23
Geometric Quantization on Manifolds

23.1 Introduction

Geometric quantization is a type of *quantization*, which is a general term for a procedure that associates a quantum system with a given classical system. In practical terms, if one is trying to deduce what sort of quantum system should model a given physical phenomenon, one often begins by observing the classical limit of the system. Electromagnetic radiation, for example, is describable on a macroscopic scale by Maxwell's equations. On a finer scale, quantum effects (photons) become important. How should one determine the correct *quantum* theory of electromagnetism? It seems that the only reasonable way to proceed is to "quantize" Maxwell's equations—and then to compare the resulting quantum system to experiment.

Meanwhile, not every physically interesting system has $\mathbb{R}^{2n}$ as its phase space. Geometric quantization, then, is an attempt to construct a quantum Hilbert space, together with appropriate operators, starting from a physical system having an arbitrary $2n$-dimensional symplectic manifold N as its phase space. To perform geometric quantization on N, one must first choose a polarization, that is, roughly, a choice of n directions on N in which the wave functions will be constant. If $N = T^*M$, then one may use the "vertical polarization," in which the wave functions are constant along the fibers of T^*M. For cotangent bundles with the vertical polarization, geometric quantization reproduces the "half-density quantization" of Blattner [4]. (See Examples 23.45 and 23.48.) Even for cotangent bundles, however, it is of interest to use polarizations other than the vertical polarization, as

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we have seen already in the $\mathbb{R}^n$ case. In the case of the cotangent bundle of a compact Lie group, for example, the paper [20] shows how quantization with a complex polarization gives rise to a generalized Segal–Bargmann transform.

Some phase spaces, meanwhile, may not even be in the form of a cotangent bundle. In the orbit method in representation theory, for example, the relevant symplectic manifolds are "coadjoint orbits," which typically are not cotangent bundles. [In the $\mathsf{SU}(2)$ case, for instance, these orbits are 2-spheres with the natural rotationally invariant symplectic form.] In quantum field theory, meanwhile, one encounters Lagrangians that are linear, rather than quadratic, in the "velocity" variables. In such cases, the initial velocity is determined by the initial position, and one cannot think of the space of initial conditions as a (co)tangent bundle. Systems of this form *can* still be symplectic, but they are not cotangent bundles. Furthermore, it is common to think of compact symplectic manifolds (such as S^2 with a rotationally invariant symplectic form) as classical models of internal degrees of freedom, such as spin.

To quantize these more general symplectic manifolds, one needs a more general approach to quantization. Given a symplectic manifold (N, ω) satisfying a certain integrality condition, one can construct a line bundle L over N along with a connection ∇ on L which has a curvature of $\omega/\hbar$. One can then define "prequantum" operators, acting on sections of L, in much the same way we did in the Euclidean case in Chap. 22, and these operators will have the desired relationship between Poisson brackets and commutators. One then chooses a polarization on N and defines the quantum Hilbert space to be the space of sections that are covariantly constant in the directions of that polarization. If the Hamiltonian flow generated by a function f preserves the relevant polarization, then $Q_{\text{pre}}(f)$ will preserve the quantum Hilbert space. In the case of real polarizations, there may fail to be any nonzero *square-integrable* sections that are covariantly constant in the directions of the polarization, a possibility that forces us to introduce the machinery of "half-forms."

Let us end this introduction with a brief critique of the framework of geometric quantization. In the first place, geometric quantization has too many definitions (bundles, connections, curvature, polarizations, half-forms) and too few theorems. In the second place, the class of functions that geometric quantization allows us to quantize—those functions for which the associated Hamiltonian flow preserves the polarization—is often dishearteningly small. In the case $N = T^*M$, for example, with the natural "vertical" polarization, geometric quantization does not allow us to quantize the kinetic energy function, at least not by the "standard procedure" of geometric quantization. Nevertheless, geometric quantization is the only game in town if one wants to quantize general symplectic manifolds in a way that produces an actual Hilbert space and operators thereon.

This chapter lays out in an orderly fashion all the ingredients needed to "do" geometric quantization. Furthermore, although this approach increases length, the chapter fills in the details of several arguments that are only sketched in the standard reference on the subject, the book [45] of Woodhouse. The presentation assumes basic results about symplectic manifolds from Chap. 21. Besides the basic results about manifolds reviewed in Sect. 21.1, we will make use of the Frobenius theorem (see, e.g., Chap. 19 of [29]).

As we have noted already in the introduction to Chap. 22, sign conventions in the subject of geometric quantization are not consistent from author to author.

23.2 Line Bundles and Connections

In this section, we develop the necessary machinery to extend the prequantization construction of Sect. 22.2 to arbitrary symplectic manifolds. We introduce the notion of a line bundle over a manifold and sections thereof, which look locally like complex-valued functions. We then introduce the notion of covariant derivatives of sections of a line bundle, where locally these covariant derivatives take the form $\nabla_X = X - i\theta(X)$ for a certain 1-form θ. We then introduce the curvature 2-form, which is a globally defined, closed 2-form that can be computed locally as $d\theta$. We continue to observe the summation convention, in which repeated indices are always summed on.

Definition 23.1 *If X is a smooth manifold, a **complex line bundle** over X is a smooth manifold L together with the following additional structures. First, we have a smooth, surjective map $\pi : L \to X$. Second, for each $x \in X$, the set $\pi^{-1}(\{x\})$ is equipped with the structure of a complex vector space of dimension 1. For each $x \in N$, the vector space $\pi^{-1}(\{x\})$ is called the **fiber** of L over x.*

*These structures are assumed to satisfy the **local triviality property**, namely that each $x \in X$ has a neighborhood U such that there exists a diffeomorphism $\chi : \pi^{-1}(U) \to U \times \mathbb{C}$ with the following properties. First,*

$$\pi(p) = \pi_1(\chi(p)),$$

where $\pi_1 : U \times \mathbb{C} \to U$ is projection onto the first factor. Second, for each $x \in U$, the map $p \mapsto \pi_2(\chi(p))$ is a vector space isomorphism of $\pi^{-1}(\{x\})$ with $\mathbb{C}$.

*A **section** of a line bundle L over X is a map $s : X \to L$ such that $\pi(s(p)) = p$ for all $p \in X$.*

For any manifold X, we can form the trivial line bundle $X \times \mathbb{C}$, where $\pi(x, z) = z$ and where the vector space structure on $\{x\} \times \mathbb{C}$ is just the usual vector space structure on $\mathbb{C}$. The local triviality property for a general line bundle L means that L "looks" locally like the trivial line bundle.

Definition 23.2 *A* ***connection*** ∇ *on a line bundle L over N is a map associating to each vector field X on N and section s of L another section $\nabla_X(s)$ of L satisfying the following properties. First, for each smooth function f on N, we have*

$$\nabla_{fX}(s) = f\nabla_X(s) \tag{23.1}$$

for all vector fields X and sections s. Second, for each smooth function f on N, we have the ***product rule***

$$\nabla_X(fs) = (X(f))s + f\nabla_X(s) \tag{23.2}$$

for all vector fields X and sections s.

Note that for any section s of L and any function f on N, the quantity fs is a section of s. Given a connection ∇ and a vector field X, the operator ∇_X is called the *covariant derivative* in the direction of X.

Definition 23.3 *A* ***Hermitian structure*** *on a line bundle L over N is a choice of an inner product $(\cdot,\cdot)$ on each fiber $\pi^{-1}(\{x\})$ of L such that for each smooth section s of L, (s,s) is a smooth function on N. A line bundle L together with a choice of a Hermitian structure on L will be called a* ***Hermitian line bundle****. A connection ∇ on a Hermitian line bundle L is called* ***Hermitian*** *if for every vector field on X, we have*

$$(\nabla_X(s_1), s_2) + (s_1, \nabla_X(s_2)) = X(s_1, s_2) \tag{23.3}$$

for all smooth sections s_1 and s_2 of L.

We will let the expression "Hermitian line bundle with connection" refer to a Hermitian line bundle L together with a Hermitian connection on L; that is, in this expression, "Hermitian" applies both to the bundle and to the connection.

Given a Hermitian line bundle L with connection, it is always possible to choose a locally defined smooth section s_0 near any point such that $(s_0, s_0) \equiv 1$. We call s_0 a *local isometric trivialization* of L. Any section s of L can be written locally as $s = fs_0$ for a unique complex-valued function f. Given a vector field X, let $\theta(X)$ be the unique function such that

$$\nabla_X(s_0) = -i\theta(X)s_0.$$

Using the assumption $\nabla_{fX} = f\nabla_X$, it can be shown (Exercise 1) that the value of $\theta(X)$ at a point p depends only on the value of X at p. Thus, θ defines a 1-form on N. Using the assumption that ∇ is Hermitian, it can be shown (Exercise 2) that $\theta(X)$ is always real valued.

Now, using the product rule (23.2) for covariant derivatives, we have

$$\begin{aligned}\nabla_X(fs_0) &= X(f)s_0 + f\nabla_X(s_0)\\ &= (X(f) - i\theta(X)f)s_0.\end{aligned}$$

Thus, if we identify sections of L locally with the coefficient function f, we have

$$\nabla_X(f) = X(f) - i\theta(X)f, \tag{23.4}$$

as in Sect. 22.2. We call θ the *connection 1-form* associated to the particular local isometric trivialization.

Definition 23.4 *For any Hermitian line bundle* (L, ∇) *with connection, define the* ***curvature 2-form*** ω *of* ∇ *by requiring that*

$$\omega(X, Y)s = i\left(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\right)(s)$$

for all sections s *and vector fields* X *and* Y*.*

Of course, one should check that the given expression for ω is really a 2-form, meaning that the value of $\omega(X, Y)$ at a point z depends only on the values of X and Y at z, and that it does not depend on the choice of section s, provided only that $s(z) \neq 0$. One way to do this is to compute ω in a local isometric trivialization, as in the following result. (See Exercise 3 for a different approach.)

Proposition 23.5 *Let* s_0 *be a local isometric trivialization of* L *and let* θ *be the associated connection 1-form. Then the curvature 2-form* ω *of* ∇ *is expressed locally as*

$$\omega = d\theta.$$

In particular, ω *is a* closed *2-form.*

Proof. The computation is precisely the same as in the proof of Proposition 22.3 in the Euclidean case. ■

A locally defined 1-form θ satisfying $d\theta = \omega$ is called a (local) *symplectic potential* for ω. Our next result says that *every* symplectic potential is the connection 1-form for some local isometric trivialization of L.

Proposition 23.6 *Let* (L, ∇) *be a Hermitian line bundle with connection over* N *with curvature 2-form* ω*. For each point* $z_0 \in N$ *and 1-form* θ *defined in a neighborhood* U *of* z_0 *satisfying* $d\theta = \omega$*, there is a subneighborhood* $V \subset U$ *of* z_0 *and a local isometric trivialization of* L *over* V *such that the connection 1-form of the trivialization is* θ*.*

Proof. Let s_0 be any isometric trivializing section defined in a neighborhood of z_0 and let η be the associated connection 1-form. Since $d(\eta - \theta) = 0$, there is a subneighborhood $V \subset U$ of z_0 on which $\eta - \theta = df$, for some smooth function f. If $s_1 = e^{if} s_0$, then

$$\begin{aligned}\nabla_X(s_1) &= iX(f)e^{if}s_0 + e^{if}\nabla_X(s_0)\\ &= iX(f)e^{if}s_0 - i\eta(X)e^{if}s_0\\ &= -i(\eta(X) - df(X))s_1.\end{aligned}$$

Thus, the connection 1-form associated with the local isometric trivialization s_1 is $\eta - df = \theta$. ■

Proposition 23.7 *If (L_1, ∇^1) and (L_2, ∇^2) are Hermitian line bundles with connection over N, let $L_1 \otimes L_2$ denote the line bundle over N for which the fiber over x is $L_{1,x} \otimes L_{2,x}$, with the natural inner product induced by the inner products on $L_{1,x}$ and $L_{2,x}$. Then there is a unique Hermitian connection ∇ on $L_1 \otimes L_2$ with the property that*

$$\nabla_X(s_1 \otimes s_2) = (\nabla^1_X s_1) \otimes s_2 + s_1 \otimes (\nabla^2_X s_2),$$

for all vector fields X on N and all smooth sections s_1 of L_1 and s_2 of L_2. The curvature 2-form ω for $(L_1 \otimes L_2, \nabla)$ is given by

$$\omega = \omega_1 + \omega_2,$$

where ω_1 and ω_2 are the curvature 2-forms for (L_1, ∇^1) and (L_2, ∇^2), respectively.

The proof of this proposition is a straightforward exercise in "definition chasing" and is left as an exercise to the reader.

Suppose that L is a Hermitian line bundle over N with connection ∇ and curvature 2-form ω. Given a loop $\gamma : [a, b] \to N$, we can construct a section s of L that is defined over γ such that the covariant derivative of s in the directions along γ is zero. Indeed, in a local isometric trivialization, such a section can be constructed as

$$s(\gamma(T)) = \exp\left\{ i \int_{\gamma(a)}^{\gamma(T)} \theta(\gamma(t))\ dt \right\}. \tag{23.5}$$

The value of s at the endpoint of the loop will in general not agree with the value at the starting point, but will differ by multiplication by a constant of absolute value 1.

Definition 23.8 *The **holonomy** of a loop $\gamma : [a, b] \to N$ is the unique constant α (of absolute value 1) such that $s(\gamma(b)) = \alpha s(\gamma(a))$, where s is a nonzero section defined over γ that is covariantly constant in the directions of γ.*

The value of the holonomy of γ is easily seen to be independent of the value of s at the starting point, provided this starting value is nonzero.

Suppose that S is a compact, oriented surface with boundary in N whose boundary ∂S is a loop. It is not hard to show that the holonomy around ∂S can be computed as

$$\text{holonomy}(\partial S) = \exp\left\{ i \int_S \omega \right\}. \tag{23.6}$$

Indeed, if S is contained in the domain of a local isometric trivialization, then this result follows from (23.5) by means of Stoke's theorem (Sect. 21.1.2).

Now, if S is a closed (i.e., boundaryless) surface, its boundary is the trivial loop, which has a holonomy that is trivial, that is, equal to 1. (Think of approximating S by a surface for which the boundary is a very small loop.) Thus, for any closed surface S, (23.6) gives

$$\exp\left\{ i \int_S \omega \right\} = 1, \quad \partial S = \varnothing. \tag{23.7}$$

Equivalently, we have

$$\frac{1}{2\pi} \int_S \omega \in \mathbb{Z}. \tag{23.8}$$

The condition (23.8) says that $\omega/(2\pi)$ is an *integral* 2-form. Clearly, not every closed 2-form satisfies this property.

The closedness of ω (Proposition 23.5) and the condition (23.8) represent necessary conditions that the curvature of a Hermitian connection must satisfy. It turns out that these two conditions are also sufficient.

Theorem 23.9 *Suppose ω is a closed 2-form on a manifold N for which $\omega/(2\pi)$ is integral in the sense of (23.8). Then there exists a Hermitian line bundle L over N with Hermitian connection ∇ such that the curvature of ∇ is equal to ω. If, in addition, N is simply connected, then (L, ∇) is unique up to equivalence.*

See Sect. 8.3 of [45] for a proof of this result. An equivalence of two Hermitian line bundles L_1 and L_2 with Hermitian connection over N is a diffeomorphism $\Phi : L_1 \to L_2$ such that for each $x \in N$, the restriction of Φ to $\pi_1^{-1}(\{x\})$ is an isometric linear map onto $\pi_2^{-1}(\{x\})$ and such that for each section s of L_1, we have

$$\Phi(\nabla_X(s)) = \nabla_X(\Phi(s)).$$

We now have the necessary tools to proceed with the program of geometric quantization on symplectic manifolds.

23.3 Prequantization

The first step in the program of geometric quantization for a symplectic manifold (N, ω) is to construct a Hermitian line bundle L over N with Hermitian connection for which the curvature 2-form is equal to $\omega/\hbar$. Theorem 23.9 gives the condition for the existence of such a bundle.

Definition 23.10 *A symplectic manifold* (N, ω) *is* ***quantizable*** *(for a particular value of* $\hbar$*) if*

$$\frac{1}{2\pi\hbar}\int_S \omega \in \mathbb{Z}$$

for every closed surface S *in* N*.*

Note that if (N, ω) is quantizable for a given value $\hbar_0$ of Planck's constant, then (N, ω) is also quantizable for $\hbar = \hbar_0/k$ for every positive integer k. Indeed, according to Proposition 23.7, if L is a Hermitian line bundle with connection having curvature $\omega/\hbar_0$, then $L^{\otimes k}$ (the tensor product of L with itself k times) is a Hermitian line bundle with connection having curvature $\omega/(\hbar_0/k)$.

For the remainder of this chapter, we will assume that N is a quantizable symplectic manifold with symplectic form ω and that (L, ∇) is a fixed Hermitian line bundle with connection of N with curvature $\omega/\hbar$.

If L is a Hermitian line bundle over a symplectic manifold N, we say that a measurable section s of L is *square integrable* if

$$\|s\| := \left(\int_N (s_1(x), s_1(x))\ \lambda(x)\right)^{1/2}$$

is finite, where λ is the Liouville volume form on N. Given two square-integrable sections s_1 and s_2 of L, we define the *inner product* of s_1 and s_2 by

$$\langle s_1, s_2\rangle = \int_N (s_1(x), s_2(x))\ \lambda(x). \tag{23.9}$$

We use parentheses to denote the *pointwise* inner product $(s_1(x), s_2(x))$ of two sections s_1 and s_2, which is a function on N, and we use angled brackets to denote the *global* inner product $\langle s_1, s_2\rangle$ of the sections, which is a number.

Definition 23.11 *The* ***prequantum Hilbert space*** *for* N *is the space of equivalence classes of square-integrable sections of* L*, where two sections are equivalent if they are equal almost everywhere with respect to the Liouville volume measure.*

Definition 23.12 *If* f *is a smooth complex-valued function on* N*, the* ***prequantum operator*** $Q_{\text{pre}}(f)$ *is the unbounded operator on the prequantum*

Hilbert space given by

$$Q_{\mathrm{pre}}(f) = i\hbar \nabla_{X_f} + f,$$

where f represents the operation of multiplying a section by f.

Proposition 23.13 *If f is real-valued, then $Q_{\mathrm{pre}}(f)$ is symmetric on the space of smooth compactly supported sections of L.*

Proof. Let s_1 and s_2 be smooth, compactly supported sections of L and let Φ^f denote the Hamiltonian flow generated by f. For all sufficiently small t, every point in the supports of s_1 and s_2 will contained in the domain of Φ_t^f. Furthermore, by Liouville's theorem, the value of

$$\int_N [(s_1, s_2) \circ \Phi_t] \; \lambda$$

is independent of t. If we differentiate this relation with respect to t and evaluate at $t = 0$, we obtain, by (23.3),

$$0 = \int_N [(\nabla_{X_f}(s_1), s_2) + (s_1, \nabla_{X_f}(s_2))] \; \lambda.$$

Thus, ∇_{X_f} is a skew-symmetric operator on the space of smooth, compactly supported sections, from which it follows that $Q_{\mathrm{pre}}(f)$ is symmetric. ∎

By the product rule for covariant derivatives and the identity $X_f(f) = \{f, f\} = 0$, we see that the two terms in the definition of $Q_{\mathrm{pre}}(f)$ commute. We would then expect the exponential $e^{itQ_{\mathrm{pre}}(f)}$ to decompose as a product of two exponentials. One of these exponentials is just e^{itf} and the other may be constructed as "parallel transport along the flow generated by X_f." Thus, if the flow generated by X_f is complete, it is possible to use Stone's theorem to construct $Q_{\mathrm{pre}}(f)$ as a self-adjoint operator on a domain that includes the space of smooth compactly supported sections.

Proposition 23.14 *For any $f, g \in C^\infty(X)$, we have*

$$\frac{1}{i\hbar}[Q_{\mathrm{pre}}(f), Q_{\mathrm{pre}}(g)] = Q_{\mathrm{pre}}(\{f, g\}),$$

where the equality holds as operators on the space of smooth sections of L.

Proof. The argument is precisely the same as in Proposition 22.1 in the $\mathbb{R}^{2n}$ case. ∎

As we have seen already in Sect. 22.3 in the $\mathbb{R}^{2n}$ case, the prequantum Hilbert space is "too large" to be considered the quantization of N.

23.4 Polarizations

In the $\mathbb{R}^n$ case, we have the position, momentum, and holomorphic subspaces (Definition 22.7), consisting of functions that depend only on $\mathbf{x}$, $\mathbf{p}$, or $\mathbf{z}$, in the sense that the covariant derivatives of functions in the directions of $\mathbf{p}$, $\mathbf{x}$, and $\bar{\mathbf{z}}$ are zero. In each case, the "basic observables" of the particular representation (the x_j's, the p_j's, and the z_j's, respectively) act simply as multiplication operators.

To generalize this to a symplectic manifold N of dimension $2n$, we may think of choosing n functions $\alpha_1, \ldots, \alpha_n$ on N that are "independent," in the sense that $d\alpha_1, \ldots, d\alpha_n$ are linearly independent at each point. We assume that the functions α_j Poisson commute ($\{\alpha_j, \alpha_k\} = 0$), which makes it reasonable to hope that the quantizations of the α_j's could act as (commuting) multiplication operators. For each $z \in N$, we let P_z be the n-dimensional space of directions in which the α_j's are constant, that is, the intersection of the kernels of $d\alpha_1, \ldots, d\alpha_n$. Since we wish to allow the functions α_j to be complex valued, P_z should be thought of as a subspace of the *complexified* tangent space $T_z^{\mathbb{C}}(N)$. The idea is that our quantum Hilbert space should consist of sections of a prequantum line bundle that are covariantly constant in the directions of P.

Now, at each point z, the Hamiltonian vector field X_{α_j} will belong to P_z, because

$$d\alpha_j(X_{\alpha_k}) = X_{\alpha_k}(\alpha_j) = \{\alpha_k, \alpha_j\} = 0.$$

Furthermore, since the $d\alpha_j$'s are linearly independent, the X_{α_j}'s are also independent, since X_{α_j} is obtained from $d\alpha_j$ by an isomorphism of tangent and cotangent spaces. Thus, the X_{α_j}'s must actually span P_z at each point, by a dimension count. Since also $\omega(X_{\alpha_j}, X_{\alpha_k}) = -\{\alpha_j, \alpha_k\} = 0$, we conclude that ω is identically zero on P_z. Furthermore, if X and Y are vector fields lying in P at each point, we can express them as

$$X = a_j(z)X_{\alpha_j}, \quad Y = b_j(z)X_{\alpha_j},$$

for some smooth functions a_j and b_j. Then

$$[X, Y] = a_j(z)X_{\alpha_j}(b_k)X_{\alpha_k} - b_k(z)X_{\alpha_k}(a_j)X_{\alpha_j},$$

because $[X_{\alpha_j}, X_{\alpha_k}] = X_{\{\alpha_j, \alpha_k\}} = 0$. Thus, the commutator of two vector fields lying in P will again lie in P.

Definition 23.15 *For any $z \in N$, a subspace P of T_zN is said to be* ***Lagrangian*** *if $\dim P = n$ and $\omega(X, Y) = 0$ for all $X, Y \in P$.*

Definition 23.16 *A* ***polarization*** *of a symplectic manifold N is a choice at each point $z \in N$ of a Lagrangian subspace $P_z \subset T_z^{\mathbb{C}}(X)$, satisfying the following two conditions.*

1. *If two complex vector fields X and Y lie in P_z at each point z, then so does $[X, Y]$.*

2. *The dimension of $P_z \cap \overline{P_z}$ is constant.*

The first condition is called *integrability*, and we have motivated this condition in the discussion preceding the definition. The second condition is a technical one that prevents problems with certain constructions, such as the pairing map. (Although, in practice, one sometimes needs to work with "polarizations" in which the second condition is violated, extra care is needed in such cases.)

There is one small inaccuracy in our discussion of polarizations: For purely conventional reasons, the quantum Hilbert space is defined as the space of sections that are covariantly constant in the direction of $\bar{P}$, rather than P. Thus, P should really be the *complex conjugate* of the space of directions in which the sections are constant. This convention, however, makes no difference to the definition of a polarization, since if P satisfies the conditions of Definition 23.16, so does $\bar{P}$.

Example 23.17 *If M is any smooth manifold, let $N = T^*M$ be the cotangent bundle of M, equipped with the canonical 2-form ω (Example 21.2). For each $z \in T^*M$, let P_z be the complexification of the tangent space to the fiber T_z^*M. Then P is a polarization on T^*M, called the* ***vertical polarization****.*

Proof. If $\{x_j\}$ is any local coordinate system on M, let $\{x_j, p_j\}$ be the associated local coordinate system on T^*M. The canonical 2-form is given by $\omega = dp_j \wedge dx_j$. At each point $z \in T^*M$, the vertical subspace P_z is spanned by the vectors $\partial/\partial p_j$. Since $\omega(\partial/\partial p_j, \partial/\partial p_k) = 0$, we see that P_z is Lagrangian. Furthermore, $P_z = \bar{P}_z$ at every point, and so $\dim P_z \cap \overline{P_z}$ has the constant value $n = \dim M$. Finally, the integrability of P follows by computing the commutator of two vector fields of the form $f_j(x,p)\ \partial/\partial p_j$, which will again be a linear combination of the $\partial/\partial p_j$'s. Integrability also follows from the easy direction of the Frobenius theorem, since the fibers of T^*M are integral submanifolds for P. ∎

We may identify two special classes of polarizations, those that are *purely real* (i.e., $\overline{P_z} = P_z$ for all $z \in N$) and those that are *purely complex* (i.e., $P_z \cap \overline{P_z} = \{0\}$ for all $z \in N$). The vertical polarization, for example, is purely real.

If P is purely real, the integrability of P implies, by the Frobenius theorem, that every point in N is contained in a unique submanifold R that is maximal in the class of connected integral submanifolds for P. [An integral submanifold R for P is submanifold for which $T_z^{\mathbb{C}}(R) = P_z$ for all $z \in R$.] We will refer to the maximal connected, integral submanifolds of a purely real polarization as the *leaves* of the polarization.

In general, the leaves may not be embedded submanifolds of N. Suppose, for example, that $N = S^1 \times S^1$, with $\omega = d\theta \wedge d\phi$, where θ and ϕ are angular coordinates on the two copies of S^1. Then the tangent space to N at any point may be identified with $\mathbb{R}^2$ by means of the basis $\{\partial/\partial\theta, \partial/\partial\phi\}$. We may define a polarization P on N by defining P_z to be the span of the vector

$$\frac{\partial}{\partial\theta} + a\frac{\partial}{\partial\phi},$$

for some fixed irrational number a. Each leaf of P is then a set of the form

$$\left\{(e^{i\theta_0}e^{it}, e^{iat}) \in S^1 \times S^1 \middle| t \in \mathbb{R}\right\},$$

for some θ_0, which is an "irrational line" in $S^1 \times S^1$. Each leaf is then dense in $S^1 \times S^1$ and, thus, not embedded. We will need to avoid such pathological examples if we hope to successfully carry out the program of geometric quantization with respect to a real polarization. Much more information about the structure of real polarizations may be found in Sects. 4.5–4.7 of [45].

We now consider some elementary results concerning purely complex polarizations.

Proposition 23.18 *Suppose P is a purely complex polarization on N. For each $z \in N$, let $J_z : T_z^{\mathbb{C}}N \to T_z^{\mathbb{C}}N$ be the unique linear map such that $J_z = iI$ on P_z and $J_z = -iI$ on $\overline{P_z}$. Then J_z is real (i.e., it maps the real tangent space to itself) and ω is J_z-invariant [i.e., $\omega(J_zX_1, J_zX_2) = \omega(X_1, X_2)$ for all $X_1, X_2 \in T_z^{\mathbb{C}}N$].*

Proof. Since the restriction of J_z to $\overline{P_z}$ is the complex-conjugate of its restriction to P_z, the map J_z commutes with complex conjugation and thus maps real vectors (those satisfying $\bar{X} = X$) to real vectors. Meanwhile, since P_z is Lagrangian and ω is real, $\overline{P_z}$ is also Lagrangian. Given two vectors $X_1 = Y_1 + Z_1$ and $X_2 = Y_2 + Z_2$, with $Y_j \in P_z$ and $Z_j \in \overline{P_z}$, we compute that

$$\begin{aligned}
&\omega(J_zX_1, J_zX_2) \\
&= \omega(iY_1, iY_2) + \omega(iY_1, -iZ_2) + \omega(-iZ_1, iY_2) + \omega(-iZ_1, -iZ_2) \\
&= \omega(Y_1, Z_2) + \omega(Z_1, Y_2).
\end{aligned}$$

A similar calculation gives the same value for $\omega(X_1, X_2)$, showing that ω is J_z-invariant. ■

A *complex structure* on a $2n$-dimensional manifold N is a collection of "holomorphic" coordinate systems that cover N and such that the transition maps between coordinate systems are holomorphic as maps between open sets in $\mathbb{R}^{2n} \cong \mathbb{C}^n$. At each point $z \in N$, there is a linear map $J_z : T_zN \to T_zN$ defined by the expression

$$J_z\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}; \quad J_z\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j},$$

where the x_j's and y_j's are the real and imaginary parts of holomorphic coordinates. This map is independent of the choice of holomorphic coordinates and satisfies $J_z^2 = -I$. At each point $z \in N$, the complexified tangent space $T_z^{\mathbb{C}}N$ can be decomposed into eigenspaces for J_z with eigenvalues i and $-i$; these are called the $(1,0)$- and $(0,1)$-tangent spaces, respectively.

Meanwhile, if N is any $2n$-dimensional manifold and J is a smoothly varying family of linear maps on each tangent space satisfying $J_z^2 = -I$ for all z, then J is called an *almost-complex structure*. Given an almost complex structure, we can divide the complexified tangent space into $\pm i$ eigenspaces for J. The Newlander–Nirenberg theorem asserts that if the family of $+i$ eigenspaces is integrable (in the sense of Point 1 of Definition 23.16), then there exists a unique complex structure on N for which these are the $(1,0)$-tangent spaces.

A purely complex polarization P gives rise to a complex structure on N, as follows. By Proposition 23.18 and the Newlander–Nirenberg theorem, there is a unique complex structure on N for which P_z is the $(1,0)$-tangent space, for all $z \in N$.

Now, we have already seen in the $\mathbb{R}^{2n}$ case that some purely complex polarizations behave better than others. [Compare (22.11) to (22.13)]. The geometric condition that characterizes the "good" polarizations is the following.

Definition 23.19 *For any purely complex polarization P, let J be the unique almost-complex structure on N such that $J_z = iI$ on P_z and $J_z = -iI$ on $\overline{P_z}$. We say that P is a* ***Kähler polarization*** *if the bilinear form*

$$g(X, Y) := \omega(X, J_z Y) \tag{23.10}$$

is positive definite for each $z \in N$.

For any purely complex polarization, the bilinear form g in (23.10) is symmetric, as the reader may easily verify using the J_z-invariance of ω.

Suppose, for example, that we identify $\mathbb{R}^2$ with $\mathbb{C}$ by the map $z = x - i\alpha p$, for some fixed $\alpha > 0$. If we define a purely complex polarization on $\mathbb{R}^2$ by taking P_z to be the span of the vector $\partial/\partial z$ in (22.9), then (Exercise 4), P is a Kähler polarization.

23.5 Quantization Without Half-Forms

To construct a prequantum Hilbert space, we must choose a line bundle (L, ∇) over (N, ω) having curvature $\omega/\hbar$. Such a bundle exists if $\omega/\hbar$ is an integral 2-form and is unique (up to equivalence) if N is simply connected. To pass to the *quantum* Hilbert space, we must make a substantial additional choice, that of a polarization P on N. In our first attempt at defining the quantum Hilbert space associated with P, we consider the space of sections of (L, ∇) that are covariantly constant in the directions of $\overline{P}$. Although this approach works reasonably well for a purely complex polarization, in the case of a purely real polarization, there typically are no square-integrable sections satisfying this condition. (Indeed, we have seen this problem already in the $\mathbb{R}^{2n}$ case, in Sect. 22.4.) In the next section, we will introduce half-forms to address this problem.

In the remainder of the chapter, we will let P denote a fixed polarization on N.

23.5.1 The General Case

As we have remarked, it is customary to consider sections that are covariantly constant in the directions of $\bar{P}$ rather than in the directions of P.

Definition 23.20 *A smooth section s of L is* ***polarized*** *(with respect to P) if*

$$\nabla_X s = 0 \tag{23.11}$$

for every vector field X lying in $\overline{P}$. The ***quantum Hilbert space*** *associated with P is the closure in the prequantum Hilbert space of the space of smooth, square-integrable, polarized sections of L.*

As in the Euclidean case, we will simply restrict the prequantum operators to the quantum Hilbert space, in those cases where $Q_{\text{pre}}(f)$ preserves the space of polarized sections.

Definition 23.21 *A smooth, complex-valued function f on N is* ***quantizable*** *with respect to P if $Q_{\text{pre}}(f)$ preserves the space of smooth sections that are polarized with respect to P.*

The following definition will provide a natural geometric condition guaranteeing quantizability of a function.

Definition 23.22 *A possibly complex vector field X* ***preserves*** *a polarization P if for every vector field Y lying in P, the vector field $[X, Y]$ also lies in P.*

Note that if X lies in P, then X preserves P, by the integrability assumption on P. There will typically be, however, many vector fields that do not lie in P but nevertheless preserve P.

If X is a real vector field, then $[X, Y]$ is the same as the Lie derivative $\mathcal{L}_X(Y)$. It is then not hard to show that X preserves P if and only if the flow generated by X preserves P, that is, if and only if $(\Phi_t)_*(P_z) = P_{\Phi_t(z)}$ for all z and t, where Φ is the flow of X. Furthermore, if X is real, then X preserves P if and only if X preserves $\bar{P}$.

Example 23.23 *If $N = T^*M$ for some manifold M and P is the vertical polarization on N, then a Hamiltonian vector field X_f preserves P if and only if $f = f_1 + f_2$, where f_1 is constant on each fiber and f_2 is linear on each fiber.*

Proof. In local coordinates $\{x_j, p_j\}$, a vector field X lying in P has the form $X = g_j\ \partial/\partial p_j$. Thus,

$$[X_f, X] = \left[\frac{\partial f}{\partial p_j}\frac{\partial}{\partial x_j}, g_k\frac{\partial}{\partial p_k}\right] - \left[\frac{\partial f}{\partial x_j}\frac{\partial}{\partial p_j}, g_k\frac{\partial}{\partial p_k}\right].$$

This commutator will consist of three "good" terms, which involve only p-derivatives, along with the following "bad" term:

$$-g_k\frac{\partial^2 f}{\partial p_k \partial p_j}\frac{\partial}{\partial x_j}.$$

If $\partial^2 f/\partial p_k \partial p_j$ is 0 for all j and k, then the bad term vanishes and $[X_f, X]$ again lies in P. Conversely, if we want the bad term to vanish for each choice of the coefficient functions g_j, we must have $\partial^2 f/\partial p_k \partial p_j = 0$ for all j and k. Thus, for each fixed value of x, f must contain only terms that are independent of p and terms that are linear in p. ∎

We now identify the condition for quantizability of functions.

Theorem 23.24 *For any smooth, complex-valued function f on N, if the Hamiltonian vector field X_f preserves $\bar{P}$, then f is quantizable.*

Since we do not assume that f is real-valued, the condition that X_f preserve $\bar{P}$ is not equivalent to the condition that X_f preserve P.

Proof. Given a polarized section s, we apply $Q_{\mathrm{pre}}(f)$ to s and then test whether $Q_{\mathrm{pre}}(f)s$ is still polarized, by applying ∇_X for some vector field X lying in $\bar{P}$. To this end, it is useful to compute the commutator of ∇_X and $Q_{\mathrm{pre}}(f)$, as follows:

$$\begin{aligned}[\nabla_X, Q_{\mathrm{pre}}(f)] &= i\hbar\left[\nabla_X, \nabla_{X_f}\right] + [\nabla_X, f] \\ &= i\hbar\left(\nabla_{[X,X_f]} - \frac{i}{\hbar}\omega(X, X_f)\right) + X(f) \\ &= i\hbar\nabla_{[X,X_f]}, \end{aligned} \tag{23.12}$$

where we have used that

$$\omega(X, X_f) = -\omega(X_f, X) = -df(X) = -X(f),$$

by Definition 21.6. Since X_f preserves $\bar{P}$, the vector field $[X, X_f]$ again lies in $\bar{P}$ and, thus,

$$\nabla_X(Q_{\mathrm{pre}}(f)s) = Q_{\mathrm{pre}}(f)\nabla_X s + i\hbar\nabla_{[X,X_f]}s = 0,$$

for every polarized section s, showing that $Q_{\text{pre}}(f)s$ is again polarized. ■

The converse of Theorem 23.24 is false in general. After all, as we will see in the following subsections, for a given polarization, there may not be any nonzero globally defined polarized sections, in which case, *any* function is quantizable. On the other hand, it can be shown that if $Q_{\text{pre}}(f)$ preserves the space of *locally defined* polarized sections, then the Hamiltonian flow generated by f must preserve $\bar{P}$. This result follows by the same reasoning as in the proof of Theorem 23.24, once we know that there are sufficiently many locally defined polarized sections. We will establish such an existence result for purely real and purely complex polarizations in the following subsections; for the general case, see the discussion following Definition 9.1.1 in [45].

A special case of Theorem 23.24 is provided by "polarized functions," that is, functions f for which $X(f) = 0$ for all vector fields X lying in $\bar{P}$. For such an f, the action of $Q_{\text{pre}}(f)$ on the quantum space is simply multiplication by f, as we anticipated in the introductory discussion in Sect. 23.4.

Proposition 23.25 *If f is a smooth, complex-valued function on N and the derivatives of f in the $\bar{P}$ directions are zero, then $Q_{\text{pre}}(f)$ preserves the space P-polarized sections, and the restriction of $Q_{\text{pre}}(f)$ to this space is simply multiplication by f.*

We have already seen special cases of this result in the $\mathbb{R}^{2n}$ case; see the discussion following Proposition 22.11.

Proof. If the derivatives of f in the direction of $\bar{P}$ are zero, then for $X \in \bar{P}$, we have

$$0 = X(f) = df(X) = \omega(X_f, X),$$

meaning that X_f is in the ω-orthogonal complement of $\bar{P}$. But since $\bar{P}$ is Lagrangian, this complement is just $\bar{P}$. Thus, X_f belongs to $\bar{P}$ and, in particular, X_f preserves $\bar{P}$, so that f is quantizable, by Theorem 23.24. Furthermore, $\nabla_{X_f} s = 0$ for any P-polarized section s, leaving only the fs term in the formula for $Q_{\text{pre}}(f)s$. ■

23.5.2 The Real Case

In the $\mathbb{R}^{2n}$ case, we have already computed the space of polarized sections for the vertical polarization in Proposition 22.8. As we observed there, there are no nonzero polarized sections that are square integrable over $\mathbb{R}^{2n}$. The same difficulty is easily seen to arise for the vertical polarization on any cotangent bundle $N = T^*M$. In Sect. 23.6, we will introduce half-forms to deal with this failure of square integrability.

We now examine properties of general real polarizations. We will see that polarized sections always exist locally, but not always globally.

Proposition 23.26 *If P is a purely real polarization on N, then for any $z_0 \in N$, there exist a neighborhood U of z_0 and a P-polarized section s of L defined over U such that $s(z_0) \neq 0$.*

Proof. According to the local form of the Frobenius theorem, we can find a neighborhood U of z_0 and a diffeomorphism Φ of U with a neighborhood V of the origin in $\mathbb{R}^n \times \mathbb{R}^n$ such that under Φ, the polarization P looks like the vertical polarization. That is to say, for each $z \in U$, the image of P_z under $\Phi_*(z)$ is just the span of the vectors $\partial/\partial y_1, \ldots, \partial/\partial y_n$, where the y's are the coordinates on the second copy of $\mathbb{R}^n$. By shrinking U if necessary, we can assume that L can be trivialized over U and that the open set V is the product of a ball B_1 centered at the origin in the first copy of $\mathbb{R}^n$ with a ball B_2 centered at the origin in the second copy of $\mathbb{R}^n$.

Let θ be the connection 1-form for an isometric trivialization of L over U and let $\tilde{\theta} = (\Phi^{-1})^*(\theta)$. Since the subspaces P_z are Lagrangian, the restriction of $\tilde{\theta}$ to the each set of the form $\{\mathbf{x}\} \times B_2$ is closed. Since B_2 is simply connected, there exists, for each $\mathbf{x} \in B_1$, a function $f_{\mathbf{x}}$ on B_2 such that the restriction of $\tilde{\theta}$ to $\{\mathbf{x}\} \times B_2$ equals $df_{\mathbf{x}}$. If we assume that $f_{\mathbf{x}}(0) = 0$, then $f_{\mathbf{x}}(\mathbf{y})$ will be smooth as a function of $(\mathbf{x}, \mathbf{y})$, since it is obtained simply by integrating $\tilde{\theta}$ from 0 to $\mathbf{y}$ in the vertical directions.

Now, let ϕ be any smooth function on B_1 with $\phi(0) \neq 0$ and define a function ψ on $B_1 \times B_2$ by

$$\psi(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{x}) e^{i f_{\mathbf{x}}(\mathbf{y})/\hbar}.$$

For any "vertical" vector field X (i.e., one where X is a linear combination of $\partial/\partial y_1, \ldots, \partial/\partial y_n$ with smooth coefficients), we compute that

$$X\psi = \frac{i}{\hbar}(X f_{\mathbf{x}})\psi = \frac{i}{\hbar} df_{\mathbf{x}}(X)\psi = \frac{i}{\hbar}\tilde{\theta}(X)\psi.$$

Thus,

$$\left(X - \frac{i}{\hbar}\tilde{\theta}(X)\right)\psi = 0,$$

from which it follows that the function $\hat{\psi} := \psi \circ \Phi$ represents a polarized section on U in the given local trivialization of L. ■

The existence of nonzero *global* polarized sections for a purely real polarization P is a more delicate question. If the leaves of P are not embedded, there is little chance of finding global polarized sections. Even if the leaves are embedded, there are obstructions. Since the tangent spaces to the leaves of P are Lagrangian subspaces, the restriction of L to R has zero curvature. There may, nevertheless, be loops in R for which the holonomy (Definition 23.8) is nontrivial. After all, if a loop γ in R is not the boundary of a surface S in R, then we cannot apply (23.6) to conclude that the holonomy of γ is trivial. The collection of holonomies for a leaf R of P can be understood as a homomorphism of $\pi_1(R)$ into S^1. If there is any loop in R with nontrivial holonomy, any polarized section of L must vanish on R.

Definition 23.27 *A submanifold R of N is said to be* ***Lagrangian*** *if* dim $R = n$ *and T_zR is a Lagrangian subspace of T_zN for each $z \in R$. A Lagrangian submanifold R of N is said to be* ***Bohr–Sommerfeld*** *(with respect to L) if the holonomy in L of every loop in R is trivial.*

We may summarize the preceding discussion as follows.

Conclusion 23.28 *For a purely real polarization P with embedded leaves, a polarized section vanishes on every leaf of P that is not Bohr–Sommerfeld.*

Our next example suggests that when the leaves are compact, the Bohr–Sommerfeld leaves typically form a discrete set within the set of all leaves.

Example 23.29 *Let $N = S^1 \times \mathbb{R}$, equipped with the symplectic form $\omega = dx \wedge d\phi$, where x is the linear coordinate on $\mathbb{R}$ and ϕ is the angular coordinate on S^1. Let L be the trivial line bundle on N, with sections that are identified with smooth functions. Let $\theta = x\ d\phi$ and define a connection ∇ on L by $\nabla_X = X - (i/\hbar)\theta(X)$, and let P be the purely real polarization of N for which the leaves are the sets of the form $S^1 \times \{x\}$, for $x \in \mathbb{R}$. Then a leaf $S^1 \times \{x\}$ is Bohr–Sommerfeld if and only if $x/\hbar$ is an integer.*

In particular, there are no nonzero, smooth polarized sections of L.

Proof. If we define a section locally on a given leaf $S^1 \times \{x\}$ as

$$s(\phi) = ce^{ix\phi/\hbar}$$

for some nonzero constant c, then it is easily verified that $\nabla_{\partial/\partial\phi} s = 0$. After one trip around the circle, the value of this section will be the starting value times $e^{2\pi i x/\hbar}$. Thus, the holonomy around $S^1 \times \{x\}$ is trivial if and only if $x/\hbar$ is an integer. A polarized section, then, would have to vanish on all the leaves where $x/\hbar$ is not an integer. Since such leaves form a dense subset of N, any smooth polarized section must be identically zero. ■

Even in cases, such as Example 23.29, where there are no smooth polarized sections, one may still consider "distributional" polarized sections that are supported on the Bohr–Sommerfeld leaves, as on pp. 251–252 of [45].

23.5.3 The Complex Case

In Proposition 22.8, we computed the space of polarized sections for a certain positive, translation-invariant polarization on $\mathbb{R}^{2n}$, namely the one for which P_z is spanned by the vectors $\partial/\partial z_j$ in (22.9). The situation here is better than that for the vertical polarization, in that there are nonzero polarized sections that are square integrable over $\mathbb{R}^{2n}$. Recall, however, that if we take our polarization to be spanned by the vectors $\partial/\partial \bar{z}_j$, then [see (22.13)], then there are no nonzero square-integrable polarized sections. This example indicates the importance of the positivity condition in Definition 23.19.

For our next example, we consider the example of the unit disk D, equipped with the unique (up to a constant) symplectic form that is invariant under the group of fractional linear transformations that map D onto D. In this case, the quantum Hilbert space can be identified with a *weighted Bergman space*, that is, an L^2 space of holomorphic functions on D with respect to a measure of the form $(1-|z|^2)^\nu dx\ dy$.

Example 23.30 *Let N be the unit disk $D \subset \mathbb{R}^2$ equipped with the following symplectic form:*

$$\omega = 4(1-|z|^2)^{-2}\ dx \wedge dy = (1-r^2)^{-2} r\ dr \wedge d\phi,$$

where (r,ϕ) are the usual polar coordinates. Let L be the trivial line bundle over D with connection $\nabla_X = X - (i/\hbar)\theta$, where θ is the symplectic potential for ω given by

$$\theta = 2\frac{r^2}{1-r^2}\ d\phi.$$

Define a complex polarization on D by letting $P_z = \mathrm{Span}(\partial/\partial z)$, where $z = x - iy$. In that case, holomorphic sections s have the form

$$s(z) = F(z)(1-|z|^2)^{1/\hbar},$$

where F is holomorphic. The norm of such a section is computed as

$$\|s\|^2 = 4\int_D |F(z)|^2 (1-|z|^2)^{2/\hbar-2}\ dx\ dy.$$

As in the case of the plane, the seemingly unnatural definition $z = x - iy$ is necessary to obtain a Kähler polarization. If we used $z = x + iy$ instead, the holomorphic sections would have the form $F(z)(1-|z|^2)^{-1/\hbar}$, in which case there would be no nonzero, square-integrable holomorphic sections.
Proof. See Exercise 8. ■

We now consider general purely complex polarizations. Recall that, by Proposition 23.18 and the Newlander–Nirenberg theorem, N has a unique complex structure for which P_z is the $(1,0)$-subspace of $T_z^{\mathbb{C}}N$, for all $z \in N$. As in the purely real case, there always exist local polarized sections.

Theorem 23.31 *Suppose P is a purely complex polarization on N. Then for each $z_0 \in N$, there exists a P-polarized section s of L, defined in a neighborhood of z_0, such that $s(z_0) \neq 0$.*

We defer the proof of Theorem 23.31 until the end of this subsection.

Suppose s is as in the theorem and s' is any other locally defined P-polarized section. Then $s' = fs$ for some unique complex-valued function f, and by the product rule for covariant derivatives, $X(f) = 0$ for all $X \in \bar{P}_z$. This means that f is holomorphic with respect to the complex structure on N for which P is the $(1,0)$-tangent space. Thus, we have a preferred family of local trivializations of L (the ones given by nonvanishing local polarized sections) such that the "ratio" of any two such trivializations is a holomorphic function. This means that we have given L the structure of a "holomorphic line bundle" over the complex manifold N in such a way that the holomorphic sections of L are precisely the polarized sections with respect to P.

Arguing as in the proof of Proposition 14.15, it is not hard to show that for a purely complex polarization, the space of square-integrable polarized sections of L forms a closed subspace of the prequantum Hilbert space. For any $z \in N$, if we choose a linear identification of the fiber of L over z with $\mathbb{C}$, then the map $s \mapsto s(z)$ is a linear functional on the quantum Hilbert space. It is not hard to show, as in the proof of Proposition 14.15, that this linear functional is continuous, and can therefore be represented as an inner product with a unique element of the quantum Hilbert space.

Definition 23.32 *Let P be a purely complex polarization on N. For each $z \in N$, choose a linear identification of the fiber of L over z with $\mathbb{C}$. Then the **coherent state** χ_z is the unique element of the quantum Hilbert space with respect to P such that*

$$s(z) = \langle \chi_z, s \rangle$$

for all s.

Suppose $N = \mathbb{R}^2$ with a polarization given by $P_z = \mathrm{Span}(\partial/\partial z)$, where $z = x - i\alpha p$. If we use the symplectic potential $\theta = (p\ dx - x\ dp)/2$, then, as in the proof of Proposition 22.14, the quantum Hilbert space is naturally identifiable with the Segal–Bargmann space. In this case, the coherent states can be read off from Proposition 14.17.

It could happen that $\chi_z = 0$ for some $z \in N$, or even for *all* $z \in N$, depending on the choice of P. Even if χ_z is nonzero, χ_z is only well defined up to multiplication by a constant, because we must choose an identification of $L^{-1}(\{z\})$ with $\mathbb{C}$. But if $\chi_z \neq 0$, the one-dimensional subspace spanned by χ_z is independent of this choice. That is to say, whenever $\chi_z \neq 0$, the span of χ_z is a well-defined element of the projective space $\mathcal{P}(\mathbf{H})$, where $\mathbf{H}$ is the quantum Hilbert space.

Recall, meanwhile, that if (L, ∇) is a Hermitian line bundle with connection having curvature $\omega/\hbar$, then for any positive integer n, there is a natural Hermitian connection on $L^{\otimes k}$ having curvature $k\omega/\hbar$. This means that if L is a prequantum line bundle with one value $\hbar_0$ of Planck's constant, then $L^{\otimes k}$ is a prequantum line bundle with Planck's constant equal to $\hbar_0/k$. The following result shows that in the case of compact symplectic manifolds with Kähler polarizations, things behave nicely when k tends to infinity.

Theorem 23.33 *Assume N is compact and let P be a Kähler polarization on N. For each positive integer k, let $\mathbf{H}_k$ denote the space of polarized* *sections of* $L^{\otimes k}$*. Then for all* k, $\mathbf{H}_k$ *is finite dimensional. Furthermore, for all sufficiently large* k*, we have the following results. First, the coherent state* $\chi_z \in \mathbf{H}_k$ *is nonzero for each* $z \in N$*. Second, the map*

$$z \mapsto \mathrm{Span}(\chi_z)$$

is an antiholomorphic embedding of N *into* $\mathcal{P}(\mathbf{H}_k)$*.*

The finite dimensionality of $\mathbf{H}_k$ is a standard result in the theory of compact, complex manifolds. The embedding of N into $\mathcal{P}(\mathbf{H}_k)$ is the *Kodaira embedding theorem*, which we will not prove here. The Kodaira embedding theorem implies, in particular, that there exist nonzero, globally defined polarized sections of $L^{\otimes k}$, at least for large k. Since the value of Planck's constant for $L^{\otimes k}$ is $\hbar_0/k$, Planck's constant tends to zero as k tends to infinity. Thus, the study of holomorphic sections of $L^{\otimes k}$ for large k can be understood as being part of semiclassical analysis.

We now turn to the proof of Theorem 23.31, in which we will make use of basic properties of complex-valued differential forms on complex manifolds. ("Complex-valued" means that we allow the value of a k-form on a collection of k tangent vectors to be a complex number.) In a holomorphic local coordinate system $z_1, \ldots, z_n$, each form can be written as a wedge product of the dz_j's and $d\bar{z}_j$'s. A form is called a (p,q)-form if it is a linear combination of wedge products of p factors involving the dz_j's and q factors involving the $d\bar{z}_j$'s. Each form can be decomposed uniquely as a linear combination of (p,q)-forms for various values of p and q, and this decomposition does not depend on the choice of holomorphic coordinate system. If α is a (p,q)-form, then $d\alpha$ will be a linear combination of a $(p+1,q)$-form and a $(p,q+1)$-form. We define operators ∂ and $\bar{\partial}$ in such a way that ∂ maps (p,q)-forms to $(p+1,q)$-forms, $\bar{\partial}$ maps (p,q)-forms to $(p,q+1)$ forms, and $d = \partial + \bar{\partial}$. In particular,

$$\begin{aligned}
&\partial(f\ dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge \cdots \wedge dz_{k_q}) \\
&= \sum_l \frac{\partial f}{\partial z_l} dz_l \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge \cdots \wedge dz_{k_q}
\end{aligned}$$

and similarly for $\bar{\partial}$ with $(\partial f/\partial z_l)\ dz_l$ replaced by $(\partial f/\partial \bar{z}_l)\ d\bar{z}_l$.

The maps ∂ and $\bar{\partial}$ satisfy the identities:

$$\begin{aligned}
\partial\partial &= \bar{\partial}\bar{\partial} = 0 \\
\partial\bar{\partial} &= -\bar{\partial}\partial.
\end{aligned}$$

The Dolbeault lemma states that if a (p,q)-form α satisfies $\partial\alpha = 0$, then α can be expressed locally as $\partial\beta$ for some $(p-1,q)$-form, and if $\bar{\partial}\alpha = 0$, then α can be expressed locally as $\bar{\partial}\beta$ for some $(p,q-1)$-form. A $(p,0)$-form α is said to be *holomorphic* if it can be expressed in holomorphic coordinates as a sum of terms of the form

$$f(z)\ dz_{j_1} \wedge \cdots \wedge dz_{j_p},$$

where the coefficient functions f is holomorphic. A $(p,0)$-form α is holomorphic if and only if $\bar{\partial}\alpha = 0$. If a holomorphic $(p,0)$-form α satisfies $d\alpha = 0$ (or, equivalently, $\partial\alpha = 0$), then α can be written locally as $\alpha = d\beta$, for some holomorphic $(p-1,0)$-form.

Let P be a purely complex polarization on N and let J be the almost-complex structure for which P_z is the $(1,0)$-tangent space at z. Since (Proposition 23.18), ω is J-invariant, it follows (Exercise 6) that ω is a $(1,1)$-form.

Lemma 23.34 *Let N be a complex manifold with almost-complex structure J and let ω be a closed, J-invariant, real-valued (1,1)-form on N. Then for every point $z_0 \in N$, there exists a smooth, real-valued function κ defined in a neighborhood of z_0 such that $i\partial\bar{\partial}\kappa = \omega$.*

In the case that N is Kähler [i.e., the case where $\omega(X, JX) \geq 0$], a function κ as in the lemma is called a (local) *Kähler potential* for N.

Proof. By assumption, $d\omega = (\partial + \bar{\partial})\omega = 0$, from which it follows that $\partial\omega = \bar{\partial}\omega = 0$, because $\partial\omega$ is a $(2,1)$-form and $\bar{\partial}\omega$ is a $(1,2)$ form. Thus, by the Dolbeault lemma, there exists a $(1,0)$-form α, defined in a neighborhood of z_0, such that $\bar{\partial}\alpha = \omega$. Then $\partial\alpha$ is a $(2,0)$-form that satisfies

$$\bar{\partial}\partial\alpha = -\partial\bar{\partial}\alpha = -\partial\omega = 0.$$

This shows that $\partial\alpha$ is actually a *holomorphic* $(2,0)$-form.

Since also $\partial\partial\alpha = 0$, we see that $\partial\alpha$ is closed, which means that there exists a holomorphic 1-form η, defined in a possibly smaller neighborhood of z_0, such that $d\eta = \partial\eta = \partial\alpha$. Thus, $\partial(\alpha - \eta) = 0$, and so by the Dolbeault lemma, there exists a function g, defined in a neighborhood of z_0, such that $\partial g = \alpha - \eta$. Thus, $\alpha = \eta + \partial g$ and so

$$\omega = \bar{\partial}\alpha = \bar{\partial}\partial g = -\partial\bar{\partial}g$$

since $\bar{\partial}\eta = 0$. The function $\kappa := ig$ then satisfies $i\partial\bar{\partial}\kappa = \omega$.

Now, a calculation in coordinates (Exercise 7) shows that the map $\kappa \mapsto i\partial\bar{\partial}f$ is real, that is, it maps real-valued functions to real-valued 2-forms. Since ω is real, the operator $i\partial\bar{\partial}$ must map the imaginary part of κ to zero. Thus, $i\partial\bar{\partial}\kappa$ is unchanged if κ is replaced by its real part. ∎

Proof of Theorem 23.31. Let κ be as in Lemma 23.34 and let θ be the real-valued 1-form given by

$$\theta = \operatorname{Im}(\partial\kappa) = \frac{1}{2i}\left(\partial\kappa - \bar{\partial}\kappa\right). \tag{23.13}$$

Then because $\partial^2 = \bar{\partial}^2 = 0$, we have

$$d\theta = (\partial + \bar{\partial})\theta = \frac{1}{2i}(\bar{\partial}\partial\kappa - \partial\bar{\partial}\kappa) = \omega.$$

That is to say, θ is a symplectic potential for ω. Thus, by Proposition 23.6, we can find a local isometric trivialization s_0 of L for which the connection 1-form is $\theta/\hbar$.

For any vector X, we have

$$\nabla_X \left(e^{-\kappa/(2\hbar)} s_0 \right) = \left(-\frac{1}{2\hbar} X(\kappa) - \frac{i}{\hbar}\theta(X) \right) e^{-\kappa/\hbar} s_0, \tag{23.14}$$

where $X(\kappa) = d\kappa(X) = \partial\kappa(X) + \bar{\partial}\kappa(X)$. Now, if X is of type $(0,1)$, then $\partial\kappa(X) = 0$, in which case, if we use (23.13), we find that the two terms on the right-hand side of (23.14) cancel. Thus, $e^{-\kappa/(2\hbar)} s_0$ is the desired local polarized section. ∎

23.6 Quantization with Half-Forms: The Real Case

In this section, we introduce a concept known as *half-forms*, which are designed to work around the problem that, in the case of real polarizations, there often do not exist any nonzero square-integrable polarized sections.

A polarized section s for a real polarization P tends to have infinite norm, because we may get infinity from integrating $|s|^2$ along the leaves of the polarization. To illustrate how half-forms work around this problem, consider the case of the vertical polarization on $\mathbb{R}^2 \cong T^*\mathbb{R}$. Elements of the half-form Hilbert space will be representable in the form $s \otimes \sqrt{dx}$, where s is a polarized section of L and where $\sqrt{dx}$ will be interpreted as a "section of the square root of the canonical bundle." To compute the norm of such an object, we first square it at each point to obtain the quantity $|s|^2\ dx$. Since s is polarized, $|s|^2$ is a function of x only, independent of p. Thus, $|s|^2\ dx$ may be thought of as a 1-form on $\mathbb{R}$, rather than on $\mathbb{R}^2$, which we may then integrate to obtain

$$\|s\|^2 := \int_{\mathbb{R}} |s|^2 (x)\ dx.$$

This procedure has two advantages over the one we used in Sect. 22.4, where we simply integrated $|s|^2$ itself over $\mathbb{R}$. First, a version of this procedure works for real polarizations on general symplectic manifolds. Second, the half-form approach will allow quantized observables to be self-adjoint, which was not the case in Sect. 22.5 when we simply restricted prequantized observables to the polarized subspace. (See the discussion following Proposition 22.12.)

Throughout this section, we assume that N is a quantizable symplectic manifold, that L is a fixed prequantum line bundle over N, and that P is a fixed purely real polarization on N.

23.6.1 The Space of Leaves

Recall that a *leaf* of P is a maximal connected, integral submanifold of P. We may then form the *leaf space* Ξ (the set of all leaves of P) and a quotient map $q : N \to \Xi$ sending each point $z \in N$ to the unique leaf containing z. We may topologize Ξ by defining a set U in Ξ to be open if $q^{-1}(U)$ is open in N.

In order to be able to carry out the program of geometric quantization with respect to P, we must assume that Ξ can be given the structure of a smooth, n-dimensional manifold in such a way that $q : N \to \Xi$ is smooth and such that the kernel of $q_{*,z}$ is equal to $P_z^{\mathbb{R}}$, the intersection of P_z with the real tangent space of P_z. We abbreviate this assumption on Ξ by saying that Ξ *is a smooth manifold.* In the case $N = T^*M$ with the vertical polarization (Example 23.17), the leaf space Ξ is a smooth manifold diffeomorphic to M.

It should be emphasized that even if Ξ is a smooth manifold, there is no canonical "volume measure" on Ξ. Thus, our half-form Hilbert space will be defined in such a way that the pointwise "square" of an element will be an n-form, rather than a function, on the leaf space, which can then be integrated over the n-manifold Ξ.

23.6.2 The Canonical Bundle

We now introduce the canonical bundle of a purely real polarization P, with sections that are a special sort of n-form on N, along with a notion of polarized section of the canonical bundle. If the leaf space Ξ is a smooth manifold, the space of polarized sections of the canonical bundle can be identified with the space of all n-forms on the n-manifold Ξ.

Definition 23.35 *The* ***canonical bundle*** $\mathcal{K}_P$ *of* P *is the* real *line bundle with sections that are* n*-forms* α *having the property that*

$$X \lrcorner \alpha = 0 \tag{23.15}$$

for every vector field X *lying in* P*. A section* α *of* $\mathcal{K}_P$ *is* ***polarized*** *if*

$$X \lrcorner (d\alpha) = 0 \tag{23.16}$$

for every vector field X *lying in* P*.*

If an n-form α satisfies (23.15), then $\alpha(X_1, \ldots, X_n) = 0$ if any of the X_j's belongs to P. Thus, the value of α at any point z can be viewed as an n-linear, alternating functional on the quotient vector space $T_zN/P_z^{\mathbb{R}}$, where $P_z^{\mathbb{R}}$ is the intersection of P_z with the real tangent space. Since this quotient space is n-dimensional, we see that at each point, the space of possible values for α is one dimensional.

Meanwhile, if α satisfies (23.16), then at each point, $d\alpha$ is an $(n+1)$-linear, alternating functional on $T_zN/P_z^{\mathbb{R}}$, which must be zero. Thus, for sections of $\mathcal{K}_P$, (23.16) is equivalent to the condition

$$d\alpha = 0. \tag{23.17}$$

We can also introduce the *complexified canonical bundle* $\mathcal{K}_P^{\mathbb{C}}$, the sections of which are complex-valued n-forms satisfying (23.15). We define a section of $\mathcal{K}_P^{\mathbb{C}}$ to be polarized if it satisfies (23.16).

Example 23.36 *Let $N = T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ and let P be the vertical polarization on N. Then an n-form α on $\mathbb{R}^{2n}$ is a section of $\mathcal{K}_P$ if and only if α is of the form*

$$\alpha = f(\mathbf{x}, \mathbf{p})\ dx_1 \wedge \cdots \wedge dx_n, \tag{23.18}$$

and α is a polarized section of $\mathcal{K}_P$ if and only if α is of the form

$$\alpha = g(\mathbf{x})\ dx_1 \wedge \cdots \wedge dx_n, \tag{23.19}$$

for smooth functions f on $\mathbb{R}^{2n}$ and g on $\mathbb{R}^n$.

Proof. If α contained any term involving dp_j, the contraction of α with $\partial/\partial p_j$ would not be zero, leaving (23.18) as the only possible form for a section of $\mathcal{K}_P$. Assuming α is of the form (23.18), if f is not independent of $\mathbf{p}$, then $d\alpha$ will contain a nonzero term of the form $dp_j \wedge dx_1 \wedge \cdots \wedge dx_n$, leaving (23.19) as the only possible form for a polarized section of $\mathcal{K}_P$. ■

In Example 23.36, the polarized sections of $\mathcal{K}_P$ are effectively just n-forms on the configuration space $\mathbb{R}^n$. This conclusion is a special case of the following result.

Proposition 23.37 *If the leaf space Ξ of P is a smooth manifold and α is a polarized section of $\mathcal{K}_P$, then there exists a unique n-form $\tilde{\alpha}$ on Ξ such that*

$$\alpha = q^*(\tilde{\alpha}),$$

where $q : N \to \Xi$ is the quotient map. Conversely, if β is any n-form on Ξ, then $\alpha := q^*(\beta)$ is a polarized section of $\mathcal{K}_P$.

Proof. Suppose, first, that $\alpha = q^*(\beta)$, for an n-form β on Ξ. Then $X \lrcorner \alpha = 0$ whenever X lies in P, since P is the kernel of q_*. Furthermore, $d\alpha = q^*(d\beta) = 0$, since β is an n-form on an n-manifold, showing that α is a polarized section of $\mathcal{K}_P$.

In the other direction, we have already noted in the proof of Proposition 23.26 that N can be identified locally with a neighborhood $U \times V$ of the origin $\mathbb{R}^n \times \mathbb{R}^n$ in such a way that leaves of P correspond to the sets of the form $\{\mathbf{x}\} \times V$. We can use q to identify $U \cong U \times \{0\}$ with an open set $\tilde{U}$ in Ξ. Thus, P looks locally just like the vertical polarization on $\mathbb{R}^{2n}$, and so, by Example 23.36, any polarized section α of $\mathcal{K}_P$ will be of the form

(23.19). Thus, α determines an n-form $\hat{\alpha}$ on U and α is the pullback of $\hat{\alpha}$ by the projection map of $U \times V$ onto U. It follows that α is locally the pullback by q of an n-form $\tilde{\alpha}$ on $\tilde{U}$. We leave it to the reader to check that overlapping neighborhoods in N give the same form $\tilde{\alpha}$ on Ξ and that the desired result holds globally. ■

Recall from Theorem 23.24 that $Q_{\text{pre}}(f)$ preserves the space of polarized sections with respect to P, provided that the flow of X_f preserves $\bar{P}$ (which equals P, in this case). We now establish that for any such f, the Lie derivative $\mathcal{L}_{X_f}$ preserves the space of polarized sections of $\mathcal{K}_P$. This result will eventually allow us to define a quantum operator $Q(f)$ on the half-form Hilbert space associated to P.

Proposition 23.38 *Suppose X is a vector field on N that preserves P, in the sense of Definition 23.22, and suppose α is a smooth section of $\mathcal{K}_P$. Then the Lie derivative $\mathcal{L}_X\alpha$ is another section of $\mathcal{K}_P$ and if α is polarized, $\mathcal{L}_X\alpha$ is also polarized.*

Proof. Suppose $X_1, \ldots, X_n$ are smooth vector fields, with X_1 lying in $\bar{P} = P$. Then, by a standard formula for the Lie derivative,

$$\begin{aligned} &(\mathcal{L}_X\alpha)(X_1, \ldots, X_n) \\ &= X(\alpha(X_1, \ldots, X_n)) - \alpha([X, X_1], X_2, \ldots, X_n) \\ &- \sum_{j=2}^{n} \alpha(X_1, \ldots, X_{j-1}, [X, X_j], X_{j+1}, \ldots, X_n). \end{aligned} \tag{23.20}$$

Now, because α is a section of $\mathcal{K}_P$, the first and third terms on the right-hand side of (23.20) vanish. Because X preserves P, $[X, X_1]$ will again lie in P, and so the second term vanishes as well. Thus, $X_1 \lrcorner (\mathcal{L}_X\alpha) = 0$, which means that $\mathcal{L}_X\alpha$ is again a section of $\mathcal{K}_P$.

Since $\mathcal{L}_X\alpha = X \lrcorner d\alpha + d(X \lrcorner \alpha)$, if α satisfies (23.17), we have

$$d(\mathcal{L}_X\alpha) = d^2(X \lrcorner \alpha) = 0,$$

showing that α is again polarized. ■

Proposition 23.39 *Suppose the leaf space Ξ of P is a smooth manifold and that a vector field X on N preserves P. Then there exists a unique vector field Y on Ξ such that*

$$q_{*,z}(X) = Y \tag{23.21}$$

for all $z \in N$. Furthermore, if $\alpha = q^*(\beta)$ is a polarized section of $\mathcal{K}_P$, as in Proposition 23.37, then

$$\mathcal{L}_X(q^*(\beta)) = q^*(\mathcal{L}_Y(\beta)). \tag{23.22}$$

That is to say, under the identification in Proposition 23.37 of polarized sections of $\mathcal{K}_P$ with n-forms on Ξ, the operator $\mathcal{L}_X$ corresponds to the Lie derivative on Ξ in the direction of Y.

Proof. By Definition 23.22, $[X, Z]$ lies in P whenever the vector field Z lies in P. Thus, if a function ϕ is constant along P (i.e., annihilated by every vector field Z lying in P), the same will be true of $X\phi$. Thus, if ϕ is of the form $\phi = \psi \circ q$ for some function ψ on Ξ, then $X\phi$ is of the form $\hat{\psi} \circ q$ for some other function $\hat{\psi}$ on Ξ. The map $\psi \mapsto \hat{\psi}$ is easily seen to be a vector field, that is, a derivation of $C^\infty(\Xi)$. We conclude, then, that there is a unique vector field Y on Ξ such that

$$X(\psi \circ q) = (Y\psi) \circ q \tag{23.23}$$

for every smooth function ψ on Ξ. It then follows from the definition of the differential that (23.21) holds for all $z \in N$. From (23.21), it follows easily that for any n-form β on Ξ, we have

$$X \lrcorner (q^*(\beta)) = q^*(Y \lrcorner \beta). \tag{23.24}$$

Since β, being a top-degree form, is closed, $q^*(\beta)$ is also closed. Thus, one of the terms in the formula (21.7) for the Lie derivative of β and $q^*(\beta)$ is zero. Applying d to both sides of (23.24) then gives (23.22). ∎

Given a vector field Y and a nowhere-vanishing n-form β on Ξ, let $\operatorname{div}_\beta Y$ be the unique function on Ξ such that

$$\mathcal{L}_Y(\beta) = (\operatorname{div}_\beta Y)\beta.$$

Then by (23.22), we have

$$\mathcal{L}_X(q^*(\beta)) = ((\operatorname{div}_\beta Y) \circ q)q^*(\beta). \tag{23.25}$$

The expression (23.25) will be helpful in analyzing the quantization of observables in Sect. 23.6.5.

23.6.3 Square Roots of the Canonical Bundle

We now assume that the leaf space Ξ of P is an *orientable* manifold, and we choose on particular orientation of Ξ.

Definition 23.40 *Choose a nowhere-vanishing, oriented n-form β on Ξ, so that $\alpha := q^*(\beta)$ is (Proposition 23.37) a nowhere-vanishing section of $\mathcal{K}_P$. A section of $\mathcal{K}_P$ is* ***non-negative*** *if it is, at each point, a non-negative multiple of α. This notion does not depend on the choice of oriented n-form β.*

Since Ξ is orientable, the canonical bundle $\mathcal{K}_P$ is trivializable, since the section α in Definition 23.40 is a globally trivializing section. Thus, we can find a square root of $\mathcal{K}_P$, that is, a line bundle δ_P such that $\delta_P \otimes \delta_P$ is isomorphic to $\mathcal{K}_P$. (We may, for example, take δ_P to be the trivial bundle.) When we speak of a *square root* of $\mathcal{K}_P$, we will mean, more precisely, a bundle δ_P together with a *particular* isomorphism of $\delta_P \otimes \delta_P$ with $\mathcal{K}_P$. Thus, if s_1 and s_2 are sections of δ_P, we think of $s_1 \otimes s_2$ as being a section of $\mathcal{K}_P$. We assume, further, that the isomorphism of $\delta_P \otimes \delta_P$ with $\mathcal{K}_P$ is chosen so that for any section s of δ_P, the section $s \otimes s$ of $\mathcal{K}_P$ is non-negative. (If the initial isomorphism of $\delta_P \otimes \delta_P$ with $\mathcal{K}_P$ does not have this property, compose it with $-I$ in the fibers of $\mathcal{K}_P$.)

We may consider the *complexification* of δ_P, that is, the line bundle $\delta_P^{\mathbb{C}}$ whose fiber at each point is the complexification of the fiber of δ_P. There is then a notion of *complex conjugation* for sections of $\delta_P^{\mathbb{C}}$, which fixes the fiber of δ_P inside the fiber of $\delta_P^{\mathbb{C}}$ at each point. If s_1 and s_2 are sections of $\delta_P^{\mathbb{C}}$, we think of $s_1 \otimes s_2$ as a section of the complexified canonical bundle $\mathcal{K}_P^{\mathbb{C}}$.

If α is a section of $\mathcal{K}_P$ and X is a vector field lying in P, let us define an n-form $\nabla_X \alpha$ by

$$\nabla_X \alpha = X \lrcorner (d\alpha). \tag{23.26}$$

Since α is a section of $\mathcal{K}_P$, we have $X \lrcorner \alpha = 0$, which means that $\nabla_X \alpha$ actually coincides with $\mathcal{L}_X \alpha$, by (21.7). Since it lies in P, the vector field X preserves P, and thus $\nabla_X \alpha = \mathcal{L}_X \alpha$ is again a section of $\mathcal{K}_P$, by Proposition 23.38. The operator ∇ in (23.26) has all the properties of a connection on $\mathcal{K}_P$ except that it is only defined in the directions of P. [Note that $\mathcal{L}_X$ does not, in general, satisfy the condition $\mathcal{L}_{fX} = f\mathcal{L}_X$, as required by Definition 23.2. Since, however, $\mathcal{L}_X \alpha$ can also be computed as in (23.26), for any section α of $\mathcal{K}_P$, the map ∇ does satisfy $\nabla_{fX} = f\nabla_X$.]

We call ∇ the natural *partial connection* on $\mathcal{K}_P$. According to Definition 23.35, a section α of $\mathcal{K}_P$ is polarized if and only if $\nabla_X \alpha = 0$ for each vector field X lying in P. We now show that both the partial connection and the Lie derivative "descend" to sections of δ_P in a natural way. This result will, in particular, allow us to define a notion of polarized sections of δ_P.

Proposition 23.41 *Let δ_P be a fixed square root of $\mathcal{K}_P$. For any vector field X lying in P, there is a unique linear operator ∇_X mapping sections of δ_P to sections of δ_P, such that*

$$\nabla_X (fs_1) = X(f)s_1 + f\nabla_X s_1 \tag{23.27}$$

$$\nabla_X (s_1 \otimes s_2) = (\nabla_X s_1) \otimes s_2 + s_1 \otimes (\nabla_X s_2) \tag{23.28}$$

for all smooth functions f and all sections s_1 and s_2 of δ_P. On the left-hand side of (23.28), ∇_X is the partial connection on $\mathcal{K}_P$ given by (23.26).

If X is a vector field on N that preserves P, then there is a unique linear operator $\mathcal{L}_X$, mapping sections of δ_P to sections of δ_P such that

$$\mathcal{L}_X(fs_1) = X(f)s_1 + f\mathcal{L}_X s_1$$
$$\mathcal{L}_X(s_1 \otimes s_2) = (\mathcal{L}_X s_1) \otimes s_2 + s_1 \otimes (\mathcal{L}_X s_2)$$

for all smooth functions f and all sections s_1 and s_2 of δ_P.

Both of these constructions extend naturally from sections of δ_P to sections of $\delta_P^{\mathbb{C}}$.

We may then say that a section s of $\delta_P^{\mathbb{C}}$ is *polarized* if $\nabla_X s = 0$ for every smooth vector field X lying in P.

Proof. If V is a one-dimensional vector space, then the map $\otimes : V \times V \to V \otimes V$ is commutative: $u \otimes v = v \otimes u$ for all $u, v \in V$. Furthermore, if u_0 is a nonzero element of V, then the map $u \mapsto u \otimes u_0$ is an invertible linear map of V to $V \otimes V$. Suppose s_0 is a local nonvanishing section of δ_P. Applying (23.28) with $s_1 = s_2 = s_0$, we want

$$2(\nabla_X s_0) \otimes s_0 = \nabla_X (s_0 \otimes s_0). \tag{23.29}$$

Since the operation of tensoring with s_0 is invertible, there is a unique section "$\nabla_X s_0$" of δ_P for which (23.29) holds.

Locally, any section s of δ_P can be written as $s = gs_0$ for a unique function g. We then define $\nabla_X s$ by

$$\nabla_X s = X(g)s_0 + g\nabla_X s_0, \tag{23.30}$$

in which case, (23.27) is easily seen to hold. If $s_1 = g_1 s_0$ and $s_2 = g_2 s_0$, then using (23.29) and the symmetry of the tensor product, it is easy to verify that (23.28) holds, with both sides of the equation equal to

$$X(g_1 g_2)\nabla_X (s_0 \otimes s_0).$$

Uniqueness of ∇_X holds because both (23.29) and (23.30) are required by the definition of ∇_X. The action of ∇_X extends to sections of $\delta_P^{\mathbb{C}}$, by writing such sections as complex-valued functions times s_0. The analysis of the Lie derivative is similar and is omitted. ■

23.6.4 The Half-Form Hilbert Space

We continue to assume that the leaf space Ξ of P is an orientable manifold, and that we have chosen an orientation on Ξ. We assume that we have chosen a square root δ_P of $\mathcal{K}_P$, as in Sect. 23.6.3. If L is a prequantum line bundle over N, we now form the tensor product bundle $L \otimes \delta_P^{\mathbb{C}}$. Given two sections s_1 and s_2 of $L \otimes \delta_P^{\mathbb{C}}$, we decompose them locally as $s_j = \mu_j \otimes \nu_j$, where μ_j is a section of L and ν_j is a section of $\delta_P^{\mathbb{C}}$, and where, say, the

μ_j's are taken to be nonvanishing. Then we can combine these sections to form the quantity

$$(s_1, s_2) := (\mu_1, \mu_2)\overline{\nu_1} \otimes \nu_2, \tag{23.31}$$

where (μ_1, μ_2) is the pointwise inner product given by the Hermitian structure on L. Since (μ_1, μ_2) is a scalar-valued function and $\overline{\nu_1} \otimes \nu_2$ is a section of $\mathcal{K}_P^{\mathbb{C}}$, the quantity (s_1, s_2) is a section of $\mathcal{K}_P^{\mathbb{C}}$. Any other decomposition of s_j as the tensor product of a nonvanishing section of a L and a section of δ_P is of the form $(f\mu_j) \otimes (\nu_j/f)$ for some nonvanishing function f, and the value of (s_1, s_2) is the same as for the original decomposition. Since it is independent of the choice of local decomposition, (s_1, s_2) is actually defined globally.

Given the connection on L and the partial connection (23.41) on $\delta_P^{\mathbb{C}}$, we can form a partial connection on $L \otimes \delta_P^{\mathbb{C}}$ with the following property. For any vector field X lying in P, and any section s of $L \otimes \delta_P^{\mathbb{C}}$, if we decompose s locally as $s = \mu \otimes \nu$, where μ is a nonvanishing section of L and ν is a section of δ_P, then

$$\nabla_X(s) = (\nabla_X \mu) \otimes \nu + \mu \otimes (\nabla_X \nu). \tag{23.32}$$

The reader may verify that if $\mu \otimes \nu$ is replaced by $(f\mu) \otimes (\nu/f)$ for some nonvanishing function f, the value of $\nabla_X(s)$ is unchanged. Thus, as with the quantity (s_1, s_2) in (23.31), $\nabla_X(s)$ is defined globally. We then define a section s of $L \otimes \delta_P^{\mathbb{C}}$ to be *polarized* if $\nabla_X s = 0$ for each vector field X lying in P. If s_1 and s_2 are polarized sections of $L \otimes \delta_P^{\mathbb{C}}$, then the section (s_1, s_2) in (23.31) is easily seen to be a polarized section of $\mathcal{K}_P$.

As in the case without half-forms there is an obstruction to the existence of globally defined polarized sections of $L \otimes \delta_P^{\mathbb{C}}$. We say that a leaf R is *Bohr–Sommerfeld* (in the half-form sense, with respect to a particular choice of δ_P) if there exists a nonzero section s of $L \otimes \delta_P^{\mathbb{C}}$ defined over R such that $\nabla_X s = 0$ for each tangent vector to R. As in the case without half-forms, if the leaves are topologically nontrivial, the Bohr–Sommerfeld leaves will in general be a discrete set in the space of all leaves.

The Bohr–Sommerfeld leaves in the half-form sense need not be the same as the Bohr–Sommerfeld leaves in the sense of Definition 23.27. In the setting of Example 23.29, for instance, the canonical bundle $\mathcal{K}_P$ is trivial, but the square-root bundle δ_P may be chosen to be nontrivial, by putting in a twist by 180 degrees over each copy of S^1. (That is to say, we think of S^1 as the interval $[0, 2\pi]$ with the ends identified, and we attach a copy of $\mathbb{R}$ to each point. But when identifying the fiber at 2π with the fiber at 0, we use the negative of the identity map.) As Exercise 9 shows, in this example, the Bohr–Sommerfeld leaves are the sets of the form $\{x\} \times S^1$, where $x/\hbar = n + 1/2$ for some integer n.

Definition 23.42 *For any purely real polarization P and any square root δ_P of $\mathcal{K}_P$, the* ***half-form space*** *is the space of smooth, polarized sections* *of $L \otimes \delta_P^{\mathbb{C}}$. For a polarized section s of $L \otimes \delta_P^{\mathbb{C}}$, define the norm of s by*

$$\|s\|^2 = \int_\Xi \widetilde{(s,s)}, \tag{23.33}$$

where (s,s) is as in (23.31) and where $\widetilde{(s,s)}$ is the n-form on Ξ given by Proposition 23.37. If s_1 and s_2 are elements of the half-form space with $\|s_1\| < \infty$ and $\|s_2\| < \infty$, define the inner product of s_1 and s_2 by

$$\langle s_1, s_2 \rangle = \int_\Xi \widetilde{(s_1, s_2)}.$$

*The **half-form Hilbert space** is the completion with respect to the norm (23.33) of the space of polarized sections s for which $\|s\|^2 < \infty$.*

The integral of n-forms on Ξ is taken with respect to the chosen orientation on Ξ. We can always decompose s locally as $s = \mu \otimes \nu$ with ν being a section of δ_P (as opposed to $\delta_P^{\mathbb{C}}$) and μ being a section of L. Then

$$(s,s) = (\mu,\mu)\nu \otimes \nu,$$

from which we see that (s,s) is a non-negative section of $\mathcal{K}_P$ (Definition 23.40). (Recall that we have chosen the identification of $\delta_P \otimes \delta_P$ with $\mathcal{K}_P$ in a particular way, so that $\nu \otimes \nu$ is always the pullback by q of an oriented form on Ξ.) Thus, the integral on the right-hand side of (23.33) is non-negative, but possibly infinite.

Example 23.43 *Let $N = T^*\mathbb{R} \cong \mathbb{R}^2$ and let L be the trivial bundle on N, with connection $\nabla_X = X - (i/\hbar)\theta(X)$, where $\theta = p\ dx$. Let P be the vertical polarization on N and orient $\mathbb{R}$ so that oriented 1-forms are positive multiples of dx. Let δ_P to be the trivial bundle and with a trivializing section "$\sqrt{dx}$" of δ_P such that $\sqrt{dx} \otimes \sqrt{dx} = dx$. Then every polarized section s of $L \otimes \delta_P^{\mathbb{C}}$ has the form*

$$s = \psi(x) \otimes \sqrt{dx} \tag{23.34}$$

for some function ψ on $\mathbb{R}$. The norm of such a section is computed as

$$\|s\|^2 = \int_{\mathbb{R}} |\psi(x)|^2\ dx.$$

Proof. The sections of $\mathcal{K}_P$ are 1-forms that are zero on $\partial/\partial p$, that is, 1-forms of the form $\alpha = f(x,p)\ dx$. Such a 1-form satisfies $d\alpha = 0$ if and only if f is independent of p. Thus, dx is a globally defined polarized section of $\mathcal{K}_P$. If we choose δ_P to be trivial and let $\sqrt{dx}$ be such that $\sqrt{dx} \otimes \sqrt{dx} = dx$, then $\sqrt{dx}$ will be a polarized section of δ_P. Every section s of $L \otimes \delta_P^{\mathbb{C}}$ can be written uniquely as $s = \psi(x,p) \otimes \sqrt{dx}$ for some function ψ. Since $\sqrt{dx}$ is polarized and $\theta(\partial/\partial p) = 0$, we see that s is polarized if and only if ψ is independent of p. For a section of the form (23.34), we have $(s,s) = |\psi(x)|^2\ dx$, in which case, $\widetilde{(s,s)}$ is given by the same formula as (s,s), but now interpreted as a 1-form on $\Xi \cong \mathbb{R}$ rather than $\mathbb{R}^2$. ∎

23.6.5 Quantization of Observables

Suppose f is a function on N for which X_f preserves P in the sense of Definition 23.22. We will now associate with f a self-adjoint (or, at least, symmetric) operator $Q(f)$ on the half-form Hilbert space of P. Operators of this sort will satisfy exactly the desired commutation relations.

Definition 23.44 *For any function f on N for which X_f preserves P, let $Q(f)$ be the operator on the half-form space of P given by*

$$Q(f)s = (Q_{\text{pre}}(f)\mu) \otimes \nu + i\hbar\ \mu \otimes \mathcal{L}_{X_f}\nu,$$

where s is decomposed locally as $s = \mu \otimes \nu$, with μ being a section of L and ν a section of $\delta_P^{\mathbb{C}}$.

The operator $Q(f)$ is well defined (i.e., independent of the choice of local trivialization) as may easily be verified. This independence holds, however, only because the coefficient $i\hbar$ of ∇_{X_f} in the first term exactly matches the coefficient $i\hbar$ of $\mathcal{L}_{X_f}$ in the second term.

Before describing the general properties of the operators $Q(f)$, we consider a simple example that illustrates the essential role of the Lie derivative term in Definition 23.44.

Example 23.45 *Let the notation be as in Example 23.43, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be of the form*

$$f(x,p) = a(x) + b(x)p,$$

for some smooth functions a and b on $\mathbb{R}$. Then X_f preserves P and

$$Q(f)(\psi(x) \otimes \sqrt{dx}) = \tilde{\psi}(x) \otimes \sqrt{dx},$$

where

$$\tilde{\psi}(x) = -i\hbar\left(b(x)\psi'(x) + \frac{1}{2}b'(x)\psi(x)\right) + a(x)\psi(x).$$

In particular, if $f(x,p) = x$, then $\tilde{\psi}(x) = x\psi(x)$ and if $f(x,p) = p$, then $\tilde{\psi}(x) = -i\hbar\ \partial\psi/\partial x$. More generally, if a and b are polynomials, then the action of $Q(f)$ on ψ coincides with the Weyl quantization of f (Exercise 8 in Chap. 13).

The term involving $b'(x)$ comes from the presence of half-forms and is absent in the formula (22.15) for $Q_{\text{pre}}(f)$. The b' term, with the exact coefficient of 1/2, is necessary for $Q(f)$ to be self-adjoint (or, at least, symmetric); see Exercise 10. Example 23.45 is actually quite representative of the general case. [Compare (23.38) in the proof of Theorem 23.47 and Example 23.48.]

Proof. We have computed $Q_{\text{pre}}(f)$ in (22.15) in the proof of Proposition 22.12. We compute that X_f is equal to $-b(x)\ \partial/\partial x$ plus a term involving $\partial/\partial p$. Since the 1-form dx is closed, we obtain, by (21.7),

$$\mathcal{L}_{X_f}(dx) = d(X_f \lrcorner dx) = -db(x) = -b'(x)\ dx.$$

Using Proposition 23.41, we then obtain

$$\mathcal{L}_{X_f}\left(\sqrt{dx}\right)\otimes\sqrt{dx} = -\frac{1}{2}b'(x)\ dx = -\frac{1}{2}b'(x)\sqrt{dx}\otimes\sqrt{dx}, \tag{23.35}$$

which gives

$$\mathcal{L}_{X_f}\left(\sqrt{dx}\right) = -\frac{1}{2}b'(x)\sqrt{dx}.$$

Adding the $\mathcal{L}_{X_f}$ term to the previously computed expression for $Q_{\text{pre}}(f)$ gives the desired result. ∎

Returning now to the setting of general real polarizations, we establish two key results for the quantized observables $Q(f)$, that they satisfy the desired commutation relations and that they are self-adjoint (or, at least, symmetric) whenever f is real valued. It can also be shown that when f is a polarized function (i.e., constant along each leaf of P), then $Q(f)$ acts on the quantum Hilbert space simply as multiplication by f. See Exercise 11.

Theorem 23.46 *Suppose f and g are functions on N for which X_f and X_g preserve P. Then the operators $Q(f)$ and $Q(g)$ satisfy*

$$\frac{1}{i\hbar}\left[Q(f),Q(g)\right] = Q(\{f,g\})$$

on the space of smooth, polarized sections of $L\otimes\delta_P^{\mathbb{C}}$.

Proof. Since $Q(h)$ is a local operator for any function h, it suffices to prove the result locally. Let us choose, then, a local nonvanishing section ν_0 of $\delta_P^{\mathbb{C}}$, so that, locally, each section s of $L\otimes\delta_P^{\mathbb{C}}$ can be decomposed uniquely as $s=\mu\otimes\nu_0$. For any vector field preserving P, we let $\gamma(X)$ be the function such that

$$\mathcal{L}_X(\nu_0) = \gamma(X)\nu_0.$$

We then have $Q(f)(\mu\otimes\nu_0) = \tilde{\mu}\otimes\nu_0$, where

$$\tilde{\mu} = [Q_{\text{pre}}(f) + i\hbar\gamma(X_f)]\mu.$$

We now compute that

$$\begin{aligned}&[Q_{\text{pre}}(f) + i\hbar\gamma(X_f), Q_{\text{pre}}(g) + i\hbar\gamma(X_g)]\\&= [Q_{\text{pre}}(f), Q_{\text{pre}}(g)] + i\hbar[Q_{\text{pre}}(f),\gamma(X_g)] + i\hbar[\gamma(X_g), Q_{\text{pre}}(f)]\\&= i\hbar Q_{\text{pre}}(\{f,g\}) + (i\hbar)^2\left(X_f(\gamma(X_g)) - X_g(\gamma(X_f))\right).\end{aligned}$$

The desired result will follow if we can verify that

$$X_f(\gamma(X_g)) - X_g(\gamma(X_f)) = \gamma(X_{\{f,g\}}). \tag{23.36}$$

To verify (23.36), we use a standard identity for the Lie derivative on forms: $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X,\mathcal{L}_Y]$. Using Proposition 23.41, we can easily show that this identity holds also on sections of $\delta_P^{\mathbb{C}}$, for vector fields that preserve P. It is then a simple calculation (Exercise 12) to verify (23.36). ∎

Theorem 23.47 *If $f \in C^\infty(N)$ is real valued and X_f preserves P, then the operator $Q(f)$ is symmetric on the space of smooth sections s in the half-form space for which $\widetilde{(s,s)}$ has compact support on Ξ.*

Proof. Suppose $\alpha = q^*(\beta)$ is polarized section of $\mathcal{K}_P$, so that there is, at least locally, a corresponding polarized section $\sqrt{q^*(\beta)}$ of δ_P. If X_f preserves P, then by Proposition 23.39, there is a unique vector field Y_f on Ξ such that $q_{*,z}(X_f) = Y_f$ for all $z \in N$. Using (23.25) and Proposition 23.41, we get

$$\mathcal{L}_{X_f}\left(\sqrt{q^*(\beta)}\right) = \frac{1}{2}((\operatorname{div}_\beta Y_f)\circ q)\sqrt{q^*(\beta)}.$$

Meanwhile, it is not hard to show (Exercise 13) that it is possible to choose a local symplectic potential θ that is zero in the directions of P. Thus, we can trivialize L locally in such a way that sections that are covariantly constant along P are simply functions that are constant along P in the ordinary sense. Thus, elements s of the half-form space have, locally, the form

$$s = (\psi \circ q) \otimes \sqrt{q^*(\beta)} \tag{23.37}$$

for some function ψ and n-form β on Ξ. Thus, if X_f preserves P, and a section s is decomposed locally as in (23.37), we have

$$Q(f)(s) = (\tilde{\psi}\circ q) \otimes \sqrt{q^*(\beta)},$$

where

$$\tilde{\psi} = i\hbar\left(Y_f\psi + \frac{1}{2}(\operatorname{div}_\beta Y_f)\psi\right) + (-\theta(X_f) - f)\psi. \tag{23.38}$$

It can be verified (Exercise 14) that the function $-\theta(X_f) - f$ is constant along P and thus may be thought of as a function on Ξ.

By multiplying elements of the half-form space by functions of the form $\chi \circ q$, with χ having compact support in Ξ, we can "localize" the calculations on Ξ. Suppose s_1 and s_2 are two elements of the half-form space decomposed as in (23.37) near a point $z \in N$, with the same β and two different functions ψ_1 and ψ_2 on Ξ. Then $\widetilde{(s_1, s_2)}$ has the form $\overline{\psi_1}\psi_2\beta$ in a neighborhood U of $q(z)$. By localization, we may assume that $\widetilde{(s_1, s_2)}$ has compact support in U, and we then have

$$\langle s_1, Q(f)s_2\rangle = -i\hbar\int_\Xi \overline{\psi_1}\tilde{\psi}_2\ \beta,$$

where $\tilde{\psi}_2$ is as in (23.38). "Integration by parts" (Exercise 15) with respect to β then shows that this quantity coincides with $\langle Q(f)s_1, s_2\rangle$. ∎

Example 23.48 (Cotangent Bundles) *Let $N = T^*M$ for an oriented manifold M, let θ be the canonical 1-form on N, and let L be the trivial* *line bundle on N, with connection $\nabla_X = X - (i/\hbar)\theta(X)$. Let P be the vertical polarization on N, so that $\mathcal{K}_P$ is trivial, and let δ_P be chosen to be trivial. Let β be an arbitrary nowhere-vanishing, oriented n-form on M, so that $\alpha := \pi^*(\beta)$ is a nowhere-vanishing section of $\mathcal{K}_P$, and choose a trivializing section $\sqrt{\alpha}$ of δ_P with $\sqrt{\alpha}\otimes\sqrt{\alpha} = \alpha$. In that case, elements s of the half-form Hilbert space have the form $s = (\psi\circ\pi)\otimes\sqrt{\alpha}$, where ψ is a function on M, and*

$$\|s\|^2 = \int_M |\psi|^2\,\beta.$$

The half-form Hilbert space may, thus, be identified with $L^2(M,\beta)$.

Suppose now that f is a function on T^*M of the form $f = f_1 + f_2$, where f_1 is constant on each fiber of T^*M and f_2 is linear on each fiber. Then f_2 may be thought of as a section of $T^{**}M \cong TM$, that is, as a vector field Y_f on M. In that case, X_f preserves P and $Q(f)$ acts on elements of the half-forms space as

$$Q(f)\left((\psi\circ\pi)\otimes\sqrt{\alpha}\right) = (\tilde{\psi}\circ\pi)\otimes\sqrt{\alpha},$$

where

$$\tilde{\psi} = i\hbar\left(Y_f\psi + \frac{1}{2}(\mathrm{div}_\beta\, Y_f)\psi\right) + f_1\psi.$$

Here $\mathrm{div}_\beta\, Y_f$ is the unique function such that $\mathcal{L}_{Y_f}\beta = (\mathrm{div}_\beta\, Y_f)\beta$.

A simple calculation in coordinates shows that the vector field Y_f in the example satisfies $X_f(\psi\circ\pi) = (Y_f\psi)\circ\pi$, so that our notation is consistent with that in Proposition 23.39 [see (23.23)].

Proof. The calculation is precisely the same as in the proof of Theorem 23.47, except that the decomposition in (23.37) is now global. The claimed form of $Q(f)$ is nothing but the expression (23.38), where the reader may easily compute, using local coordinates, that $-\theta(X_f) - f = f_1$. ■

It is an unfortunate feature of geometric quantization that in the case of the vertical polarization on cotangent bundles, it only permits us to quantize functions that are at most linear in the momentum variables. In a typical physical system having T^*M as its phase space, there will be a "kinetic energy" term in the classical Hamiltonian that is quadratic in p. To quantize such a system, one has to find a way to quantize the kinetic energy term, "by hook or by crook."

One approach to this problem is to allow the exponentiated quantized Hamiltonian to change the polarization, and then to use pairing maps (Sect. 23.8) to "project" back to the Hilbert space for the original polarization. As explained in Sect. 9.7 of [45], this approach succeeds in the case that the kinetic energy term is $g(p,p)/(2m)$, where g is the Riemannian structure on T^*M induced by a Riemannian structure on TM. The quantized kinetic energy operator turns out to be given by the map

$$\psi \mapsto -\frac{\hbar^2}{2m}\left((\Delta\psi)(x) - \frac{1}{6}R(x)\psi(x)\right), \qquad (23.39)$$

where Δ is the Laplacian for M (taken to be a negative operator) and where $R(x)$ is the scalar curvature of the Riemannian structure on TM. The calculation in [45] glosses over one technical issue, which is that the time-evolved polarizations may not be everywhere transverse to the original polarization. Nevertheless, the calculation provides a reasonable geometric motivation for the formula (23.39).

It should be emphasized that, because of the projections involved in the computation of the quantized kinetic energy operator, it does *not* satisfy the desired commutation relations with the quantizations of functions whose flow preserves the vertical polarization. Nevertheless, this approach to quantizing the kinetic energy may simply be the best one can do.

23.7 Quantization with Half-Forms: The Complex Case

In the case of a purely complex polarization, half-forms are not "necessary," in that we typically have a nonzero Hilbert space even without them. Nevertheless, their inclusion gives advantages. In the first place, using half-forms makes the complex case more parallel to the real case. In the second place, complex quantization with half-forms simply gives better results than without half-forms. In the case of the harmonic oscillator, for example, the inclusion of half-forms allows (Example 23.53) geometric quantization to reproduce precisely the spectrum $(n+1/2)\hbar\omega$, $n = 0, 1, 2, \ldots$, that we found in the traditional treatment. This result should be compared to Proposition 22.14 without half-forms, where the spectrum is found to be $n\hbar\omega$.

Throughout this section, we assume that (N, ω) is a $2n$-dimensional quantizable symplectic manifold, that (L, ∇) is prequantum line bundle over N, and that P is a Kähler polarization on N (Definition 23.19). Since the definitions in the complex case are very similar to those in the real case (with a few important differences), we will run through them quickly. Since $\bar{P}$ is no longer equal to P, we need to replace P by $\bar{P}$ in may of the formulas from Sect. 23.6.

The *canonical bundle* $\mathcal{K}_P$ of P is the complex line bundle for which the sections are n-forms α satisfying

$$X \lrcorner \alpha$$

for each vector field X lying in $\bar{P}$. Sections of $\mathcal{K}_P$ are precisely the $(n, 0)$-forms on N. A section of $\mathcal{K}_P$ is said to be *polarized* if

$$X \lrcorner (d\alpha) = 0 \tag{23.40}$$

for every vector field lying in $\bar{P}$, or, equivalently, if $d\alpha = 0$. Polarized sections of $\mathcal{K}_P$ are precisely the holomorphic $(n, 0)$-forms on N. By a *square* *root* of $\mathcal{K}_P$ we will mean a complex line bundle δ_P over N such that $\delta_P \otimes \delta_P$ is isomorphic with $\mathcal{K}_P$, together with a particular isomorphism of $\delta_P \otimes \delta_P$ with $\mathcal{K}_P$. Thus, if s_1 and s_2 are sections of δ_P, we think of $s_1 \otimes s_2$ as being a section of $\mathcal{K}_P$. We assume that such a square root exists and we fix for the remainder of this section one particular square root δ_P.

If X is a vector field that preserves $\bar{P}$, in the sense of Definition 23.22, then $\mathcal{L}_X$ preserves the space of sections of $\mathcal{K}_P$ and also the space of polarized sections of $\mathcal{K}_P$. The condition (23.40) defining polarized sections of $\mathcal{K}_P$ can be understood as the vanishing of a *partial connection* $\nabla.$, defined for vector fields lying in $\bar{P}$, and given by $\nabla_X \alpha = X \lrcorner (d\alpha)$. Both the partial connection (for vector fields lying in $\bar{P}$) and the Lie derivative (for vector fields preserving $\bar{P}$) descend from $\mathcal{K}_P$ to δ_P, as in Proposition 23.41 in the real case. The connection on L and the partial connection on δ_P combine to give a partial connection on $L \otimes \delta_P$. A section s of $L \otimes \delta_P$ is said to be *polarized* if $\nabla_X s = 0$ for all vector fields X lying in $\bar{P}$.

Notation 23.49 *If β is any $2n$-form on N, let the expression*

$$\frac{\beta}{\lambda}$$

denote the unique function on N such that $\beta = (\beta/\lambda)\lambda$, where λ is the Liouville form in Definition 21.16.

Unlike the canonical bundle in the real case, the canonical bundle in the purely complex case carries a natural Hermitian structure.

Proposition 23.50 *If α is an $(n,0)$-form on N, then at each point the $2n$-form*

$$(-1)^{n(n-1)/2}(-i)^n \; \bar{\alpha} \wedge \alpha$$

is a non-negative multiple of the Liouville form λ. There is then a unique Hermitian structure on δ_P with the property that for each section s of δ_P we have

$$|s|^2 = \left(\frac{(-1)^{n(n-1)/2}(-i)^n}{2^n} \frac{\overline{(s \otimes s)} \wedge (s \otimes s)}{\lambda} \right)^{1/2}. \tag{23.41}$$

The factor of 2^n in the denominator in (23.41) is inserted for convenience, to make certain formulas come out more nicely.

Proof. See Exercise 17. ∎

Since, by assumption, there is Hermitian structure on L, the above Hermitian structure on δ_P gives rise in a natural way to a Hermitian structure on $L \otimes \delta_P$.

Definition 23.51 *The **half-form Hilbert space** for a Kähler polarization P on N is the space of square-integrable polarized sections of $L \otimes \delta_P$.*

In the $\mathbb{C}^n$ case, using the canonical 1-form as our symplectic potential, elements of the half-form Hilbert space take the form

$$e^{-|\mathrm{Im}\,\mathbf{z}|^2/(2\alpha\hbar)}F(\mathbf{z})\otimes\sqrt{dz_1\wedge\cdots\wedge dz_n}.$$

In this special case, the norm of the half-form factor $\sqrt{dz_1\wedge\cdots\wedge dz_n}$ is constant and the half-form Hilbert space is still identifiable with the space in Conclusion 22.10. In the case of the unit disk, on the other hand, the presence of half-forms alters the inner product; see Exercise 16.

We now define quantized observables on the half-form Hilbert space, using the same formula as in the real case.

Definition 23.52 *If f is a function on N for which X_f preserves $\bar{P}$, let $Q(f)$ be the operator on the half-form Hilbert space of P given by*

$$Q(f)s=(Q_{\mathrm{pre}}(f)\mu)\otimes\nu-i\hbar\;\mu\otimes\mathcal{L}_{X_f}\nu,$$

where s is decomposed locally as $s=\mu\otimes\nu$, with μ being a section of L and ν a section of δ_P.

These operators satisfy $[Q(f),Q(g)]/(i\hbar)=Q(\{f,g\})$ on the space of smooth polarized sections of $L\otimes\delta_P$, with the proof of this result being identical to the proof of Theorem 23.46 in the real case. If f is real-valued and X_f preserves $\bar{P}$, then $Q(f)$ will be at least symmetric, assuming we can find a dense subspace of the half-form Hilbert space consisting of "nice" functions. (Finding dense subspaces is more difficult in the holomorphic case than in the real case.) A proof of this claim is sketched in Exercise 18.

Example 23.53 *Consider $\mathbb{R}^2\cong T^*\mathbb{R}$ with the Kähler polarization P given by the global complex coordinate $z=(x-ip/(m\omega))$, for some positive number ω. Take δ_P to be trivial with trivializing section $\sqrt{dz}$. Consider also the harmonic oscillator Hamiltonian $H:=(p^2+(m\omega x)^2)/(2m)$. Then X_H preserves the P and the operator $Q(H)$ on the half-form Hilbert space has spectrum consisting of numbers of the form $(n+1/2)\hbar\omega$, where $n=0,1,2,\ldots$.*

In this example, ω is the frequency of the oscillator and not the canonical 2-form.

Proof. The calculation is the same as in the proof of Proposition 22.14, except for the addition of the Lie derivative term. A simple calculation shows that $\mathcal{L}_{X_H}(dz)=i\omega\;dz$, from which it follows that $\mathcal{L}_{X_H}\sqrt{dz}=(i\omega/2)\sqrt{dz}$. It is then easy to see that the set of elements of the form $e^{-m\omega|\mathrm{Im}\,z|^2/(2\hbar)}z^n\otimes\sqrt{dz}$ form an orthonormal basis of eigenvectors for $Q(H)$, with eigenvalues $(n+1/2)\hbar\omega$. ∎

23.8 Pairing Maps

Pairing maps are designed to allow us to compare the results of quantizing with respect to two different polarizations. We consider mainly the case of two "transverse" real polarizations; the case of two complex polarizations or one real and one complex polarization can be treated with minor modifications.

Suppose that P and P' are two purely real polarizations and that the associated leaf spaces Ξ_1 and Ξ_2 are oriented manifolds. Suppose also that P and P' are *transverse* at each point $z \in N$, meaning that $P_z \cap P'_z = \{0\}$. If α and β are polarized sections of $\mathcal{K}_P$ and $\mathcal{K}_{P'}$, respectively, the transversality assumption is easily shown to imply that $\alpha \wedge \beta$ is a nowhere-vanishing $2n$-form on N. Thus, for any point $z \in N$, we can define a bilinear "pairing" from $\delta_{P,z} \times \delta_{P',z} \to \mathbb{R}$ by

$$(\nu_1, \nu_2) = \left(\frac{(\nu_1 \otimes \nu_1) \wedge (\nu_2 \otimes \nu_2)}{\lambda} \right)^{1/2}. \tag{23.42}$$

(Recall Notation 23.49.) We can extend this pairing to a pairing $\delta^{\mathbb{C}}_{P,z} \times \delta^{\mathbb{C}}_{P',z} \to \mathbb{C}$ that is conjugate linear in the first factor and linear in the second factor. Finally, we extend to a pairing of $(L_z \otimes \delta^{\mathbb{C}}_{P,z}) \times (L_z \otimes \delta^{\mathbb{C}}_{P',z}) \to \mathbb{C}$ by setting $(\mu_1 \otimes \nu_1, \mu_2 \otimes \nu_2)$ equal to $(\mu_1, \mu_2)(\nu_1, \nu_2)$, where (μ_1, μ_2) is computed with respect to the Hermitian structure on L.

Let $\mathbf{H}_1$ and $\mathbf{H}_2$ denote the half-form Hilbert spaces for P and P', respectively. Given $s_1 \in \mathbf{H}_1$ and $s_2 \in \mathbf{H}_2$, we define the *pairing* of s_1 and s_2 by

$$\langle s_1, s_2 \rangle_{P,P'} := c \int_N (s_1, s_2)\ \lambda,$$

provided that the integral is absolutely convergent. Here (s_1, s_2) is the *pointwise* pairing of s_1 and s_2 defined in the previous paragraph and c is a certain "universal" constant, depending only on $\hbar$ and the dimension of n, that can be chosen to make certain examples work out nicely. We now look for a *pairing map* $\Lambda_{P,P'} : \mathbf{H}_1 \to \mathbf{H}_2$ with the property that

$$\langle s_1, s_2 \rangle_{P,P'} = \langle \Lambda_{P,P'} s_1, s_2 \rangle_{\mathbf{H}_2}. \tag{23.43}$$

If the pairing is bounded (i.e., it satisfies $|\langle s_1, s_2 \rangle_{P,P'}| \leq C \|s_1\| \|s_2\|$ for some constant C), there is a unique bounded operator $\Lambda_{P,P'}$ satisfying (23.43). Even if the pairing is unbounded, we may be able to define $\Lambda_{P,P'}$ as an unbounded operator.

If we were optimistic, we might hope that the pairing map for any two transverse polarizations would be unitary, or at least a constant multiple of a unitary map. *If* this were the case, it would suggest that quantization is independent of the choice of polarization, in the sense that there would be a natural unitary map between the Hilbert spaces for two different polarizations. As it turns out, however, the typical pairing map is *not* a constant multiple of a unitary map. Nevertheless, there are certain special cases where the pairing map is unitary (up to a constant), including the case of translation-invariant polarizations on $\mathbb{R}^{2n}$. See also [20] for an example of a pairing map between a real and a complex polarization that is a constant multiple of a unitary map.

We compute just one very special case of the pairing map between two real polarizations.

Example 23.54 *Consider $N = \mathbb{R}^2 \cong T^*\mathbb{R}$ and take L to be trivial with connection 1-form $\theta = p\ dx$. Let P be the vertical polarization, spanned at each point by $\partial/\partial p$, and let P' be the horizontal polarization, spanned at each point by $\partial/\partial x$. Then elements s_1 of the half-form space for P have the form*

$$s_1(x,p) = \phi(x) \otimes \sqrt{dx} \tag{23.44}$$

and elements s_2 of the half-form space for P' have the form

$$s_2(x,p) = \psi(p)e^{ixp/\hbar} \otimes \sqrt{dp}, \tag{23.45}$$

where ϕ and ψ are functions on $\mathbb{R}$. If $c = 1$, the pairing is computed as

$$\langle s_1, s_2 \rangle_{P,P'} = -\int_{\mathbb{R}^2} \overline{\phi(x)}\psi(p)e^{ixp/\hbar}\ dx\ dp. \tag{23.46}$$

If s_1 has the form (23.44), then $\Lambda_{P,P'}(s_1)$ has the form (23.45), where

$$\psi(p) = -\int_{\mathbb{R}} \phi(x)e^{-ixp/\hbar}\ dx.$$

Thus, $\Lambda_{P,P'}$ is a scaled version of the Fourier transform and is, in particular, a constant multiple of a unitary map.

The pairing should be defined initially on some dense subspace of the Hilbert spaces, such as the subspaces where ϕ and ψ are Schwartz functions. The pairing map can also be defined initially on the Schwartz space, recognized as being unitary (up to a constant), and then extended by continuity to all of $\mathbf{H}_1$. Once the pairing map is extended to $\mathbf{H}_1$, the pairing itself can be defined for all $s_1 \in \mathbf{H}_1$ and $s_2 \in \mathbf{H}_2$ by taking (23.43) as the definition of $\langle s_1, s_2 \rangle_{P,P'}$. Even though it is possible, as just described, to extend the pairing to all of $\mathbf{H}_1 \times \mathbf{H}_2$, the integral in (23.46) is not always absolutely convergent.

Proof. The forms (23.44) and (23.45) are obtained by a simple modification of the argument in the proof of Proposition 22.8. We can compute that the pointwise pairing of $\sqrt{dx}$ and $\sqrt{dp}$ is -1, which gives the indicated form of the pairing in (23.46). The pairing may be rewritten as

$$\int_{\mathbb{R}} \overline{\int_{\mathbb{R}} \phi(x)e^{-ixp/\hbar}\ dx}\ \psi(p)\ dp,$$

which gives the indicated form of the pairing map. ∎

23.9 Exercises

1. Let L be a line bundle with connection ∇ over N. Let s be a section of L and let X_1 and X_2 be two vector fields on N such that $X_1(z) = X_2(z)$ for some fixed point $z \in N$. Show that
$$\nabla_{X_1}(s)(z) = \nabla_{X_2}(s)(z).$$
Hint: Use the assumption that $\nabla_{fX} = f\nabla_X$.

2. Let L be a Hermitian line bundle with Hermitian connection ∇ and let s_0 be a locally defined section of L such that $(s_0, s_0) \equiv 1$. Given a vector field X, let $\theta(X)$ be the unique function such that
$$\nabla_X s_0 = -i\theta(X)s_0.$$
Show that $\theta(X)$ is real valued.

   Hint: Use the Hermitian property of the connection.

3. Consider the definition of the curvature 2-form $\omega(X, Y)$ in Definition 23.4.

   (a) Show that the expression for ω is C^∞-linear in each of the variables X, Y, and s. That is to say, show that for all smooth functions f, we have $\omega(fX, Y)s = f\omega(X, Y)s$, and similarly for the variables Y and s.

   (b) Show that the value of $\omega(X, Y)s$ at a point z depends only on the values of X, Y, and s at the point z.

   (c) Show that the value of $\omega(X, Y)$ at a point z does not depend on the value of s at z, provided that $s(z) \neq 0$.

4. Consider the symplectic form $\omega = dp \wedge dx$ on $\mathbb{R}^2$. Define a purely complex polarization on $\mathbb{R}^2$ by taking P_z to be the span of the vector $\partial/\partial z$ in (22.9), for some fixed $\alpha > 0$. Show that P is a Kähler polarization.

5. Let P be the polarization on $\mathbb{R}^2$ in Exercise 4. Show that the function $\kappa(x, p) := \alpha p^2$ is a Kähler potential for P.

6. Suppose that ω is a J-invariant 2-form on a complex manifold N. Show that ω is a $(1, 1)$-form. (Recall the definitions preceding Lemma 23.34.)

   Hint: Write $\omega = \omega^1 + \omega^2$, where ω^1 is a $(1, 1)$-form and ω^2 is a sum of a $(2, 0)$-form and a $(0, 2)$-form. Show that
$$\omega^2(JX, JY) = -\omega^2(X, Y)$$
for all tangent vectors X and Y.

7. Suppose that κ is a smooth, real-valued function on a complex manifold N. Show that the 2-form $i\partial\bar{\partial}\kappa$ is a real-valued 2-form.

8. In Example 23.30, verify that θ is a symplectic potential for ω, and compute $\theta(\partial/\partial\bar{z})$, where, with $z = x - iy$, we have $\partial/\partial\bar{z} = (\partial/\partial x - i\partial/\partial y)/2$. Then verify that $s_0(z) := (1 - |z|^2)^{1/\hbar}$ satisfies $\nabla_{\partial/\partial\bar{z}} s_0 = 0$ and thus constitutes a global trivializing holomorphic section.

9. Consider the situation in Example 23.29. Show that the canonical bundle for P is trivial, with trivializing section dx. Let δ_P be the (nontrivial) bundle described in the paragraph preceding Definition 23.42. Since the tensor product of any real line bundle with itself is trivial, $\delta_P \otimes \delta_P$ is isomorphic to $\mathcal{K}_P$. Let $\sqrt{dx}$ denote a *discontinuous* section defined over the set $0 < \phi < 2\pi$ such that $\sqrt{dx} \otimes \sqrt{dx} = dx$. Show that $\nabla_X(dx) = 0$ and $\nabla_X \sqrt{dx} = 0$ for every vector field lying in P. Now show that the Bohr–Sommerfeld leaves (in the half-form sense, for this choice of δ_P) are the sets of the form $\{x\} \times S^1$, where $x/\hbar = n + 1/2$ for some integer n.

10. Let b be a smooth, real-valued function on $\mathbb{R}$ and let c be a real constant. Show that an operator of the form

$$\psi \mapsto -i\hbar \left(b(x)\psi'(x) + cb'(x)\psi(x) \right)$$

is symmetric on $C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$ if and only if $c = 1/2$.

11. Let P be a real polarization and let f be a smooth polarized function on N, that is, one for which derivatives in the direction of P are zero. Show that $Q(f)$ acts on the half-form Hilbert space simply as multiplication by f. (Compare Proposition 23.25 in the case without half-forms.)

    Hint: Show that $\mathcal{L}_{X_f}\alpha = 0$ whenever α is a polarized section of $\mathcal{K}_P$.

12. Using the identities $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$ and $X_{\{f,g\}} = [X_f, X_g]$, verify the identity (23.36).

13. Prove that if P is a real polarization on N, it is possible to choose a symplectic potential θ locally in such a way that θ is zero on P.

    Hint: Use functions $f_{\mathbf{x}}$ as in the proof of Proposition 23.26.

14. Suppose that P is a purely real polarization on N and θ is a local symplectic potential that vanishes on P. Suppose also that f is a real-valued function for which X_f preserves P. Show that the function $-\theta(X_f) - f$ is constant along the leaves of P.

    Hint: If X is a vector field lying in P, use (21.6) to show that $X(\theta(X_f)) = d\theta(X, X_f)$.

15. Suppose that β is a nowhere vanishing n-form on an oriented manifold Ξ, that X is a real vector field on Ξ, and that ϕ and ψ are smooth, compactly supported functions on Ξ. Verify the following formula for "integration by parts":
$$\int_\Xi (X\phi)\psi\ \beta = -\int_\Xi \phi(X\psi)\ \beta - \int_\Xi \phi\psi(\operatorname{div}_\beta X)\ \beta,$$
where $\operatorname{div}_\beta X$ is the function such that $\mathcal{L}_X\beta = (\operatorname{div}_\beta X)\beta$.

Hint: If Φ_t is the flow generated by X, then for all sufficiently small t, $\Phi_t(x)$ is defined for all x in the support of $\phi\psi$ and the integral of $(\Phi_t)^*(\phi\psi\beta)$ over Ξ is independent of t.

16. Let the notation be as in Exercise 8. Then the canonical bundle for P is trivial, with trivializing section dz. Take δ_P to be trivial, with trivializing section $\sqrt{dz}$. Show that every polarized section s of $L\otimes\delta_P$ is of the form
$$s = F(z)s_0(z)\otimes\sqrt{dz},$$
where F is holomorphic. Show that the norm of such a section is, up to a constant, the L^2 norm of F with respect to a measure of the form $(1-|z|^2)^\nu$, but that the value of ν is not the same as when half-forms are not included.

17. Let P be a Kähler polarization on N, let $z_1,\ldots,z_n$ be holomorphic local coordinates on N, and let A be the matrix given by
$$A_{jk} = \omega\left(\frac{\partial}{\partial\bar{z}_j},\frac{\partial}{\partial z_k}\right).$$
   (a) Show that the matrix iA is positive definite.
   (b) Show that $\omega = A_{jk}\ d\bar{z}_j\wedge dz_k$.
   (c) Show that the quantity $\omega^{\otimes n}/n!$ may be computed as
$$\det(iA)(-1)^{n(n-1)/2}(-i)^n d\bar{z}_1\wedge\cdots\wedge d\bar{z}_n\wedge dz_1\wedge\cdots\wedge dz_n.$$
   (d) Verify Proposition 23.50.

18. Let P be a Kähler polarization on N, let δ_P be a fixed square root of $\mathcal{K}_P$, and let f be a smooth, real-valued function such that X_f preserves $\bar{P}$. Throughout this problem, if s_1 and s_2 are local sections of a line bundle, with s_2 nonvanishing, s_1/s_2 will denote the unique function such that $s_1 = (s_1/s_2)s_2$.
   (a) Show that for any continuous compactly supported function ψ on N, we have
$$\int_N X_f(\psi)\ \lambda = 0.$$

Hint: Use Liouville's theorem.

Note: The same result holds if ψ is not compactly supported but is "sufficiently nice."

(b) If ν is a local nonvanishing section of δ_P, show that

$$\frac{\mathcal{L}_{X_f}\nu}{\nu} = \frac{1}{2}\frac{\mathcal{L}_{X_f}(\nu \otimes \nu)}{\nu \otimes \nu}.$$

(c) If α is any $2n$-form on N, show that

$$\frac{\mathcal{L}_{X_f}\alpha}{\lambda} = X_f\left(\frac{\alpha}{\lambda}\right).$$

(d) Suppose s_1 and s_2 are polarized sections of $L \otimes \delta_P$, decomposed locally as $s_j = \mu_j \otimes \nu_j$, $j = 1, 2$. Show that

$$\begin{aligned} iX_f(s_1, s_2) &= (i(\nabla_{X_f}\mu_1) \otimes \nu_1, s_2) + (i\mu_1 \otimes (\mathcal{L}_{X_f}\nu_1) \otimes s_2) \\ &\quad + (s_1, i(\nabla_{X_f}\mu_2) \otimes \nu_2) + (s_1, i\mu_2 \otimes (\mathcal{L}_{X_f}\nu_2)), \end{aligned}$$

where $(\cdot, \cdot)$ is computed with respect to the Hermitian structure on $L \otimes \delta_P$ described in Sect. 23.7.

Hint: Use the identity $\mathcal{L}_{X_f}(\alpha \wedge \beta) = (\mathcal{L}_{X_f}\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_{X_f}\beta)$.

(e) Suppose s_1 and s_2 are polarized sections of $L \otimes \delta_P$ belonging to the domain of $Q(f)$ and such that (s_1, s_2) is "sufficiently nice." Show that

$$\langle s_1, Q(f)s_2 \rangle = \langle Q(f)s_1, s_2 \rangle.$$