Chapter 2
Overview of *Mathematica*

2.1 Symbols

Let's assign an expression containing a symbol x to a variable a. No value is assigned to x.

In[1] := $a = x$^2 − 1

Out[1] $= -1 + x^2$

Now let's assign some value to x and see what happens to a.

In[2] := $x = z + 1$; a

Out[2] $= -1 + (1 + z)^2$

The value of a has not really changed. It is still the same expression containing x. What if we assign another value to x?

In[3] := $x = z - 1$; a

Out[3] $= -1 + (-1 + z)^2$

We can delete the value of the variable x, thus returning it to its initial state in which it means just the symbol x. We see that indeed the value of a has not changed.

In[4] := Clear[x]; a

Out[4] $= -1 + x^2$

Now let's try to assign an expression containing x to the variable x.

In[5] := $RecursionLimit = 32; $x = x + 1$

$RecursionLimit :: reclim : Recursion depth of 32 exceeded.

Out[5] = 1 + (1 + (1 + (1 + (1 + (1 + (1 + (1 + (1 + (1 + (1 + (1 + (1 + (1 + (1 + (1+ (1+(1+(1+(1+(1+(1+(1+(1+(1+(1+(1+(1+(1+(1+(1+Hold[1+ x]))))))))))))))))))))))))))))))))

Mathematica complains. What has happened? *Mathematica* wants to print x, and to this end it calculates the value of x. It sees x in this value and substitutes the value of x. In this value, it again sees x and substitutes its value. And so on ad infinitum. In reality, the depth of such substitutions is limited; the default value of the limit is 1,024 (we have temporarily changed it to 32). The value of the expression $1 + x$ which failed to evaluate is returned as the function Hold; we shall discuss it in Sect. 6.9.

In[6] := $RecursionLimit = 1024; Clear[x]

A. Grozin, *Introduction to Mathematica® for Physicists*, Graduate Texts in Physics,
DOI 10.1007/978-3-319-00894-3_2,

2.2 Numbers

Mathematica can work with arbitrarily long integer numbers.

In[7] := Factorial[100]

Out[7] = 933262154439441526816992388562667004907159682643816214685\
    9296389521759999322991560894146397615651828625369792082722375\
    82511852109168640000000000000000000000000

When working with a rational number, the greatest common divisors of its numerator and denominator are canceled.

In[8] := *a* = 1234567890/987654321

Out[8] $= \dfrac{137174210}{109739369}$

Calculations with rational numbers are exact.

In[9] := *a*^5

Out[9] = 48569355286282885522765185491603110100000/
    1591520706534578461823798623667024590 7849

How much is this numerically? Say, with 30 significant digits?

In[10] := *N*[*a*, 30]

Out[10] = 1.24999998860937500014238281250

Mathematica can work with real (floating-point) numbers having arbitrarily high precision.

In[11] := *a* = 1234567890987654321.1234567890987654321

Out[11] = $1.2345678909876543211234567890987654 32 \times 10^{18}$

In[12] := *a*^5

Out[12] = $2.8679718717716056727592153172536 3508 \times 10^{90}$

Here are π and e with 100 significant digits.

In[13] := *N*[Pi, 100]

Out[13] = 3.14159265358979323846264338327950288419716939937510582097\
    4944592307816406286208998628034825342117068

In[14] := *N*[*E*, 100]

Out[14] = 2.71828182845904523536028747135266249775724709369995957496\
    6967627724076630353547594571382178525166427

2.3 Polynomials and Rational Functions

Let's take a polynomial.

In[15] := *a* = (*x* + *y* + *z*)^6

Out[15] = $(x+y+z)^6$

Expand it.

In[16] := a = Expand[a]

Out[16] = $x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6+6x^5z+30x^4yz+60x^3y^2z+60x^2y^3z+30xy^4z+6y^5z+15x^4z^2+60x^3yz^2+90x^2y^2z^2+60xy^3z^2+15y^4z^2+20x^3z^3+60x^2yz^3+60xy^2z^3+20y^3z^3+15x^2z^4+30xyz^4+15y^2z^4+6xz^5+6yz^5+z^6$

The degree in x.

In[17] := Exponent[a,x]

Out[17] = 6

The coefficient of x^2.

In[18] := Coefficient[a,x,2]

Out[18] = $15y^4+60y^3z+90y^2z^2+60yz^3+15z^4$

Collect terms with the same power of x together.

In[19] := Collect[a,x]

Out[19] = $x^6+y^6+6y^5z+15y^4z^2+20y^3z^3+15y^2z^4+6yz^5+z^6+x^5(6y+6z)+x^4\left(15y^2+30yz+15z^2\right)+x^3\left(20y^3+60y^2z+60yz^2+20z^3\right)+x^2\left(15y^4+60y^3z+90y^2z^2+60yz^3+15z^4\right)+x\left(6y^5+30y^4z+60y^3z^2+60y^2z^3+30yz^4+6z^5\right)$

Factorize it.

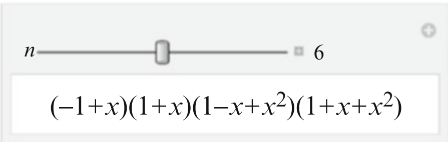
In[20] := a = Factor[a]

Out[20] = $(x+y+z)^6$

Suppose we want to factorize polynomials x^n-1 with various n. The parameter n can be varied from 2 to 10 by dragging the marker with the mouse.

In[21] := Manipulate[Factor[x^n − 1], {n, 2, 10, 1, Appearance−>"Labeled"}]

Out[21] =



There exists an algorithm which completely factorizes any polynomial with integer coefficients into factors which also have integer coefficients.

In[22] := Factor[x^4 − 1]

Out[22] = $(-1+x)(1+x)\left(1+x^2\right)$

If we want to get factors whose coefficients come from an extension of the ring of integers, say, by the imaginary unit i, we should say so explicitly.

In[23] := Factor[x^4 − 1, Extension−>I]

Out[23] = $(-1+x)(-\mathrm{i}+x)(\mathrm{i}+x)(1+x)$

This polynomial factorizes into two factors with integer coefficients.

In[24] := a = x^4 − 4; Factor[a]

Out[24] = $\left(-2+x^2\right)\left(2+x^2\right)$

If coefficients from the extension of the ring of integers by $\sqrt{2}$ are allowed—into three factors.

In[25] := Factor[a, Extension−>Sqrt[2]]

Out[25] = $-\left(\sqrt{2}-x\right)\left(\sqrt{2}+x\right)\left(2+x^2\right)$

And if the ring of coefficients is extended by both $\sqrt{2}$ and i—into four factors.

In[26] := Factor[*a*,Extension−>{Sqrt[2],*I*}]

Out[26] $= -\left(\sqrt{2}-x\right)\left(\sqrt{2}-\mathrm{i}x\right)\left(\sqrt{2}+\mathrm{i}x\right)\left(\sqrt{2}+x\right)$

And this is a rational function.

In[27] := (*x*^3 − *y*^3)/(*x*^2 − *y*^2)

Out[27] $= \dfrac{x^3-y^3}{x^2-y^2}$

It is not canceled by the greatest common divisor of its numerator and denominator; this should be done explicitly.

In[28] := Cancel[%]

Out[28] $= \dfrac{x^2+xy+y^2}{x+y}$

(% means the result of the previous calculation). A sum of rational functions.

In[29] := *a* = *x*/(*x* + *y*) + *y*/(*x* − *y*)

Out[29] $= \dfrac{y}{x-y}+\dfrac{x}{x+y}$

Let's put it over the common denominator.

In[30] := *a* = Together[*a*]

Out[30] $= \dfrac{x^2+y^2}{(x-y)(x+y)}$

Partial fraction decomposition with respect to x.

In[31] := Apart[*a*,*x*]

Out[31] $= 1+\dfrac{y}{x-y}-\dfrac{y}{x+y}$

In[32] := Clear[*a*]

2.4 Elementary Functions

Mathematica knows some simple properties of elementary functions.

In[33] := Sin[−*x*]

Out[33] $= -\mathrm{Sin}[x]$

In[34] := Cos[Pi/4]

Out[34] $= \dfrac{1}{\sqrt{2}}$

In[35] := Sin[5 ∗ Pi/6]

Out[35] $= \dfrac{1}{2}$

In[36] := Log[1]

Out[36] $= 0$

In[37] := Log[*E*]

Out[37] $= 1$

In[38] := Exp[Log[*x*]]

Out[38] $= x$

In[39] := Log[Exp[*x*]]

Out[39] $= \mathrm{Log}\left[\mathrm{e}^x\right]$
And why not x? Because this simplification is not always correct. Try to substitute $2\pi\mathrm{i}$.
In[40] := Sqrt[0]
Out[40] $= 0$
In[41] := Sqrt[*x*]^2
Out[41] $= x$
In[42] := Sqrt[*x*^2]
Out[42] $= \sqrt{x^2}$
And why not x? Try to substitute -1.
In[43] := *a* = Sqrt[12 * *x*^2 * *y*]
Out[43] $= 2\sqrt{3}\sqrt{x^2 y}$
This result can be improved, if we know that $x > 0$.
In[44] := Simplify[*a*, *x* > 0]
Out[44] $= 2\sqrt{3}x\sqrt{y}$
And this is the case $x < 0$.
In[45] := Simplify[*a*, *x* < 0]
Out[45] $= -2\sqrt{3}x\sqrt{y}$
Expansion of trigonometric functions of multiple angles, sums, and differences:
In[46] := TrigExpand[Cos[2 * *x*]]
Out[46] $= \mathrm{Cos}[x]^2 - \mathrm{Sin}[x]^2$
In[47] := TrigExpand[Sin[*x* − *y*]]
Out[47] $= \mathrm{Cos}[y]\,\mathrm{Sin}[x] - \mathrm{Cos}[x]\,\mathrm{Sin}[y]$
The inverse operation—transformation of products and powers of trigonometric functions into linear combinations of such functions—is used more often. Let's take a truncated Fourier series.
In[48] := *a* = a1 * Cos[*x*] + a2 * Cos[2 * *x*] + b1 * Sin[*x*] + b2 * Sin[2 * *x*]
Out[48] $= \mathrm{a1}\,\mathrm{Cos}[x] + \mathrm{a2}\,\mathrm{Cos}[2x] + \mathrm{b1}\,\mathrm{Sin}[x] + \mathrm{b2}\,\mathrm{Sin}[2x]$
Its square is again a truncated Fourier series.
In[49] := TrigReduce[*a*^2]

$$\mathrm{Out[49]} = \frac{1}{2}\left(\mathrm{a1}^2 + \mathrm{a2}^2 + \mathrm{b1}^2 + \mathrm{b2}^2 + 2\,\mathrm{a1}\,\mathrm{a2}\,\mathrm{Cos}[x] + 2\,\mathrm{b1}\,\mathrm{b2}\,\mathrm{Cos}[x] + \mathrm{a1}^2\,\mathrm{Cos}[2x] - \mathrm{b1}^2\,\mathrm{Cos}[2x] + 2\,\mathrm{a1}\,\mathrm{a2}\,\mathrm{Cos}[3x] - 2\,\mathrm{b1}\,\mathrm{b2}\,\mathrm{Cos}[3x] + \mathrm{a2}^2\,\mathrm{Cos}[4x] - \mathrm{b2}^2\,\mathrm{Cos}[4x] - 2\,\mathrm{a2}\,\mathrm{b1}\,\mathrm{Sin}[x] + 2\,\mathrm{a1}\,\mathrm{b2}\,\mathrm{Sin}[x] + 2\,\mathrm{a1}\,\mathrm{b1}\,\mathrm{Sin}[2x] + 2\,\mathrm{a2}\,\mathrm{b1}\,\mathrm{Sin}[3x] + 2\,\mathrm{a1}\,\mathrm{b2}\,\mathrm{Sin}[3x] + 2\,\mathrm{a2}\,\mathrm{b2}\,\mathrm{Sin}[4x]\right)$$

2.5 Calculus

Let's take a function.
In[50] := *f* = Log[*x*^5 + *x* + 1] + 1/(*x*^5 + *x* + 1)

$$\mathrm{Out[50]} = \frac{1}{1+x+x^5} + \mathrm{Log}\left[1+x+x^5\right]$$

Calculate its derivative.

In[51] := g = D[f,x]

$$\text{Out}[51] = -\frac{1+5x^4}{(1+x+x^5)^2} + \frac{1+5x^4}{1+x+x^5}$$

Put over the common denominator.

In[52] := g = Together[g]

$$\text{Out}[52] = \frac{(1+5x^4)(x+x^5)}{(1+x+x^5)^2}$$

A stupid integration algorithm would try to solve the fifth degree equation in the denominator, in order to decompose the integrand into partial fractions. *Mathematica* is more clever than that.

In[53] := Integrate[g,x]

$$\text{Out}[53] = \frac{1}{1+x+x^5} + \text{Log}\left[1+x+x^5\right]$$

Let's expand our function in x at 0 up to x^{10}.

In[54] := Series[f, {x,0,10}]

$$\text{Out}[54] = 1 + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{3x^4}{4} - \frac{4x^5}{5} + \frac{11x^6}{6} - \frac{20x^7}{7} + \frac{31x^8}{8} - \frac{44x^9}{9} + \frac{32x^{10}}{5} + O[x]^{11}$$

Mathematica can calculate many definite integrals even when the corresponding indefinite integral cannot be taken. Here is an integral from 0 to 1.

In[55] := Integrate[Log[x]^2/(x+1), {x,0,1}]

$$\text{Out}[55] = \frac{3\,\text{Zeta}[3]}{2}$$

Mathematica knows how to sum many series.

In[56] := Sum[1/n^4, {n,1,Infinity}]

$$\text{Out}[56] = \frac{\pi^4}{90}$$

Let's clear all the garbage we have generated—a very good habit.

In[57] := Clear[f,g]

2.6 Lists

We have already encountered this construct several times:

In[58] := a = {x,y,z}

Out[58] = $\{x,y,z\}$

This is a list. And here are its elements.

In[59] := a[[1]]

Out[59] = x

In[60] := a[[2]]

Out[60] = y

In[61] := a[[3]]

Out[61] = z

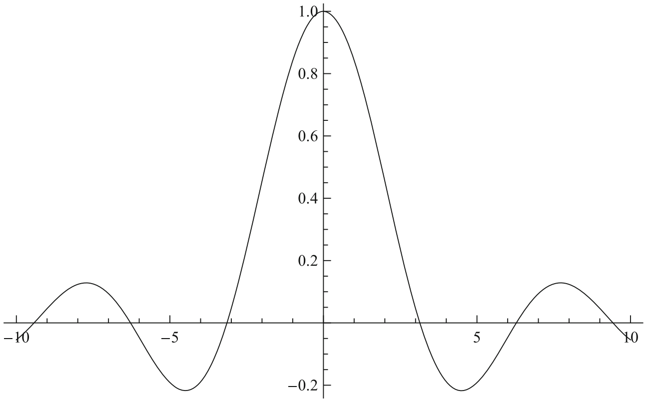
In[62] := Clear[a]

2.7 Plots

A simple plot of a function.

In[63] := Plot[Sin[*x*]/*x*, {*x*, −10, 10}]

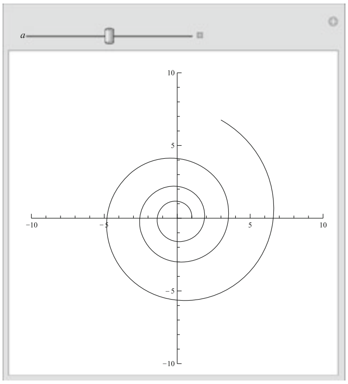
Out[63] =



A curve given parametrically—x and y are functions of t. This particular curve contains a parameter a, which can be adjusted by the mouse. If you click the small plus sign near the marker, a control panel will open. There you can start (and stop) animation.

In[64] := Manipulate[ParametricPlot[{Exp[*a* ∗ *t*] ∗ Cos[*t*], Exp[*a* ∗ *t*] ∗ Sin[*t*]}, {*t*, 0, 20}, PlotRange−>{{−10, 10}, {−10, 10}}], {{*a*, 0.1}, 0, 0.2}]

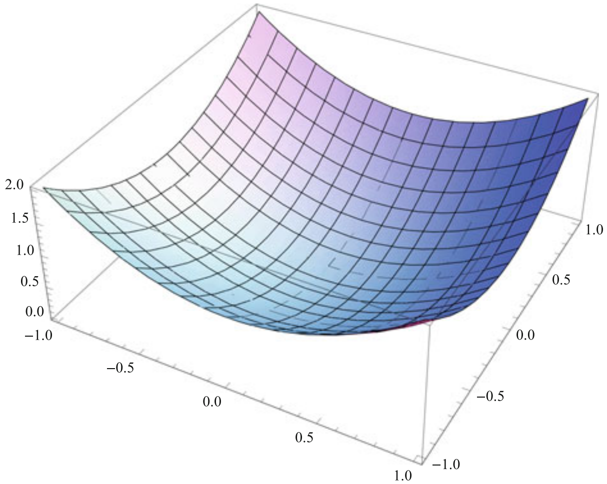
Out[64] =



A three-dimensional plot of a function of two variables. It can be rotated by the mouse.

In[65] := Plot3D[*x*^2 + *y*^2, {*x*, −1, 1}, {*y*, −1, 1}]

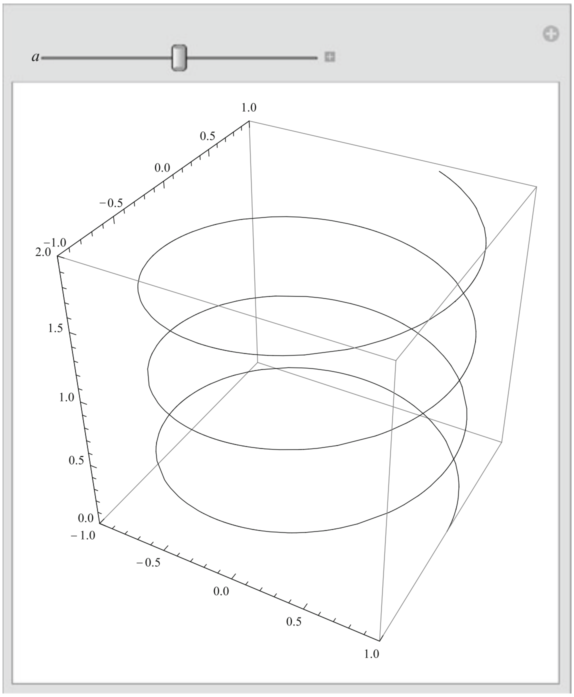
Out[65] =



A three-dimensional curve given parametrically. The parameter *a* can be adjusted by the mouse.

**In[66] := Manipulate[ParametricPlot3D[{Cos[*t*], Sin[*t*], *a* ∗ *t*}, {*t*, 0, 20},
PlotRange−>{{−1, 1}, {−1, 1}, {0, 2}}], {{*a*, 0.1}, 0, 0.2}]**

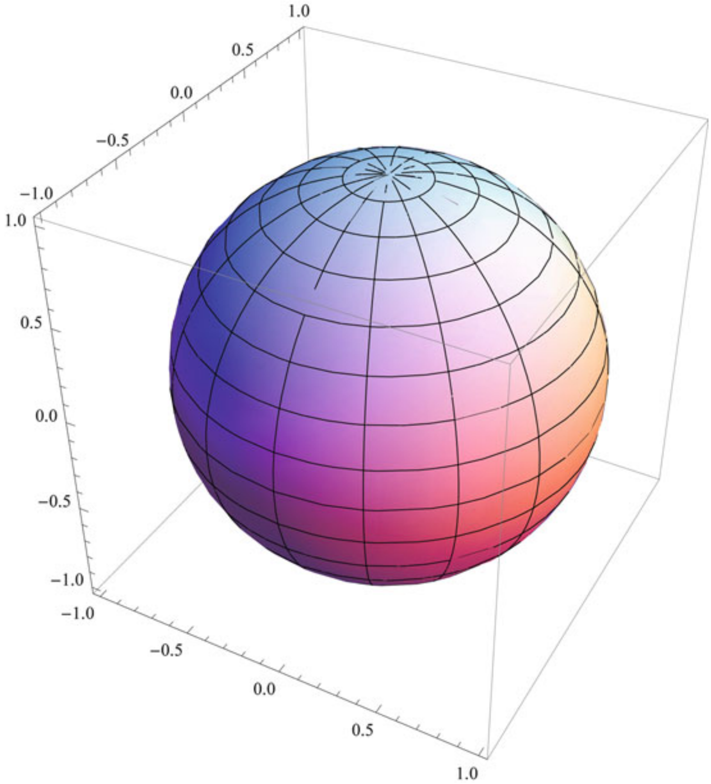
Out[66] =



A surface given parametrically.

In[67] := ParametricPlot3D[{Sin[*t*] * Cos[*u*], Sin[*t*] * Sin[*u*], Cos[*t*]}, {*t*, 0, Pi}, {*u*, 0, 2 * Pi}]

Out[67] =



2.8 Substitutions

Substitutions are a fundamental concept in *Mathematica*, its main working instrument. This substitution replaces $f[x]$ by x^2.

In[68] := *S* = *f*[*x*]−>*x*^2

Out[68] = $f[x] \to x^2$

Let's apply it to the expression $f[x]$.

In[69] := *f*[*x*]/.*S*

Out[69] = x^2

We've got x^2, as expected. And what if we apply it to $f[y]$?

In[70] := *f*[*y*]/.*S*

Out[70] = $f[y]$

It hasn't triggered. The following substitution replaces the function f with an arbitrary argument by the square of this argument.

In[71] := S = f[x_]−>x^2
Out[71] $= f[x\_] \to x^2$
Let's check.
In[72] := {f[x], f[y], f[2]}/.S
Out[72] $= \{x^2, y^2, 4\}$
In[73] := Clear[S]

2.9 Equations

Here is an equation.
In[74] := Eq = a∗x+b == 0
Out[74] $= b + ax == 0$
Let's solve it for x.
In[75] := S = Solve[Eq, x]
Out[75] $= \left\{\left\{x \to -\frac{b}{a}\right\}\right\}$
We've got a list of solutions, in this particular case having a single element. Each solution is a list of substitutions, which replaces our unknowns by the corresponding expressions. And how can we extract the value of x from this result? Let's take the first (and the only) element of the list S.
In[76] := S1 = First[S]
Out[76] $= \left\{x \to -\frac{b}{a}\right\}$
And now we apply this list of substitutions (in this particular case, it's single element) to the unknown x.
In[77] := x/.S1
Out[77] $= -\frac{b}{a}$

Here is a more advanced example—a quadratic equation. It has two solutions.
In[78] := S = Solve[a∗x^2+b∗x+c == 0, x]
Out[78] $= \left\{\left\{x \to \frac{-b-\sqrt{b^2-4ac}}{2a}\right\}, \left\{x \to \frac{-b+\sqrt{b^2-4ac}}{2a}\right\}\right\}$
How can we extract the value of x in the second solution? Let's apply the second element of the solutions list S (which is a single-element list of substitutions) to the unknown x.
In[79] := x/.S[[2]]
Out[79] $= \frac{-b+\sqrt{b^2-4ac}}{2a}$

And here is a system of 2 linear equations.
In[80] := Eq = {a∗x+b∗y == e, c∗x+d∗y == f}
Out[80] $= \{ax + by == e, cx + dy == f\}$
It has a single solution.

In[81] := S = Solve[Eq, {x, y}]

$$\text{Out[81]} = \left\{\left\{x \to -\frac{de-bf}{bc-ad}, y \to -\frac{-ce+af}{bc-ad}\right\}\right\}$$

This (first and the only) solution is a list of two substitutions.

In[82] := S1 = S[[1]]

$$\text{Out[82]} = \left\{x \to -\frac{de-bf}{bc-ad}, y \to -\frac{-ce+af}{bc-ad}\right\}$$

How to find the values of x and y in this solution? Apply this list of substitutions to the unknowns x and y.

In[83] := {x/.S1, y/.S1}

$$\text{Out[83]} = \left\{-\frac{de-bf}{bc-ad}, -\frac{-ce+af}{bc-ad}\right\}$$

In[84] := Clear[Eq, S, S1]