^2 + \tau^2 = r^2 \,. \tag{2.17}$$

This is the equation of a circle in the σ, τ-plane: the points (σ, τ) lie on the *stress circle* , also called *Mohr's circle* (Otto Mohr, 1835–1918). It is centered at $(\sigma_M, 0)$ and has the radius r (Fig. 2.8a).

Equation (2.16) can be rewritten as

$$r^2 = \frac{1}{4}\left[(\sigma_x + \sigma_y)^2 - 4(\sigma_x\sigma_y - \tau_{xy}^2)\right] \,.$$

Note that this equation for the radius coincides with the absolute value of the maximum shear stress given by (2.12a). That is, the radius of Mohr's circle graphically indicates the maximum shear stress at a point. Moreover, since the expressions in the round brackets are invariant (cf. Section 2.2.1), r is also an invariant.

The stress circle in the σ, τ-plane can be constructed directly if the stresses σ_x, σ_y and τ_{xy} are known, thereby avoiding the need to calculate σ_M and r. For this purpose, the stresses σ_x and σ_y, including their signs, are marked on the σ-axis. At these points the shear stress τ_{xy} is plotted according to the following rule: with the correct sign at σ_x and with the reversed sign at σ_y. This determines two points of the circle, P and P' (Fig. 2.8a). The intersection of their connecting line with the σ-axis yields the center of the circle. The circle now can be drawn using this point as its center and extending to pass through P and P'.

The stress state at a point of a disk is fully described by Mohr's circle; each section is represented by a point on the circle. For example, point P corresponds to the section where the stresses σ_x and τ_{xy} act while point P' represents the section perpendicular to the former one. The stresses in arbitrary sections as well as the extreme stresses and associated directions can be determined from the stress circle. In particular, the principal stresses σ_1, σ_2 and the maximum shear stress $\tau_{\max}$ can be directly identified (Fig. 2.8b).

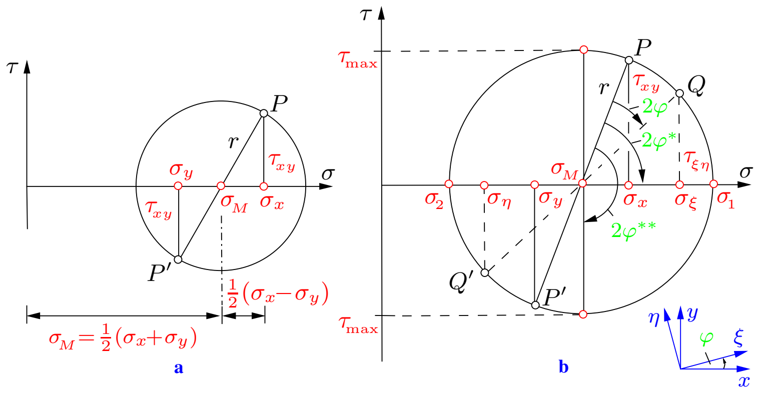

Fig. 2.8

We will now show that the stresses σ_ξ, σ_η and $\tau_{\xi\eta}$ in a ξ,η-system which is rotated with respect to the x,y-system by an angle φ (positive *counterclockwise*) are identified on Mohr's circle as follows: point Q, corresponding to a section with the stresses σ_ξ and $\tau_{\xi\eta}$ is found by plotting the *doubled* angle – i.e. 2φ – in the *reversed* sense of rotation (Fig. 2.8b); point Q' corresponding to a section perpendicular to the first one lies opposite to Q. The principal directions and the directions of maximum shear stress finally are given by the angles φ^* and φ^{**}.

To proof these statements we first find from Fig. 2.8a,b:

$$\tan 2\,\varphi^* = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y}\,,$$

$$\frac{1}{2}\,(\sigma_x - \sigma_y) = r\,\cos 2\varphi^*\,, \quad \tau_{xy} = r\,\sin 2\varphi^*\,.$$

Introducing these equations into the transformation relations (2.6) for σ_ξ and σ_η yields

$$\begin{aligned}\sigma_\xi &= \frac{1}{2}(\sigma_x + \sigma_y) + r\cos 2\varphi^*\cos 2\varphi + r\sin 2\varphi^*\sin 2\varphi \\ &= \frac{1}{2}(\sigma_x + \sigma_y) + r\cos\left(2\varphi^* - 2\varphi\right),\end{aligned}$$

$$\tau_{\xi\eta} = -\,r\cos 2\varphi^*\sin 2\varphi + r\sin 2\varphi^*\cos 2\varphi = r\sin\left(2\varphi^* - 2\varphi\right).$$

The same result follows from geometric relations in Fig. 2.8b, i.e. Mohr's circle is nothing other than the geometric representation of the transformation relations.

If Mohr's circle is used for the solution of specific problems, three quantities must be known (e.g. σ_x, τ_{xy}, σ_1) in order to draw the circle. In graphical solutions an appropriate scale for the stresses must be chosen.

In the following we finally consider three special cases. *Uniaxial tension* (Fig. 2.9a) is characterized by $\sigma_x = \sigma_0 > 0$, $\sigma_y = 0$, $\tau_{xy} = 0$. Since the shear stress is zero in the respective sections, the stresses $\sigma_1 = \sigma_x = \sigma_0$ and $\sigma_2 = \sigma_y = 0$ are the principal stresses. Mohr's circle lies just to the right of the τ-axis so that this vertical axis is its tangent. The maximum shear stress $\tau_{\max} = \sigma_0/2$ acts in sections rotated 45° with respect to the x-axis (see also Section 1.1).

The stress state characterized by $\sigma_x = 0$, $\sigma_y = 0$ and $\tau_{xy} = \tau_0$ is called *pure shear*. On account of $\sigma_M = 0$, the center of Mohr's

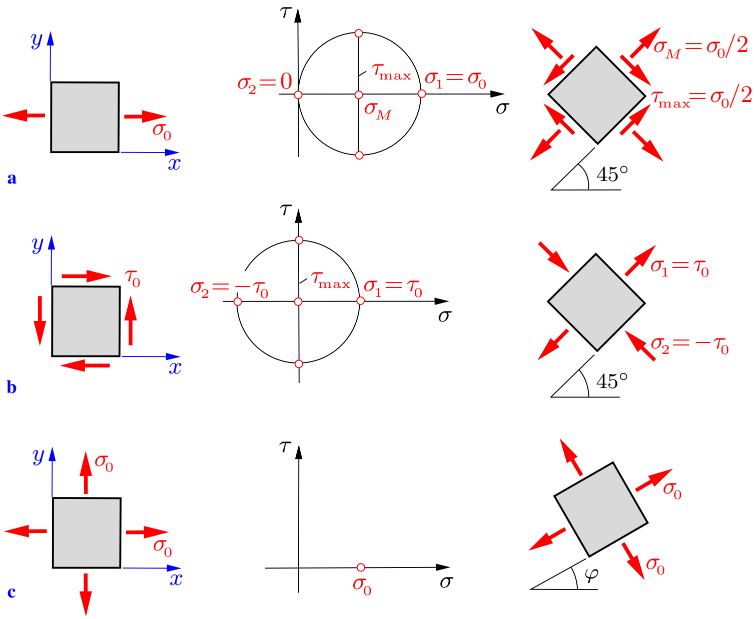

Fig. 2.9

circle in this case coincides with the origin of the coordinate system (Fig. 2.9b). The principal stresses are $\sigma_1 = \tau_0$ and $\sigma_2 = -\tau_0$; they act in sections at 45° with respect to the x-axis.

In the case of a *hydrostatic stress state* the stresses are $\sigma_x = \sigma_y = \sigma_0$ and $\tau_{xy} = 0$. Mohr's circle then is reduced to a single point on the σ-axis (Fig. 2.9c). The normal stresses for all section directions have the same value $\sigma_\xi = \sigma_\eta = \sigma_0$ and no shear stresses appear (cf. Section 2.2.1).

E2.2 **Example 2.2** A plane stress state is given by $\sigma_x = 50$ MPa, $\sigma_y = -20$ MPa and $\tau_{xy} = 30$ MPa.

Using Mohr's circle, determine

a) the principal stresses and principal directions,

b) the normal and shear stress acting in a section whose normal forms the angle $\varphi = 30°$ with the x-axis.

Display the results in sketches of the sections.

Solution a) After having chosen a scale, Mohr's circle can be constructed from the given stresses (in Fig. 2.10a the given stresses are marked by green circles). From the circle, the principal stresses and directions can be directly identified:

$$\underline{\underline{\sigma_1 = 61 \text{ MPa}}}\,, \qquad \underline{\underline{\sigma_2 = -31 \text{ MPa}}}\,, \qquad \underline{\underline{\varphi^* = 20°}}\,.$$

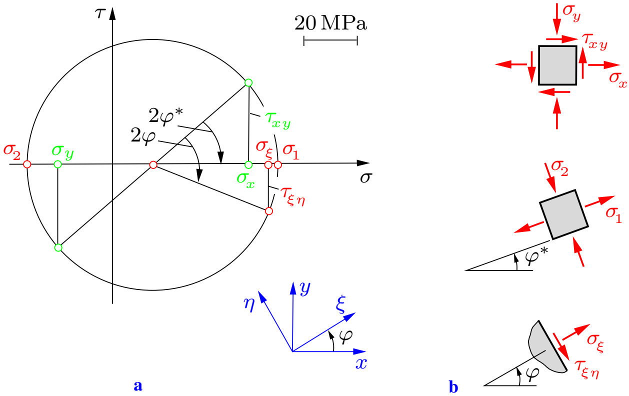

Fig. 2.10

b) To determine the stresses in the inclined section we introduce a ξ, η-coordinate system whose ξ-axis coincides with the normal of the section. The unknown stresses σ_ξ and $\tau_{\xi\eta}$ are obtained by plotting in Mohr's circle the angle 2φ in the reversed direction to φ. Doing so we obtain:

$$\underline{\underline{\sigma_\xi = 58.5 \text{ MPa}}}\,, \qquad \underline{\underline{\tau_{\xi\eta} = -15.5 \text{ MPa}}}\,.$$

The stresses with their true directions and the associated sections are displayed in Fig. 2.10b.

Example 2.3 The two principal stresses $\sigma_1 = 40$ MPa and $\sigma_2 = -20$ MPa of a plane stress state are known. **E2.3**

Determine the orientation of a x, y-coordinate system with respect to the principal axes for which $\sigma_x = 0$ and $\tau_{xy} > 0$. Calculate the stresses σ_y and τ_{xy}.

Solution Using the given principal stresses σ_1 and σ_2, the properly scaled Mohr's circle can be drawn (Fig. 2.11a). From the circle the orientation of the unknown x, y-system can be obtained: the counterclockwise angle 2φ (from point σ_1 to point P) in Mohr's circle corresponds to the clockwise angle φ between the 1-axis and the x-axis. The angle and the stresses are found as

$$2\,\varphi = 110° \quad \rightarrow \quad \underline{\underline{\varphi = 55°}}\,, \quad \underline{\underline{\sigma_y = 20 \text{ MPa}}}\,, \quad \underline{\underline{\tau_{xy} = 28 \text{ MPa}}}\,.$$

The stresses and the coordinate systems are shown in Fig. 2.11b.

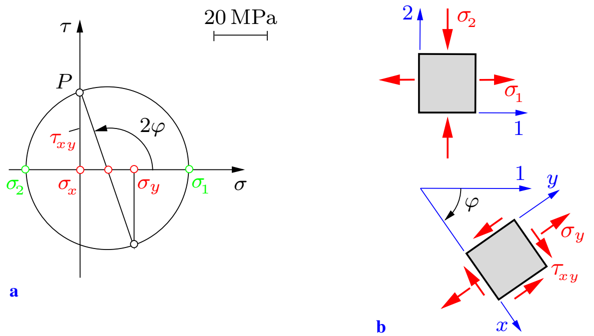

Fig. 2.11

2.2.4 The Thin-Walled Pressure Vessel

As an important application of plane stress we first consider a *thin-walled* cylindrical vessel with radius r and wall thickness $t \ll r$ (Fig. 2.12a). The vessel is subjected to an internal gage pressure p that causes stresses in its wall which need to be determined (Fig. 2.12b).

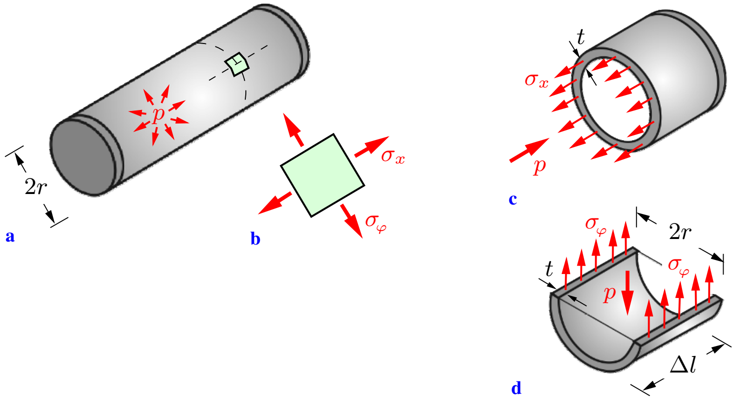

Fig. 2.12

At a sufficient distance from the end caps of the vessel, the stress state is independent of the location (homogeneous stress state). Given that $t \ll r$, the stresses in radial directions can be neglected. Thus, within a good approximation a plane stress state acts locally in the wall of the vessel (note: although the element in Fig. 2.12b is curved, it is replaced by a plane element in the tangent plane). The stress state can be described by the stresses in two sections perpendicular to each other.

First, the vessel is cut perpendicularly to its longitudinal axis (Fig. 2.12c). Since the gas or fluid pressure is independent of the location, the pressure on the section area πr^2 (of the gas or fluid) has the constant value p. Assuming that the *longitudinal stress* σ_x is constant across the wall thickness because of $t \ll r$, the equilibrium condition yields (Fig. 2.12c)

$$\sigma_x 2\,\pi\,r\,t - p\,\pi\,r^2 = 0 \qquad \to \qquad \boxed{\sigma_x = \frac{1}{2}\,p\,\frac{r}{t}\,.} \tag{2.18}$$

As illustrated in Fig. 2.12d we now separate a half-circular part of length Δl from the vessel. The horizontal sections of the wall are subjected to the *circumferential stress* σ_φ, also called *hoop stress*, which again is constant across the thickness. These stresses will counteract the force $p\,2\,r\Delta l$, exerted from the gas onto the half-circular part of the vessel. Equilibrium in the vertical direction yields

$$2\,\sigma_\varphi\,t\,\Delta\,l - p\,2\,r\Delta l = 0 \qquad \to \qquad \boxed{\sigma_\varphi = p\,\frac{r}{t}\,.} \tag{2.19}$$

We notice that the hoop stress is twice the longitudinal stress. This is why a cylindrical vessel under internal pressure usually fails by cracking in the longitudinal direction. A simple example is an overcooked hot dog which splits in the longitudinal direction first.

The two equations (2.18) and (2.19) for σ_x and σ_φ sometimes are called *vessel formulas*. Because of $t \ll r$ it can be seen that σ_x, $\sigma_\varphi \gg p$. Therefore, the initially made assumption that the stresses σ_r in radial direction may be neglected is justified ($|\,\sigma_r\,| \leq p$). Generally, a vessel may be called *thin-walled* when it fulfills the condition $r > 5\,t$.

The vessel formulas are also applicable to a vessel subjected to external pressure. In this case only the sign of p has to be changed, i.e. the wall is then under a compressive stress state.

Since no shear stresses are present in both sections (symmetry), the stresses σ_x and σ_φ are principal stresses: $\sigma_1 = \sigma_\varphi = p\ r/t$, $\sigma_2 = \sigma_x = p\ r/(2t)$. According to (2.12b) the maximum shear stress is given by

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{4}\,p\,\frac{r}{t}\,;$$

it acts in sections inclined under 45°. It should be noted that in the vicinity of the end caps more complex stress states are present which cannot be determined with an elementary theory.

Now we consider a thin-walled spherical vessel of radius r, subjected to a gage pressure p (Fig. 2.13a). Here, the stresses σ_t and σ_φ act in the wall (Fig. 2.13b). When we cut the vessel into half (Fig. 2.13c), we obtain σ_t from the equilibrium condition:

$$\sigma_t\, 2\,\pi\, r\, t - p\,\pi\, r^2 = 0 \quad \rightarrow \quad \sigma_t = \frac{1}{2}\, p\, \frac{r}{t}\,.$$

A cut, perpendicular to the first one, similarly leads to

$$\sigma_\varphi\, 2\,\pi\, r\, t - p\,\pi\, r^2 = 0 \quad \rightarrow \quad \sigma_\varphi = \frac{1}{2}\, p\, \frac{r}{t}\,.$$

Thus,

$$\sigma_t = \sigma_\varphi = \frac{1}{2}\, p\, \frac{r}{t}\,. \tag{2.20}$$

Therefore, the stress in the wall of a thin-walled spherical vessel has the value $p\, r/(2\, t)$ in any arbitrary direction. As in the foregoing case, this formula is also valid for an external pressure in which case p is negative.

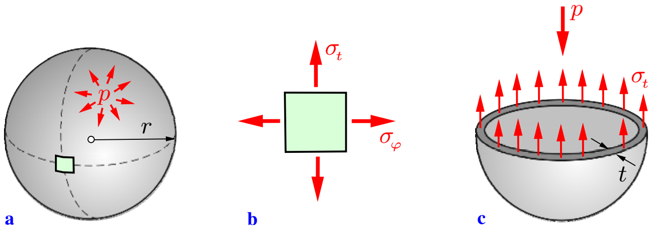

Fig. 2.13

2.3

2.3 Equilibrium Conditions

According to Section 2.1 the stress state at a material point of a body is determined by the stress tensor; its components are shown in Fig. 2.2a. In general, these components vary from point to point and these variations are not independent of each other: they are connected via the *equilibrium conditions*.

To derive the equilibrium conditions we first consider in Fig. 2.14 the stresses acting on an infinitesimal element under plane stress which is cut out from a disk of thickness t. Since the stresses in general depend on x and y, they are not the same at the opposite sections: they differ by infinitesimal increments. For example, the left face is subjected to the normal stress σ_x whereas the stress $\sigma_x + \dfrac{\partial \sigma_x}{\partial x}\,\mathrm{d}x$ (first terms of the Taylor-expansion, see e.g. Section 3.1) acts on the right face. The symbol $\partial/\partial x$ denotes the partial derivative with respect to x. Furthermore, the element may be loaded by the volume force $\boldsymbol{f}$ with the components f_x and f_y.

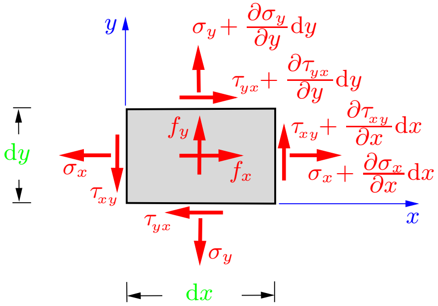

Fig. 2.14

The equilibrium condition in x-direction yields

$$-\sigma_x\,\mathrm{d}y\,t - \tau_{yx}\,\mathrm{d}x\,t + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\,\mathrm{d}x\right)\mathrm{d}y\,t + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\,\mathrm{d}y\right)\mathrm{d}x\,t + f_x\,\mathrm{d}x\;\mathrm{d}y\,t = 0$$

i.e., after division by $\mathrm{d}x\mathrm{d}y$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0\,. \qquad (2.21\text{a})$$

Similarly, from the equilibrium condition in y-direction we obtain

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0\,. \qquad (2.21\text{b})$$

Equations (2.21a, b) are called *equilibrium conditions*. In the considered case of plane stress, they consist of *two* coupled partial differential equations for the *three* components σ_x, σ_y and $\tau_{xy} = \tau_{yx}$ of the stress tensor. The stress state cannot be uniquely determined from these equations: the problem is statically indeterminate.

For a *spatial* (three dimensional) stress state the corresponding equilibrium conditions are obtained as

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x &= 0\,,\\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y &= 0\,,\\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z &= 0\,.
\end{aligned} \tag{2.22}$$

These are three coupled partial differential equations for the six components of the stress tensor.

The components of the stress tensor are constant in a *homogeneous* stress state . In this case all partial derivatives in (2.21a, b) and (2.22), respectively, vanish. The equilibrium conditions are then only fulfilled if $f_x = f_y = f_z = 0$. Thus, a homogeneous stress state under the action of volume forces is not possible.

It should be mentioned that from the equilibrium of moments, applied to the element, the symmetry of the stress tensor follows even when the stress increments are taken into account (cf. Section 2.1).

2.4 Supplementary Examples

2.4

Detailed solutions to the following examples are given in (**A**) D. Gross et al. *Mechanics of Materials - Formulas and Problems*, Springer 2017 or (**B**) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer 2017.

E2.4

Example 2.4 The stresses $\sigma_x = 20$ MPa, $\sigma_y = 30$ MPa and $\tau_{xy} = 10$ MPa in a metal sheet are known (Fig. 2.15).

Determine the principal stresses and their directions.

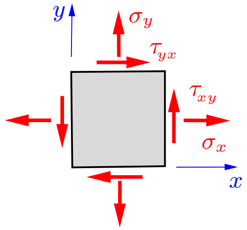

Fig. 2.15

Results: see (**A**) $\sigma_1 = 36.2$ MPa, $\sigma_2 = 13.8$ MPa,

$$\varphi_1^* = 58.3°, \quad \varphi_2^* = 148.3°.$$

E2.5

Example 2.5 A plane stress state is given by the principal stresses $\sigma_1 = 30$ MPa and $\sigma_2 = -10$ MPa (Fig. 2.16).

a) Determine the stress components in a ξ, η-coordinate system which is inclined by 45° with respect to the principal axes.

b) Using Mohr's circle, determine the rotation angle α of an x, y-coordinate system where $\sigma_y = 0$ and $\tau_{xy} < 0$. Calculate σ_x and τ_{xy}.

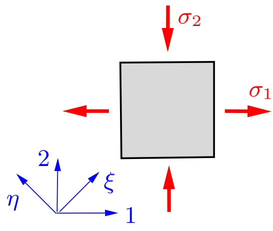

Fig. 2.16

Results: see (**B**)

a) $\sigma_\xi = 10$ MPa, $\sigma_\eta = 10$ MPa, $\tau_{\xi\eta} = -20$ MPa.

b) $\alpha = 30°$, $\sigma_x = 20$ MPa, $\tau_{xy} = -17.3$ MPa.

E2.6

Example 2.6 A thin-walled tube is subjected to bending and torsion such that the following stresses act at points A and B:

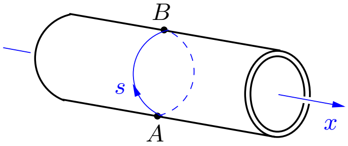

Fig. 2.17

$$\sigma_x^{A,B} = \pm 25\,\text{MPa}, \quad \sigma_s^{A,B} = 50\,\text{MPa}, \quad \tau_{xs}^{A,B} = 50\,\text{MPa}\,.$$

Determine the principal stresses and their directions at A and B.

Results: see (**A**) Point A

$\sigma_1 = 89.0$ MPa, $\sigma_2 = -14.0$ MPa, $\varphi_1^* = 52.0°$, $\varphi_2^* = -38.0°$.

Point B

$\sigma_1 = 75.0$ MPa, $\sigma_2 = -50.0$ MPa, $\varphi_1^* = 63.4°$, $\varphi_2^* = -26.6°$.

E2.7

Example 2.7 A slender bar (weight W, cross sectional area A) is suspended from the ceiling (Fig. 2.18).

Given: $\rho = 10^4$ kg/m^3, $g = 10$ m/s^2, $l = 1$ m.

Using Mohr's circle, determine the normal stress and the shear stress acting in the section characterized by the angle $\varphi = 20°$ as shown in the figure.

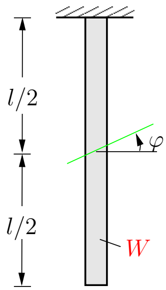

Fig. 2.18

Results: $\sigma_\xi = 4.4 \cdot 10^2$ MPa, $\tau_{\xi\eta} = -1.6 \cdot 10^2$ MPa.
(ξ upwards normal to the section, η downwards tangential to the section)

E2.8

Example 2.8 A thin-walled bathysphere (radius $r = 500$ mm, wall-thickness $t = 12.5$ mm) is lowered to a depth of 500 m under the water surface (pressure $p = 5$ MPa).

Determine the stresses in the wall.

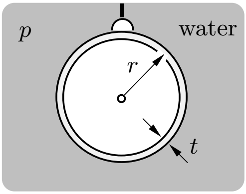

Fig. 2.19

Result: see (**A**) $\sigma_t = -100$ MPa (in any section).

E2.9

Example 2.9 A thin-walled cylindrical vessel has the radius $r = 1$ m and wall-thickness $t = 10$ mm.

Determine the maximum internal pressure $p_{\max}$ so that the maximum stress in the wall does not exceed the allowable stress $\sigma_{\text{allow}} = 150$ MPa.

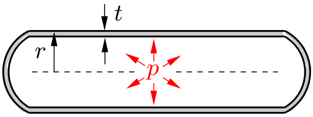

Fig. 2.20

Result: see (**A**) $p_{\max} = 1.5$ MPa.

2.5

2.5 Summary

- The stress state at a point of a body is determined by the stress tensor $\boldsymbol{\sigma}$. In the spatial case it has 3×3 components (note the symmetry). In the plane stress state it reduces to

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} \quad \text{where} \quad \tau_{xy} = \tau_{yx} \, .$$

- Sign convention: positive stresses at a positive (negative) face point in positive (negative) directions of the coordinates.
- Transformation relations (plane stress):

$$\begin{aligned} \sigma_\xi &= \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi + \tau_{xy}\sin 2\varphi \, , \\ \sigma_\eta &= \tfrac{1}{2}(\sigma_x + \sigma_y) - \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi - \tau_{xy}\sin 2\varphi \, , \\ \tau_{\xi\eta} &= -\tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\varphi + \tau_{xy}\cos 2\varphi \, . \end{aligned}$$

  The axes ξ, η are rotated with respect to x, y by the angle φ.
- Principal stresses and directions (plane stress):

$$\begin{aligned} \sigma_{1,2} &= \tfrac{1}{2}(\sigma_x + \sigma_y) \pm \sqrt{\tfrac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2} \, , \\ \tan 2\varphi^* &= \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \rightarrow \quad \varphi_1^*, \ \varphi_2^* = \varphi_1^* \pm \pi/2 \, . \end{aligned}$$

  Principal stresses are extreme stresses; the shear stresses vanish in the corresponding sections.
- Maximum shear stresses and their directions (plane stress):

$$\tau_{\max} = \sqrt{\tfrac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2} \, , \qquad \varphi^{**} = \varphi^* \pm \pi/4 \, .$$

- Mohr's circle allows the geometric representation of the coordinate transformation.
- Equilibrium conditions for the stresses (plane stress):

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0 \, , \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0 \, .$$

  In the spatial case there are three equilibrium conditions.