

# 21

## Hamiltonian Mechanics on Manifolds

In this chapter, we generalize the Hamiltonian approach to mechanics (introduced already in the Euclidean case in Sect. 2.5) to general manifolds. The chapter assumes familiarity with the basic notions of smooth manifolds, including tangent and cotangent spaces, vector fields, and differential forms. These notions are reviewed very briefly in Sect. 21.1, mainly in the interest of fixing the notation. See, for example, Chap. 2 of [40] for a concise treatment of manifolds and [29] for a detailed account. Throughout the chapter, we will use the *summation convention*, that repeated indices are always summed on.

### 21.1 Calculus on Manifolds

Throughout this section,  $M$  will denote a smooth,  $n$ -dimensional manifold.

#### 21.1.1 Tangent Spaces, Vector Fields, and Flows

For each  $x \in M$ , we have the *tangent space* to  $M$  at  $x$ , denoted  $T_x M$ . Given a smooth coordinate system  $x_1, \dots, x_n$  on  $M$ , the vectors

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \tag{21.1}$$

form a basis for the tangent space at each point. A *vector field*  $X$  on  $M$  is map assigning to each point  $x \in M$  an element  $X_x$  of  $T_x M$ . A vector

field  $X$  is *smooth* if the coefficients of  $X$  in a basis of the form (21.1) are smooth functions, for every smooth coordinate system. As in Exercise 14 in Chap. 2, we think of a vector field as a first-order differential operator satisfying the Leibniz rule:

$$X(fg) = X(f)g + fX(g).$$

Given a smooth vector field  $X$  on  $M$  and a point  $x \in M$ , there exists a curve  $\gamma_x : (a, b) \rightarrow M$  such that  $\gamma_x(0) = x$  and

$$\frac{d\gamma_x}{dt} = X_{\gamma_x(t)}.$$

Any two such curves agree on the intersection of their intervals of definition. There is a largest interval  $(a_x^{\max}, b_x^{\max})$  on which such a curve can be defined. If, for each  $x \in M$ , we have  $a_x^{\max} = -\infty$  and  $b_x^{\max} = +\infty$ , we say that the vector field  $X$  is *complete*. If  $M$  is compact, then each smooth vector field on  $M$  is complete. We may assemble the curves  $\gamma_x$  into the *flow*  $\Phi$  generated by  $X$ , defined as

$$\Phi_t(x) = \gamma_x(t),$$

whenever  $a_x^{\max} < t < b_x^{\max}$ . If  $t$  does not belong to  $(a_x^{\max}, b_x^{\max})$ , then  $\Phi_t(x)$  is not defined. The flow  $\Phi$  satisfies

$$\Phi_0(x) = x. \tag{21.2}$$

Furthermore, if  $x$  is in the domain of  $\Phi_t$  and  $\Phi_t(x)$  is in the domain of  $\Phi_s$ , then  $x$  is in the domain of  $\Phi_{s+t}$  and

$$\Phi_s(\Phi_t(x)) = \Phi_{s+t}(x). \tag{21.3}$$

In the other direction, given a family of maps  $\Phi$  satisfying (21.2) and (21.3) and appropriate domain properties, there is a unique vector field  $X$  such that  $\Phi$  is the flow generated by  $X$ . In particular, if  $\Phi_t(x)$  is defined for all  $x$  and  $t$ , is smooth as a map of  $M \times \mathbb{R}$  into  $M$ , and satisfies (21.2) and (21.3), there is a unique complete vector field  $X$  such that  $\Phi$  is the flow generated by  $X$ .

### 21.1.2 Differential Forms

For each  $x$ , the tangent space  $T_x M$  is an  $n$ -dimensional real vector space. The dual vector space to  $T_x M$  is the *cotangent space* to  $M$  at  $x$ , denoted  $T_x^* M$ . Given a smooth function  $f$  on  $M$  and a point  $x \in M$ , the *differential* of  $f$  at  $x$  is the element of  $T_x^* M$  given by

$$df(X) = X(f)$$

for each  $X \in T_x f$ . In particular, in any local coordinate system  $x_1, \dots, x_n$ , the elements  $dx_1, \dots, dx_n$  satisfy

$$dx_j \left( \frac{\partial}{\partial x_k} \right) = \delta_{jk}.$$

Thus, the elements  $dx_1, \dots, dx_n$  form a basis for  $T_x^* M$  at each point. For any smooth function  $f$ , we have

$$df = \frac{\partial f}{\partial x_j} dx_j. \quad (21.4)$$

A  $k$ -form  $\alpha$  on  $M$  is a mapping assigning to each point  $x \in M$  a  $k$ -linear, alternating functional  $\alpha_x$  on  $T_x M$ . A  $k$ -form is *smooth* if  $\alpha(X_1, \dots, X_k)$  is a smooth function on  $M$  for each  $k$ -tuple of smooth vector fields  $X_1, \dots, X_k$  on  $M$ . In particular, if  $f$  is a smooth function, then  $df$  is a smooth 1-form. If  $\alpha$  is a smooth  $k$ -form and  $X$  a smooth vector field, we may define the *contraction* of  $\alpha$  with  $X$ , which is the  $(k-1)$ -form  $i_X \alpha$  given by

$$(i_X \alpha)(X_1, \dots, X_{k-1}) = \alpha(X, X_1, \dots, X_{k-1}).$$

Given a  $k$ -linear form  $\phi$  on a vector space  $V$ , define the *antisymmetrization*  $\text{AS}(\phi)$  of  $\phi$  by

$$\text{AS}(\phi)(v_1, \dots, v_k) = \sum_{\sigma \in S_k} \text{sign}(\sigma) \phi(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k)}),$$

where  $S_k$  denotes the permutation group on  $k$  elements. Given a  $k$ -form  $\alpha$  and an  $l$ -form  $\beta$  on  $M$ , let  $\alpha \otimes \beta$  be the  $(k+l)$ -linear form on each  $T_x M$  given by

$$(\alpha \otimes \beta)(X_1, \dots, X_{k+l}) = \alpha(X_1, \dots, X_k) \beta(X_{k+1}, \dots, X_{k+l}).$$

Then let  $\alpha \wedge \beta$  denote the  $(k+l)$ -form given by

$$\alpha \wedge \beta = \text{AS}(\alpha \otimes \beta).$$

In particular, if  $\alpha$  and  $\beta$  are 1-forms, then  $\alpha \wedge \beta$  is the 2-form given by

$$(\alpha \wedge \beta)(X, Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X).$$

In a smooth coordinate system  $x_1, \dots, x_n$ , a smooth  $k$ -form  $\alpha$  can be expressed uniquely as

$$\alpha = a_{j_1, \dots, j_k}(x) dx_{j_1} \wedge \dots \wedge dx_{j_k}.$$

A 2-form  $\omega$  on  $M$  is said to be *nondegenerate* if  $\omega$  defines a nondegenerate bilinear form on each  $T_x M$ . More explicitly, this means that for each  $x \in M$  and each nonzero  $X \in T_x M$ , there exists a  $Y \in T_x M$  such that

$$\omega(X, Y) \neq 0.$$

Suppose  $\alpha$  is a smooth  $k$ -form on  $M$  and  $S$  is a compact, oriented,  $k$ -dimensional submanifold-with-boundary of  $M$ . Then one can define the *integral* of  $\alpha$  over  $M$ . There is a map  $d$ , called the *exterior derivative*, mapping smooth  $k$ -forms to smooth  $(k + 1)$ -forms and having the property that

$$\int_S d\beta = \int_{\partial S} \beta \tag{21.5}$$

for every compact, oriented,  $k$ -dimensional submanifold-with-boundary  $S$  of  $M$  and every  $(k - 1)$ -form  $\beta$  on  $M$ . Here  $\partial S$  is the boundary of  $S$ , with the natural orientation induced by the orientation on  $M$ . The relation (21.5) is known as *Stoke's theorem*. A  $k$ -form  $\alpha$  is said to be *closed* if  $d\alpha = 0$ .

The exterior derivative may be computed in coordinates by the formula

$$d(f dx_{j_1} \wedge \cdots \wedge dx_{j_k}) = \frac{\partial f}{\partial x_l} dx_l \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}.$$

A coordinate-invariant formula for the exterior derivative of a  $k$ -form  $\alpha$  is:

$$\begin{aligned} d\alpha(X_1, \dots, X_{k+1}) &= \sum_{j=1}^{k+1} (-1)^{j+1} \alpha(X_1, \dots, \widehat{X}_j, \dots, X_{k+1}) \\ &+ \sum_{j < l} (-1)^{j+l} \alpha([X_j, X_l], X_1, \dots, \widehat{X}_j, \dots, X_{k+1}), \end{aligned}$$

where  $\widehat{X}_j$  indicates that the  $X_j$  term is omitted and where  $[X_j, X_l]$  is the commutator of  $X_j$  and  $X_l$  as first-order differential operators. In particular, if  $\alpha$  is a 1-form, we have

$$(d\alpha)(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]). \tag{21.6}$$

A key identity satisfied by the exterior derivative is

$$d(d\alpha) = 0$$

for all  $k$ -forms  $\alpha$ . Conversely, if  $\beta$  is a closed  $(k + 1)$ -form (i.e.,  $d\beta = 0$ ), then  $\beta$  can be expressed *locally* in the form  $\beta = d\alpha$  for some  $k$ -form  $\alpha$ . More precisely, if  $\beta$  is closed, then for any  $x \in M$  there exists a neighborhood  $U$  of  $x$  and a  $k$ -form  $\alpha$  defined on  $U$  such that  $\beta = d\alpha$  on  $U$ . If  $M$  satisfies certain topological conditions, then each closed  $k$ -form  $\alpha$  on  $M$  can be expressed *globally* in the form  $\alpha = d\beta$ . In particular, if  $M$  is simply connected, then each closed 1-form  $\beta$  can be expressed globally in the form  $\beta = df$  for some smooth function (i.e., 0-form)  $f$ .

If  $X$  is a vector field and  $\alpha$  is a  $k$ -form, we may define the *Lie derivative* of  $\alpha$  in the direction of  $X$ , denoted  $\mathcal{L}_X\alpha$ , as follows:

$$\mathcal{L}_X\alpha = \left. \frac{d}{dt}(\Phi_t^*)(\alpha) \right|_{t=0},$$

where  $\Phi_t$  is the flow generated by  $X$  and  $(\Phi_t^*)(\alpha)$  is the *pullback* of  $\alpha$  by  $\Phi_t$ . The Lie derivative may be computed using the formula

$$\mathcal{L}_X = i_X \circ d + d \circ i_X. \quad (21.7)$$

## 21.2 Mechanics on Symplectic Manifolds

The reader is warned that sign conventions in the subject of Hamiltonian mechanics are not consistent from author to author.

### 21.2.1 Symplectic Manifolds

A symplectic manifold is, roughly, a manifold with enough additional structure to allow one to define the Poisson bracket of two functions.

**Definition 21.1** A **symplectic manifold** is a smooth manifold  $N$  together with a closed, nondegenerate 2-form  $\omega$  on  $N$ . If  $(N_1, \omega_1)$  and  $(N_2, \omega_2)$  are symplectic manifolds, a map  $\Phi : N_1 \rightarrow N_2$  is a **symplectomorphism** if  $\Phi$  is a diffeomorphism and in addition

$$\Phi^*(\omega_2) = \omega_1.$$

It is not hard to see that every symplectic manifold must be even dimensional, for the simple reason that an odd-dimensional vector space does not admit a nondegenerate, skew-symmetric bilinear form.

Throughout this chapter,  $N$  will always denote a symplectic manifold of dimension  $2n$  with symplectic form  $\omega$ .

We now show that the cotangent bundle of any manifold has the structure of a symplectic manifold in a canonical way. Suppose  $x_1, \dots, x_n$  is a coordinate system defined on an open set  $U \subset M$ . Then at each point  $x \in U$ , an element  $\phi$  of  $T_x^*M$  can be expressed uniquely in the form

$$\phi = p_j dx_j$$

for a sequence  $p_1, \dots, p_n$  of real numbers. The quantities  $x_1, \dots, x_n$  and  $p_1, \dots, p_n$  constitute a coordinate system on  $\pi^{-1}(U)$ . We refer to a coordinate system of this sort as a *standard coordinate system* on  $T^*M$ .

**Example 21.2** For any smooth manifold  $M$ , define a 1-form  $\theta$  on the cotangent bundle  $T^*M$  by

$$\theta(X)_{(x,\phi)} = \phi(\pi_*(X))$$

for each tangent vector  $X \in T_{(x,\phi)}(T^*M)$ , where  $\pi : T^*M \rightarrow M$  is the canonical projection. Then the 2-form  $\omega := d\theta$  is closed and nondegenerate. We refer to  $\theta$  and  $\omega$  as the *canonical 1-form* and the *canonical 2-form* on  $T^*M$ , respectively.

**Proof.** Using a coordinate system  $\{x_j\}$  on  $X$  and the associated standard coordinate system  $\{x_j, p_j\}$  on  $T^*M$ , the projection  $\pi$  is given by  $\pi(x, p) = x$ . Meanwhile, a tangent vector  $X$  to  $T^*M$  is expressible as a linear combination the  $\partial/\partial x_j$ 's and  $\partial/\partial p_j$ 's. Thus,

$$\theta \left( a_k \frac{\partial}{\partial x_k} + b_k \frac{\partial}{\partial p_k} \right) = (p_j dx_j) \left( a_k \frac{\partial}{\partial x_k} \right).$$

What this means is that

$$\theta = p_j dx_j,$$

where the  $x_j$ 's are now viewed as functions on  $T^*M$  rather than on  $M$ . We have, then,

$$\omega = d\theta = dp_j \wedge dx_j.$$

It is now easy to see that  $\omega$  is nondegenerate (Exercise 1). ■

### 21.2.2 Poisson Brackets and Hamiltonian Vector Fields

If  $\omega$  is nondegenerate, then it gives a canonical identification of  $T_z N$  with  $T_z^* N$  at each point, by identifying a vector  $X$  in  $T_z N$  with the linear functional  $\omega(X, \cdot)$  in  $T_z^* N$ . We can then transfer the bilinear form  $\omega$  from  $T_z N$  to  $T_z^* N$  by means of this identification. We denote the resulting bilinear form on  $T_z^* N$  by  $\omega^{-1}$ .

**Definition 21.3** If  $f$  and  $g$  are smooth functions on  $N$ , define the **Poisson bracket**  $\{f, g\}$  of  $f$  and  $g$  by

$$\{f, g\} = -\omega^{-1}(df, dg).$$

In particular, if  $\mathbf{1}$  denotes the constant function on  $N$ , then  $\{\mathbf{1}, f\} = \{f, \mathbf{1}\} = 0$  for all smooth functions  $f$ .

**Example 21.4** If  $\omega$  is the canonical 2-form on  $T^*M$ , then the associated Poisson bracket may be computed in standard coordinates as

$$\{f, g\} = \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_j}$$

for all smooth functions  $f$  and  $g$  on  $T^*M$ .

**Proof.** The linear functional

$$\omega \left( \frac{\partial}{\partial x_j}, \cdot \right)$$

has a value of  $-1$  on the vector  $\partial/\partial p_j$  and a value of  $0$  on all the other basic partial derivatives. This means that  $\omega(\partial/\partial x_j, \cdot) = -dp_j$ . Similarly,

$\omega(\partial/\partial p_j, \cdot) = dx_j$ . We may thus compute, for example, that

$$\begin{aligned} -1 &= \omega\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial p_j}\right) \\ &= \omega^{-1}(-dp_j, dx_j) \\ &= \omega^{-1}(dx_j, dp_j). \end{aligned}$$

Meanwhile,  $\omega^{-1}(dx_j, dx_k) = \omega^{-1}(dp_j, dp_k) = 0$  and  $\omega^{-1}(dp_j, dx_k) = 0$  when  $j \neq k$ . Thus, we compute that

$$\begin{aligned} \{f, g\} &= -\omega^{-1}\left(\frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial p_j} dp_j, \frac{\partial g}{\partial x_k} dx_k + \frac{\partial g}{\partial p_k} dp_k\right) \\ &= \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_k} \delta_{jk} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_k} \delta_{jk}, \end{aligned}$$

which reduces to the claimed expression. ■

**Proposition 21.5** *For any smooth functions  $f, g, h$  on  $N$ , we have*

$$\{g, f\} = -\{f, g\}$$

and

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

**Proof.** Since  $\omega$  is skew-symmetric on the tangent space to  $N$  at each point and  $\omega^{-1}$  is obtained from  $\omega$  by means of an isomorphism of tangent and cotangent space,  $\omega^{-1}$  is a skew-symmetric form on the cotangent space. The skew-symmetry of the Poisson bracket follows. The second relation follows from the Leibniz product rule for  $d(gh)$  together with the bilinearity of  $\omega^{-1}$ . ■

**Definition 21.6** *If  $f$  is a smooth function on  $N$ , let  $X_f$  be the unique vector field on  $N$  such that*

$$df = \omega(X_f, \cdot). \tag{21.8}$$

We call  $X_f$  the **Hamiltonian vector field** associated to  $f$ .

That is to say,  $X_f$  corresponds to  $df$  under the isomorphism between tangent and cotangent spaces established by  $\omega$ .

**Proposition 21.7** *For all  $f$  and  $g$ ,*

$$X_f(g) = \{f, g\} = -X_g(f).$$

Furthermore,

$$\omega(X_f, X_g) = -\{f, g\}.$$

**Proof.** For each  $z \in N$ , we are using  $\omega$  to identify  $T_z N$  with  $T_z^* N$ . Equation (21.8) says that under this identification,  $X_f$  is identified with  $df$ . Thus,

$$-\omega^{-1}(df, dg) = -\omega(X_f, X_g) = -df(X_g) = -X_g(f).$$

Thus,  $\{f, g\} = -X_g(f)$ , as claimed. A similar argument with the roles of  $f$  and  $g$  reversed gives the claimed relationship between  $X_f(g)$  and  $\{g, f\}$ . Finally,

$$\omega(X_f, X_g) = df(X_g) = X_g(f) = -\{f, g\},$$

as claimed. ■

**Definition 21.8** For any smooth function  $f$  on  $N$ , the **Hamiltonian flow** generated by  $f$ , denoted  $\Phi^f$ , is the flow generated by the vector field  $-X_f$ .

In the case  $N = T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ , this definition agrees with our notation in Sect. 2.5.

**Proposition 21.9** For any smooth function  $f$  on  $N$ , the Hamiltonian flow  $\Phi^f$  preserves  $\omega$ .

**Proof.** In general, a flow  $\Phi$  preserves a differential form  $\alpha$  if and only if the Lie derivative  $L_X \alpha = 0$ , where  $X$  is the vector field generating  $\Phi$ . In our case, since  $\omega$  is closed, we have, by (21.7),

$$\mathcal{L}_{X_f} \omega = d[i_{X_f} \omega] = d^2 f = 0,$$

since  $i_{X_f} \omega$  is, by the definition of  $X_f$ , equal to  $df$ . ■

**Proposition 21.10** For any smooth functions  $f, g, h$  on  $N$ , the Jacobi identity holds:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

This result shows that the space of smooth function on  $N$  forms a Lie algebra under the Poisson bracket. The proof of Proposition 21.10 relies on Proposition 21.9, which in turn relies on the fact that  $\omega$  is closed.

**Proof.** Since the Hamiltonian flow  $\Phi^f$  preserves  $\omega$ , it also preserves  $\omega^{-1}$  and thus

$$\omega^{-1}(d(g \circ \Phi_t^f), d(h \circ \Phi_t^f)) = \omega^{-1}(dg, dh) \circ \Phi_t^f,$$

or, equivalently,

$$\{g \circ \Phi_t^f, h \circ \Phi_t^f\} = \{g, h\} \circ \Phi_t^f.$$

Differentiating this relation with respect to  $t$  at  $t = 0$  gives

$$\{-X_f(g), h\} + \{g, -X_f(h)\} = -X_f(\{g, h\}),$$

or, equivalently,

$$-\{\{f, g\}, h\} + \{g, \{f, h\}\} = -\{f, \{g, h\}\}.$$

After moving  $-\{f, \{g, h\}\}$  to the left-hand side of the equation and using the skew-symmetry of the Poisson bracket, we obtain the Jacobi identity. ■

**Proposition 21.11** *For any smooth functions  $f$  and  $g$  on  $N$ , the Hamiltonian vector fields  $X_f$  and  $X_g$  satisfy*

$$[X_f, X_g] = X_{\{f, g\}}.$$

**Proof.** See Exercise 3. ■

### 21.2.3 Hamiltonian Flows and Conserved Quantities

We have seen (Proposition 21.9) that if  $f$  is a smooth function, then the flow generated by  $X_f$  preserves  $\omega$ . We have the following partial converse to this result.

**Proposition 21.12** *Suppose  $\Phi$  is the flow generated by a vector field  $-X$  on  $N$ . If  $\Phi$  preserves  $\omega$  then  $X$  can be represented locally in the form  $X = X_f$  for some smooth function  $f$  on  $N$ . If  $N$  is simply connected, the function  $f$  exists globally on  $N$ .*

**Proof.** The statement that  $\Phi$  preserves  $\omega$  can be expressed infinitesimally as

$$\mathcal{L}_X \omega = 0.$$

Since also  $\omega$  is closed, (21.7) tells us that

$$d(i_X \omega) = 0.$$

Since  $i_X \omega$  is closed, this 1-form can be expressed locally as  $i_X \omega = df$  for some smooth function  $f$ , which says precisely that  $X = X_f$ . If  $N$  is simply connected, then every closed 1-form can be expressed globally as  $df$ , for some smooth function  $f$ . ■

A flow of the sort in Proposition 21.12 is said to be *locally Hamiltonian*. Such a flow is said to be (globally) *Hamiltonian* if the function  $f$  in the proposition can be defined on all of  $N$ . (Compare Definition 21.8.) If  $\Phi$  is a Hamiltonian flow, the function  $f$  such that  $\Phi = \Phi^f$  is called a *Hamiltonian generator* of  $\Phi$ . If  $N$  is connected, then any two Hamiltonian generators of  $\Phi$  must differ by a constant.

To see that, in general,  $f$  is only defined locally, consider the symplectic manifold  $S^1 \times \mathbb{R}$ , with symplectic form  $\omega = d\phi \wedge dx$ , where  $\phi$  is the angular coordinate on  $S^1$  and  $x$  is the linear coordinate on  $\mathbb{R}$ . Note that the 1-form

$d\phi$  is independent of the choice of a local angle variable on  $S^1$ , since any two such angle functions differ by a constant (an integer multiple of  $2\pi$ ). Thus,  $d\phi$  is a globally defined, smooth 1-form, even though there is no globally defined, smooth angle function  $\phi$ . Define a flow  $\Phi$  by

$$\Phi_t(\phi, x) = (\phi, x + t).$$

This flow certainly preserves  $\omega$ , since  $dx$  is invariant under translations.

The flow  $\Phi$  is generated by the vector field  $-X = \partial/\partial x$ , and

$$\omega(-\partial/\partial x, \cdot) = d\phi.$$

As we have already noted, however, there is no globally defined function  $\phi$  whose differential is  $d\phi$ .

Although any smooth function on a symplectic manifold  $N$  generates a Hamiltonian flow, in physical examples there is usually one distinguished function with a Hamiltonian flow that is thought of as “the” time-evolution of the system.

**Definition 21.13** A **Hamiltonian system** is a symplectic manifold  $N$  together with a distinguished Hamiltonian flow  $\Phi^H$ , generated by smooth function  $H$  on  $N$ , called the **Hamiltonian** of the system. A function  $f$  is called a **conserved quantity** for a Hamiltonian system  $(N, \Phi^H)$  if  $f(\Phi_t^H(x))$  is independent of  $t$  for each fixed  $x \in N$ .

As in the  $\mathbb{R}^{2n}$  case, conserved quantities are useful in understanding the nature of the dynamics. See the discussion following Corollary 2.26.

**Proposition 21.14** For any Hamiltonian system  $(N, \Phi^H)$ , we have

$$\frac{d}{dt}f(\Phi_t^H(z)) = \{f, H\}(\Phi_t^H(z)),$$

for all  $z \in N$ , or, more concisely,

$$\frac{df}{dt} = \{f, H\}.$$

In particular, a smooth function  $f$  on  $N$  is a conserved quantity for a Hamiltonian system  $\Phi^H$  if and only if  $\{f, H\} = 0$ .

**Proof.** For the flow generated by any vector field  $X$ , we have

$$\frac{d}{dt}f(\Phi_t(z)) = X_{\Phi_t(z)}f.$$

If  $X = -X_f$ , then by Proposition 21.7, we have the claimed result. ■

**Proposition 21.15** A smooth function  $f$  is a conserved quantity for a Hamiltonian system  $(N, \Phi^H)$  if and only if  $H$  is invariant under the Hamiltonian flow  $\Phi^f$  generated by  $f$ .

**Proof.** By the previous proposition,  $H$  is invariant under the flow generated by  $f$  if and only if  $\{H, f\} = 0$ , which holds if and only if  $\{f, H\} = 0$ , which holds if and only if  $f$  is a conserved quantity. ■

### 21.2.4 The Liouville Form

A symplectic manifold  $N$  has a natural volume form, which allows us to formulate an analog on  $N$  of Liouville's theorem (Theorem 2.27).

**Definition 21.16** *If  $N$  is a  $2n$ -dimensional symplectic manifold, the Liouville form on  $N$  is the  $2n$ -form  $\lambda$  given by*

$$\lambda = \frac{1}{n!} \omega^n,$$

where  $\omega^n = \omega \wedge \cdots \wedge \omega$ .

Since  $\omega$  is, by assumption, a nondegenerate form on each tangent space  $T_z N$ , it is not hard to check that  $\lambda$  is a nonvanishing  $(2n)$ -linear form on each  $T_z N$ . Thus,  $\lambda$  determines an orientation on  $N$ . Given a compactly supported continuous function  $f$  on  $N$ , we can define the integral of  $f$  over  $N$ , computed with respect to the orientation determined by  $\lambda$  itself. Using the version of the Riesz representation theorem for *locally* compact topological spaces, one can show that there is a unique *measure*, called the Liouville volume measure, for which the integral of every continuous compactly supported function  $f$  is given by  $\int_N f \lambda$ .

We are now ready to state the general form of Liouville's theorem.

**Theorem 21.17 (Liouville's Theorem)** *For any smooth function  $f$  on  $N$ , the Hamiltonian flow  $\Phi^f$  preserves  $\lambda$ .*

**Proof.** The flow  $\Phi^f$  will preserve  $\lambda$  if and only if the vector field  $X_f$  satisfies  $\mathcal{L}_{X_f} \lambda = 0$ . But

$$\begin{aligned} \mathcal{L}_{X_f} \lambda &= \frac{1}{n!} [(\mathcal{L}_{X_f} \omega) \wedge \omega \wedge \cdots \wedge \omega \\ &\quad + \omega \wedge (\mathcal{L}_{X_f} \omega) \wedge \omega \wedge \cdots \wedge \omega \\ &\quad + \cdots + \omega \wedge \cdots \wedge \omega \wedge (\mathcal{L}_{X_f} \omega)]. \end{aligned}$$

Since we have already shown (Proposition 21.9) that  $\mathcal{L}_{X_f} \omega = 0$ , we see that  $\mathcal{L}_{X_f} \lambda = 0$ . ■

## 21.3 Exercises

1. Show that the canonical 2-form  $\omega$  on  $T^*M$  is nondegenerate.

*Hint:* Work in standard coordinates  $\{x_j, p_j\}$ .

2. Show that if  $\Phi : M \rightarrow M$  is a diffeomorphism, then the induced map  $\Phi^* : T^*M \rightarrow T^*M$  is a symplectomorphism.
3. Using Proposition 21.7 and the Jacobi identity for the Poisson bracket, verify that

$$[X_f, X_g] = X_{\{f, g\}}$$

for all smooth functions  $f$  and  $g$  on  $N$ .

4. If  $N$  is compact, show that

$$\int_N \{f, g\} \lambda = 0$$

for all smooth function  $f$  and  $g$  on  $N$ .

*Hint:* Apply Liouville's theorem to the flow  $\Phi_t^f$ .