

22

Geometric Quantization on Euclidean Space

22.1 Introduction

In this chapter, we consider the geometric quantization program in the setting of the symplectic manifold \mathbb{R}^{2n} , with the canonical 2-form $\omega = dp_j \wedge dx_j$. We begin with the “prequantum” Hilbert space $L^2(\mathbb{R}^{2n})$ and define “prequantum” operators $Q_{\text{pre}}(f)$. These operators satisfy

$$Q_{\text{pre}}(\{f, g\}) = \frac{1}{i\hbar} [Q_{\text{pre}}(f), Q_{\text{pre}}(g)]$$

for *all* f and g . Nevertheless, there are several undesirable aspects to the prequantization map that make it physically unreasonable to interpret it as “quantization.” To obtain the quantum Hilbert space, we reduce the number of variables from $2n$ to n . Depending on how we do this reduction, we will obtain either the position Hilbert space, the momentum Hilbert space, or the Segal–Bargmann space. Each of these subspaces is preserved by the prequantized position and momentum operators, and by certain other operators of the form $Q_{\text{pre}}(f)$.

Although the material in this chapter is a special case of what we do in Chap. 23, doing this case first allows us to get a feeling for the methods and results of geometric quantization quickly, without needing to develop the full machinery of line bundles, connections, and polarizations over general symplectic manifolds. In any case, we would need to carry out most of the calculations in this chapter eventually, as standard examples of the general theory.

Although this chapter does not require the full machinery of symplectic manifolds, we will make use of the notions of 1-forms and 2-forms on \mathbb{R}^{2n} , along with the notion of the differential of a 1-form. In particular, the expression (21.6) for the differential of a 1-form will be used.

The reader should be warned that sign conventions in geometric quantization are not consistent from author to author. The sign conventions used here are chosen to maintain consistency with the physics literature. In particular, we could eliminate an annoying minus sign in the definition of the holomorphic subspace if we were willing to allow the function p_j to quantize to $i\hbar \partial/\partial x_j$. Since, however, the convention $P_j = -i\hbar \partial/\partial x_j$ is universal in the physics literature, we have chosen to be consistent with that convention and to accept some slightly inconvenient sign choices elsewhere. We continue to follow the *summation convention*, in which repeated indices are always summed on.

22.2 Prequantization

Ideally, a quantization procedure Q , mapping functions on a symplectic manifold N to operators on some Hilbert space \mathbf{H} , should satisfy the following properties. First, $Q(f)$ should be self-adjoint whenever f is real valued. Second, we should have $Q(\mathbf{1}) = I$, where $\mathbf{1}$ is the constant function. Third, $Q(\{f, g\})$ should be equal to $[Q(f), Q(g)]/(i\hbar)$. Fourth, there should be some sort of “smallness” assumption. In the case $N = \mathbb{R}^{2n}$, for example, we may require that \mathbf{H} should be irreducible under the action of the (exponentiated) position and momentum operators. (See Definition 14.6.) Although Groenewold’s theorem (Theorem 13.13) suggests that it is unrealistic to expect to find a quantization procedure that satisfies all of these properties exactly, we try to come as close as possible.

Throughout this chapter, we follow the convention of thinking of a “vector field” on \mathbb{R}^N as a first-order differential operator, as in Exercise 14 in Chap. 2. Given, for example, the vector-valued function

$$X = (2x_1 + x_2, x_1x_2)$$

on \mathbb{R}^2 , we identify X with the operator of “differentiation in the direction of X ,” that is, with the following first-order differential operator:

$$X = (2x_1 + x_2) \frac{\partial}{\partial x_1} + x_1x_2 \frac{\partial}{\partial x_2}.$$

In particular, given a smooth function f on \mathbb{R}^{2n} , the Hamiltonian vector field X_f associated to f is thought of as a differential operator:

$$X_f = \{f, \cdot\} = \frac{\partial f}{\partial x_j} \frac{\partial}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial}{\partial x_j}, \quad (22.1)$$

acting on $C^\infty(\mathbb{R}^{2n})$. (Compare Proposition 21.7.) By Proposition 21.11, the commutator (as differential operators) of two Hamiltonian vector fields X_f and X_g is $X_{\{f,g\}}$. Thus, the operators $i\hbar X_f$ satisfy the desired commutation relations:

$$[i\hbar X_f, i\hbar X_g] = (i\hbar)^2 X_{\{f,g\}} = (i\hbar)(i\hbar X_{\{f,g\}}).$$

It is tempting, then, to define a (pre)quantization map simply by taking $Q(f) = i\hbar X_f$, viewed as a self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^{2n})$. This map, however, does not satisfy $Q(\mathbf{1}) = I$. If we to correct our definition to $Q(f) = i\hbar X_f + f$, where f means the operator of multiplication by f , then $Q(\mathbf{1}) = I$ but the desired commutation property is destroyed.

It is possible to achieve both $Q(\mathbf{1}) = I$ and the desired commutation relations by adding one more term as follows. If $\omega = dp_j \wedge dx_j$ is the canonical 2-form on \mathbb{R}^{2n} , let θ be any *symplectic potential* for ω , that is, any one-form with

$$d\theta = \omega. \quad (22.2)$$

(We may, e.g., take $\theta = p_j dx_j$.) For a smooth function f on \mathbb{R}^{2n} , define an operator $Q_{\text{pre}}(f)$, acting on $C^\infty(\mathbb{R}^{2n})$, by

$$Q_{\text{pre}}(f) = i\hbar \left(X_f - \frac{i}{\hbar} \theta(X_f) \right) + f. \quad (22.3)$$

The expression f on the right-hand side of (22.3) means, more precisely, the operator of multiplication by f , and similarly for the function $\theta(X_f)$. Note that since θ is a 1-form and X_f is a vector field, $\theta(X_f)$ is a function on \mathbb{R}^{2n} . The operator $Q_{\text{pre}}(f)$ is the *prequantization* of f and is to be viewed as an unbounded operator on $L^2(\mathbb{R}^{2n})$, where we refer to $L^2(\mathbb{R}^{2n})$ as the *prequantum Hilbert space*.

According to Exercise 1, any divergence free vector field on \mathbb{R}^N is a skew-symmetric operator on $C_c^\infty(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$. Meanwhile, each Hamiltonian vector field is divergence free, as we have already remarked in the proof of Liouville's theorem (Theorem 2.27). Thus, for any smooth, real-valued function f on \mathbb{R}^{2n} , the operator $Q_{\text{pre}}(f)$ is at least symmetric. It can be shown that if X_f is complete, meaning that the associated Hamiltonian flow is defined for all times, then $Q_{\text{pre}}(f)$ is actually self-adjoint on a natural domain. (See the discussion following the proof of Proposition 23.13.)

As it turns out, the $\theta(X_f)$ term in (22.3) is precisely what is needed to restore the desired commutation relations, while still allowing $Q_{\text{pre}}(\mathbf{1})$ to equal the identity.

Proposition 22.1 *For all $f, g \in C^\infty(\mathbb{R}^{2n})$, we have*

$$\frac{1}{i\hbar} [Q_{\text{pre}}(f), Q_{\text{pre}}(g)] = Q_{\text{pre}}(\{f, g\}),$$

where the identity is to be understood as an equality of operators on $C^\infty(\mathbb{R}^{2n})$.

Before proving this result, it is useful to understand the behavior of the expression $X_f - (i/\hbar)\theta(X_f)$ occurring in the definition of $Q_{\text{pre}}(f)$.

Definition 22.2 For any symplectic potential θ and vector field X on \mathbb{R}^{2n} , let ∇_X denote the **covariant derivative operator**, acting on $C^\infty(\mathbb{R}^{2n})$, given by

$$\nabla_X = X - \frac{i}{\hbar}\theta(X). \tag{22.4}$$

Note that our prequantized operators can be written as

$$Q_{\text{pre}}(f) = i\hbar\nabla_{X_f} + f.$$

Proposition 22.3 For any symplectic potential θ , let ∇_X denote the associated covariant derivative in (22.4). Then for all smooth vector fields X and Y on \mathbb{R}^{2n} , we have

$$[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} - \frac{i}{\hbar}\omega(X, Y). \tag{22.5}$$

In particular, if $X = X_f$ and $Y = X_g$, we have

$$[\nabla_{X_f}, \nabla_{X_g}] = \nabla_{X_{\{f,g\}}} + \frac{i}{\hbar}\{f, g\}.$$

According to standard differential geometric definitions, the 2-form ω/\hbar on the right-hand side of (22.5) is the *curvature* of the covariant derivative ∇ . For our purposes, the fact that $[\nabla_{X_f}, \nabla_{X_g}]$ is not simply $\nabla_{X_{\{f,g\}}}$ is an advantage. The extra term in the formula for the commutator is just what we need to compensate for the failure of the operators $i\hbar X_f + f$ to have the desired commutation relations.

Proof. Using the easily verified identity $[\nabla_X, f] = X(f)$, we obtain

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = -\frac{i}{\hbar}[X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y])].$$

In light of (21.6), the right-hand side becomes $-(i/\hbar)(d\theta)(X, Y)$, where $d\theta = \omega$. ■

We may now easily prove Proposition 22.1.

Proof of Proposition 22.1. Using Proposition 22.3, we obtain

$$\begin{aligned} & \frac{1}{i\hbar} [i\hbar\nabla_{X_f} + f, i\hbar\nabla_{X_g} + g] \\ &= (i\hbar) \left(\nabla_{X_{\{f,g\}}} + \frac{i}{\hbar}\{f, g\} \right) + X_f(g) - X_g(f) \\ &= i\hbar\nabla_{X_{\{f,g\}}} - \{f, g\} + \{f, g\} + \{f, g\}, \end{aligned}$$

which reduces to what we want. ■

Example 22.4 If $\theta = p_j dx_j$, the prequantized position and momentum operators are given by

$$Q_{\text{pre}}(x_j) = x_j + i\hbar \frac{\partial}{\partial p_j}$$

$$Q_{\text{pre}}(p_j) = -i\hbar \frac{\partial}{\partial x_j}.$$

These operators are essentially self-adjoint on $C_c^\infty(\mathbb{R}^{2n})$ and their self-adjoint extensions satisfy the exponentiated commutation relations of Definition 14.2.

Proof. We compute that $X_{x_j} = \partial/\partial p_j$ and that $\theta(X_{x_j}) = 0$, giving the indicated expression for $Q_{\text{pre}}(x_j)$. Meanwhile, $X_{p_j} = -\partial/\partial x_j$ and $\theta(X_{p_j}) = -p_j$. There is a cancellation of the $\theta(X_{p_j})$ term in the definition of $Q_{\text{pre}}(p_j)$ with the p_j term, leaving $Q_{\text{pre}}(p_j) = i\hbar X_{p_j}$.

The essential self-adjointness of the operators follows from Proposition 9.40. To verify the exponentiated commutation relations, we calculate the associated one-parameter unitary groups as

$$(e^{itQ_{\text{pre}}(x_j)}\psi)(\mathbf{x}, \mathbf{p}) = e^{itx_j}\psi(\mathbf{x}, \mathbf{p} - t\mathbf{h}e_j)$$

$$(e^{itQ_{\text{pre}}(p_j)}\psi)(\mathbf{x}, \mathbf{p}) = \psi(\mathbf{x} + t\mathbf{h}e_j, \mathbf{p}), \tag{22.6}$$

where we now let $Q_{\text{pre}}(x_j)$ and $Q_{\text{pre}}(p_j)$ denote the unique self-adjoint extensions of the given operators on $C_c^\infty(\mathbb{R}^{2n})$. (Compare Proposition 13.5.) The exponentiated commutation relations can now be easily verified by direct calculation. ■

As we have presented things so far, the concept of covariant derivative, and thus also of prequantization, depends on the choice of symplectic potential θ . This dependence is, however, illusory; we will now show that the prequantum maps obtained with two different symplectic potentials are unitarily equivalent.

Proposition 22.5 Suppose that θ_1 and θ_2 are two different symplectic potentials for the canonical 2-form ω , so that $d(\theta^1 - \theta^2) = 0$. Let the associated covariant derivatives be denoted by ∇^1 and ∇^2 . Choose a real-valued function γ so that $d\gamma = \theta^1 - \theta^2$ and let U_γ be the unitary map of $L^2(\mathbb{R}^{2n})$ to itself given by

$$U_\gamma\psi = e^{-i\gamma/\hbar}\psi.$$

Then for every vector field X , we have

$$U_\gamma\nabla_X^1U_\gamma^{-1} = \nabla_X^2. \tag{22.7}$$

If $Q_{\text{pre}}^j(f)$, $j = 1, 2$, are the associated prequantization maps, it follows that

$$U_\gamma Q_{\text{pre}}^1(f)U_\gamma^{-1} = Q_{\text{pre}}^2(f). \tag{22.8}$$

The map U_γ is called a **gauge transformation**.

Proof. The operation of multiplication by $\theta^1(X)$ commutes with multiplication by $e^{-i\gamma/\hbar}$, whereas

$$X(e^{i\gamma/\hbar}\psi) = e^{i\gamma/\hbar}X\psi + \frac{i}{\hbar}e^{i\gamma/\hbar}X(\gamma)\psi.$$

Since $X(\gamma) = (d\gamma)(X) = \theta^1(X) - \theta^2(X)$, we obtain

$$\begin{aligned} \nabla_X^1(e^{i\gamma/\hbar}\psi) &= e^{i\gamma/\hbar} \left(X + \frac{i}{\hbar}X(\gamma) - \frac{i}{\hbar}\theta^1(X_f) \right) \psi \\ &= e^{i\gamma/\hbar} \left(X - \frac{i}{\hbar}\theta^2(X_f) \right) \psi \\ &= e^{i\gamma/\hbar} \nabla_X^2 \psi. \end{aligned}$$

Multiplying both sides of this equality by $e^{-i\gamma/\hbar}$ gives (22.7). Equation (22.8) follows by observing that multiplication by f commutes with multiplication by $e^{-i\gamma/\hbar}$. ■

22.3 Problems with Prequantization

Given the naturalness of the prequantization construction, it is tempting to think that prequantization could actually be considered as quantization. Why not take our Hilbert space to be $L^2(\mathbb{R}^{2n})$ and the quantized operators to be $Q_{\text{pre}}(f)$? To answer this question, we now examine some undesirable properties of prequantization.

In the first place, the Hilbert space $L^2(\mathbb{R}^{2n})$ is very far from irreducible under the action of the quantized position and momentum operators, in contrast to the ordinary Schrödinger Hilbert space $L^2(\mathbb{R}^n)$, which is irreducible, by Proposition 14.7. Indeed, in Sect. 22.4, we will construct a large family of invariant subspaces. (See Proposition 22.13.)

In the second place, the prequantization map is very far from being multiplicative. Of course, since quantum operators do not commute, we cannot expect any quantization scheme Q to satisfy $Q(fg) = Q(f)Q(g)$ for all f and g . Nevertheless, the standard quantization schemes we have considered in Chap. 13 do satisfy this relation for certain classes of observables f and g . In the Weyl quantization, for example, we have multiplicativity if f and g are both functions of \mathbf{x} only, independent of \mathbf{p} (or functions of \mathbf{p} , independent of \mathbf{x}). For the prequantization map, however, we almost never have multiplicativity, for the simple reason that $Q_{\text{pre}}(fg)$ is a first-order differential operator, whereas $Q_{\text{pre}}(f)Q_{\text{pre}}(g)$ is second-order, provided there is at least one point where X_f and X_g are both nonzero.

In the third place, the prequantization map badly fails to map positive functions to positive operators. Although most of the quantization schemes in Chap. 13 do not *always* map positive functions to positive operators, they

somehow come close to doing so. Indeed, Q_{Weyl} , Q_{Wick} , and $Q_{\text{anti-Wick}}$ all map the harmonic oscillator Hamiltonian to a non-negative operator, since $a^*a + (1/2)I$, a^*a , and aa^* are all non-negative. (See Exercise 4 in Chap. 13.) By contrast, the prequantized harmonic oscillator Hamiltonian has spectrum that is unbounded below, as we now demonstrate.

Proposition 22.6 *Consider a harmonic oscillator Hamiltonian of the form*

$$H(x, p) = \frac{1}{2m} (p^2 + (m\omega x)^2).$$

Then for each integer n , the number $n\hbar\omega$ is an eigenvalue for $Q_{\text{pre}}(H)$.

Note that n in the proposition is allowed to be negative, so that the spectrum of $Q_{\text{pre}}(H)$ is not even bounded below. On the other hand, in Sect. 22.5, we will consider a certain closed subspace \mathbf{H}_α of the prequantum Hilbert space, which is one candidate for the *quantum* Hilbert space. For appropriate choice of α , the space \mathbf{H}_α is invariant under $Q_{\text{pre}}(H)$ and the restriction of $Q_{\text{pre}}(H)$ is self-adjoint with spectrum $n\hbar\omega$, where n ranges over the *non-negative* integers. See Proposition 22.14. And finally, when we introduce half-forms in Sect. 23.7, we will finally restore the spectrum $(n + 1/2)\hbar\omega$, where n ranges over the non-negative integers, that we found in Chap. 11.

Proof. We can write H as

$$H(x, p) = \frac{1}{2m}(p^2 + y^2),$$

where $y = m\omega x$. The flow associated to this Hamiltonian consists of rotations in the (y, p) -plane. If we choose our symplectic potential to be

$$\theta = \frac{1}{2}(p \, dx - x \, dp) = \frac{1}{2m\omega}(p \, dy - y \, dp),$$

then the $\theta(X_H)$ term in $Q_{\text{pre}}(H)$ cancels with the H term, leaving

$$\begin{aligned} Q_{\text{pre}}(H) &= i\hbar X_H \\ &= i\hbar \left(m\omega^2 x \frac{\partial}{\partial p} - \frac{p}{m} \frac{\partial}{\partial x} \right) \\ &= i\hbar\omega \left(y \frac{\partial}{\partial p} - p \frac{\partial}{\partial y} \right). \end{aligned}$$

Now, if ϕ denotes the angular variable for polar coordinates in the (y, p) -plane, then $y \, \partial/\partial p - p \, \partial/\partial y$ is just $\partial/\partial\phi$. Thus, we can find eigenvectors for $Q_{\text{pre}}(H)$ of the form

$$\psi_n(r, \phi) = f(r)e^{-in\phi}$$

where n is an integer and f is an arbitrary function with $\int_0^\infty |f(r)|^2 r \, dr < \infty$.

■

The conclusion of the matter is that it is not physically reasonable to use prequantization as our quantization scheme. Instead, we will pass to a “smaller” Hilbert space on which the position and momentum operators act irreducibly.

22.4 Quantization

To obtain a Hilbert space that can be thought of as giving us a “quantization” (as opposed to a prequantization) of \mathbb{R}^{2n} , we restrict ourselves to a subspace of the prequantum Hilbert space. The idea is that we should be using only half of the variables on \mathbb{R}^{2n} . We might, for example, restrict ourselves to functions that depend only on the position variables and are independent of the momentum variables. Now, the space of functions ψ that are, say, independent of \mathbf{p} in the ordinary sense (i.e., $\psi(\mathbf{x}, \mathbf{p}) = \psi(\mathbf{x}, \mathbf{p}')$) is not invariant under gauge transformations (the maps U_γ in Proposition 22.5). The gauge-invariant notion of being independent of \mathbf{p} is that the *covariant* derivatives of ψ should be zero in the \mathbf{p} -directions. Similarly, we may consider spaces of functions with covariant derivatives that are zero in some other set of n directions.

Definition 22.7 Fix a symplectic potential θ . Define the **position subspace** as the subspace of $C^\infty(\mathbb{R}^{2n})$ consisting of functions ψ for which

$$\nabla_{\partial/\partial p_j} \psi = 0$$

for all j . Similarly, define the **momentum subspace** as the subspace of $C^\infty(\mathbb{R}^{2n})$ consisting of functions ψ for which

$$\nabla_{\partial/\partial x_j} \psi = 0$$

for all j . Finally, define the **holomorphic subspace** with parameter α to be the subspace of $C^\infty(\mathbb{R}^{2n})$ consisting of functions ψ for which

$$\nabla_{\partial/\partial \bar{z}_j} \psi = 0$$

for all j , where $z_j = x_j - i\alpha p_j$ and where $\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$ are defined by

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + \frac{i}{\alpha} \frac{\partial}{\partial p_j} \right); \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - \frac{i}{\alpha} \frac{\partial}{\partial p_j} \right), \quad (22.9)$$

The operators $\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$ are nothing but the usual complex derivative operators on \mathbb{C}^n written in terms of the variables \mathbf{x} and \mathbf{p} , where we identify \mathbb{R}^{2n} with \mathbb{C}^n by the map $(\mathbf{x}, \mathbf{p}) \mapsto \mathbf{x} - i\alpha \mathbf{p}$.

Of course, the exact form of the various subspaces in Definition 22.7 depends on the choice of symplectic potential. It is convenient to use the symplectic potential $\theta = p_j \, dx_j$.

Proposition 22.8 *Take the symplectic potential $\theta = p_j dx_j$. Then the position, momentum, and holomorphic subspaces may be computed as follows. The position subspace consists of smooth functions ψ on \mathbb{R}^{2n} of the form*

$$\psi(\mathbf{x}, \mathbf{p}) = \phi(\mathbf{x}),$$

where ϕ is an arbitrary smooth function on \mathbb{R}^n . The momentum subspace consists of smooth functions ψ of the form

$$\psi(\mathbf{x}, \mathbf{p}) = e^{i\mathbf{x}\cdot\mathbf{p}/\hbar}\phi(\mathbf{p}), \tag{22.10}$$

where ϕ is an arbitrary smooth function on \mathbb{R}^n . Finally, the holomorphic subspace consists of functions of the form

$$\psi(\mathbf{x}, \mathbf{p}) = F(z_1, \dots, z_n)e^{-\alpha|\mathbf{p}|^2/(2\hbar)}, \tag{22.11}$$

where F is an arbitrary holomorphic function on \mathbb{C}^n and where $z_j = x_j - i\alpha p_j$.

Proof. Since $\theta(\partial/\partial p_j) = 0$, we have $\nabla_{\partial/\partial p_j} = \partial/\partial p_j$, so that functions that are covariantly constant in the \mathbf{p} -directions are actually constant in the \mathbf{p} -directions. Meanwhile, $\theta(\partial/\partial x_j) = p_j$ and so

$$\nabla_{\partial/\partial x_j} = \frac{\partial}{\partial x_j} - \frac{i}{\hbar}p_j.$$

Now, any function ψ on \mathbb{R}^{2n} can be written in the form $e^{i\mathbf{x}\cdot\mathbf{p}/\hbar}\phi(\mathbf{x}, \mathbf{p})$ for some other function ϕ . If we use this form to compute $\nabla_{\partial/\partial p_j}\psi$, there is a convenient cancellation, giving

$$(\nabla_{\partial/\partial x_j}\psi)(\mathbf{x}, \mathbf{p}) = e^{i\mathbf{x}\cdot\mathbf{p}/\hbar}\frac{\partial\phi}{\partial x_j}.$$

Thus, $\nabla_{\partial/\partial x_j}\psi = 0$ for all j if and only if ϕ is independent of \mathbf{x} .

Finally, we note that $\theta(\partial/\partial \bar{z}_j) = p_j/2$, so that

$$\nabla_{\partial/\partial \bar{z}_j} = \frac{\partial}{\partial \bar{z}_j} - \frac{i}{2\hbar}p_j.$$

Any function ψ on \mathbb{R}^{2n} can be written in the form $\psi(\mathbf{x}, \mathbf{p}) = e^{-\alpha|\mathbf{p}|^2/(2\hbar)}F$ for some other function F , where we note that

$$e^{-\alpha|\mathbf{p}|^2/(2\hbar)} = \exp\left(\sum_j (\bar{z}_j - z_j)^2/(8\alpha\hbar)\right).$$

Thus,

$$\frac{\partial}{\partial \bar{z}_j}e^{-\alpha|\mathbf{p}|^2/(2\hbar)} = \frac{\bar{z}_j - z_j}{4\alpha\hbar}e^{-\alpha|\mathbf{p}|^2/(2\hbar)} = \frac{i}{2\hbar}p_j e^{-\alpha|\mathbf{p}|^2/(2\hbar)}.$$

When we compute $\nabla_{\partial/\partial\bar{z}_j}\psi$ using the indicated form, there is another convenient cancellation, giving

$$(\nabla_{\partial/\partial\bar{z}_j}\psi)(\mathbf{x}, \mathbf{p}) = e^{-\alpha|\mathbf{p}|^2/(2\hbar)} \frac{\partial F}{\partial\bar{z}_j}.$$

Thus, $\nabla_{\partial/\partial\bar{z}_j}\psi = 0$ for all j if and only if F is holomorphic as a function of the variables $z_j = x_j - i\alpha p_j$. ■

From the physical standpoint, we do not merely want a vector space of functions, but a Hilbert space. It is natural, then, to look at functions of the forms computed in Proposition 22.8 that belong to $L^2(\mathbb{R}^{2n})$. In the case of the position and momentum subspaces, we encounter a serious problem: There are no nonzero functions of the indicated form that are square integrable over \mathbb{R}^{2n} . After all, if ψ is in the position subspace, then $\psi(\mathbf{x}, \mathbf{p})$ is independent of \mathbf{p} and the integral of $|\psi|^2$ over the \mathbf{p} -variables will be infinite, unless ψ is zero almost everywhere. If ψ is in the momentum subspace, $|\psi|^2$ is independent of \mathbf{x} and we have a similar problem.

The solution to this problem is to integrate not over \mathbb{R}^{2n} but over \mathbb{R}^n . Although the “proper” way to make this change of integration is to introduce the notion of “half-forms,” as in Chap. 23, we will content ourselves in this chapter with the following simplistic rule: integrate only over the variables on which $|\psi|^2$ depends. If we want to get a Hilbert space (not just an inner product space), we must also allow functions of the specified form that are square integrable but not necessarily smooth. We may therefore identify the *position Hilbert space* and *momentum Hilbert space* as follows.

Conclusion 22.9 *The position Hilbert space is the space of functions on \mathbb{R}^{2n} of the form*

$$\psi(\mathbf{x}, \mathbf{p}) = \phi(\mathbf{x}),$$

where $\phi \in L^2(\mathbb{R}^n)$. The norm of such a function is computed as

$$\|\psi\|^2 = \int_{\mathbb{R}^n} |\phi(\mathbf{x})|^2 d\mathbf{x}.$$

The momentum Hilbert space is the space of functions on \mathbb{R}^{2n} of the form

$$\psi(\mathbf{x}, \mathbf{p}) = e^{i\mathbf{x}\cdot\mathbf{p}/\hbar}\phi(\mathbf{p}),$$

where $\phi \in L^2(\mathbb{R}^n)$. The norm of such a function is computed as

$$\|\psi\|^2 = \int_{\mathbb{R}^n} |\phi(\mathbf{p})|^2 d\mathbf{p}.$$

If we consider the holomorphic subspace, we find that it behaves better than the position and momentum subspaces, in that there exist nonzero functions of the form (22.11) that are square integrable over \mathbb{R}^{2n} , as we will see shortly. Furthermore, the space of functions of the form (22.11) that are square integrable over \mathbb{R}^{2n} form a closed subspace of $L^2(\mathbb{R}^{2n})$, by the same argument as in the proof of Proposition 14.15.

Conclusion 22.10 *The holomorphic Hilbert space consists of those functions ψ of the form (22.11) that are square integrable over \mathbb{R}^{2n} . If ψ is identified with the holomorphic function F in (22.11), then this Hilbert space may be identified with $\mathcal{HL}^2(\mathbb{C}^n, \nu)$, where*

$$\nu(\mathbf{z}) = e^{-|\operatorname{Im} \mathbf{z}|^2 / (\alpha \hbar)}.$$

The space $\mathcal{HL}^2(\mathbb{C}^n, \nu)$ is nothing but an invariant form of the Segal–Bargmann space (Definition 14.14), where here “invariant” means that the density ν is invariant under translations in the real directions. This space can be identified unitarily with the ordinary Segal–Bargmann space $\mathcal{HL}^2(\mathbb{C}^n, \mu_{2\alpha\hbar})$ as follows. Define a map $\Psi : \mathcal{HL}^2(\mathbb{C}^n, \mu_{2\alpha\hbar}) \rightarrow \mathcal{HL}^2(\mathbb{C}^n, \nu)$ by

$$\Psi(F)(\mathbf{z}) = (2\pi\alpha\hbar)^{-n/2} e^{-\mathbf{z}^2 / (4\alpha\hbar)} F(\mathbf{z}), \quad (22.12)$$

where $\mathbf{z}^2 = z_1^2 + \cdots + z_n^2$. Then a simple calculation shows that

$$\|\Psi(F)\|_{L^2(\mathbb{C}^n, \nu)}^2 = \int_{\mathbb{C}^n} |F(\mathbf{z})|^2 \mu_{2\alpha\hbar}(\mathbf{z}) \, d\mathbf{z}.$$

Since also $e^{-\mathbf{z}^2 / (4\alpha\hbar)}$ is holomorphic as a function of \mathbf{z} , we see that Ψ maps $\mathcal{HL}^2(\mathbb{C}^n, \mu_{2\alpha\hbar})$ isometrically into $\mathcal{HL}^2(\mathbb{C}^n, \nu)$. The map Ψ has an inverse given by multiplication by $(2\pi\alpha\hbar)^{n/2} e^{\mathbf{z}^2 / (4\alpha\hbar)}$, showing that Ψ is actually unitary. In particular, there exist many nonzero holomorphic functions on \mathbb{C}^n that belong to $\mathcal{HL}^2(\mathbb{C}^n, \nu)$.

We will regard any of the Hilbert spaces in Conclusions 22.9 and 22.10 as our *quantum Hilbert space*. These spaces are to be compared to the prequantum Hilbert space $L^2(\mathbb{R}^{2n})$, which is in some sense “bigger,” consisting of functions of twice as many variables. Note there are multiple possibilities for the quantum Hilbert space. To reduce from the prequantum Hilbert space to the quantum Hilbert space, we have to *choose* a set of n variables, and then we look a functions that depend only on those n variables. Indeed, there are many other possibilities for the quantum Hilbert space; we have considered only the most common choices. We defer a discussion of the general theory until Chap. 23.

The reader may wonder why we are using the definition $z_j = x_j - i\alpha p_j$ ($\alpha > 0$) rather than $z_j = x_j + i\alpha p_j$. If we repeated the preceding calculations with $z_j = x_j + i\alpha p_j$, with a corresponding sign change in the definition of $\partial/\partial \bar{z}_j$, we would find that ψ satisfies $\nabla_{\partial/\partial \bar{z}_j} \psi$ for all j if and only if ψ is of the form

$$\psi(\mathbf{x}, \mathbf{p}) = F(z_1, \dots, z_n) e^{\alpha|\mathbf{p}|^2 / (2\hbar)}, \quad (22.13)$$

where F is holomorphic on \mathbb{C}^n . The change in sign in the exponent between (22.11) and (22.13) has a drastic effect: There are no nonzero holomorphic functions F for which the function ψ in (22.13) is square integrable over \mathbb{R}^{2n} . (See Exercise 3.) Unlike the situation with the position and momentum

Hilbert spaces, there is no natural way to alter the domain of integration to make a function of the form (22.13) have finite norm.

We see, then, that there is a big difference between the definitions $z_j = x_j - i\alpha p_j$ and $z_j = x_j + i\alpha p_j$. In the general framework of geometric quantization, we will have a similar distinction, where complex structures satisfying a certain positivity condition behave well, whereas the “opposite” complex structures behave badly. (See Definition 23.19 in Sect. 23.4.)

22.5 Quantization of Observables

Now that we have constructed our quantum (as opposed to prequantum) Hilbert spaces, we need to construct operators on these spaces. According to the standard geometric quantization program, the quantum operator associated with a function f is supposed to be simply the restriction to the quantum Hilbert space of the prequantum operator $Q_{\text{pre}}(f)$, provided that $Q_{\text{pre}}(f)$ leaves the quantum Hilbert space invariant.

Proposition 22.11 *The position, momentum, and holomorphic subspaces in Definition 22.7 are all invariant under the prequantum operators $Q_{\text{pre}}(x_j)$ and $Q_{\text{pre}}(p_j)$. Specifically, in the position subspace, we have*

$$\begin{aligned} Q_{\text{pre}}(x_j)\phi(\mathbf{x}) &= x_j\phi(\mathbf{x}) \\ Q_{\text{pre}}(p_j)\phi(\mathbf{x}) &= -i\hbar\frac{\partial\phi}{\partial x_j}, \end{aligned}$$

in the momentum subspace, we have

$$\begin{aligned} Q_{\text{pre}}(x_j)(e^{i\mathbf{x}\cdot\mathbf{p}/\hbar}\phi(\mathbf{p})) &= e^{i\mathbf{x}\cdot\mathbf{p}/\hbar}\left(i\hbar\frac{\partial\phi}{\partial p_j}(\mathbf{p})\right) \\ Q_{\text{pre}}(p_j)(e^{i\mathbf{x}\cdot\mathbf{p}/\hbar}\phi(\mathbf{p})) &= e^{i\mathbf{x}\cdot\mathbf{p}/\hbar}(p_j\phi(\mathbf{p})), \end{aligned}$$

and in the holomorphic subspace, we have

$$\begin{aligned} Q_{\text{pre}}(x_j)(F(\mathbf{z})e^{-\alpha|\mathbf{p}|^2/(2\hbar)}) &= \left(\alpha\hbar\frac{\partial F}{\partial z_j} + z_jF(\mathbf{z})\right)e^{-\alpha|\mathbf{p}|^2/(2\hbar)} \\ Q_{\text{pre}}(p_j)(F(\mathbf{z})e^{-\alpha|\mathbf{p}|^2/(2\hbar)}) &= \left(-i\hbar\frac{\partial F}{\partial z_j}\right)e^{-\alpha|\mathbf{p}|^2/(2\hbar)}. \end{aligned}$$

Proof. See Exercise 4. ■

The invariance of the three subspaces under the prequantized position and momentum operators follows from a general result in geometric quantization, that for a real-valued function f , the prequantum operator $Q_{\text{pre}}(f)$ preserves a given quantum space if and only if the Hamiltonian flow generated by f preserves the polarization defining the quantum space. The

term “polarization” refers to the set of directions in which the elements of the quantum space are covariantly constant. In the case of the position, momentum, and holomorphic spaces, the set of such directions is the same at every point, which means that the polarization is invariant under translations. But the Hamiltonian flows generated by x_j and p_j are nothing but translations in the $-p_j$ -directions and the x_j -directions, respectively. Of course, in this simple example, we can verify the invariance by direct computation, which also gives the indicated form of the operators on each subspace.

Note also that in each case, the “preferred” functions act simply as multiplication operators. In the position subspace, for example, the position operator $Q_{\text{pre}}(x_j)$ acts simply as multiplication by x_j , whereas in the momentum subspace, the operator $Q_{\text{pre}}(p_j)$ acts as multiplication by p_j . Finally, in the holomorphic subspace, the operator

$$Q_{\text{pre}}(z_j) \left(F(\mathbf{z}) e^{-\alpha|\mathbf{p}|^2/(2\hbar)} \right) = (z_j F(\mathbf{z})) e^{-\alpha|\mathbf{p}|^2/(2\hbar)},$$

where $z_j = x_j - i\alpha p_j$, since the terms involving $\partial F/\partial z_j$ cancel.

We now focus on the position Hilbert space and look for operators of the form $Q_{\text{pre}}(f)$ that leave the position subspace invariant.

Proposition 22.12 *The position subspace is invariant under $Q_{\text{pre}}(f)$ whenever f is of the form*

$$f(\mathbf{x}, \mathbf{p}) = a(\mathbf{x}) + b_j(\mathbf{x})p_j \quad (22.14)$$

for some smooth functions a and b_1, \dots, b_n on \mathbb{R}^n . On the other hand, the position subspace is not invariant under the operator $Q_{\text{pre}}(p_j^2)$.

Proof. If f is of the form (22.14), calculation shows that $\theta(X_f) + f = a(\mathbf{x})$. If we drop any terms in X_f involving $\partial/\partial p_j$, since these are zero on the position subspace, we end up with

$$Q_{\text{pre}}(f)(\phi(\mathbf{x})) = -i\hbar b_j(\mathbf{x}) \frac{\partial \phi}{\partial x_j} + a(\mathbf{x})\phi(\mathbf{x}), \quad (22.15)$$

which is again in the position subspace. [There is no \mathbf{p} -dependence in the coefficient of $\partial/\partial x_j$ in (22.15) because $\partial f/\partial p_j$ is independent of \mathbf{p} .] On the other hand, direct calculation shows that the restriction to the position subspace of $Q_{\text{pre}}(f)$ is

$$-2i\hbar p_j \frac{\partial}{\partial x_j} - p_j^2,$$

which does not preserve the space of functions on \mathbb{R}^{2n} that are independent of \mathbf{p} . ■

It should be noted that the expression on the right-hand side of (22.15) is not a self-adjoint, or even symmetric, operator on $L^2(\mathbb{R}^n)$, unless the vector field $\mathbf{b}(\mathbf{x})$ happens to be divergence free. (Even though the vector field X_f is divergence free on \mathbb{R}^{2n} , the way X_f acts on functions that are independent of \mathbf{p} is not necessarily a divergence free vector field on \mathbb{R}^n .) This undesirable feature of our quantization scheme is the result of our simplistic method of passing from $L^2(\mathbb{R}^{2n})$ to $L^2(\mathbb{R}^n)$ in our derivation of Conclusion 22.9. When we do this reduction properly, using half-forms, we will obtain a self-adjoint operator. See Sect. 23.6.

We now consider the behavior of the holomorphic subspace under the prequantized position and momentum operators.

Proposition 22.13 *For any $\alpha > 0$, let \mathbf{H}_α be the subspace of $L^2(\mathbb{R}^{2n})$ consisting of smooth functions ψ that satisfy $\nabla_{\partial/\partial\bar{z}_j}\psi = 0$, where $\partial/\partial\bar{z}_j$ is as in (22.9). Then \mathbf{H}_α is a closed subspace of $L^2(\mathbb{R}^{2n})$ and \mathbf{H}_α is invariant under the one-parameter unitary groups generated by $Q_{\text{pre}}(x_j)$ and $Q_{\text{pre}}(p_j)$. Furthermore, $Q_{\text{pre}}(x_j)$ and $Q_{\text{pre}}(p_j)$ act irreducibly on \mathbf{H}_α in the sense of Definition 14.6.*

For each $\alpha > 0$, the holomorphic Hilbert space is a subspace of the prequantum Hilbert space invariant under the exponentiated position and momentum operators. Thus, the prequantum Hilbert space is far from being irreducible under the action of those operators.

Proof. The invariance of \mathbf{H}_α is a simple calculation (Exercise 5). Irreducibility can be established by reducing to the previously established irreducibility of the Segal–Bargmann space under the operators $T_{\mathbf{a}}$ in Theorem 14.16. To this end, we should check that the unitary map Ψ in (22.12) intertwines products of exponentials of $Q_{\text{pre}}(x_j)$ and $Q_{\text{pre}}(p_j)$ with operators of the form $T_{\mathbf{a}}$ (with \hbar replaced by $2\alpha\hbar$). This is a straightforward but tedious calculation, and we omit the details. ■

We conclude this section with an example of a quantum subspace that is invariant under the (pre)quantized Hamiltonian of a harmonic oscillator.

Proposition 22.14 *Consider a harmonic oscillator with Hamiltonian*

$$H = \frac{1}{2m} (p^2 + (m\omega x)^2).$$

Consider also the subspace \mathbf{H}_α in Proposition 22.13, with $\alpha = 1/(m\omega)$. Then the operator $Q_{\text{pre}}(H)$ leaves \mathbf{H}_α invariant. Furthermore, the restriction of $Q_{\text{pre}}(H)$ to \mathbf{H}_α has non-negative spectrum consisting of eigenvalues of the form $n\hbar\omega$, where n ranges over the non-negative integers.

Proposition 22.14 is a much more physically reasonable result for the spectrum of the quantization of the non-negative function H than on the full prequantum Hilbert space, where (Proposition 22.6) the spectrum of $Q_{\text{pre}}(H)$ is not even bounded below. When we introduce the “half-form

correction” in Sect. 23.7, we will finally be able to obtain the “correct” spectrum for the quantum harmonic oscillator, consisting of numbers of the form $(n + 1/2)\hbar\omega$, $n = 0, 1, 2, \dots$. See Example 23.53.

Proof. As in the proof of Proposition 22.6, we introduce the variable $y = m\omega x$. With $\alpha = 1/(m\omega)$, this gives $z = (y - ip)/(m\omega)$. We use the symplectic potential

$$\theta = \frac{1}{2}(p dx - x dp) = \frac{1}{2m\omega}(p dy - y dp).$$

Then

$$\theta\left(\frac{\partial}{\partial \bar{z}}\right) = \frac{1}{2}\left(p + \frac{i}{\alpha}x\right) = \frac{i}{2\alpha}z$$

and so $\nabla_{\partial/\partial \bar{z}} = \partial/\partial \bar{z} + z/(2\alpha\hbar)$. From this, we can easily check that the holomorphic subspace consists of functions of the form

$$F(\mathbf{z})e^{-|z|^2/(2\alpha\hbar)} = F(\mathbf{z})\exp\left\{-\frac{(y^2 + p^2)}{2m\omega\hbar}\right\}, \quad (22.16)$$

where F is holomorphic.

Meanwhile, as in the proof of Proposition 22.6, we have

$$Q_{\text{pre}}(H) = i\hbar\omega\left(y\frac{\partial}{\partial p} - p\frac{\partial}{\partial y}\right),$$

which is just an angular derivative in the (y, p) -plane. Since the exponential factor in (22.16) is rotationally invariant, $Q_{\text{pre}}(H)$ only hits F . Meanwhile,

$$\begin{aligned} \left(y\frac{\partial}{\partial p} - p\frac{\partial}{\partial y}\right)F\left(\frac{y - ip}{m\omega}\right) &= y\frac{dF}{dz}\left(-\frac{i}{m\omega}\right) - p\frac{dF}{dz}\frac{1}{m\omega} \\ &= -\frac{i}{m\omega}(y - ip)\frac{dF}{dz} \\ &= -iz\frac{dF}{dz}. \end{aligned}$$

Thus,

$$Q_{\text{pre}}(H)(F(\mathbf{z})e^{-|z|^2/(2\alpha\hbar)}) = \left(\hbar\omega z\frac{dF}{dz}\right)e^{-|z|^2/(2\alpha\hbar)},$$

which is again in the holomorphic subspace.

Finally, as in Proposition 14.15, the functions z^n , $n = 0, 1, 2, \dots$, form an orthogonal basis for the Hilbert space \mathbf{H}_α . Each monomial z^n is an eigenvector for the operator $z d/dz$ with eigenvalue n . This establishes the claim about the spectrum of the restriction to \mathbf{H}_α of $Q_{\text{pre}}(H)$. ■

The operator $F \mapsto \hbar\omega z dF/dz$ is self-adjoint on the holomorphic Hilbert space, in contrast to the operators in (22.15) in the case of the position Hilbert space. Indeed, self-adjointness is “automatic” in this case, because the holomorphic Hilbert space is actually a subspace of the prequantum Hilbert space, and the restriction of a self-adjoint operator to an invariant subspace is self-adjoint.

22.6 Exercises

1. Consider the vector field

$$X := a_j(\mathbf{x}) \frac{\partial}{\partial x_j}$$

on \mathbb{R}^{2n} , where the a_j 's are smooth, real-valued functions. Show that X is skew-self-adjoint on $C_c^\infty(\mathbb{R}^N)$ if and only if the divergence of X (i.e., the quantity $\partial a_j / \partial x_j$) is identically zero.

2. Using the symplectic potential $\theta = p \, dx$, compute $Q_{\text{pre}}(xp^2)$. Show that $Q_{\text{pre}}(xp^2)$ is not in the algebra of operators generated by $Q_{\text{pre}}(x)$ and $Q_{\text{pre}}(p)$.

Hint: Consider how $Q_{\text{pre}}(xp^2)$ acts on functions that are independent of p .

3. (a) Suppose F is a holomorphic function on \mathbb{C} such that

$$\int_{\mathbb{C}} |F(z)|^2 \, dz < \infty,$$

where here dz denotes the 2-dimensional Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$. Show that F is identically zero.

Hint: If F is not identically zero, use a power series argument to show that the L^2 norm of F over a disk of radius R tends to infinity as R tends to infinity.

- (b) Show that if a function of the form (22.13), with F holomorphic on \mathbb{C}^n , is square integrable, then F must be identically zero.

4. Prove Proposition 22.11, using the explicit form of $Q_{\text{pre}}(x_j)$ and $Q_{\text{pre}}(p_j)$ in Example 22.4.

Hint: In the case of the holomorphic subspace, express the operators $\partial/\partial x_j$ and $\partial/\partial p_j$ in terms of the operators $\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$ in (22.9).

5. Show that the space of functions of the form in (22.11), where F is holomorphic on \mathbb{C}^n , is invariant under the operators $e^{itQ_{\text{pre}}(x_j)}$ and $e^{itQ_{\text{pre}}(p_j)}$ computed in (22.6), for all $t \in \mathbb{R}$ and $j = 1, 2, \dots, n$.