Chapter 15
Trigonometric Functions

Statement of the Problem

Mathematica can calculate Cos and Sin for many arguments equal to π times rational numbers. For example,

$\text{In}[1] := \text{Table}[\text{Cos}[\text{Pi}/n], \{n, 1, 12\}]$

$\text{Out}[1] = \left\{ -1, 0, \dfrac{1}{2}, \dfrac{1}{\sqrt{2}}, \dfrac{1}{4}\left(1+\sqrt{5}\right), \dfrac{\sqrt{3}}{2}, \text{Cos}\left[\dfrac{\pi}{7}\right], \text{Cos}\left[\dfrac{\pi}{8}\right], \text{Cos}\left[\dfrac{\pi}{9}\right], \right.$

$\left. \sqrt{\dfrac{5}{8}+\dfrac{\sqrt{5}}{8}}, \text{Cos}\left[\dfrac{\pi}{11}\right], \dfrac{1+\sqrt{3}}{2\sqrt{2}} \right\}$

But it does not apply the half-angle formulas in all possible cases. We'll write our own cos function which does this; then

$\text{In}[2] := \sin[x\_] := \cos[x - \text{Pi}/2]$

Simple Cases

These cases should be considered separately because *Mathematica* does not treat them as rational numbers times π.

$\text{In}[3] := \cos[0] = 1; \cos[\text{Pi}] = -1;$

$\text{In}[4] := \cos[n\_\text{Integer} * \text{Pi}] := (-1)\wedge n$

General Case

$\text{In}[5] := \cos[r\_\text{Rational} * \text{Pi}] := \text{Which}[r < 0, \cos[-r * \text{Pi}],$
$\quad r > 2, \cos[\text{FractionalPart}[r/2] * 2 * \text{Pi}], r > 1, \cos[(2-r) * \text{Pi}],$
$\quad r > 1/2, -\cos[(1-r) * \text{Pi}],$

EvenQ[Denominator[r]], Simplify[Sqrt[(1 + cos[2 ∗ r ∗ Pi])/2]],
True, Cos[r ∗ Pi]]

For example,

In[6] := {cos[Pi/32], cos[−65 ∗ Pi/32], cos[3 ∗ Pi/32], cos[33 ∗ Pi/32]}

$$\text{Out}[6] = \left\{ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}, \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}, \right.$$

$$\left. \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2}}}}, -\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} \right\}$$

In[7] := {cos[Pi/48], cos[5 ∗ Pi/48], cos[7 ∗ Pi/48]}

$$\text{Out}[7] = \left\{ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}, \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{3}}}}, \right.$$

$$\left. \frac{1}{2}\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{3}}}} \right\}$$

In[8] := {cos[Pi/40], cos[3 ∗ Pi/40]}

$$\text{Out}[8] = \left\{ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}}}, \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{\frac{1}{2}\left(5-\sqrt{5}\right)}}} \right\}$$

Check

In[9] := check[$d\_, n\_$] := Module[{$e = 0, x, \text{ex}$},
　　Do[$x = i/d$ ∗ Pi; ex = Abs[N[cos[x] − Cos[x]]]; If[ex > $e, e =$ ex],
　　　{$i, -d ∗ n, d ∗ n$}];
　　e]

In[10] := {check[128, 5], check[192, 5], check[320, 5]}

$$\text{Out}[10] = \left\{ 2.220446049250313 \times 10^{-16}, 6.106226635438361 \times 10^{-16}, \right.$$

$$\left. 1.861358289723114 \times 10^{-15} \right\}$$