Chapter 1

1

Tension and Compression in Bars

1 Tension and Compression in Bars

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Objectives: In this textbook about the *Mechanics of Materials* we investigate the stressing and the deformations of elastic structures subjected to applied loads. In the first chapter we will restrict ourselves to the simplest structural members, namely, bars under tension or compression.

In order to treat such problems, we need kinematic relations and a constitutive law to complement the equilibrium conditions which are known from Volume 1. The kinematic relations represent the geometry of the deformation, whereas the behaviour of the elastic material is described by the constitutive law. The students will learn how to apply these equations and how to solve statically determinate as well as statically indeterminate problems.

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1.1 Stress

Let us consider a straight bar with a constant cross-sectional area A. The line connecting the centroids of the cross sections is called the *axis* of the bar. The ends of the bar are subjected to the forces F whose common line of action is the axis (Fig. 1.1a).

The *external* load causes *internal* forces. The internal forces can be visualized by an imaginary cut of the bar (compare Volume 1, Section 1.4). They are distributed over the cross section (see Fig. 1.1b) and are called *stresses*. Being area forces, they have the dimension force per area and are measured, for example, as multiples of the unit MPa (1 MPa = 1 N/mm^2). The unit "Pascal" (1 Pa = 1 N/m^2) is named after the mathematician and physicist Blaise Pascal (1623–1662); the notion of "stress" was introduced by Augustin Louis Cauchy (1789–1857). In Volume 1 (Statics) we only dealt with the resultant of the internal forces (= normal force) whereas now we have to study the internal forces (= stresses).

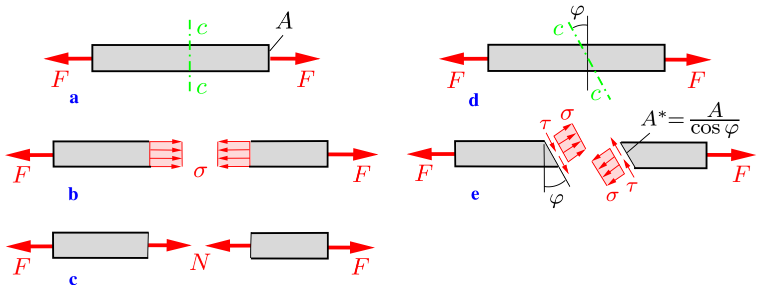

Fig. 1.1

In order to determine the stresses we first choose an imaginary cut $c - c$ perpendicular to the axis of the bar. The stresses are shown in the free-body diagram (Fig. 1.1b); they are denoted by σ. We assume that they act perpendicularly to the exposed surface A of the cross section and that they are uniformly distributed. Since they are normal to the cross section they are called *normal stresses*. Their resultant is the normal force N shown in Fig. 1.1c (compare Volume 1, Section 7.1). Therefore we have $N = \sigma A$ and the stresses σ can be calculated from the normal force N:

$$\sigma = \frac{N}{A} \,. \tag{1.1}$$

In the present example the normal force N is equal to the applied force F. Thus, we obtain from (1.1)

$$\sigma = \frac{F}{A} \,. \tag{1.2}$$

In the case of a positive normal force N (tension) the stress σ is then positive (tensile stress). Reversely, if the normal force is negative (compression) the stress is also negative (compressive stress).

Let us now imagine the bar being sectioned by a cut which is not orthogonal to the axis of the bar so that its direction is given by the angle φ (Fig. 1.1d). The internal forces now act on the exposed surface $A^* = A/\cos\varphi$. Again we assume that they are uniformly distributed. We resolve the stresses into a component σ perpendicular to the surface (the normal stress) and a component τ tangential to the surface (Fig. 1.1e). The component τ which acts *in* the direction of the surface is called *shear stress*.

Equilibrium of the forces acting on the left portion of the bar yields (see Fig. 1.1e)

$$\rightarrow: \qquad \sigma A^* \cos\varphi + \tau A^* \sin\varphi - F = 0 \,,$$

$$\uparrow: \qquad \sigma A^* \sin\varphi - \tau A^* \cos\varphi = 0 \,.$$

Note that we have to write down the equilibrium conditions for the *forces*, *not* for the *stresses*. With $A^* = A/\cos\varphi$ we obtain

$$\sigma + \tau \tan\varphi = \frac{F}{A} \,, \qquad \sigma \tan\varphi - \tau = 0 \,.$$

Solving these two equations for σ and τ yields

$$\sigma = \frac{1}{1+\tan^2\varphi}\frac{F}{A} \,, \qquad \tau = \frac{\tan\varphi}{1+\tan^2\varphi}\frac{F}{A} \,.$$

It is practical to write these equations in a different form. Using the standard trigonometric relations

$$\frac{1}{1+\tan^2\varphi} = \cos^2\varphi\,, \qquad \cos^2\varphi = \frac{1}{2}(1+\cos 2\,\varphi)\,,$$
$$\sin\varphi\cos\varphi = \frac{1}{2}\sin 2\,\varphi$$

and the abbreviation $\sigma_0 = F/A$ (= normal stress in a section perpendicular to the axis) we finally get

$$\sigma = \frac{\sigma_0}{2}(1+\cos 2\,\varphi)\,, \qquad \tau = \frac{\sigma_0}{2}\sin 2\,\varphi\,. \tag{1.3}$$

Thus, the stresses depend on the direction of the cut. If σ_0 is known, the stresses σ and τ can be calculated from (1.3) for arbitrary values of the angle φ. The maximum value of σ is obtained for $\varphi = 0$, in which case $\sigma_{max} = \sigma_0$; the maximum value of τ is found for $\varphi = \pi/4$ for which $\tau_{max} = \sigma_0/2$.

If we section a bar near an end which is subjected to a concentrated force F (Fig. 1.2a, section $c - c$) we find that the normal stress is not distributed uniformly over the cross-sectional area. The concentrated force produces high stresses near its point of application (Fig. 1.2b). This phenomenon is known as *stress concentration*. It can be shown, however, that the stress concentration is restricted to sections in the proximity of the point of application of the concentrated force: the high stresses decay rapidly towards the average value σ_0 as we increase the distance from the end of the bar. This fact is referred to as *Saint-Venant's principle* (Adhémar Jean Claude Barré de Saint-Venant, 1797–1886).

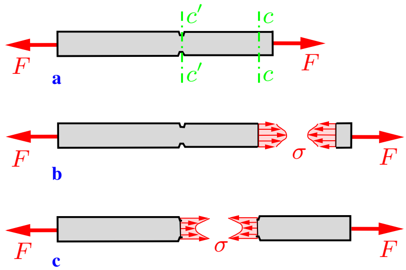

Fig. 1.2

The uniform distribution of the stress is also disturbed by holes, notches or any abrupt changes (discontinuities) of the geometry. If, for example, a bar has notches the remaining cross-sectional area (section $c' - c'$) is also subjected to a stress concentration (Fig. 1.2c). The determination of these stresses is not possible with the elementary analysis presented in this textbook.

Let us now consider a bar with only a *slight* taper (compare Example 1.1). In this case the normal stress may be calculated from (1.1) with a sufficient accuracy. Then the cross-sectional area A and the stress σ depend on the location along the axis. If volume forces act in the direction of the axis in addition to the concentrated forces, then the normal force N also depends on the location. Introducing the coordinate x in the direction of the axis we can write:

$$\sigma(x) = \frac{N(x)}{A(x)} . \qquad (1.4)$$

Here it is also assumed that the stress is uniformly distributed over the cross section at a fixed value of x.

In statically determinate systems we can determine the normal force N from equilibrium conditions alone. If the cross-sectional area A is known, the stress σ can be calculated from (1.4). Statically indeterminate systems will be treated in Section 1.4.

In engineering applications structures have to be designed in such a way that a given maximum stressing is not exceeded. In the case of a bar this requirement means that the absolute value of the stress σ must not exceed a given *allowable stress* σ_{allow} : $|\sigma| \le \sigma_{\text{allow}}$. (Note that the allowable stresses for tension and for compression are different for some materials.) The required cross section A_{req} of a bar for a given load and thus a known normal force N can then be determined from $\sigma = N/A$:

$$A_{\text{req}} = \frac{|N|}{\sigma_{\text{allow}}} . \qquad (1.5)$$

This is referred to as *dimensioning* of the bar. Alternatively, the allowable load can be calculated from $|N| \le \sigma_{\text{allow}} A$ in the case of a given cross-sectional area A.

Note that a slender bar which is subjected to compression may fail due to buckling before the stress attains an inadmissibly large value. We will investigate buckling problems in Chapter 7.

Example 1.1 A bar (length l) with a circular cross section and a slight taper (linearly varying from radius r_0 to $2\,r_0$) is subjected to the compressive forces F as shown in Fig. 1.3a. **E1.1**

Determine the normal stress σ in an arbitrary cross section perpendicular to the axis of the bar.

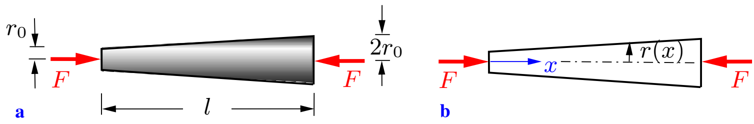

Fig. 1.3

Solution We introduce the coordinate x, see Fig. 1.3b. Then the radius of an arbitrary cross section is given by

$$r(x) = r_0 + \frac{r_0}{l}\,x = r_0\left(1 + \frac{x}{l}\right).$$

Using (1.4) with the cross section $A(x) = \pi\, r^2(x)$ and the constant normal force $N = -F$ yields

$$\underline{\underline{\sigma}} = \frac{N}{A(x)} = \underline{\underline{\frac{-F}{\pi r_0^2\left(1 + \frac{x}{l}\right)^2}}}.$$

The minus sign indicates that σ is a compressive stress. Its value at the left end ($x = 0$) is four times the value at the right end ($x = l$).

Example 1.2 A water tower (height H, density ϱ) with a cross section in the form of a circular ring carries a tank (weight W_0) as shown in Fig. 1.4a. The inner radius r_i of the ring is constant. **E1.2**

Determine the outer radius r in such a way that the normal stress σ_0 in the tower is constant along its height. The weight of the tower cannot be neglected.

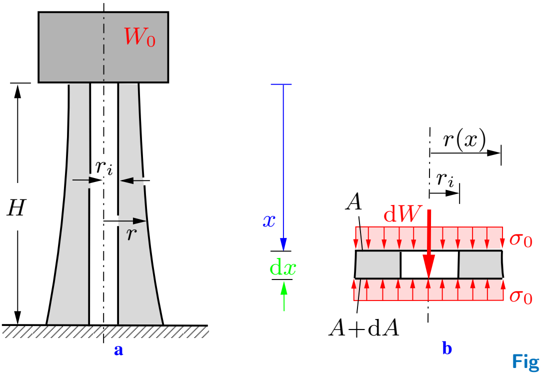

Fig. 1.4

Solution We consider the tower to be a slender bar. The relationship between stress, normal force and cross-sectional area is given by (1.4). In this example the constant compressive stress $\sigma = \sigma_0$ is given; the normal force (here counted positive as compressive force) and the area A are unknown.

The equilibrium condition furnishes a second equation. We introduce the coordinate x as shown in Fig. 1.4b and consider a slice element of length $\mathrm{d}x$. The cross-sectional area of the circular ring as a function of x is

$$A = \pi(r^2 - r_i^2) \tag{a}$$

where $r = r(x)$ is the unknown outer radius. The normal force at the location x is given by $N = \sigma_0 A$ (see 1.4). At the location $x + \mathrm{d}x$, the area and the normal force are $A + \mathrm{d}A$ and $N + \mathrm{d}N = \sigma_0(A + \mathrm{d}A)$.

The weight of the element is $\mathrm{d}W = \varrho\, g\, \mathrm{d}V$ where $\mathrm{d}V = A\,\mathrm{d}x$ is the volume of the element. Note that terms of higher order are neglected (compare Volume 1, Section 7.2.2). Equilibrium in the vertical direction yields

$$\uparrow: \quad \sigma_0(A + \mathrm{d}A) - \varrho\, g\, \mathrm{d}V - \sigma_0 A = 0 \quad \rightarrow \quad \sigma_0\, \mathrm{d}A - \varrho\, g\, A\, \mathrm{d}x = 0\,.$$

Separation of variables and integration lead to

$$\int \frac{\mathrm{d}A}{A} = \int \frac{\varrho\, g}{\sigma_0}\, \mathrm{d}x \quad \rightarrow \quad \ln \frac{A}{A_0} = \frac{\varrho\, g\, x}{\sigma_0} \quad \rightarrow \quad A = A_0\, \mathrm{e}^{\frac{\varrho\, g\, x}{\sigma_0}}\,. \tag{b}$$

The constant of integration A_0 follows from the condition that the stress at the upper end of the tower (for $x = 0$ we have $N = W_0$) also has to be equal to σ_0:

$$\frac{W_0}{A_0} = \sigma_0 \qquad \rightarrow \qquad A_0 = \frac{W_0}{\sigma_0}\,. \tag{c}$$

Equations (a) to (c) yield the outer radius:

$$\underline{\underline{r^2(x) = r_i^2 + \frac{W_0}{\pi\,\sigma_0} e^{\frac{\varrho\, g\, x}{\sigma_0}}}}\,.$$

1.2 Strain

1.2

We will now investigate the deformations of an elastic bar. Let us first consider a bar with a constant cross-sectional area which has the undeformed length l. Under the action of tensile forces (Fig. 1.5) it gets slightly longer. The elongation is denoted by Δl and is assumed to be much smaller than the original length l. As a measure of the amount of deformation, it is useful to introduce, in addition to the elongation, the ratio between the elongation and the original (undeformed) length:

$$\varepsilon = \frac{\Delta l}{l}\,. \tag{1.6}$$

The dimensionless quantity ε is called *strain*. If, for example, a bar of the length $l = 1$ m undergoes an elongation of $\Delta l = 0.5$ mm then we have $\varepsilon = 0.5 \cdot 10^{-3}$. This is a strain of 0.05%. If the bar gets longer ($\Delta l > 0$) the strain is positive; it is negative in the case of a shortening of the bar. In what follows we will consider only small deformations: $|\Delta l| \ll l$ or $|\varepsilon| \ll 1$, respectively.

The definition (1.6) for the strain is valid only if ε is constant over the entire length of the bar. If the cross-sectional area is not

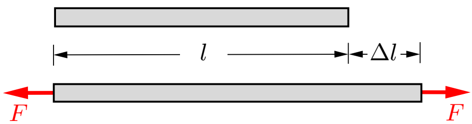

Fig. 1.5

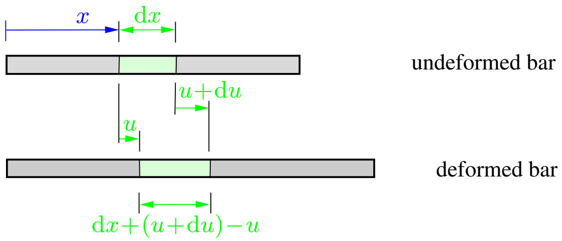

Fig. 1.6

constant or if the bar is subjected to volume forces acting along its axis, the strain may depend on the location. In this case we have to use a *local* strain which will be defined as follows. We consider an element of the bar (Fig. 1.6) instead of the whole bar. It has the length $\mathrm{d}x$ in the undeformed state. Its left end is located at x, the right end at $x + \mathrm{d}x$. If the bar is elongated, the cross sections undergo displacements in the x-direction which are denoted by u. They depend on the location: $u = u(x)$. Thus, the displacements are u at the left end of the element and $u + \mathrm{d}u$ at the right end. The length of the elongated element is $\mathrm{d}x+(u+\mathrm{d}u)-u = \mathrm{d}x+\mathrm{d}u$. Hence, the elongation of the element is given by $\mathrm{d}u$. Now the local strain can be defined as the ratio between the elongation and the undeformed length of the element:

$$\varepsilon(x) = \frac{\mathrm{d}u}{\mathrm{d}x}\,. \qquad (1.7)$$

If the displacement $u(x)$ is known, the strain $\varepsilon(x)$ can be determined through differentiation. Reversely, if $\varepsilon(x)$ is known, the displacement $u(x)$ is obtained through integration.

The displacement $u(x)$ and the strain $\varepsilon(x)$ describe the geometry of the deformation. Therefore they are called *kinematic quantities*. Equation (1.7) is referred to as a kinematic relation.

1.3

1.3 Constitutive Law

Stresses are quantities derived from statics; they are a measure for the stressing in the material of a structure. On the other hand, strains are kinematic quantities; they measure the deformation of a body. However, the deformation depends on the load which acts on the body. Therefore, the stresses and the strains are not independent. The physical relation that connects these quantities is called *constitutive law*. It describes the behaviour of the material of the body under a load. It depends on the material and can be obtained only with the aid of experiments.

One of the most important experiments to find the relationship between stress and strain is the tension or compression test. Here, a small specimen of the material is placed into a testing machine and elongated or shortened. The force F applied by the machine onto the specimen can be read on the dial of the machine; it causes the normal stress $\sigma = F/A$. The change Δl of the length l of the specimen can be measured and the strain $\varepsilon = \Delta l/l$ can be calculated.

The graph of the relationship between stress and strain is shown schematically (not to scale) for a steel specimen in Fig. 1.7. This graph is referred to as *stress-strain diagram*. One can see that for small values of the strain the relationship is linear (straight line) and the stress is proportional to the strain. This behaviour is valid until the stress reaches the *proportional limit* σ_P. If the stress exceeds the proportional limit the strain begins to increase more rapidly and the slope of the curve decreases. This continues until the stress reaches the *yield stress* σ_Y. From this point of the stress-strain diagram the strain increases at a practically constant stress: the material begins to *yield*. Note that many materials do

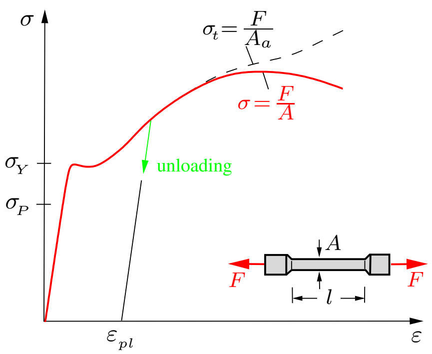

Fig. 1.7

not exhibit a pronounced yield point. At the end of the yielding the slope of the curve increases again which shows that the material can sustain an additional load. This phenomenon is called *strain hardening*.

Experiments show that an elongation of the bar leads to a reduction of the cross-sectional area A. This phenomenon is referred to as *lateral contraction*. Whereas the cross-sectional area decreases uniformly over the entire length of the bar in the case of small stresses, it begins to decrease locally at very high stresses. This phenomenon is called *necking*. Since the actual cross section A_a may then be considerably smaller than the original cross section A, the stress $\sigma = F/A$ does not describe the real stress any more. It is therefore appropriate to introduce the stress $\sigma_t = F/A_a$ which is called *true stress* or *physical stress*. It represents the true stress in the region where necking takes place. The stress $\sigma = F/A$ is referred to as *nominal* or *conventional* or *engineering stress*. Fig. 1.7 shows both stresses until fracture occurs.

Consider a specimen being first *loaded* by a force which causes the stress σ. Assume that σ is smaller than the yield stress σ_Y, i.e., $\sigma < \sigma_Y$. Subsequently, the load is again removed. Then the specimen will return to its original length: the strain returns to zero. In addition, the curves during the loading and the unloading coincide. This behaviour of the material is called *elastic*; the behaviour in the region $\sigma \leq \sigma_P$ is referred to as *linearly elastic*. Now assume that the specimen is loaded beyond the yield stress, i.e., until a stress $\sigma > \sigma_Y$ is reached. Then the curve during the unloading is a straight line which is parallel to the straight line in the linear-elastic region, see Fig. 1.7. If the load is completely removed the strain does not return to zero: a *plastic strain* ε_{pl} remains after the unloading. This material behaviour is referred to as *plastic*.

In the following we will always restrict ourselves to a linearly-elastic material behaviour. For the sake of simplicity we will refer to this behaviour shortly as elastic, i.e., in what follows "elastic" always stands for "linearly elastic". Then we have the linear relationship

Table 1.1 Material Constants

Material	E in MPa	α_T in 1/°C
Steel	$2.1{\cdot}10^5$	$1.2{\cdot}10^{-5}$
Aluminium	$0.7{\cdot}10^5$	$2.3{\cdot}10^{-5}$
Concrete	$0.3{\cdot}10^5$	$1.0{\cdot}10^{-5}$
Wood (in fibre direction)	$0.7 \ldots 2.0{\cdot}10^4$	$2.2 \ldots 3.1{\cdot}10^{-5}$
Cast iron	$1.0{\cdot}10^5$	$0.9{\cdot}10^{-5}$
Copper	$1.2{\cdot}10^5$	$1.6{\cdot}10^{-5}$
Brass	$1.0{\cdot}10^5$	$1.8{\cdot}10^{-5}$

$$\sigma = E\,\varepsilon \tag{1.8}$$

between the stress and the strain. The proportionality factor E is called *modulus of elasticity* or *Young's modulus* (Thomas Young, 1773–1829). The constitutive law (1.8) is called *Hooke's law* after Robert Hooke (1635–1703). Note that Robert Hooke could not present this law in the form (1.8) since the notion of stress was introduced only in 1822 by Augustin Louis Cauchy (1789–1857).

The relation (1.8) is valid for tension and for compression: the modulus of elasticity has the same value for tension and compression. However, the stress must be less than the proportional limit σ_P which may be different for tension or compression.

The modulus of elasticity E is a constant which depends on the material and which can be determined with the aid of a tension test. It has the dimension of force/area (which is also the dimension of stress); it is given, for example, in the unit MPa. Table 1.1 shows the values of E for several materials at room temperature. Note that these values are just a guidance since the modulus of elasticity depends on the composition of the material and on the temperature.

A tensile or a compressive force, respectively, causes the strain

$$\varepsilon = \sigma / E \tag{1.9}$$

in a bar, see (1.8). Changes of the length and thus strains are not only caused by forces but also by changes of the temperature. Experiments show that the *thermal strain* ε_T is proportional to the change ΔT of the temperature if the temperature of the bar is changed uniformly across its section and along its length:

$$\varepsilon_T = \alpha_T \Delta T \, . \tag{1.10}$$

The proportionality factor α_T is called *coefficient of thermal expansion*. It is a material constant and is given in the unit 1/°C. Table 1.1 shows several values of α_T.

If the change of the temperature is not the same along the entire length of the bar (if it depends on the location) then (1.10) represents the local strain $\varepsilon_T(x) = \alpha_T \Delta T(x)$.

If a bar is subjected to a stress σ as well as to a change ΔT of the temperature, the total strain ε is obtained through a superposition of (1.9) and (1.10):

$$\varepsilon = \frac{\sigma}{E} + \alpha_T \Delta T \, . \tag{1.11}$$

This relation can also be written in the form

$$\sigma = E(\varepsilon - \alpha_T \Delta T) \, . \tag{1.12}$$

1.4

1.4 Single Bar under Tension or Compression

There are three different types of equations that allow us to determine the stresses and the strains in a bar: the equilibrium condition, the kinematic relation and Hooke's law. Depending on the problem, the equilibrium condition may be formulated for the entire bar, a portion of the bar (see Section 1.1) or for an element of the bar. We will now derive the equilibrium condition for an

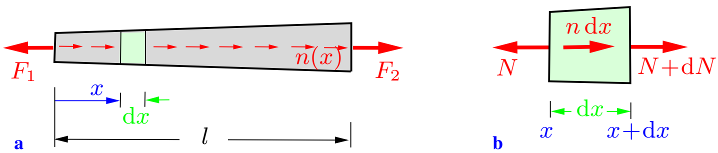

Fig. 1.8

element. For this purpose we consider a bar which is subjected to two forces F_1 and F_2 at its ends and to a line load $n = n(x)$, see Fig. 1.8a. The forces are assumed to be in equilibrium. We imagine a slice element of infinitesimal length $\mathrm{d}x$ separated from the bar as shown in Fig. 1.8b. The free-body diagram shows the normal forces N and $N + \mathrm{d}N$, respectively, at the ends of the element; the line load is replaced by its resultant $n\mathrm{d}x$ (note that n may be considered to be constant over the length $\mathrm{d}x$, compare Volume 1, Section 7.2.2). Equilibrium of the forces in the direction of the axis of the bar

$$\rightarrow: \qquad N + \mathrm{d}N + n\,\mathrm{d}x - N = 0$$

yields the *equilibrium condition*

$$\frac{\mathrm{d}N}{\mathrm{d}x} + n = 0\,. \tag{1.13}$$

In the special case of a vanishing line load ($n \equiv 0$) the normal force in the bar is constant.

The *kinematic relation* for the bar is (see (1.7))

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x}\,,$$

and *Hooke's law* is given by (1.11):

$$\varepsilon = \frac{\sigma}{E} + \alpha_T \Delta T\,.$$

If we insert the kinematic relation and $\sigma = N/A$ into Hooke's law we obtain

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{N}{EA} + \alpha_T \Delta T\,. \tag{1.14}$$

This equation relates the displacements $u(x)$ of the cross sections and the normal force $N(x)$. It may be called the *constitutive law for the bar*. The quantity EA is known as *axial rigidity*. Equations (1.13) and (1.14) are the basic equations for a bar under tension or compression.

The displacement u of a cross section is found through integration of the strain:

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x} \quad \rightarrow \quad \int \mathrm{d}u = \int \varepsilon\,\mathrm{d}x \quad \rightarrow \quad u(x) - u(0) = \int_0^x \varepsilon\,\mathrm{d}\bar{x}\,.$$

The elongation Δl follows as the difference of the displacements at the ends $x = l$ and $x = 0$ of the bar:

$$\Delta l = u(l) - u(0) = \int_0^l \varepsilon\,\mathrm{d}x\,. \tag{1.15}$$

With $\varepsilon = \mathrm{d}u/\mathrm{d}x$ and (1.14) this yields

$$\Delta l = \int_0^l \left(\frac{N}{EA} + \alpha_T \Delta T\right) \mathrm{d}x\,. \tag{1.16}$$

In the special case of a bar (length l) with constant axial rigidity ($EA = \text{const}$) which is subjected only to forces at its end ($n \equiv 0, N = F$) and to a uniform change of the temperature ($\Delta T = \text{const}$), the elongation is given by

$$\Delta l = \frac{F\,l}{EA} + \alpha_T \Delta T\,l\,. \tag{1.17}$$

If, in addition, $\Delta T = 0$ we obtain

$$\Delta l = \frac{F\,l}{EA}\,, \tag{1.18}$$

and if $F = 0$, (1.17) reduces to

$$\Delta l = \alpha_T \Delta T\, l\,. \tag{1.19}$$

If we want to apply these equations to specific problems, we have to distinguish between statically determinate and statically indeterminate problems. In a *statically determinate* system we can always calculate the normal force $N(x)$ with the aid of the equilibrium condition. Subsequently, the strain $\varepsilon(x)$ follows from $\sigma = N/A$ and Hooke's law $\varepsilon = \sigma/E$. Finally, integration yields the displacement $u(x)$ and the elongation Δl. A change of the temperature causes only *thermal strains* (no stresses!) in a statically determinate system.

In a *statically indeterminate* problem the normal force cannot be calculated from the equilibrium condition alone. In such problems the basic equations (equilibrium condition, kinematic relation and Hooke´s law) are a system of *coupled* equations and have to be solved simultaneously. A change of the temperature in general causes additional stresses; they are called *thermal stresses*.

Finally we will reduce the basic equations to a single equation for the displacement u. If we solve (1.14) for N and insert into (1.13) we obtain

$$(EA\,u')' = -\,n + (EA\,\alpha_T \Delta T)'\,. \tag{1.20a}$$

Here, the primes denote derivatives with respect to x. Equation (1.20a) simplifies in the special case $EA = \text{const}$ and $\Delta T = \text{const}$ to

$$EA\,u'' = -\,n\,. \tag{1.20b}$$

If the functions $EA(x)$, $n(x)$ and $\Delta T(x)$ are given, the displacement $u(x)$ of an arbitrary cross section can be determined through integration of (1.20a). The constants of integration are calculated from the boundary conditions. If, for example, one end of the bar is fixed then $u = 0$ at this end. If, on the other hand, one end of the bar can move and is subjected to a force F_0, then applying (1.14) and $N = F_0$ yields the boundary condition $u' = F_0/EA + \alpha_T \Delta T$. This reduces to the boundary condition $u' = 0$ in the special case of a stress-free end ($F_0 = 0$) of a bar whose temperature is not changed ($\Delta T = 0$).

Frequently, one or more of the quantities in (1.20) are given through different functions of x in different portions of the bar (e.g., if there exists a jump of the cross section). Then the bar must be divided into several regions and the integration has to be performed separately in each of theses regions. In this case the constants of integration can be calculated from boundary conditions and matching conditions (compare Volume 1, Section 7.2.4).

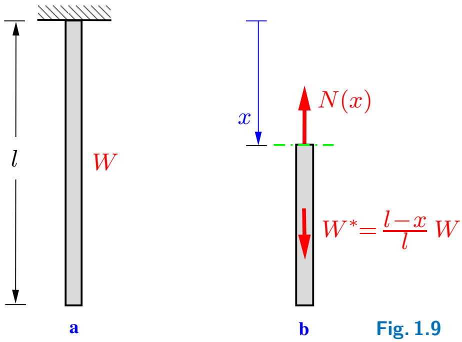

Fig. 1.9

As an illustrative example of a statically determinate system let us consider a slender bar (weight W, cross-sectional area A) that is suspended from the ceiling (Fig. 1.9a). First we determine the normal force caused by the weight of the bar. We cut the bar at an arbitrary position x (Fig. 1.9b). The normal force N is equal to the weight W^* of the portion of the bar below the imaginary cut. Thus, it is given by $N(x) = W^*(x) = W(l - x)/l$. Equation (1.4) now yields the normal stress

$$\sigma(x) = \frac{N(x)}{A} = \frac{W}{A}\left(1 - \frac{x}{l}\right) .$$

Accordingly, the normal stress in the bar varies linearly; it decreases from the value $\sigma(0) = W/A$ at the upper end to $\sigma(l) = 0$ at the free end.

The elongation Δl of the bar due to its own weight is obtained from (1.16):

$$\Delta l = \int_0^l \frac{N}{EA}\,\mathrm{d}x = \frac{W}{EA}\int_0^l \left(1 - \frac{x}{l}\right)\mathrm{d}x = \frac{1}{2}\frac{W\,l}{EA}\,.$$

It is half the elongation of a bar with negligible weight which is subjected to the force W at the free end.

We may also solve the problem by applying the differential equation (1.20b) for the displacements $u(x)$ of the cross sections of the bar. Integration with the constant line load $n = W/l$ yields

$$\begin{aligned} EA\,u'' &= -\frac{W}{l}\,, \\ EA\,u' &= -\frac{W}{l}\,x + C_1\,, \\ EA\,u &= -\frac{W}{2\,l}\,x^2 + C_1\,x + C_2\,. \end{aligned}$$

The constants of integration C_1 and C_2 can be determined from the boundary conditions. The displacement of the cross section at the upper end of the bar is equal to zero: $u(0) = 0$. Since the stress σ vanishes at the free end, we have $u'(l) = 0$. This leads to $C_2 = 0$ and $C_1 = W$. Thus, the displacement and the normal force are given by

$$u(x) = \frac{1}{2}\frac{W\,l}{EA}\left(2\,\frac{x}{l} - \frac{x^2}{l^2}\right), \quad N(x) = EA\,u'(x) = W\left(1 - \frac{x}{l}\right).$$

Since $u(0) = 0$, the elongation is equal to the displacement of the free end:

$$\Delta l = u(l) = \frac{1}{2}\frac{W\,l}{EA}\,.$$

The stress is obtained as

$$\sigma(x) = \frac{N(x)}{A} = \frac{W}{A}\left(1 - \frac{x}{l}\right) .$$

As an illustrative example of a statically indeterminate system let us consider a bar which is placed stress-free between two rigid walls (Fig. 1.10a). It has the cross-sectional areas A_1 and A_2, respectively. We want to determine the support reactions if the temperature of the bar is raised uniformly by an amount ΔT in region ①.

The free-body diagram (Fig. 1.10b) shows the *two* support reactions B and C. They cannot be calculated from only *one* equilibrium condition:

$$\rightarrow: \qquad B - C = 0 .$$

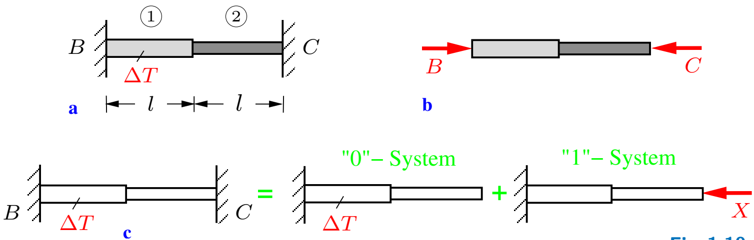

Fig. 1.10

Therefore we have to take into account the deformation of the bar. The elongations in the regions ① and ② are given by (1.16) with $N = -B = -C$:

$$\Delta l_1 = \frac{N\,l}{EA_1} + \alpha_T \Delta T\,l , \qquad \Delta l_2 = \frac{N\,l}{EA_2}$$

(the temperature in region ② is not changed).

The bar is placed between two *rigid* walls. Thus, its total elongation Δl has to vanish:

$$\Delta l = \Delta l_1 + \Delta l_2 = 0 .$$

This equation expresses the fact that the geometry of the deformation has to be compatible with the restraints imposed by the supports. Therefore it is called *compatibility condition*.

The equilibrium condition and the compatibility condition yield the unknown support reactions:

$$\frac{N\,l}{EA_1}+\alpha_T\Delta T\,l+\frac{N\,l}{EA_2}=0\;\rightarrow B=C=-N=\frac{EA_1A_2\,\alpha_T\Delta T}{A_1+A_2}\,.$$

The problem may also be solved in the following way. In a first step we generate a statically determinate system. This is achieved by removing one of the supports, for example support C. The action of this support on the bar is replaced by the action of the force $C = X$ which is as yet unknown. Note that one of the supports, for example B, is needed to have a statically determinate system. The other support, C, is in excess of the necessary support. Therefore the reaction C is referred to as being a *redundant reaction*.

Now we need to consider two different problems. First, we investigate the statically determinate system subjected to the given load (here: the change of the temperature in region ①) which is referred to as "0"-system or *primary system* (Fig. 1.10c). In this system the change of the temperature causes the thermal elongation $\Delta l_1^{(0)}$ (normal force $N = 0$) in region ①; the elongation in region ② is zero. Thus, the displacement $u_C^{(0)}$ of the right end point of the bar is given by

$$u_C^{(0)}=\Delta l_1^{(0)}=\alpha_T\Delta T\,l\,.$$

Secondly we consider the statically determinate system subjected only to force X. It is called "1"-system and is also shown in Fig. 1.10c. Here the displacement $u_C^{(1)}$ of the right end point is

$$u_C^{(1)}=\Delta l_1^{(1)}+\Delta l_2^{(1)}=-\frac{X\,l}{EA_1}-\frac{X\,l}{EA_2}\,.$$

Both the applied load (here: ΔT) as well as the force X act in the given problem (Fig. 1.10a). Therefore, the total displacement u_C at point C follows through *superposition*:

$$u_C=u_C^{(0)}+u_C^{(1)}\,.$$

Since the rigid wall in the original system prevents a displacement at C, the geometric condition

$$u_C = 0$$

has to be satisfied. This leads to

$$\alpha_T \Delta T\, l - \frac{X\, l}{EA_1} - \frac{X\, l}{EA_2} = 0 \quad \rightarrow \quad X = C = \frac{EA_1\, A_2\, \alpha_T \Delta T}{A_1 + A_2}\,.$$

Equilibrium at the free-body diagram (Fig. 1.10b) yields the second support reaction $B = C$.

E1.3

Example 1.3 A solid circular steel cylinder (cross-sectional area A_S, modulus of elasticity E_S, length l) is placed inside a copper tube (cross-sectional area A_C, modulus of elasticity E_C, length l). The assembly is compressed between a rigid plate and the rigid floor by a force F (Fig. 1.11a).

Determine the normal stresses in the cylinder and in the tube. Calculate the shortening of the assembly.

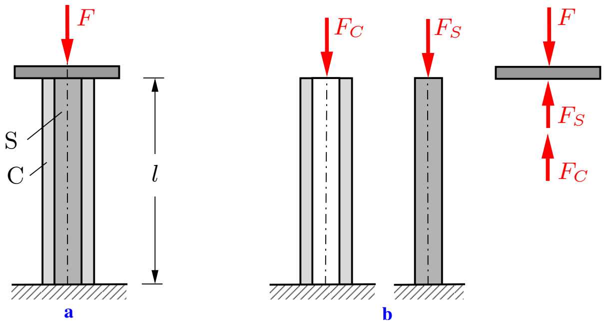

Fig. 1.11

Solution We denote the compressive forces in the steel cylinder and in the copper tube by F_S and F_C, respectively (Fig. 1.11b). Equilibrium at the free-body diagram of the plate yields

$$F_C + F_S = F\,. \qquad \text{(a)}$$

Since equilibrium furnishes only one equation for the two unknown forces F_S and F_C, the problem is statically indeterminate. We obtain a second equation by taking into account the deformation of the system. The shortenings (here counted positive) of the two parts are given according to (1.18) by

$$\Delta l_C = \frac{F_C\, l}{EA_C}\,, \qquad \Delta l_S = \frac{F_S\, l}{EA_S} \tag{b}$$

where, for simplicity, we have denoted the axial rigidity $E_C A_C$ of the copper tube byEA_C and the axial rigidity $E_S A_S$ of the steel cylinder by EA_S.

The plate and the floor are assumed to be rigid. Therefore the geometry of the problem requires that the shortenings of the copper tube and of the steel cylinder coincide. This gives the compatibility condition

$$\Delta l_C = \Delta l_S\,. \tag{c}$$

Solving the Equations (a) to (c) yields the forces

$$F_C = \frac{EA_C}{EA_C + EA_S}\, F\,, \qquad F_S = \frac{EA_S}{EA_C + EA_S}\, F\,. \tag{d}$$

The compressive stresses follow according to (1.2):

$$\underline{\underline{\sigma_C = \frac{E_C}{EA_C + EA_S} F}}\,, \qquad \underline{\underline{\sigma_S = \frac{E_S}{EA_C + EA_S} F}}\,.$$

Inserting (d) into (b) leads to the shortening:

$$\underline{\underline{\Delta l_C = \Delta l_S = \frac{F\, l}{EA_C + EA_S}}}\,.$$

Example 1.4 A copper tube ② is placed over a threaded steel bolt ① of length l. The pitch of the threads is given by h. A nut fits snugly against the tube without generating stresses in the system (Fig. 1.12a). Subsequently, the nut is given n full turns and the temperature of the entire assembly is increased by the amount ΔT. The axial rigidities and the coefficients of thermal expansion of the bolt and the tube are given. **E1.4**

Determine the force in the bolt.

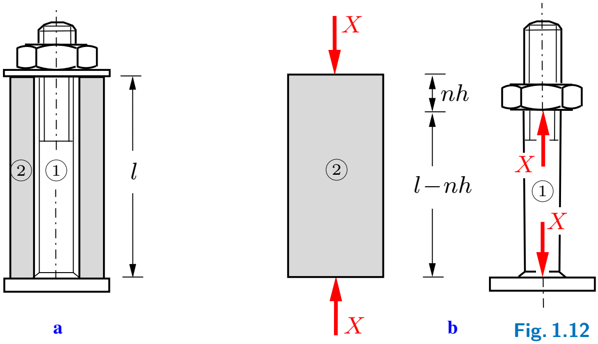

Fig. 1.12

Solution After the nut has been turned it exerts a compressive force X on the tube which causes a shortening of the tube. According to Newton's third axiom (action = reaction) a force of equal magnitude and opposite direction acts via the nut on the bolt which elongates. The free-body diagrams of bolt and tube are shown in Fig. 1.12b.

The problem is statically indeterminate since force F cannot be determined from equilibrium alone. Therefore we have to take into account the deformations. The length of the bolt after the nut has been turned, see the free-body diagram in Fig. 1.12b, is given by $l_1 = l - n\,h$. Its elongation Δl_1 follows from

$$\Delta l_1 = \frac{X(l - n\,h)}{EA_1} + \alpha_{T1}\Delta T(l - n\,h)\,.$$

Since $n\,h \ll l$, this can be reduced to

$$\Delta l_1 = \frac{X\,l}{EA_1} + \alpha_{T1}\Delta T\,l\,.$$

The change of length Δl_2 of the tube ($l_2 = l$) is obtained from

$$\Delta l_2 = -\,\frac{X\,l}{EA_2} + \alpha_{T2}\Delta T\,l\,.$$

The length of the bolt and the length of the tube have to coincide after the deformation. This yields the compatibility condition

$$l_1 + \Delta l_1 = l_2 + \Delta l_2 \qquad \rightarrow \qquad \Delta l_1 - \Delta l_2 = l_2 - l_1 = n\,h\,.$$

Solving the equations leads to the force in the bolt:

$$X\left(\frac{l}{EA_1}+\frac{l}{EA_2}\right)+(\alpha_{T1}-\alpha_{T2})\Delta T\, l = n\,h$$

$$\rightarrow \quad \underline{\underline{X=\frac{n\,h-(\alpha_{T1}-\alpha_{T2})\Delta T\, l}{\left(\frac{1}{EA_1}+\frac{1}{EA_2}\right)l}}}\,.$$

1.5 Statically Determinate Systems of Bars

1.5

In the preceding section we calculated the stresses and deformations of single slender bars. We will now extend the investigation to trusses and to structures which consist of bars and rigid bodies. In this section we will restrict ourselves to statically determinate systems where we can first calculate the forces in the bars with the aid of the equilibrium conditions. Subsequently, the stresses in the bars and the elongations are determined. Finally, the displacements of arbitrary points of the structure can be found. Since it is assumed that the elongations are small as compared with the lengths of the bars, we can apply the equilibrium conditions to the *undeformed* system.

As an illustrative example let us consider the truss in Fig. 1.13a. Both bars have the axial rigidity EA. We want to determine the displacement of pin C due to the applied force F. First we calculate the forces S_1 and S_2 in the bars. The equilibrium conditions, applied to the free-body diagram (Fig. 1.13b), yield

$$\begin{aligned}\uparrow: &\quad S_2\sin\alpha - F = 0\\ \leftarrow: &\quad S_1 + S_2\cos\alpha = 0\end{aligned}\quad\rightarrow\quad S_1=-\frac{F}{\tan\alpha}\,,\quad S_2=\frac{F}{\sin\alpha}\,.$$

According to (1.17) the elongations Δl_i of the bars are given by

$$\Delta l_1=\frac{S_1\,l_1}{EA}=-\frac{F\,l}{EA}\frac{1}{\tan\alpha}\,,\quad \Delta l_2=\frac{S_2\,l_2}{EA}=\frac{F\,l}{EA}\frac{1}{\sin\alpha\cos\alpha}\,.$$

Bar 1 becomes shorter (compression) and bar 2 becomes longer (tension). The new position C' of pin C can be found as follows. We consider the bars to be disconnected at C. Then the system becomes movable: bar 1 can rotate about point A; bar 2 can rotate about point B. The free end points of the bars then move along circular paths with radii $l_1 + \Delta l_1$ and $l_2 + \Delta l_2$, respectively. Point C' is located at the point of intersection of these arcs of circles (Fig. 1.13c).

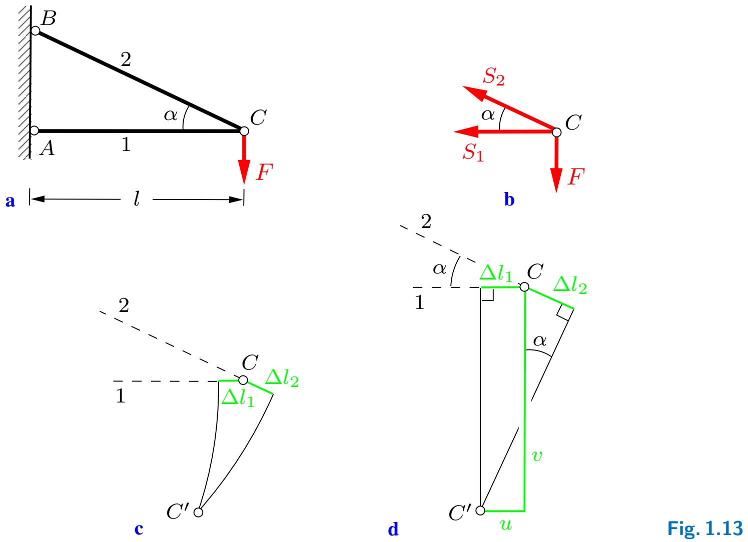

Fig. 1.13

The elongations are small as compared with the lengths of the bars. Therefore, within a good approximation the arcs of the circles can be replaced by their tangents. This leads to the *displacement diagram* as shown in Fig. 1.13d. If this diagram is drawn to scale, the displacement of pin C can directly be taken from it. We want to apply a "graphic-analytical" solution. It suffices then to draw a sketch of the diagram. Applying trigonometric relations we obtain the horizontal and the vertical components of the displacement:

$$u = |\Delta l_1| = \frac{F\,l}{EA}\frac{1}{\tan\alpha}\,,$$
$$v = \frac{\Delta l_2}{\sin\alpha} + \frac{u}{\tan\alpha} = \frac{F\,l}{EA}\frac{1+\cos^3\alpha}{\sin^2\alpha\cos\alpha}\,. \qquad (1.21)$$

To determine the displacement of a pin of a truss with the aid of a displacement diagram is usually quite cumbersome and can be recommended only if the truss has very few members. In the case of trusses with many members it is advantageous to apply an energy method (see Chapter 6).

The method described above can also be applied to structures which consist of bars and rigid bodies.

Example 1.5 A rigid beam (weight W) is mounted on three elastic bars (axial rigidity EA) as shown in Fig. 1.14a. **E1.5**

Determine the angle of slope of the beam that is caused by its weight after the structure has been assembled.

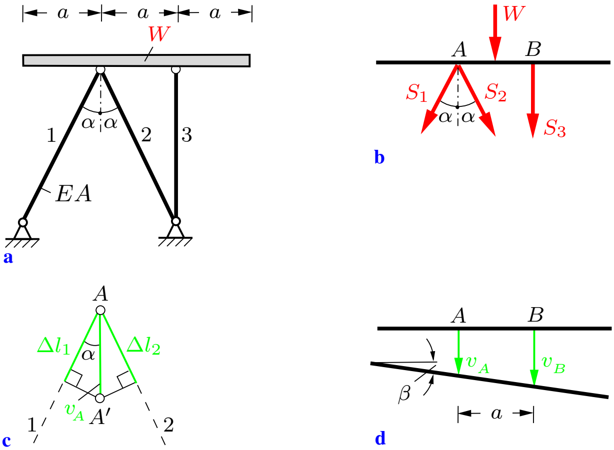

Fig. 1.14

Solution First we calculate the forces in the bars with the aid of the equilibrium conditions (Fig. 1.14b):

$$S_1 = S_2 = -\frac{W}{4\cos\alpha}\,, \qquad S_3 = -\frac{W}{2}\,.$$

With $l_1 = l_2 = l/\cos\alpha$ and $l_3 = l$ we obtain the elongations:

$$\Delta l_1 = \Delta l_2 = \frac{S_1\, l_1}{EA} = -\frac{W\, l}{4EA\cos^2\alpha}\,, \quad \Delta l_3 = \frac{S_3\, l_3}{EA} = -\frac{W\, l}{2\, EA}\,.$$

Point B of the beam is displaced downward by $v_B = |\Delta l_3|$. To determine the vertical displacement v_A of point A we sketch a displacement diagram (Fig. 1.14c). First we plot the changes Δl_1 and Δl_2 of the lengths in the direction of the respective bar. The lines perpendicular to these directions intersect at the displaced position A' of point A. Thus, its vertical displacement is given by $v_A = |\Delta l_1|/\cos\alpha$.

Since the displacements v_A and v_B do not coincide, the beam does not stay horizontal after the structure has been assembled. The angle of slope β is obtained with the approximation $\tan\beta \approx \beta$ (small deformations) and $l = a\cot\alpha$ as (see Fig. 1.14d)

$$\underline{\underline{\beta}} = \frac{v_B - v_A}{a} = \underline{\underline{\frac{2\cos^3\alpha - 1}{4\cos^3\alpha}\,\frac{W\,\cot\alpha}{EA}}}\,.$$

If $\cos^3\alpha > \frac{1}{2}$ (or $\cos^3\alpha < \frac{1}{2}$), then the beam is inclined to the right (left). In the special case $\cos^3\alpha = \frac{1}{2}$, i.e. $\alpha = 37.5°$, it stays horizontal.

E1.6 **Example 1.6** The truss in Fig. 1.15a is subjected to a force F. Given: $E = 2\cdot 10^2$ GPa, $F = 20\,\text{kN}$.

Determine the cross-sectional areas of the three members so that the stresses do not exceed the allowable stress $\sigma_{\text{allow}} = 150\,\text{MPa}$ and the displacement of support B is smaller than 0.5 ‰ of the length of bar 3.

Solution First we calculate the forces in the members. The equilibrium conditions for the free-body diagrams of pin C and support B (Fig. 1.15b) yield

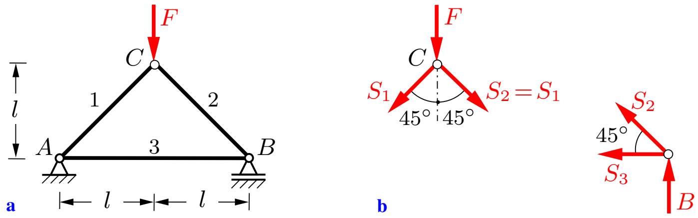

Fig. 1.15

$$S_1 = S_2 = -\frac{\sqrt{2}}{2}F\,, \qquad S_3 = \frac{F}{2}\,.$$

The stresses do not exceed the allowable stress if

$$|\sigma_1| = \frac{|S_1|}{A_1} \leq \sigma_{\text{allow}}\,, \ |\sigma_2| = \frac{|S_2|}{A_2} \leq \sigma_{\text{allow}}\,, \ \sigma_3 = \frac{S_3}{A_3} \leq \sigma_{\text{allow}}\,.$$

This leads to the cross-sectional areas

$$\underline{\underline{A_1 = A_2}} = \frac{|S_1|}{\sigma_{\text{allow}}} = \underline{\underline{94.3\ \text{mm}^2}}\,,\ A_3 = \frac{S_3}{\sigma_{\text{allow}}} = 66.7\ \text{mm}^2\,. \quad \text{(a)}$$

In addition, the displacement of support B has to be smaller than 0.5 ‰ of the length of bar 3. This displacement is equal to the elongation $\Delta l_3 = S_3\, l_3/EA_3$ of bar 3 (support A is fixed). From $\Delta l_3 < 0.5 \cdot 10^{-3}\, l_3$ we obtain

$$\frac{\Delta l_3}{l_3} = \frac{S_3}{EA_3} < 0.5{\cdot}10^{-3} \ \rightarrow\ \underline{\underline{A_3}} > \frac{2\,S_3}{E}10^3 = \frac{F}{E}10^3 = \underline{\underline{100\text{mm}^2}}\,.$$

Comparison with (a) yields the required area $A_3 = 100\,\text{mm}^2$.

1.6 Statically Indeterminate Systems of Bars

1.6

We will now investigate statically indeterminate systems for which the forces in the bars cannot be determined with the aid of the equilibrium conditions alone since the number of the unknown quantities exceeds the number of the equilibrium conditions. In such systems the basic equations (equilibrium conditions, kinematic equations (compatibility) and Hooke's law) are coupled equations.

Let us consider the symmetrical truss shown in Fig. 1.16a. It is stress-free before the load is applied. The axial rigidities EA_1, EA_2, $EA_3 = EA_1$ are given; the forces in the members are unknown. The system is statically indeterminate to the first degree (the decomposition of a force into three directions cannot be done uniquely in a coplanar problem, see Volume 1, Section 2.2). The two equilibrium conditions applied to the free-body diagram of pin K (Fig. 1.16b) yield

$$\begin{aligned} &\rightarrow: \quad -S_1 \sin\alpha + S_3 \sin\alpha = 0 \quad \rightarrow \quad S_1 = S_3\,, \\ &\uparrow: \quad S_1 \cos\alpha + S_2 + S_3 \cos\alpha - F = 0 \rightarrow \quad S_1 = S_3 = \frac{F - S_2}{2\cos\alpha}\,. \end{aligned} \tag{a}$$

The elongations of the bars are given by

$$\Delta l_1 = \Delta l_3 = \frac{S_1\, l_1}{EA_1}\,, \qquad \Delta l_2 = \frac{S_2\, l}{EA_2}\,. \tag{b}$$

To derive the compatibility condition we sketch a displacement diagram (Fig. 1.16c) from which we find

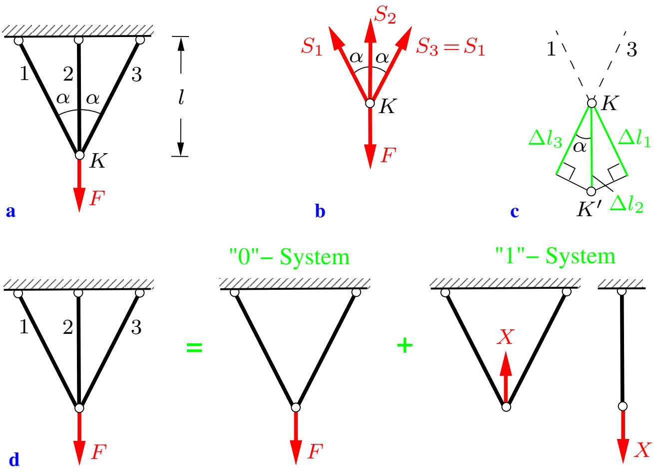

Fig. 1.16

$$\Delta l_1 = \Delta l_2 \cos\alpha\,. \tag{c}$$

With (a), (b) and $l_1 = l/\cos\alpha$ we obtain from (c)

$$\frac{(F - S_2)\,l}{2\,EA_1\cos^2\alpha} = \frac{S_2\,l}{EA_2}\cos\alpha$$

which leads to

$$S_2 = \frac{F}{1 + 2\,\dfrac{EA_1}{EA_2}\cos^3\alpha}\,.$$

The remaining two forces in the bars follow from (a):

$$S_1 = S_3 = \frac{\dfrac{EA_1}{EA_2}\cos^2\alpha}{1 + 2\,\dfrac{EA_1}{EA_2}\cos^3\alpha}\,F\,.$$

Note that now the vertical displacement v of pin K can also be written down:

$$v = \Delta l_2 = \frac{S_2\,l}{EA_2} = \frac{\dfrac{F\,l}{EA_2}}{1 + 2\,\dfrac{EA_1}{EA_2}\cos^3\alpha}\,.$$

The problem may also be solved using the method of superposition. In a first step we remove bar 2 to obtain a statically determinate system, the “0“-system. It consists of the two bars 1 and 3 and it is subjected to the given force F (Fig. 1.16d). The forces $S_1^{(0)}$ and $S_3^{(0)}$ in these bars follow from the equilibrium conditions as

$$S_1^{(0)} = S_3^{(0)} = \frac{F}{2\cos\alpha}\,.$$

The corresponding elongations are obtained with $l_1 = l/\cos\alpha$:

$$\Delta l_1^{(0)} = \Delta l_3^{(0)} = \frac{S_1^{(0)}\,l_1}{EA_1} = \frac{F\,l}{2\,EA_1\cos^2\alpha}\,. \tag{d}$$

In a second step we consider the statically determinate system under the action of an unknown force X (“1“-system, see also Fig. 1.16d). Note that this force acts in the opposite direction on bar 2 (actio = reactio). Now we get

$$\begin{aligned} S_1^{(1)} = S_3^{(1)} &= -\frac{X}{2\cos\alpha}\,, \qquad S_2^{(1)} = X\,, \\ \Delta l_1^{(1)} = \Delta l_3^{(1)} &= -\frac{Xl}{2\,EA_1\cos^2\alpha}\,, \qquad \Delta l_2^{(1)} = \frac{Xl}{EA_2}\,. \end{aligned} \tag{e}$$

The total elongation of the bars is obtained through superposition of the systems “0“ and “1“:

$$\Delta l_1 = \Delta l_3 = \Delta l_1^{(0)} + \Delta l_1^{(1)}\,, \qquad \Delta l_2 = \Delta l_2^{(1)}\,. \tag{f}$$

The compatibility condition (c) is again taken from the displacement diagram (Fig. 1.16c). It leads with (d) - (f) to the unknown force $X = S_2^{(1)} = S_2$:

$$\frac{F\,l}{2\,EA_1\cos^2\alpha} - \frac{X\,l}{2\,EA_1\cos^2\alpha} = \frac{X\,l}{EA_2}\cos\alpha$$

$$\to X = S_2 = \frac{F}{1 + 2\,\dfrac{EA_1}{EA_2}\cos^3\alpha}\,.$$

The forces S_1 and S_3 follow from superposition:

$$S_1 = S_3 = S_1^{(0)} + S_1^{(1)} = \frac{\dfrac{EA_1}{EA_2}\cos^2\alpha}{1 + 2\,\dfrac{EA_1}{EA_2}\cos^3\alpha}F\,.$$

A system of bars is statically indeterminate of degree n if the number of the unknowns exceeds the number of the equilibrium conditions by n. In order to determine the forces in the bars of such a system, n compatibility conditions are needed in addition to the equilibrium conditions. Solving this system of equations yields the unknown forces in the bars.

A statically indeterminate system of degree n can also be solved with the method of superposition. Then n bars are removed in order to obtain a statically determinate system. The action of the bars which are removed is replaced by the action of the static redundants $S_i = X_i$. Next $n+1$ different auxiliary systems are considered. The given load acts in the "0"-system, whereas the "i"-system ($i = 1, 2, ..., n$) is subjected only to the force X_i. In each of the statically determinate auxiliary problems the forces in the bars and thus the elongations can be calculated. Applying the n compatibility conditions yields a system of equations for the n unknown forces X_i. The forces in the other bars can subsequently be determined through superposition.

Example 1.7 A rigid beam (weight negligible) is suspended from three vertical bars (axial rigidity EA) as shown in Fig. 1.17a. **E1.7**

Determine the forces in the originally stress-free bars if
a) the beam is subjected to a force F ($\Delta T = 0$),
b) the temperature of bar 1 is changed by ΔT ($F = 0$).

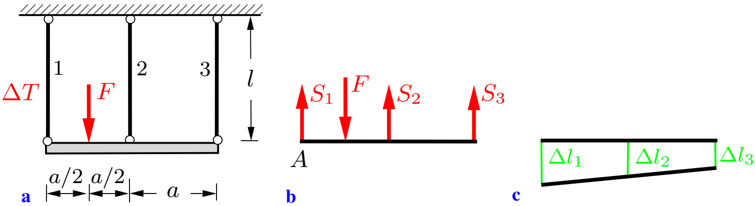

Fig. 1.17

Solution The system is statically indeterminate to the first degree: there are only two equilibrium conditions for the three unknown forces S_j (Fig. 1.17b). a) If the structure is subjected to force F the equilibrium conditions are

$$\uparrow: \quad S_1 + S_2 + S_3 - F = 0\,,$$
$$\overset{\frown}{A}: \quad -\frac{a}{2}\,F + a\,S_2 + 2\,a\,S_3 = 0\,. \qquad \text{(a)}$$

The elongations of the bars are given by ($\Delta T = 0$)

$$\Delta l_1 = \frac{S_1\,l}{EA}\,, \qquad \Delta l_2 = \frac{S_2\,l}{EA}\,, \qquad \Delta l_3 = \frac{S_3\,l}{EA}\,. \qquad \text{(b)}$$

We sketch a displacement diagram (Fig. 1.17c) and find the compatibility condition

$$\Delta l_2 = \frac{\Delta l_1 + \Delta l_3}{2} \,. \tag{c}$$

Now we have six equations for the three forces S_j and the three elongations Δl_j . Solving for the forces yields

$$\underline{\underline{S_1 = \frac{7}{12}F}}\,, \qquad \underline{\underline{S_2 = \frac{1}{3}F}}\,, \qquad \underline{\underline{S_3 = \frac{1}{12}F}}\,.$$

b) If bar 1 is heated ($F = 0$), the equilibrium conditions are

$$\begin{aligned} \uparrow : \quad & S_1 + S_2 + S_3 = 0\,, \\ \overset{\frown}{A} : \quad & aS_2 + 2aS_3 = 0\,, \end{aligned} \tag{a$'$}$$

and the elongations are given by

$$\Delta l_1 = \frac{S_1\, l}{EA} + \alpha_T \Delta T l\,, \qquad \Delta l_2 = \frac{S_2\, l}{EA}\,, \qquad \Delta l_3 = \frac{S_3\, l}{EA}\,. \tag{b$'$}$$

The compatibility condition (c) is still valid. Solving (a′), (b′) and (c) yields

$$\underline{\underline{S_1 = S_3 = -\frac{1}{6}EA\,\alpha_T \Delta T}}\,, \qquad \underline{\underline{S_2 = \frac{1}{3}EA\,\alpha_T \Delta T}}\,.$$

E1.8 **Example 1.8** To assemble the truss in Fig. 1.18a, the free end of bar 3 (length $l - \delta$, $\delta \ll l$) has to be connected with pin C.

a) Determine the necessary force F acting at pin C (Fig. 1.18b).

b) Calculate the forces in the bars after the truss has been assembled and force F has been removed.

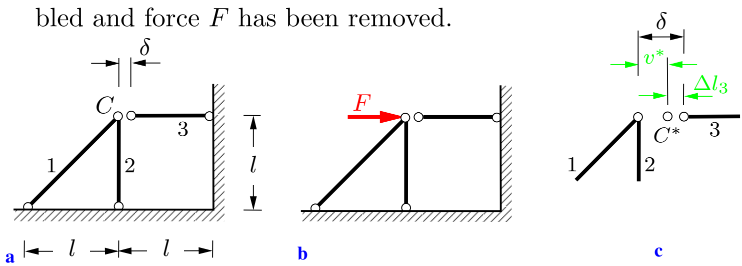

Fig. 1.18

Solution a) The force F causes a displacement of pin C. The horizontal component v of this displacement has to be equal to δ to allow assembly. The required force follows with $\alpha = 45°$ from (1.21):

$$v = \frac{F\,l}{EA}\,\frac{1+\sqrt{2}/4}{\sqrt{2}/4} = \delta \qquad \to \qquad \underline{\underline{F = \frac{EA\,\delta}{(2\sqrt{2}+1)\,l}}}\,.$$

b) The force F is removed after the truss has been assembled. Then pin C undergoes another displacement. Since now a force S_3 in bar 3 is generated, pin C does not return to its original position: it is displaced to position C^* (Fig. 1.18c). The distance between points C and C^* is given by

$$v^* = \frac{S_3\,l}{EA}\,\frac{1+\sqrt{2}/4}{\sqrt{2}/4}\,.$$

The compatibility condition

$$v^* + \Delta l_3 = \delta$$

can be taken from Fig. 1.18c. With the elongation

$$\Delta l_3 = \frac{S_3(l-\delta)}{EA} \approx \frac{S_3 l}{EA}$$

of bar 3 we reach

$$\frac{S_3\,l}{EA}\,\frac{1+\sqrt{2}/4}{\sqrt{2}/4} + \frac{S_3\,l}{EA} = \delta \qquad \to \qquad \underline{\underline{S_3 = \frac{EA\,\delta}{2(\sqrt{2}+1)l}}}\,.$$

The other two forces follow from the equilibrium condition at pin C:

$$\underline{\underline{S_1 = \sqrt{2}\;S_3}}\,, \qquad \underline{\underline{S_2 = -\,S_3}}\,.$$

1.7

1.7 Supplementary Examples

Detailed solutions to the following examples are given in (**A**) D. Gross et al. *Mechanics of Materials - Formulas and Problems*, Springer 2017 or (**B**) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer 2017.

E1.9

Example 1.9 A slender bar (density ρ, modulus of elasticity E) is suspended from its upper end as shown in Fig. 1.19. It has a rectangular cross section with a constant depth and a linearly varying width. The cross section at the upper end is A_0.

Determine the stress $\sigma(x)$ due to the force F and the weight of the bar. Calculate the minimum stress $\sigma_{\min}$ and its location.

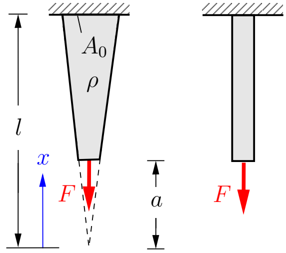

Fig. 1.19

Results: see (**A**)

$$\sigma(x) = \frac{Fl + \rho g \frac{A_0}{2}(x^2 - a^2)}{A_0 x}, \ \sigma_{\min} = \rho g x^*, \ x^* = \sqrt{\frac{2Fl}{\rho g A_0} - a^2}\,.$$

E1.10

Example 1.10 Determine the elongation Δl of the tapered circular shaft (modulus of elasticity E) shown in Fig. 1.20 if it is subjected to a tensile force F.

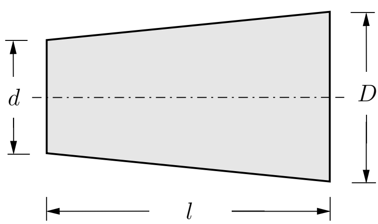

Fig. 1.20

Result: see (**A**) $\Delta l = \dfrac{4Fl}{\pi E D d}\,.$

Example 1.11 A slender bar (weight W_0, Young's modulus E, cross section A, coefficient of thermal expansion α_T) is suspended from its upper end. It just touches the ground as shown in Fig. 1.21 without generating a contact force.

E1.11

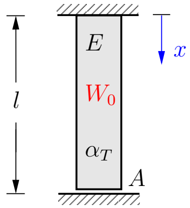

Fig. 1.21

Calculate the stress $\sigma(x)$ if the temperature of the bar is uniformly increased by ΔT. Determine ΔT so that there is compression in the whole bar.

Results: see (**A**)

$$\sigma(x) = \frac{W_0}{A}\left(1 - \frac{x}{l}\right) - E\alpha_T\Delta T, \qquad \Delta T > \frac{W_0}{EA\alpha_T}\,.$$

Example 1.12 The bar (cross sectional area A) shown in Fig. 1.22 is composed of steel and aluminium. It is placed stress-free between two rigid walls. Given: $E_{st}/E_{al} = 3, \alpha_{st}/\alpha_{al} = 1/2$.

E1.12

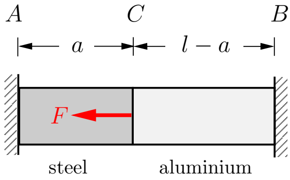

Fig. 1.22

a) Calculate the support reactions if the bar is subjected to a force F at point C.
b) Calculate the normal force in the whole bar if it is subjected only to a change of temperature ΔT ($F = 0$).

Results: see (**A**)

$$\text{a) } N_A = -F\frac{3(l-a)}{3l-2a}, \quad N_B = F\frac{a}{3l-2a},$$

$$\text{b) } N = -\frac{2l-a}{3l-2a}E_{st}\alpha_{st}A\Delta T\,.$$

E1.13 **Example 1.13** The column in Fig. 1.23 consists of reinforced concrete. It is subjected to a tensile force F. Given: $E_{st}/E_c = 6, A_{st}/A_c = 1/9$.

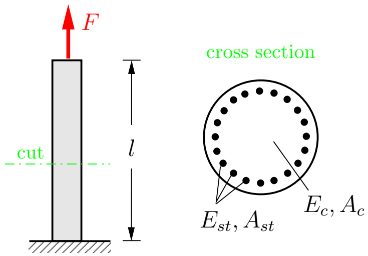

Fig. 1.23

Determine the stresses in the steel and in the concrete and the elongation Δl of the column if

a) the bonding between steel and concrete is perfect,
b) the bonding is damaged so that only the steel carries the load.

Results: see (**A**)

$$\text{a)}\ \sigma_{st} = 4\frac{F}{A}, \quad \sigma_c = \frac{2}{3}\frac{F}{A}, \quad \Delta l = \frac{2}{5}\frac{Fl}{EA_{st}},$$

$$\text{b)}\ \sigma_{st} = 10\frac{F}{A}, \quad \Delta l = \frac{Fl}{EA_{st}}.$$

E1.14 **Example 1.14** A slender bar (density ρ, modulus of elasticity E, length l) is suspended from its upper end as shown in Fig. 1.24. It has a rectangular cross section with a constant depth a. The width b varies linearly from $2b_0$ at he fixed end to b_0 at the free end.

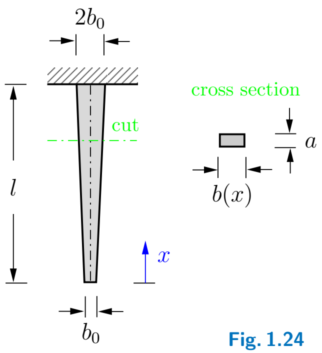

Fig. 1.24

Determine the stresses $\sigma(x)$ and $\sigma(l)$ and the elongation Δl of the bar due to its own weight.

Results: see (**B**)

$$\sigma(x) = \frac{1}{2}\rho g\frac{(2l+x)x}{l+x}, \quad \sigma(l) = \frac{3\rho g l}{4}, \quad \Delta l = \frac{\rho g l^2}{4E}(3 - 2\ln 2).$$

Example 1.15 A rigid chair (weight negligible) is supported by three bars (axial rigidity EA) as shown in Fig. 1.25. It is subjected to a force F at point B.

a) Calculate the forces S_i in the bars and the elongations Δl_i of the bars.

b) Determine the displacement of point C.

E1.15

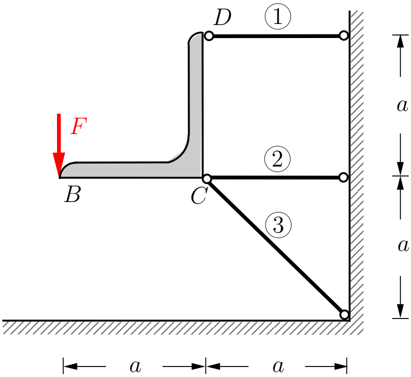

Fig. 1.25

Results: see (**A**)

a) $S_1 = F, \quad S_2 = 0, \quad S_3 = -\sqrt{2}F,$

$$\Delta l_1 = \frac{Fa}{EA}, \quad \Delta l_2 = 0, \quad \Delta l_3 = -2\frac{Fa}{EA},$$

b) $u_C = 0, \quad v_C = 2\sqrt{2}\frac{Fa}{EA}\,.$

Example 1.16 Two bars (axial rigidity EA) are pin-connected and supported at C (Fig. 1.26).

a) Calculate the support reaction at C due to the force F.

b) Determine the displacement of the support.

E1.16

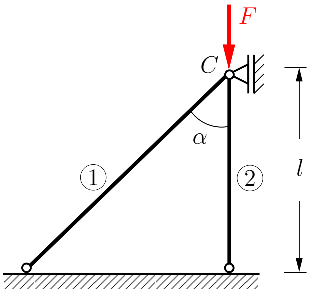

Fig. 1.26

Results: see (**A**) a) $C = \dfrac{\sin\alpha\cos^2\alpha}{1+\cos^3\alpha}F, \quad$ b) $v_C = \dfrac{1}{1+\cos^3\alpha}\dfrac{Fl}{EA}\,.$

E1.17 **Example 1.17** Consider a thin circular ring (modulus of elasticity E, coefficient of thermal expansion α_T, internal radius $r-\delta, \delta \ll r$) with a rectangular cross section (width b, thickness $t \ll r$). The ring is heated in order to increase its radius which makes it possible to place it over a rigid wheel with radius r.

Determine the necessary change of temperature ΔT. Calculate the normal stress σ in the ring and the pressure p onto the wheel after the temperature has regained its original value.

Results: see (**B**) $\Delta T = \dfrac{\delta}{\alpha_T r}, \quad \sigma = E\dfrac{\delta}{r}, \quad p = \sigma\dfrac{t}{r}\,.$

E1.18 **Example 1.18** The two rods (axial rigidity EA) shown in Fig. 1.27 are pin-connected at K. The system is subjected to a vertical force F.

Calculate the displacement of pin K.

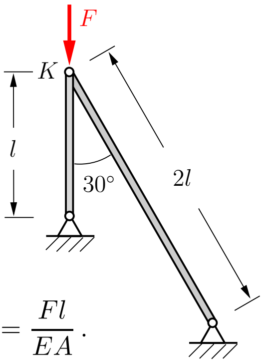

Fig. 1.27

Results: see (**B**) $u = \sqrt{3}\dfrac{Fl}{EA}, \quad v = \dfrac{Fl}{EA}\,.$

E1.19 **Example 1.19** The structure shown in Fig. 1.28 consists of a rigid beam BC and two elastic bars (axial rigidity EA). It is subjected to a force F.

Calculate the displacement of pin C.

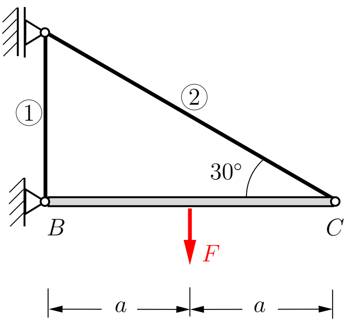

Fig. 1.28

Results: see (**B**) $u = 0, \quad v = 3\sqrt{3}\dfrac{Fa}{EA}\,.$

E1.20

Example 1.20 Fig. 1.29 shows a freight elevator. The cable (length l, axial rigidity $(EA)_1$) of the winch passes over a smooth pin K. A crate (weight W) is suspended at the end of the cable (see Example 2.13 in Volume 1). The axial rigidity $(EA)_2$ of the two bars 1 and 2 is given.

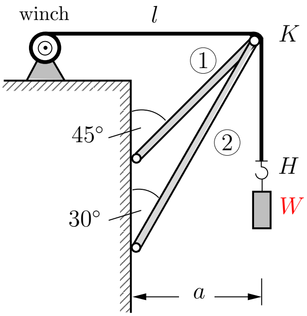

Fig. 1.29

Determine the displacements of pin K and of the end of the cable (point H) due to the weight of the crate.

Results: see (**B**)

$$u = 6.69\frac{Wa}{(EA)_2}, \quad v = 3.86\frac{Wa}{(EA)_2}, \quad f = 2.83\frac{Wa}{(EA)_2} + \frac{Wl}{(EA)_1}.$$

E1.21

Example 1.21 To assemble the truss (axial rigidity EA of the three bars) in Fig. 1.30 the end point P of bar 2 has to be connected with pin K. Assume $\delta \ll h$.

Determine the forces in the bars after the truss has been assembled.

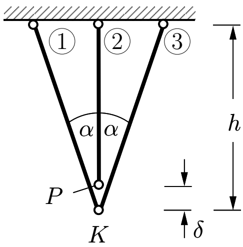

Fig. 1.30

Results: see (**B**)

$$S_1 = S_3 = -\frac{EA\delta\cos^2\alpha}{h\,(1+2\cos^3\alpha)}, \quad S_2 = \frac{2EA\delta\cos^3\alpha}{h\,(1+2\cos^3\alpha)}.$$

E1.22

Example 1.22 The depicted truss consists of three bars. The axial rigidities EA_1, EA_2 and the coefficients of thermal expansion α_{T1}, α_{T2} are given.

Determine the axial forces in the members if the temperature of the truss is raised uniformly by ΔT.

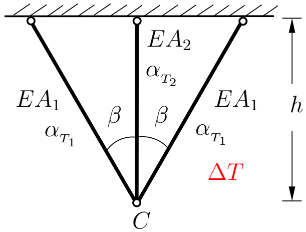

Fig. 1.31

Results: see (**A**) $S_1 = EA_1 \dfrac{\alpha_{T2}\cos^2\beta - \alpha_{T1}}{1 + 2\cos^3\beta \dfrac{EA_1}{EA_2}}\Delta T, \quad S_2 = -2\cos\beta\, S_1\,.$

E1.23

Example 1.23 A rigid beam (weight negligible) is suspended from two ropes (axial rigidity EA). It is subjected to a force F as shown in Fig. 1.32.

Determine the forces in the bars and the displacement of point B.

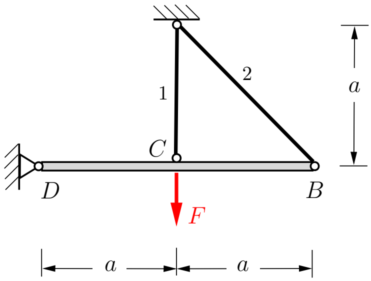

Fig. 1.32

Results: $S_1 = S_2 = \dfrac{2}{1+\sqrt{2}}\, F\,, \quad v_B = \dfrac{4}{1+\sqrt{2}}\,\dfrac{Fa}{EA}\,.$

1.8 Summary

1.8

- Normal stress in a section perpendicular to the axis of a bar:

$$\sigma = N/A\,,$$

N normal force, A cross-sectional area.

- Strain:

$$\varepsilon = \mathrm{d}u/\mathrm{d}x\,, \quad |\varepsilon| \ll 1\,,$$

u displacement of a cross section.
Special case of uniform strain: $\varepsilon = \Delta l/l$.

- Hooke's law:

$$\sigma = E\,\varepsilon\,,$$

E modulus of elasticity.

- Elongation:

$$\Delta l = \int_0^l \left(\frac{N}{EA} + \alpha_T \Delta T\right)\mathrm{d}x\,,$$

EA axial rigidity, α_T coefficient of thermal expansion,
ΔT change of temperature.
Special cases:

$$N = F, \quad \Delta T = 0, \quad EA = \mathrm{const} \quad \rightarrow \quad \Delta l = \frac{Fl}{EA}\,,$$

$$N = 0, \quad \Delta T = \mathrm{const} \quad \rightarrow \quad \Delta l = \alpha_T \Delta T\, l\,.$$

- Statically determinate system of bars: normal forces, stresses, strains, elongations and displacements can be calculated consecutively from the equilibrium conditions, Hooke's law and kinematic equations. A change of the temperature does not cause stresses.
- Statically indeterminate system: the equations (equilibrium conditions, kinematic equations and Hooke's law) are coupled equations. A change of the temperature in general causes thermal stresses.