Chapter 25
Problems for Students

1. Write a procedure which returns the hydrogen wave function (in spherical coordinates, i. e., an expression containing r, θ, ϕ) for given quantum numbers n, l, m. Write a procedure to calculate the rate of the electric dipole transition [22] from the state n, l, m to the state n', l', m'.

2. Calculate Poisson brackets of the Hamiltonian, the angular momentum components, and the Runge–Lenz vector components [19] for a particle in the Coulomb field $U = -a/r$. Calculate commutators of the same quantities in quantum mechanics [18].

3. The hypergeometric function [23, 24, 27] is defined as the sum of the series

$$F(a,b,c,x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!},$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol. In many cases it can be expressed via simpler functions. Write a list of substitutions for simplifying hypergeometric functions. It is sufficient to consider only simplifications valid for an arbitrary x (not for specific values) where results are expressed via elementary functions. More general substitutions should be near the beginning of the list, then their particular cases can be eliminated.

4. Consider indefinite integrals of the form

$$\int A(x) \log B(x)\, \mathrm{d}x,$$

where $A(x)$ and $B(x)$ are rational functions of x. *Mathematica* can calculate such integrals, but often produces results in which some terms have imaginary parts in the region of x we are interested in. It is not easy to trace their cancellations. We'll suppose that $A(x)$ and $B(x)$ contain no parameters (except x), only numbers. We'll also suppose that *Mathematica* is able to find all roots of the denominator of $A(x)$, as well as of the numerator and the denominator of $B(x)$, and all these roots are real.

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We are interested in a neighborhood of some point x_0; we want to get a result all terms of which are real near this point (if this is possible, of course). Implement the following obvious approach:

- Expand $A(x)$ into partial fractions with respect to x.
- Replace $\log B(x)$ by a combination of terms $\log(x-a_i)$ and $\log(a_i-x)$ (plus a constant) in such a way that they are all real near x_0.
- Multiply.
- Take integrals of $x^n \log(x-a)$ $(n \geq 0)$, $\log(x-a)/(x-b)^n$ $(n \geq 2)$ by parts to eliminate the logarithm. Don't use the *Mathematica* integrator—it can produce $\log(x-a)$ where $\log(a-x)$ is needed.
- We are left with the most difficult terms of the forms $\log(x-a)/(x-b)$ and $\log(a-x)/(x-b)$. By linear substitutions they reduce to 3 cases:

$$\int \frac{\log(y+1)}{y}\,\mathrm{d}y = -\mathrm{Li}_2(-y)\,,$$
$$\int \frac{\log(y-1)}{y}\,\mathrm{d}y = \log(y)\log(y-1) + \mathrm{Li}_2(1-y)\,,$$
$$\int \frac{\log(1-y)}{y}\,\mathrm{d}y = -\mathrm{Li}_2(y)\,,$$

  where y is positive near $x = x_0$ (the third formula is the definition of $\mathrm{Li}_2(y)$; the first one follows from it using the substitution $y \to -y$; the second one—using integration by parts).

The result must be real ($\log(x)$ is real at $x > 0$; $\mathrm{Li}_2(x)$—at $x < 1$). If this is impossible, print an error message.

5. Implement the algebra of Boolean expressions. They consist of the constants true and false, variables, the function not (one argument), and the functions and, or (an arbitrary number of arguments). The last two functions are commutative and associative. Take into account simplifications when one of the arguments is true or false; when two arguments coincide or equal to a and not$[a]$. Expressions should be reduced to the disjunctive normal form: "or" at the top level; its arguments can be "and"; their arguments can be "not" or variables.

6. Implement the algebra of quaternions.

7. Implement Dirac γ-matrix expressions, including trace calculations (in 4 dimensions [22] or in the general case of dimensional regularization, see, e.g., [24]). Pay no attention to efficiency.

8. Implement calculation of color factors of Feynman diagrams for the color group $SU(N_c)$ using the Cvitanović algorithm [27] (see also [24]).

9. Write a procedure to calculate two-loop massless propagator diagrams using integration by parts (see, e.g., [24]). Results should be linear combinations of the two basis integrals.

10. Hypergeometric functions whose argument is 1 and whose parameters contain a small parameters ε and tend to integers at $\varepsilon \to 0$ can be expanded in series in ε. The algorithm is described, e.g., in [24]; implement it.

11. Any polynomial over the field of complex numbers can be factorized into linear factors:

$$p(x) = \prod (x - a_i)^{d_i} ,$$

where a_i are its roots and d_i are their multiplicities (to simplify formulas, we have assumed that the leading coefficient is 1). Let's group factors with equal d_i:

$$p(x) = \prod p_i^{d_i} ,$$

where all d_i are distinct and the polynomials $p_i(x)$ have only simple zeros (are square-free). This square-free factorization can be obtained by a simple algorithm which uses only gcd (this is much simpler than the full factorization). Namely,

$$\gcd(p, p') = \prod p_i^{d_i - 1} .$$

Indeed, the polynomial $p(x)$ has zero of the order d_i at $x \to a_i$, and its derivative $p'(x)$ has zero of the order $d_i - 1$. Write a function to calculate square-free factorization using only gcd.