

16

Lie Groups, Lie Algebras, and Representations

An important concept in physics is that of *symmetry*, whether it be rotational symmetry for many physical systems or Lorentz symmetry in relativistic systems. In many cases, the group of symmetries of a system is a *continuous group*, that is, a group that is parameterized by one or more real parameters. More precisely, the symmetry group is often a *Lie group*, that is, a smooth manifold endowed with a group structure in such a way that operations of inversion and group multiplication are smooth. The tangent space at the identity in a Lie group has a natural “bracket” operation that makes the tangent space into a Lie algebra. The Lie algebra of a Lie group encodes many of the properties of the Lie group, and yet the Lie algebra is easier to work with because it is a linear space.

In quantum mechanics, the way symmetry is encoded is usually through a unitary action of the group on the relevant Hilbert space. That is, we assume we are given a *unitary representation* of the relevant symmetry group G , that is, a continuous homomorphism of G into $U(\mathbf{H})$, the group of unitary operators on the quantum Hilbert space \mathbf{H} . Actually, since two unit vectors in \mathbf{H} that differ only by a constant represent the same physical state, we should more properly consider *projective* unitary representations. A projective representation is a homomorphism of a group G into $U(\mathbf{H})/U(1)$, where $U(1)$ is the group of complex numbers of magnitude 1, thought of multiples of I in $U(\mathbf{H})$. An ordinary or projective representation of a Lie group gives rise to an ordinary or projective representation of its Lie algebra. The angular momentum operators, for example, form a representation of the Lie algebra of the rotation group.

Saying that, for example, the Hamiltonian operator of a quantum system is invariant under rotations means that \hat{H} commutes with the relevant representation of the rotation group and thus also with the associated Lie algebra operators. This commutativity, in turn, implies that the eigenspaces for \hat{H} are invariant under rotations. We will use this commutativity in Chap. 18 to help us in determining the energy eigenvectors for the hydrogen atom.

In this chapter, we will make a brief survey of Lie groups, Lie algebras, and their representations. For our purposes, it suffices to consider *matrix Lie groups*, those that can be realized as closed subgroups of the group of $n \times n$ invertible matrices. Inevitably, I have had to present some of the deeper results without proof. Proofs of all results stated here can be found in [21]. The results of this chapter will be put to use in Chap. 17, in our study of angular momentum, and in Chap. 18, in our study of the hydrogen atom.

16.1 Summary

In this chapter, we will consider a matrix Lie group G , which is, by definition, a (topologically) closed subgroup of some $\mathrm{GL}(n; \mathbb{C})$, where $\mathrm{GL}(n; \mathbb{C})$ is the group of $n \times n$ invertible matrices with complex entries. To each such G , we will associate the Lie algebra \mathfrak{g} of G , where \mathfrak{g} is a real subspace of $M_n(\mathbb{C})$, the space of all $n \times n$ matrices. We will see that G is automatically an embedded real submanifold of $M_n(\mathbb{C})$ and that \mathfrak{g} is the tangent space of G at the identity matrix.

Now, \mathfrak{g} is not just a real vector space, but comes with a “bracket” operation mapping $\mathfrak{g} \times \mathfrak{g}$ into \mathfrak{g} . Specifically, we will show that for all X and Y in \mathfrak{g} , the matrix $XY - YX$ belongs again to \mathfrak{g} . Thus, we define our bracket by setting $[X, Y]$ equal to $XY - YX$. As it turns out, the Lie algebra \mathfrak{g} , as a vector space with the bracket operation, encodes a lot of information about the group G . On the other hand, computing at the level of the Lie algebra is generally easier than computing at the group level, simply because \mathfrak{g} is a linear space.

We will be interested in unitary *representations* of our group G , that is, continuous homomorphisms of G into $\mathrm{U}(\mathbf{H})$, the group of unitary operators on a Hilbert space. If we restrict attention, at first, to the case in which \mathbf{H} is finite dimensional, then each representation Π of G gives rise to a representation π of the Lie algebra \mathfrak{g} of G . That is to say, π is a linear map of \mathfrak{g} into the space of linear maps of V to V , satisfying $\pi([X, Y]) = [\pi(X), \pi(Y)]$. A deeper question is whether every representation π of \mathfrak{g} comes from a representation Π of G . As it turns out, the answer in general is no, but the answer is yes if G is simply connected.

We may consider, for example, the case $G = \mathrm{SO}(3)$. This group is not simply connected. On the other hand, the Lie algebra $\mathfrak{so}(3)$ of $\mathrm{SO}(3)$ is isomorphic to the Lie algebra $\mathfrak{su}(2)$ of $\mathrm{SU}(2)$, and $\mathrm{SU}(2)$ is simply connected. [That is, $\mathrm{SU}(2)$ is the “universal cover” of $\mathrm{SO}(3)$.] Thus, given a representation π of $\mathfrak{so}(3)$, there may or may not be an associated representation Π of $\mathrm{SO}(3)$. Even if there is not, however, there is always a representation Π' of the group $\mathrm{SU}(2)$.

In quantum mechanics, the vector $e^{i\theta}\psi$ represents the same physical state as ψ . Thus, it is natural to consider “projective” unitary representations, that is, homomorphisms of G into the quotient group $\mathrm{U}(\mathbf{H})/\{e^{i\theta}I\}$. In the finite-dimensional case, each projective representation can be “deprojectivized” at the level of the Lie algebra \mathfrak{g} of G . We can then pass from the Lie algebra to the universal cover of G , that is, the simply connected group with Lie algebra \mathfrak{g} . In particular, in the finite-dimensional case, the irreducible *projective* unitary representations of $\mathrm{SO}(3)$ are in one-to-one correspondence with irreducible *ordinary* unitary representations of the universal cover $\mathrm{SU}(2)$ of $\mathrm{SO}(3)$. Although the Hilbert spaces of physical systems are usually infinite dimensional, for compact groups such as $\mathrm{SO}(3)$, general unitary representations can be decomposed as direct sums of finite-dimensional ones. (See, e.g., Proposition 17.19 and the discussion following it.)

16.2 Matrix Lie Groups

Let $M_n(\mathbb{C})$ denote the space of $n \times n$ matrices with complex entries. We identify $M_n(\mathbb{C})$ with \mathbb{C}^{n^2} , equipped with the usual topology. Thus, a sequence A_m in $M_n(\mathbb{C})$ converges to a matrix $A \in M_n(\mathbb{C})$ if $(A_m)_{jk}$ converges to A_{jk} as m tends to infinity, for all $1 \leq j, k \leq n$. Let $\mathrm{GL}(n; \mathbb{C})$ denote the *general linear group*, consisting of all invertible $n \times n$ matrices with complex entries. Then $\mathrm{GL}(n; \mathbb{C})$ forms a group under the operation of matrix multiplication. Furthermore, $\mathrm{GL}(n; \mathbb{C})$ —that is, the set of $A \in M_n(\mathbb{C})$ with $\det A \neq 0$ —is an open subset of $M_n(\mathbb{C})$. Since $M_n(\mathbb{C})$ is a complex vector space of dimension n^2 , it may be identified with $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$. Since $\mathrm{GL}(n; \mathbb{C})$ is an open subset of $M_n(\mathbb{C})$, it looks locally like \mathbb{R}^{2n^2} and is therefore a real manifold of dimension $2n^2$.

Definition 16.1 A subgroup G of $\mathrm{GL}(n; \mathbb{C})$ is **closed** if for each sequence A_m in G that converges to a matrix A , either A is again in G or A is not invertible. A **matrix Lie group** is a closed subgroup of some $\mathrm{GL}(n; \mathbb{C})$.

A subgroup G of $\mathrm{GL}(n; \mathbb{C})$ is closed if it is topologically closed as a subset of $\mathrm{GL}(n; \mathbb{C})$ —but not necessarily as a subset of $M_n(\mathbb{C})$. We will see that each matrix Lie group is a real embedded submanifold of $\mathrm{GL}(n; \mathbb{C})$ and thus is a Lie group.

Definition 16.2 If G_1 and G_2 are matrix Lie groups, then a **Lie group homomorphism** of G_1 to G_2 is a continuous group homomorphism of G_1 into G_2 . A Lie group homomorphism is called a **Lie group isomorphism** if it is one-to-one and onto with continuous inverse. Two matrix Lie groups are called **isomorphic** if there exists a Lie group isomorphism between them.

Example 16.3 The real **general linear group**, denoted $\mathrm{GL}(n, \mathbb{R})$, is the group of invertible $n \times n$ matrices with real entries. The groups $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{R})$ are, respectively, the groups of complex and real matrices with determinant 1. They are called the **special linear groups**.

Example 16.4 An $n \times n$ matrix $U \in M_n(\mathbb{C})$ is said to be **unitary** if $U^*U = UU^* = I$. A matrix U is unitary if and only if

$$\langle Uv, Uw \rangle = \langle v, w \rangle$$

for all $v, w \in \mathbb{C}^n$. The group of unitary matrices is denoted $\mathrm{U}(n)$ and called the $(n \times n)$ **unitary group**. The **special unitary group**, denoted $\mathrm{SU}(n)$, is the subgroup of $\mathrm{U}(n)$ consisting of unitary matrices with determinant 1.

The condition $(U^*U)_{jk} = \delta_{jk}$ is equivalent to the condition that the columns of U form an orthonormal set in \mathbb{C}^n , as can be seen by direct computation. Geometrically, the condition $U^*U = I$ is equivalent to the condition that $\langle Uv_1, Uv_2 \rangle = \langle v_1, v_2 \rangle$ for all $v_1, v_2 \in \mathbb{C}^n$, i.e., that U preserves the inner product on \mathbb{C}^n . By taking the determinant of the condition $U^*U = I$, we see that $|\det U| = 1$ for all $U \in \mathrm{U}(n)$.

In this, the finite-dimensional case, the condition $U^*U = I$ implies that U^* is the inverse of U and thus that $UU^* = I$. This result does not hold in the infinite-dimensional case.

Example 16.5 An $n \times n$ real matrix $R \in M_n(\mathbb{R})$ is said to be **orthogonal** if $R^{tr}R = RR^{tr} = I$. A matrix R is orthogonal if and only if

$$\langle Rv, Rw \rangle = \langle v, w \rangle$$

for all $v, w \in \mathbb{R}^n$. The group of orthogonal matrices is denoted $\mathrm{O}(n)$ and is called the $(n \times n)$ **orthogonal group**. The **special orthogonal group**, denoted $\mathrm{SO}(n)$, is the subgroup of $\mathrm{O}(n)$ consisting of orthogonal matrices with determinant 1.

As in the unitary case, the condition $R^{tr}R = I$ implies that $RR^{tr} = I$ and that the columns of R form an orthonormal set in \mathbb{R}^n . Geometrically, a real matrix R is in $\mathrm{O}(n)$ if and only if $\langle Rv_1, Rv_2 \rangle = \langle v_1, v_2 \rangle$ for all $v_1, v_2 \in \mathbb{R}^n$, i.e., if and only if R preserves the inner product on \mathbb{R}^n . By taking the determinant of the condition $R^{tr}R = I$ we see that $\det R = \pm 1$ for all $R \in \mathrm{O}(n)$.

It is easy to verify that all the groups in Examples 16.3, 16.4, and 16.5 are, indeed, subgroups of $\mathrm{GL}(n, \mathbb{C})$ and that they are closed.

Definition 16.6 A matrix Lie group G is **connected** if for all $A, B \in G$ there is a continuous path $A : [0, 1] \rightarrow M_n(\mathbb{C})$ such that $A(0) = A$ and $A(1) = B$ and such that $A(t)$ lies in G for all t . A matrix Lie group G is **simply connected** if it is connected and every continuous loop in G can be shrunk continuously to a point in G . A matrix Lie group G is **compact** if it is **compact** as a subset of $M_n(\mathbb{C}) \cong \mathbb{R}^{2n^2}$.

By the Heine–Borel theorem (e.g., Proposition 0.26 of [12]), a matrix Lie group G is compact if and only if it is a closed and bounded subset of $M_n(\mathbb{C})$. The condition we are calling “connected” is, more properly, the condition of being *path connected*. We will see, however, that each matrix Lie group is an embedded real submanifold of $M_n(\mathbb{C})$ and is, therefore, locally path connected. For matrix Lie groups, then, connectedness and path connectedness are equivalent.

To prove that a matrix Lie group G is connected, it suffices to prove that for all $A \in G$, there is a continuous path in G connecting A to I . After all, if both A and B can be connected to I , then they can be connected to each other.

Example 16.7 The groups $O(n)$, $SO(n)$, $U(n)$, and $SU(n)$ are compact.

Proof. The conditions defining these groups are obtained by setting certain continuous functions equal to a constant. The group $SU(n)$, for example, is defined by setting $(U^*U)_{jk} = \delta_{jk}$ for each j and k and by setting $\det U = 1$. These groups are thus closed not just as subsets of $GL(n; \mathbb{C})$ but also as subsets of $M_n(\mathbb{C})$. Furthermore, each of these groups has the property that each column of any matrix in the group is a unit vector. Thus, each group is a bounded subset of $M_n(\mathbb{C})$. ■

Example 16.8 The group $U(n)$ is connected.

Proof. If $U \in M_n(\mathbb{C})$ is unitary, then U has an orthonormal basis of eigenvectors with eigenvalues of absolute value 1. Thus, there is another unitary matrix V (the change of basis matrix) such that

$$U = V \begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_n} \end{pmatrix} V^{-1},$$

for some real numbers $\theta_1, \theta_2, \dots, \theta_n$. Thus, we can define a family $U(t)$ of unitary matrices by setting

$$U(t) = V \begin{pmatrix} e^{it\theta_1} & & & \\ & e^{it\theta_2} & & \\ & & \ddots & \\ & & & e^{it\theta_n} \end{pmatrix} V^{-1}.$$

Then $U(\cdot)$ is a continuous path lying in $U(n)$ with $U(0) = I$ and $U(1) = U$.

■

Example 16.9 *The group $SU(2)$ is simply connected.*

Proof. We claim that

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

It is easy to see that each matrix of the indicated form is indeed unitary and has determinant 1. On the other hand, if U is any element of $SU(2)$, then the first column of U is a unit vector $(\alpha, \beta) \in \mathbb{C}^2$. The second column of U must then be orthogonal to (α, β) . Since $(-\bar{\beta}, \bar{\alpha})$ is orthogonal to (α, β) and \mathbb{C}^2 is 2-dimensional, the second column of U must be a multiple of $(-\bar{\beta}, \bar{\alpha})$. But the only multiple that produces a matrix with determinant 1 is 1.

We see, then, that $SU(2)$ is, topologically, the unit sphere S^3 inside $\mathbb{C}^2 \cong \mathbb{R}^4$ and is, therefore, simply connected. ■

16.3 Lie Algebras

We now introduce the general algebraic concept of a Lie algebra. Once this is done, we will show how to associate a real Lie algebra with an arbitrary matrix Lie group.

Definition 16.10 *A **Lie algebra** over a field \mathbb{F} is a vector space \mathfrak{g} over \mathbb{F} , together with a “bracket” map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ having the following properties:*

1. $[\cdot, \cdot]$ is bilinear
2. $[Y, X] = -[X, Y]$ for all $X, Y \in \mathfrak{g}$
3. $[X, X] = 0$ for all $X \in \mathfrak{g}$
4. For all $X, Y, Z \in \mathfrak{g}$ we have the **Jacobi identity**

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

If the characteristic of \mathbb{F} is not equal to 2, then Property 3 is a consequence of Property 2. If $\mathbb{F} = \mathbb{R}$, then we say that \mathfrak{g} is a real Lie algebra. An example of a real Lie algebra is the vector space \mathbb{R}^3 with the bracket equal to the cross product. Properties 1, 2, and 3 are evident from the definition of the cross product, while the Jacobi identity is a known property of the cross product that can be verified by direct calculation.

A large class of Lie algebras may be obtained by the following procedure.

Example 16.11 Let \mathcal{A} be an associative algebra and let \mathfrak{g} be a subspace of \mathcal{A} with the property that for all x, y in \mathfrak{g} , $xy - yx$ is again in \mathfrak{g} . Then the bracket

$$[x, y] := xy - yx$$

makes \mathfrak{g} into a Lie algebra.

In Example 16.11, we may take, for example, $\mathfrak{g} = \mathcal{A}$. It is evident that this bracket satisfies Properties 1, 2, and 3 of a Lie algebra, and the Jacobi identity is easily verified by direct calculation. As it turns out, every Lie algebra is isomorphic to a Lie algebra of this type. (This claim is a consequence of the Poincaré–Birkhoff–Witt theorem, which is proved, for example, in Sect. 5.2 of [25]. The algebra \mathcal{A} in the Poincaré–Birkhoff–Witt theorem is the so-called *universal enveloping algebra* of \mathfrak{g} .)

Definition 16.12 If \mathfrak{g}_1 and \mathfrak{g}_2 are Lie algebras, a map $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is called a **Lie algebra homomorphism** if ϕ is linear and ϕ satisfies

$$\phi([X, Y]) = [\phi(X), \phi(Y)]$$

for all $X, Y \in \mathfrak{g}_1$. A Lie algebra homomorphism is called a **Lie algebra isomorphism** if it is one-to-one and onto.

Definition 16.13 If \mathfrak{g} is a Lie algebra, a **subalgebra** of \mathfrak{g} is a subspace \mathfrak{h} of \mathfrak{g} with the property that $[X, Y] \in \mathfrak{h}$ for all X and Y in \mathfrak{h} . An **ideal** in \mathfrak{g} is a subalgebra \mathfrak{h} of \mathfrak{g} with the stronger property that $[X, Y] \in \mathfrak{h}$ for all X in \mathfrak{g} and Y in \mathfrak{h} .

The notion of a subalgebra of a Lie algebra is analogous to the notion of a subgroup of a group, while the notion of an ideal in a Lie algebra is analogous to the notion of a *normal* subgroup of a group. In particular, the kernel of any Lie algebra homomorphism is an ideal, just as the kernel of a group homomorphism is a normal subgroup.

Definition 16.14 The **direct sum** of Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 , denoted $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, is the direct sum of \mathfrak{g}_1 and \mathfrak{g}_2 as a vector space, equipped with the bracket given by

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2])$$

for all $X_1, X_2 \in \mathfrak{g}_1$ and $Y_1, Y_2 \in \mathfrak{g}_2$.

16.4 The Matrix Exponential

In the next section, we will associate a Lie algebra with each matrix Lie group. To describe this association, we need the notion of the exponential

of a matrix. Given a matrix $X \in M_n(\mathbb{C})$, we define the *matrix exponential* of X , denoted by e^X or $\exp(X)$, by the usual power series,

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!},$$

where $X^0 = I$ (the identity matrix). This series converges absolutely for all $X \in M_n(\mathbb{C})$, as can easily be seen using the inequality $\|X^m\| \leq \|X\|^m$, where $\|X\|$ is the operator norm of X ; see Definition A.35. (In this, the finite-dimensional case, we could just as well use the Hilbert–Schmidt norm, which amounts to using the usual Euclidean norm on $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$. See Exercise 3.) The matrix exponential shares some but not all of the properties of the exponential of a number.

Theorem 16.15 *The matrix exponential has the following properties for all $X, Y \in M_n(\mathbb{C})$.*

1. $e^0 = I$
2. $e^{X^{tr}} = (e^X)^{tr}$ and $e^{X^*} = (e^X)^*$
3. If A is an invertible $n \times n$ matrix, then

$$e^{AXA^{-1}} = Ae^XA^{-1}.$$

4. $\det(e^X) = e^{\text{trace}(X)}$
5. If $XY = YX$ then $e^{X+Y} = e^Xe^Y$
6. e^X is invertible and $(e^X)^{-1} = e^{-X}$
7. Even if $XY \neq YX$, we have

$$e^{X+Y} = \lim_{m \rightarrow \infty} \left(e^{X/m} e^{Y/m} \right)^m.$$

Here X^{tr} and X^* denote the transpose and adjoint (conjugate transpose) of X , respectively. Property 7 is known as the Lie Product Formula and is a special case of the Trotter Product formula (Theorem 20.1). Properties 1, 2, and 3 are easily verified using term-by-term computation. Property 6 follows from Property 5 by taking $Y = -X$ and applying Property 1. The proofs of Properties 4, 5, and 7 are outlined in Exercises 5, 6, and 7.

Suppose a matrix X is diagonalizable, meaning that

$$X = A \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} A^{-1},$$

for some invertible matrix A and complex numbers $\lambda_1, \lambda_2, \dots, \lambda_n$. Then using Property 3 of Theorem 16.15, it is easy to see that

$$e^X = A \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} A^{-1}.$$

If X is not diagonalizable, e^X can be computed in terms of the SN decomposition of X . See Sect. 2.2 of [21] for details.

Example 16.16 *If*

$$X = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$$

then

$$e^X = \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}.$$

Proof. The eigenvalues of X are $\pm ia$ and the corresponding eigenvectors are $(1, \pm i)$. Thus, we may calculate that

$$\begin{aligned} e^X &= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{ia} & 0 \\ 0 & e^{-ia} \end{pmatrix} \frac{1}{(-2i)} \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} \\ &= -\frac{1}{2i} \begin{pmatrix} -i(e^{ia} + e^{-ia}) & -e^{ia} + e^{-ia} \\ e^{ia} - e^{-ia} & -i(e^{ia} + e^{-ia}) \end{pmatrix}, \end{aligned}$$

which simplifies to the desired result. ■

The relation $e^{X+Y} = e^X e^Y$ certainly does not hold for general (noncommuting) matrices X and Y . Nevertheless, for any $X \in M_n(\mathbb{C})$ we have

$$e^{(s+t)X} = e^{sX} e^{tX}$$

for all s and t in \mathbb{R} , since sX commutes with tX . Thus, for each X , the set of matrices of the form e^{tX} , $t \in \mathbb{R}$, forms a subgroup of $GL(n; \mathbb{C})$. It is not hard to show (Exercise 4), using term-by-term differentiation, that

$$\left. \frac{d}{dt} e^{tX} \right|_{t=0} = X. \quad (16.1)$$

Here, the derivative of a matrix-valued function is defined as being *entry-wise*. [That is, if $f(t)$ is a matrix-valued function, df/dt is the matrix-valued function whose (j, k) entry is $d(f(t)_{jk})/dt$.]

Definition 16.17 *A one-parameter subgroup of $GL(n; \mathbb{C})$ is a continuous homomorphism of \mathbb{R} into $GL(n; \mathbb{C})$, that is, a continuous map $A : \mathbb{R} \rightarrow GL(n; \mathbb{C})$ such that $A(0) = I$ and $A(s+t) = A(s)A(t)$ for all $s, t \in \mathbb{R}$.*

Theorem 16.18 *If $A(\cdot)$ is a one-parameter subgroup of $\mathrm{GL}(n; \mathbb{C})$, there exists a unique $X \in M_n(\mathbb{C})$ such that*

$$A(t) = e^{tX}$$

for all $t \in \mathbb{R}$.

This is Theorem 2.13 in [21].

16.5 The Lie Algebra of a Matrix Lie Group

We now associate a Lie algebra \mathfrak{g} to each matrix Lie group G .

Definition 16.19 *If $G \subset \mathrm{GL}(n; \mathbb{C})$ is a matrix Lie group, then the **Lie algebra** \mathfrak{g} of G is defined as follows:*

$$\mathfrak{g} = \{X \in M_n(\mathbb{C}) \mid e^{tX} \in G \text{ for all } t \in \mathbb{R}\}.$$

That is to say, X belongs to \mathfrak{g} if and only if the one-parameter subgroup generated by X lies entirely in G . Note that to have X belong to \mathfrak{g} , we need only have e^{tX} belong to G for all *real* numbers t .

Proposition 16.20 *For any matrix Lie group G , the Lie algebra \mathfrak{g} of G has the following properties.*

1. *The zero matrix 0 belongs to \mathfrak{g} .*
2. *For all X in \mathfrak{g} , tX belongs to \mathfrak{g} for all real numbers t .*
3. *For all X and Y in \mathfrak{g} , $X + Y$ belongs to \mathfrak{g} .*
4. *For all $A \in G$ and $X \in \mathfrak{g}$ we have $AXA^{-1} \in \mathfrak{g}$.*
5. *For all X and Y in \mathfrak{g} , the commutator $[X, Y] := XY - YX$ belongs to \mathfrak{g} .*

The first three properties of \mathfrak{g} say that \mathfrak{g} is a real vector space. Since $M_n(\mathbb{C})$ is an associative algebra under the operation of matrix multiplication, the last property of \mathfrak{g} shows that \mathfrak{g} is a real Lie algebra (Example 16.11).

Proof. Points 1 and 2 are elementary, and Point 3 follows from the Lie product formula, using the assumption that G is closed. Point 4 follows from Property 3 in Theorem 16.15. To verify Point 5, we observe that the commutator $[X, Y]$ may be computed as

$$[X, Y] = \left. \frac{d}{dt} e^{tX} Y e^{-tX} \right|_{t=0},$$

using (4) and an easily verified product rule for differentiation of matrix-valued functions. For $X, Y \in \mathfrak{g}$, $e^{tX}Y e^{-tX}$ belongs to \mathfrak{g} for all $t \in \mathbb{R}$, by Point 4. Furthermore, we have already shown that \mathfrak{g} is a real subspace of $M_n(\mathbb{C})$ and therefore a closed subset of $M_n(\mathbb{C})$. Thus,

$$[X, Y] = \lim_{h \rightarrow 0} \frac{e^{hX}Y e^{-hX} - Y}{h}$$

belongs to \mathfrak{g} . ■

Example 16.21 Let $\mathfrak{gl}(n; \mathbb{C})$, $\mathfrak{gl}(n; \mathbb{R})$, $\mathfrak{sl}(n; \mathbb{C})$, and $\mathfrak{sl}(n; \mathbb{R})$ denote the Lie algebras of $\mathrm{GL}(n; \mathbb{C})$, $\mathrm{GL}(n; \mathbb{R})$, $\mathrm{SL}(n; \mathbb{C})$, and $\mathrm{SL}(n; \mathbb{R})$, respectively. Then we have

$$\begin{aligned}\mathfrak{gl}(n; \mathbb{C}) &= M_n(\mathbb{C}) \\ \mathfrak{gl}(n; \mathbb{R}) &= M_n(\mathbb{R}) \\ \mathfrak{sl}(n; \mathbb{C}) &= \{X \in M_n(\mathbb{C}) \mid \mathrm{trace}(X) = 0\} \\ \mathfrak{sl}(n; \mathbb{R}) &= \{X \in M_n(\mathbb{R}) \mid \mathrm{trace}(X) = 0\}.\end{aligned}$$

Proof. Let us consider, for example, the case of $\mathfrak{sl}(n; \mathbb{C})$. By Property 4 of Theorem 16.15, if $\mathrm{trace}(X) = 0$, then

$$\det(e^{tX}) = e^{t\mathrm{trace}(X)} = e^0 = 1,$$

so that $e^{tX} \in \mathrm{SL}(n; \mathbb{C})$. In the other direction, if $X \in \mathfrak{sl}(n; \mathbb{C})$, then by the above calculation, we must have $e^{t\mathrm{trace}(X)} = 0$ for all $t \in \mathbb{R}$, which is possible only if $\mathrm{trace}(X) = 0$. The proofs of the other cases are similar and are omitted. ■

Example 16.22 The Lie algebras $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$ of $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are given by

$$\begin{aligned}\mathfrak{u}(n) &= \{X \in M_n(\mathbb{C}) \mid X^* = -X\} \\ \mathfrak{su}(n) &= \{X \in \mathfrak{u}(n) \mid \mathrm{trace}(X) = 0\}.\end{aligned}$$

The Lie algebra $\mathfrak{so}(n)$ of $\mathrm{SO}(n)$ is given by

$$\mathfrak{so}(n) = \{X \in M_n(\mathbb{R}) \mid X^{tr} = -X\}.$$

Finally, the Lie algebra of $\mathrm{O}(n)$ is equal to $\mathfrak{so}(n)$.

Proof. If $X^* = -X$, then by Property 2 of Theorem 16.15,

$$(e^{tX})^* = e^{tX^*} = e^{-tX} = (e^{tX})^{-1},$$

showing that e^{tX} is unitary. In the other direction, if e^{tX} is unitary for all $t \in \mathbb{R}$, then $(e^{tX})^* = (e^{tX})^{-1} = e^{-tX}$. Thus, $e^{tX^*} = e^{-tX}$. Differentiating this relation at $t = 0$, using (16.1), gives $X^* = -X$. Thus, the Lie algebra of

$U(n)$ consists exactly of the matrices with the property that $X^* = -X$. For the Lie algebra of $SU(n)$, we add the trace-zero condition, as in the proof of Example 16.21. The calculations for $SO(n)$ are similar and are omitted. Note that if $X \in M_n(\mathbb{R})$ satisfies $X^{tr} = -X$, then the diagonal entries of X are zero and, thus, $\text{trace}(X)$ is automatically 0. This observation explains why the Lie algebras of $O(n)$ and $SO(n)$ are the same. ■

Specializing Proposition 16.22 the case $n = 3$ gives

$$\mathfrak{so}(3) = \left\{ \left(\begin{array}{ccc} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{array} \right) \middle| a, b, c \in \mathbb{R} \right\}.$$

We can use the following basis for $\mathfrak{so}(3)$:

$$F_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; F_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}; F_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (16.2)$$

Direct calculation establishes the following commutation relations for the F_j 's:

$$\begin{aligned} [F_1, F_2] &= F_3 \\ [F_2, F_3] &= F_1 \\ [F_3, F_1] &= F_2. \end{aligned} \quad (16.3)$$

More concisely, we have $[F_1, F_2] = F_3$, together with relations obtained from this one by cyclic permutation of the indices. Note that all remaining commutation relations follow from (16.3) by means of the skew-symmetry of the bracket; we have, for example, $[F_2, F_1] = -F_3$ and $[F_1, F_1] = 0$.

16.6 Relationships Between Lie Groups and Lie Algebras

In this section, we explore the relationships between matrix Lie groups and their Lie algebras. In particular, we investigate the question of the extent to which a matrix Lie group is determined (up to isomorphism) by its Lie algebra. We begin by showing that every Lie group homomorphism gives rise to a Lie algebra homomorphism in a natural way.

Theorem 16.23 *Suppose G_1 and G_2 are matrix Lie groups with Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 , respectively, and suppose $\Phi : G_1 \rightarrow G_2$ is a Lie group homomorphism. Then there exists a unique linear map $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ such that*

$$\Phi(e^{tX}) = e^{t\phi(X)}$$

for all $t \in \mathbb{R}$ and $X \in \mathfrak{g}$. This linear map has the following additional properties:

1. $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$
2. $\phi(AXA^{-1}) = \Phi(A)\phi(X)\Phi(A)^{-1}$ for all $A \in G$ and $X \in \mathfrak{g}$
3. $\phi(X)$ may be computed as

$$\phi(X) = \left. \frac{d}{dt} \Phi(e^{tX}) \right|_{t=0}.$$

Point 1 shows that ϕ is a Lie algebra homomorphism. Part of the assertion of Point 3 of the theorem is that $\Phi(e^{tX})$ is a smooth function of t for each X .

To construct ϕ , note that since Φ is a continuous homomorphism, the map $t \mapsto \Phi(e^{tX})$ is a one-parameter subgroup. By Theorem 16.18, there exists a unique Y such that $\Phi(e^{tX}) = e^{tY}$ for all $t \in \mathbb{R}$. We then set $\phi(X) = Y$. An argument similar to the proof of Proposition 16.20 then establishes the desired properties of ϕ . See the proof of Theorem 2.21 in [21] for the details.

Corollary 16.24 *Suppose that G_1 and G_2 are matrix Lie groups with Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 , respectively. If G_1 is isomorphic to G_2 , then \mathfrak{g}_1 is isomorphic to \mathfrak{g}_2 .*

Proof. See Exercise 11. ■

Our next task is to show that for any matrix Lie group G , the Lie algebra \mathfrak{g} of G is large enough to capture what is happening in a neighborhood of the identity in G . This will show, for example, that for connected matrix Lie groups, a Lie group homomorphism is determined by the corresponding Lie algebra homomorphism.

Theorem 16.25 *Let G be a matrix Lie group with Lie algebra \mathfrak{g} . Then there exists a neighborhood U of 0 in $M_n(\mathbb{C})$ and a neighborhood V of I in $M_n(\mathbb{C})$ such that the matrix exponential maps U diffeomorphically onto V and such that for all $X \in U$, we have that X belongs to \mathfrak{g} if and only if e^X belongs to G .*

See Theorem 2.27 in [21]. This result has a number of important consequences.

Corollary 16.26 *Every matrix Lie group $G \subset \mathrm{GL}(n; \mathbb{C})$ is a real embedded submanifold of $M_n(\mathbb{C})$ with the dimension of G equal to the dimension of \mathfrak{g} as a real vector space.*

The claim means, more precisely, that for each $A \in G$, there exists a neighborhood U of A and a diffeomorphism Φ of U with a neighborhood V of 0 in \mathbb{R}^{2n^2} such that $\Phi(U \cap G) = V \cap \mathbb{R}^d$, where $d = \dim \mathfrak{g}$. That is to

say, after a change of coordinates, G “looks” locally like a little piece of \mathbb{R}^d sitting inside $M_n(\mathbb{C}) \cong \mathbb{R}^{2n^2}$.

Proof. We use exponential coordinates in the neighborhood V of I in $M_n(\mathbb{C})$, meaning that we write each element A of V as $A = e^X$, with $X \in U$. Theorem 16.25 says that near the identity, in these coordinates, G “looks like” the real vector space \mathfrak{g} inside $M_n(\mathbb{C})$. Given any other point $A \in G$, we can use left multiplication by A^{-1} to move the action to the identity (Exercise 17), with the result that G looks like $\mathfrak{g} \subset M_n(\mathbb{C})$ near A . Thus, G is a real embedded submanifold of dimension $d = \dim \mathfrak{g}$. ■

Corollary 16.27 *The Lie algebra \mathfrak{g} of a matrix Lie group G is the tangent space to G at I . That is to say, \mathfrak{g} coincides with the set of those X in $M_n(\mathbb{C})$ for which there exists a smooth curve $\gamma : \mathbb{R} \rightarrow M_n(\mathbb{C})$ lying entirely in G and such that $\gamma(0) = I$ and $\gamma'(0) = X$.*

Proof. If $X \in \mathfrak{g}$, then X is the derivative of e^{tX} at $t = 0$, so \mathfrak{g} is contained in the tangent space at I . In the other direction, if γ is any smooth curve in $M_n(\mathbb{C})$ that lies entirely in G and passes through I at $t = 0$, then by Theorem 16.25, we can express γ as $\gamma(t) = e^{\delta(t)}$ (at least for small t), where δ is a smooth curve in \mathfrak{g} with $\delta(0) = 0$. It is then easy to see (Exercise 8) that $\gamma'(0) = \delta'(0)$. But if δ lies in \mathfrak{g} , then $\delta'(0)$, which equals $\gamma'(0)$, also lies in \mathfrak{g} , as in the proof of Proposition 16.20. Thus, the tangent space at I is contained in \mathfrak{g} . ■

Corollary 16.28 *If a matrix Lie group G is connected, then for all $A \in G$ there exists a finite sequence X_1, X_2, \dots, X_N of elements of \mathfrak{g} such that*

$$A = e^{X_1} e^{X_2} \dots e^{X_N}.$$

Proof. If G is connected in the sense of Definition 16.6 (which really means that G is path connected), then G is certainly connected in the usual topological sense of having no nontrivial sets that are both open and closed. Let U denote the set of points in G that can be expressed as a product of exponentials of elements of \mathfrak{g} . This set is open in G because if $A \in U$ and $B \in G$ is close to A , then $A^{-1}B$ is close to I in G , and therefore $A^{-1}B = e^X$ for some $X \in \mathfrak{g}$. Thus, $B = Ae^X$, which means that B is also a product of exponentials. In the other direction, if $B \in G$ is in the closure of U , then there is some element A of U that is close to B . We then have, again, that $B = Ae^X$ for some $X \in \mathfrak{g}$, which, again, means that $B \in U$. Now, G is connected and U is both open and closed. Since U is nonempty ($I \in U$), we have $U = G$. ■

Corollary 16.29 *Suppose that G_1 and G_2 are matrix Lie groups with Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 , respectively. Suppose that $\Phi_1 : G_1 \rightarrow G_2$ and $\Phi_2 : G_1 \rightarrow G_2$ are Lie group homomorphisms, with associated Lie algebra homomorphisms ϕ_1 and ϕ_2 , respectively. If G_1 is connected and $\phi_1 = \phi_2$, then $\Phi_1 = \Phi_2$.*

Proof. The result follows from Corollary 16.28 and the condition $\Phi_j(e^X) = e^{\phi_j(X)}$, $j = 1, 2$. ■

We have seen that a homomorphism of matrix Lie groups gives rise to a homomorphism of the associated Lie algebra, and (Corollary 16.29) that if the domain group is connected, the Lie algebra homomorphism determines the Lie group homomorphism. A more difficult question is whether we can go in the opposite direction, from a Lie algebra homomorphism to a Lie group homomorphism. That is to say, given a Lie algebra homomorphism between the Lie algebras of two matrix Lie groups, does there exist a Lie group homomorphism related in the usual way to the Lie algebra homomorphism? The answer turns out to be yes, *provided* that the domain group G_1 is connected and *simply connected* (i.e., that every continuous loop in G_1 can be shrunk continuously in G_1 to a point).

Theorem 16.30 *Suppose that G_1 and G_2 are matrix Lie groups with Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 , respectively, and suppose that $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a Lie algebra homomorphism. If G_1 is connected and simply connected, then there exists a unique Lie group homomorphism $\Phi : G_1 \rightarrow G_2$ such that Φ and ϕ are related as in Theorem 16.23.*

One way to prove this deep result is to make use of the *Baker–Campbell–Hausdorff formula*. (See, e.g., Chap. 3 of [21].) This formula states that for all sufficiently small X and Y in $M_n(\mathbb{C})$ we have

$$e^X e^Y = e^{X+Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \dots}.$$

Here \dots denotes terms that are expressible in terms of repeated commutators involving X and Y , with coefficients that are “universal,” that is, independent of n (the size of the matrices) and of the choice of X and Y in $M_n(\mathbb{C})$. Given a Lie algebra homomorphism $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$, one can use the Baker–Campbell–Hausdorff formula to construct a “local homomorphism,” mapping a neighborhood of the identity in G_1 into G_2 . If G_1 is connected and simply connected, it is possible to extend this local representation to a global representation. See Sect. 3.6 of [21] for the details of this construction.

Corollary 16.31 *Suppose that G_1 and G_2 are matrix Lie groups with Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 , respectively. If G_1 and G_2 are connected and simply connected and \mathfrak{g}_1 is isomorphic to \mathfrak{g}_2 , then G_1 is isomorphic to G_2 .*

Proof. Suppose $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a Lie algebra isomorphism. Since G_1 is connected and simply connected, there exists a Lie group homomorphism $\Phi : G_1 \rightarrow G_2$ related in the usual way to ϕ . Since G_2 is connected and simply connected, there exists a Lie group homomorphism $\Psi : G_2 \rightarrow G_1$ related in the usual way to ϕ^{-1} . Consider now the homomorphism $\Psi \circ \Phi : G_1 \rightarrow G_1$.

By the composition property of Lie algebra homomorphisms (Exercise 10), the Lie algebra homomorphism associated with $\Psi \circ \Phi$ is $\phi^{-1} \circ \phi = I$. It then follows from Corollary 16.29 that $\Psi \circ \Phi = I$. A similar argument shows that $\Phi \circ \Psi = I$, which means that Φ is a Lie group isomorphism. ■

Corollary 16.31 does not hold without the assumption that both groups are simply connected, as the following important example shows.

Example 16.32 *The Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are isomorphic, but the groups $SU(2)$ and $SO(3)$ are not isomorphic.*

Since $SU(2)$ is simply connected (Example 16.9), $SO(3)$ must fail to be simply connected. Indeed, $\pi_1(SO(3)) \cong \mathbb{Z}/2$, as can be seen from Example 16.34.

Proof. The Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ is the space of 2×2 skew-self-adjoint matrices with trace zero. Explicitly,

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

We may consider the following basis for $\mathfrak{su}(2)$:

$$E_1 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \quad E_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad E_3 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (16.4)$$

Direct calculation shows that $[E_1, E_2] = E_3$ and relations obtained from this by cyclic permutation of the indices. These are the same relations as those satisfied by the basis elements F_j , $j = 1, 2, 3$, for $\mathfrak{so}(3)$ in (16.2) and (16.3). Thus, there is a Lie algebra isomorphism $\phi : \mathfrak{su}(2) \rightarrow \mathfrak{so}(3)$ such that $\phi(E_j) = F_j$, $j = 1, 2, 3$.

On the other hand, there can be no isomorphism between $SU(2)$ and $SO(3)$, since $SU(2)$ has a nontrivial center (containing at least I and $-I$), whereas the center of $SO(3)$ is trivial (Exercise 14). ■

Definition 16.33 *Suppose G is a connected matrix Lie group with Lie algebra \mathfrak{g} . A **universal cover** of G is an ordered pair (\tilde{G}, Φ) consisting of a simply connected matrix Lie group \tilde{G} and a Lie group homomorphism $\Phi : \tilde{G} \rightarrow G$ such that the associated Lie algebra homomorphism $\phi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ is an isomorphism of the Lie algebra $\tilde{\mathfrak{g}}$ of \tilde{G} with \mathfrak{g} . The map Φ is called the **covering map** for \tilde{G} .*

Although each Lie group has a universal cover that is again a Lie group, the universal cover of a *matrix* Lie group may not be isomorphic to any matrix Lie group. [The universal cover of $SL(2; \mathbb{R})$, e.g., is not a matrix Lie group.] It can be shown, however, that if a matrix Lie group G is compact, then the universal cover of G is again a matrix Lie group (not necessarily compact).

Suppose \tilde{G} is any simply connected Lie group with a Lie algebra $\tilde{\mathfrak{g}}$ that is isomorphic to \mathfrak{g} . The choice of a particular isomorphism $\phi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ gives

rise, by Theorem 16.30, to a Lie group homomorphism $\Phi : \tilde{G} \rightarrow G$, so that (\tilde{G}, Φ) is a universal cover of G .

If (\tilde{G}, Φ) is a universal cover of G , it is often convenient to use the isomorphism ϕ to identify $\tilde{\mathfrak{g}}$ with \mathfrak{g} . If we follow this convention, we may say that a universal cover of G is a simply connected group \tilde{G} having “the same” Lie algebra as G .

If (\tilde{G}_1, Φ_1) and (\tilde{G}_2, Φ_2) are two universal covers of a given matrix Lie group G , then there is a unique Lie group isomorphism $\Psi : \tilde{G}_1 \rightarrow \tilde{G}_2$ such that $\Phi_2(\Psi(A)) = \Phi_1(A)$ for all $A \in \tilde{G}_1$. (This result follows easily from Corollary 16.31.) In light of this uniqueness result, we will often speak of “the” universal cover of G .

Example 16.34 Let $\Phi : \mathrm{SU}(2) \rightarrow \mathrm{SO}(3)$ be the unique Lie group homomorphism for which the associated Lie algebra homomorphism ϕ satisfies $\phi(E_j) = F_j$, $j = 1, 2, 3$. Then $\ker \Phi = \{I, -I\}$ and $(\mathrm{SU}(2), \Phi)$ is a universal cover of $\mathrm{SO}(3)$.

Proof. Since E_1 is diagonal, it is easy to see that $e^{2\pi E_1} = -I$ in $\mathrm{SU}(2)$. On the other hand, by a trivial extension of Example 16.16, we have

$$e^{aF_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos a & -\sin a \\ 0 & \sin a & \cos a \end{pmatrix}$$

for all $a \in \mathbb{R}$. In particular, $e^{2\pi F_1} = I$. Thus,

$$\Phi(-I) = \Phi(e^{2\pi E_1}) = e^{2\pi F_1} = I.$$

This shows that $-I$ belongs to the kernel of Φ .

Now, since ϕ is injective, Φ is injective in a neighborhood of I . After all, given distinct elements A and B of $\mathrm{SU}(2)$ near I , Theorem 16.25 tells us that we can express A as e^X and B as e^Y , with X and Y being distinct small elements of $\mathfrak{su}(2)$. Then $\phi(X)$ and $\phi(Y)$ are distinct small elements of $\mathfrak{so}(3)$. Applying Theorem 16.25 again tells us that $\Phi(A) = e^{\phi(X)}$ and $\Phi(B) = e^{\phi(Y)}$ are distinct.

We see, then, that $\ker \Phi$ is a discrete normal subgroup of $\mathrm{SU}(2)$. But a standard exercise (Exercise 1) shows that a discrete normal subgroup of a connected group is automatically central. On the other hand, it is easily verified (Exercise 2) that the center of $\mathrm{SU}(2)$ is $\{I, -I\}$, so $\ker \Phi$ cannot be larger than $\{I, -I\}$.

To show that Φ maps onto $\mathrm{SO}(3)$, we first verify (Exercise 13) that each element R of $\mathrm{SO}(3)$ can be expressed as $R = e^X$, with $X \in \mathfrak{so}(3)$. Since ϕ is surjective and $\Phi(e^X) = e^{\phi(X)}$, Φ maps onto $\mathrm{SO}(3)$. ■

16.7 Finite-Dimensional Representations of Lie Groups and Lie Algebras

A *representation* of a group G is a homomorphism Π of G into $\mathrm{GL}(V)$, the group of invertible linear transformations on some vector space. If Π is injective then G is isomorphic to its image under Π ; thus, Π serves to “represent” G concretely as a group of invertible linear transformations. (We continue to use the term “representation” even if Π is not injective.) Similarly, a representation of a Lie algebra \mathfrak{g} is a Lie algebra homomorphism of \mathfrak{g} into $\mathfrak{gl}(V)$, the space of all linear transformations of V , where we equip $\mathfrak{gl}(V)$ with the bracket $[X, Y] := XY - YX$.

Recall that an *action* of a group G on a set X is a map from $G \times X$ to X , denoted $(g, x) \mapsto g \cdot x$ satisfying $e \cdot x = x$ for all $x \in X$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$. A representation Π of G on some vector space V gives rise to a *linear action* of G on V , given by $g \cdot v = \Pi(g)v$. (A linear action is an action for which the map $v \mapsto g \cdot v$ is linear for each g .) Thus, we may use $g \cdot v$ as an alternative notation to $\Pi(g)v$, when convenient.

16.7.1 Finite-Dimensional Representations

If G is a matrix Lie group, then G is already represented as a group of matrices. Nevertheless, it is of interest [as we will see in Chap. 17 in the case $G = \mathrm{SO}(3)$] to explore other representations of G . Since a matrix Lie group has a topological structure (inherited from $M_n(\mathbb{C})$), it is natural to require representations to be continuous. It is also simpler to deal at first with *finite-dimensional* representations, that is, those where the vector space in question is finite dimensional, although eventually we will need to consider infinite-dimensional representations as well. This discussion leads to the following definition.

Definition 16.35 *Let $G \subset \mathrm{GL}(n; \mathbb{C})$ be a matrix Lie group. A finite-dimensional **representation** of G is a continuous homomorphism of G into $\mathrm{GL}(V)$, the group of invertible linear transformations of a finite-dimensional vector space V .*

We will assume that all of our vector spaces are over the field \mathbb{C} , even though it is occasionally of interest to consider also representations over \mathbb{R} . The topology on $\mathrm{GL}(V)$ is defined by picking a basis, and thereby identifying the space of linear maps of V to V with $M_n(\mathbb{C})$. We then use the subset topology on $\mathrm{GL}(V) \cong \mathrm{GL}(n; \mathbb{C}) \subset M_n(\mathbb{C})$. This topology is easily seen to be independent of the choice of basis.

An important example of representations in quantum theory arises from the time-independent Schrödinger equation in \mathbb{R}^n , namely the equation $\hat{H}\psi = E\psi$, for a fixed constant $E \in \mathbb{R}$. If \hat{H} is invariant under rotations, then the *space* of solutions to this equation is invariant under rotations.

Note that an individual solution ψ to this equation may or may not be a rotationally invariant (i.e., radial) function. But if \hat{H} is rotationally invariant, then rotating a solution to $\hat{H}\psi = E\psi$ will give another solution of this equation. Even if the quantum Hilbert space is infinite dimensional, the solution spaces to $\hat{H}\psi = E\psi$ are typically finite dimensional and constitute finite dimensional representations of the group $\text{SO}(n)$ of rotations. If we can understand what all possible finite-dimensional representations of $\text{SO}(n)$ look like, we will have made a lot of progress in understanding solutions to $\hat{H}\psi = E\psi$ in the rotationally invariant case. This line of reasoning will be explored in detail in Chap. 18.

We may consider as well finite-dimensional representations of Lie algebras. Assuming our Lie algebra \mathfrak{g} is finite dimensional (which is the only case we will consider in this chapter), there is no need to impose a requirement of continuity, since a linear map of one finite-dimensional real or complex vector space to another is automatically continuous.

Definition 16.36 *A finite-dimensional representation of a Lie algebra \mathfrak{g} is a Lie algebra homomorphism of \mathfrak{g} into $\mathfrak{gl}(V)$, the space of all linear transformations of V . Here $\mathfrak{gl}(V)$ is considered as a Lie algebra with bracket given by $[X, Y] = XY - YX$.*

We typically consider Lie algebras defined over the field \mathbb{R} , since the Lie algebra of a matrix Lie group is in general only a real subspace of $M_n(\mathbb{C})$. Nevertheless, it is convenient to consider vector spaces over \mathbb{C} . If \mathfrak{g} is a real Lie algebra and V , and therefore also $\mathfrak{gl}(V)$, is a complex vector space, then we require only that $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be *real* linear, which is the only requirement that makes sense.

In the interest of simplifying the terminology, we will sometimes speak of “a representation V ,” without making explicit mention of the homomorphism Π or π .

Definition 16.37 *If $\Pi : G \rightarrow \text{GL}(V)$ is a representation of a matrix Lie group G , then a subspace W of V is called an **invariant subspace** if $\Pi(g)w \in W$ for all $g \in G$ and $w \in W$. Similarly, if $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a representation of a Lie algebra \mathfrak{g} , then a subspace W of V is called an invariant subspace if $\pi(X)w \in W$ for all $X \in \mathfrak{g}$ and $w \in W$. A representation of a group or Lie algebra is called **irreducible** if the only invariant subspaces are $W = V$ and $W = \{0\}$.*

Definition 16.38 *If (Π, V_1) and (Σ, V_2) are representations of a matrix Lie group G , a map $\Phi : V_1 \rightarrow V_2$ is called an **intertwining map** (or **morphism**) if $\Phi(\Pi(g)v) = \Sigma(g)\Phi(v)$ for all $v \in V_1$, with an analogous definition for intertwining maps of Lie algebra representations. If an intertwining map is an invertible linear map, it is called an **isomorphism**. Two representations are said to be **isomorphic** (or **equivalent**) if there exists an isomorphism between them.*

In the “action” notation, the requirement on an intertwining map Φ is that $\Phi(g \cdot v) = g \cdot \Phi(v)$, meaning that Φ commutes with the action of G . A typical goal of representation theory is to classify all finite-dimensional irreducible representations of G up to isomorphism.

Given a representation $\Pi : G \rightarrow \mathrm{GL}(V)$ of a matrix Lie group G , we can identify $\mathrm{GL}(V)$ with $\mathrm{GL}(N; \mathbb{C})$ and $\mathfrak{gl}(V)$ with $\mathfrak{gl}(n; \mathbb{C})$ by picking a basis for V . We may then apply Theorem 16.23 to obtain a representation $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ such that

$$\Pi(e^X) = e^{\pi(X)}$$

for all $X \in \mathfrak{g}$.

Proposition 16.39 *Suppose G is a connected matrix Lie group with Lie algebra \mathfrak{g} . Suppose that $\Pi : G \rightarrow \mathrm{GL}(V)$ is a finite-dimensional representation of G and $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is the associated Lie algebra representation. Then a subspace W of V is invariant under the action of G if and only if it is invariant under the action of \mathfrak{g} . In particular, Π is irreducible if and only if π is irreducible. Furthermore, two representations of G are isomorphic if and only if the associated Lie algebra representations are isomorphic.*

In general, given an representation π of \mathfrak{g} , there may be no representation Π such that π and Π are related in the usual way. If, however, G is simply connected, Theorem 16.30 tells us that there is, in fact, a Π associated with every π .

Proof. Suppose $W \subset V$ is invariant under $\pi(X)$ for all $X \in \mathfrak{g}$. Then W is invariant under $\pi(X)^m$ for all m . Since V is finite dimensional, any subspace of it is automatically a closed subset and thus W is invariant under

$$\Pi(e^X) = e^{\pi(X)} = \sum_{m=0}^{\infty} \frac{\pi(X)^m}{m!}.$$

Since G is connected, every element of G is (Corollary 16.28) a product of exponentials of elements of \mathfrak{g} , and so W is invariant under $\Pi(A)$ for all $A \in G$.

In the other direction, if W is invariant under $\Pi(A)$ for all $A \in G$, then since W is closed, it is invariant under

$$\pi(X) = \lim_{h \rightarrow 0} \frac{e^{hX} - I}{h},$$

for all $X \in \mathfrak{g}$.

Now suppose Π_1 and Π_2 are two representations of G , acting on vector spaces V_1 and V_2 , respectively. If $\Phi : V_1 \rightarrow V_2$ is an invertible linear map, then an argument similar to the above shows $\Phi \Pi_1(A) = \Pi_2(A) \Phi$ for all $A \in G$ if and only if $\Phi \pi_1(X) = \pi_2(X) \Phi$ for all $X \in \mathfrak{g}$. Thus, Φ is an isomorphism of group representations if and only if it is an isomorphism of Lie algebra representations. ■

Theorem 16.40 (Schur's Lemma) *If V_1 and V_2 are two irreducible representations of a group or Lie algebra, then the following hold.*

1. *If $\Phi : V_1 \rightarrow V_2$ is an intertwining map, then either $\Phi = 0$ or Φ is an isomorphism.*
2. *If $\Phi : V_1 \rightarrow V_2$ and $\Psi : V_1 \rightarrow V_2$ are nonzero intertwining maps, then there exists a nonzero constant $c \in \mathbb{C}$ such that $\Phi = c\Psi$. In particular, if Φ is an intertwining map of V_1 to itself then $\Phi = cI$.*

Although the first part of Schur's lemma holds for representations over an arbitrary field, the second part holds only for representations over algebraically closed fields.

Proof. It is easy to see that $\ker \Phi$ is an invariant subspace of V_1 . Since V_1 is irreducible, this means that either $\ker \Phi = V_1$, in which case $\Phi = 0$, or $\ker \Phi = \{0\}$, in which case Φ is injective. Similarly, the range of Φ is invariant, and thus equal to either $\{0\}$ or V_2 . If Φ is not zero, then the range of Φ is not zero, hence all of V_2 . Thus, if Φ is not zero, it is both injective and surjective, establishing Point 1.

For Point 2, since Φ and Ψ are nonzero, they are isomorphisms, by Point 1. It suffices to prove that $\Gamma := \Phi^{-1}\Psi$ is a multiple of the identity, where Γ is an intertwining map of V_1 to itself. Since we are working over \mathbb{C} , Γ must have at least one eigenvalue λ . If W denotes the λ -eigenspace of Γ , then W is invariant under the action of the group or Lie algebra. After all, if $\Gamma w = \lambda w$, then (in the notation of the group case) $\Gamma(\Pi(A)w) = \Pi(A)\Gamma w = \lambda \Pi(A)w$. Since λ is an eigenvalue of Γ , the invariant subspace W is nonzero and thus $W = V_1$, which means precisely that $\Gamma = \lambda I$. ■

16.7.2 Unitary Representations

In quantum mechanics, we are interested not only in vector spaces, but, more specifically, in Hilbert spaces, since expectation values are defined in terms of an inner product. We wish to consider, then, actions of a group that preserve the inner product as well as the linear structure. Although the Hilbert spaces in quantum mechanics are generally infinite dimensional, we restrict our attention in this section to the finite-dimensional case.

Definition 16.41 *Suppose V is a finite-dimensional Hilbert space over \mathbb{C} . Denote by $\mathbf{U}(V)$ the group of invertible linear transformations of V that preserve the inner product. A (finite-dimensional) **unitary representation** of a matrix Lie group G is a continuous homomorphism of $\Pi : G \rightarrow \mathbf{U}(V)$, for some finite-dimensional Hilbert space V .*

Proposition 16.42 *Let $\Pi : G \rightarrow \mathbf{GL}(V)$ be a finite-dimensional representation of a connected matrix Lie group G , and let π be the associated representation of the Lie algebra \mathfrak{g} of G . Let $\langle \cdot, \cdot \rangle$ be an inner product on V .*

Then Π is unitary with respect to $\langle \cdot, \cdot \rangle$ if and only if $\pi(X)$ is skew-self-adjoint with respect to $\langle \cdot, \cdot \rangle$ for all $X \in \mathfrak{g}$, that is, if and only if

$$\pi(X)^* = -\pi(X)$$

for all $X \in \mathfrak{g}$.

In a slight abuse of notation, we will refer to a representation π of a Lie algebra \mathfrak{g} on a finite-dimensional inner product space as *unitary* if $\pi(X)^* = -\pi(X)$ for all $X \in \mathfrak{g}$.

Proof. Suppose first that $\Pi(A)$ is unitary for all $A \in G$. Then for all $X \in \mathfrak{g}$ and $t \in \mathbb{R}$ we have

$$\Pi(e^{tX})^* = \Pi(e^{tX})^{-1} = \Pi(e^{-tX}) = e^{-t\pi(X)}.$$

On the other hand,

$$\Pi(e^{tX})^* = (e^{t\pi(X)})^* = e^{t\pi(X)^*}.$$

Thus,

$$e^{t\pi(X)^*} = e^{-t\pi(X)}$$

for all t . Differentiating at $t = 0$ yields $\pi(X)^* = -\pi(X)$.

In the other direction, if $\pi(X)^* = -\pi(X)$ for all $X \in \mathfrak{g}$, then

$$\Pi(e^X)^* = e^{\pi(X)^*} = e^{-\pi(X)} = \Pi(e^{-X}) = \Pi(e^X)^{-1},$$

meaning that $\Pi(e^X)$ is unitary. Since G is connected, Corollary 16.28 tells us that each element A of G is expressible as a product of exponentials, from which it follows that $\Pi(A)$ is unitary. ■

16.7.3 Projective Unitary Representations

In quantum mechanics, two unit vectors in the quantum Hilbert space that differ by multiplication by a constant are considered to represent the same physical state. Thus, an operator of the form $e^{i\theta}I$, with $\theta \in \mathbb{R}$, will act as the identity at the level of the physical states. Suppose that V is a Hilbert space over \mathbb{C} , assumed for the moment to be finite dimensional. Then it is natural to consider homomorphisms not into $U(V)$ but rather into the quotient group $U(V)/\{e^{i\theta}I\}$. Of course, given a homomorphism Π of G into $U(V)$, we can always turn Π into a homomorphism of G into the quotient group, just by composing Π with the quotient map. Not every homomorphism into the quotient group, however, arises from a homomorphism into $U(V)$.

Definition 16.43 Suppose V is a finite-dimensional Hilbert space over \mathbb{C} . Then the **projective unitary group** over V , denoted $PU(V)$, is the quotient group

$$PU(V) = U(V)/\{e^{i\theta}I\},$$

where $\{e^{i\theta}I\}$ denotes the group of matrices of the form $e^{i\theta}I$, $\theta \in \mathbb{R}$.

Note that $\{e^{i\theta}I\}$ is a closed normal subgroup of $U(V)$. Now, $U(V)$ is (isomorphic to) a matrix Lie group, since we can identify it with $U(n)$ by picking an orthonormal basis for V . In general, the quotient of a matrix Lie group by a closed normal subgroup may not be a matrix Lie group. In this case, however, it is not hard to realize the quotient $U(n)/\{e^{i\theta}I\}$ as a matrix Lie group.

Proposition 16.44 *If V is a finite-dimensional Hilbert space over \mathbb{C} , then $PU(V)$ is isomorphic to a matrix Lie group.*

Let $Q : U(V) \rightarrow PU(V)$ be the quotient homomorphism and let $q : \mathfrak{u}(V) \rightarrow \mathfrak{pu}(V)$ be the associated Lie algebra homomorphism. Then q maps $\mathfrak{u}(V)$ onto $\mathfrak{pu}(V)$ and the kernel of q is the space of matrices of the form iaI with $a \in \mathbb{R}$. Thus, $\mathfrak{pu}(V)$ is isomorphic to $\mathfrak{u}(V)/\{iaI\}$.

The Lie algebra $\mathfrak{u}(V)$ of $U(V)$ is the space of skew-self-adjoint operators on V . In Proposition 16.44, the space $\{iaI\}$ is an ideal in $\mathfrak{u}(V)$ and the quotient is in the sense of Lie algebras over \mathbb{R} ; see Exercise 9. If $\dim V = N$, then it is not hard to see that the Lie algebra $\mathfrak{pu}(V) \cong \mathfrak{u}(V)/\{iaI\}$ is isomorphic to the Lie algebra $\mathfrak{su}(N)$. The group $PU(V)$ is not, however, isomorphic to the group $SU(N)$. See Exercise 16.

Proof. If $\dim V = N$, then $\mathfrak{gl}(V)$, the space of all linear maps of V to V , has dimension N^2 . Given $U \in U(V)$, we can define

$$C_U : \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(V)$$

by

$$C_U(X) = UXU^{-1}.$$

(That is to say, C_U is conjugation by U .) Note that $(C_U)^{-1} = C_{U^{-1}}$ and $C_{UV} = C_U C_V$. Thus, C (i.e., the map $U \mapsto C_U$) is a homomorphism of $U(V)$ into $GL(\mathfrak{gl}(V))$, and this homomorphism is clearly continuous. If U is a multiple of the identity, then C_U is the identity operator on $\mathfrak{gl}(V)$. Conversely, if C_U is the identity, then $UX = XU$ for all $X \in \mathfrak{gl}(V)$, which implies (Exercise 18) that U is a multiple of the identity. Thus, the kernel of C consists precisely of those scalar multiples of the identity that are in $U(V)$; that is, $\ker C = \{e^{i\theta}I\}$.

We have constructed, then, a homomorphism of $U(V)$ into $GL(\mathfrak{gl}(V)) \cong GL(N^2; \mathbb{C})$ with a kernel that is precisely $\{e^{i\theta}I\}$. The image of $U(V)$ under this homomorphism is, therefore, isomorphic to the quotient group $U(V)/\{e^{i\theta}I\}$. Furthermore, since $U(V)$ is compact, the image of $U(V)$ under C is compact and thus closed. This image is, then, a matrix Lie group isomorphic to $PU(V)$.

Let c be the associated Lie algebra homomorphism associated with the homomorphism C . Using Point 3 of Theorem 16.23, we may calculate that

$$\begin{aligned} c_X(Y) &= \left. \frac{d}{dt} e^{tX} Y e^{-tX} \right|_{t=0} \\ &= XY - YX \\ &= [X, Y]. \end{aligned}$$

Using Exercise 18 again, we see that $c_X = 0$ if and only if X is a multiple of the identity. Thus, the kernel of c consists of all the scalar multiples of I in $\mathfrak{u}(V)$, namely $\{iaI\}$.

Now, the image of $\mathfrak{U}(V)$ under C is (isomorphic to) $\text{PU}(V)$; in particular, C maps $\mathfrak{U}(V)$ onto $\text{PU}(V)$. It follows that c must map $\mathfrak{u}(V)$ onto $\mathfrak{pu}(V)$. (This claim follows from Theorem 3.15 in [21].) Thus, $\mathfrak{pu}(V) \cong \mathfrak{u}(V)/\{iaI\}$.

■

Definition 16.45 A finite-dimensional **projective unitary representation** of a matrix Lie group G is a continuous homomorphism Π of G into $\text{PU}(V)$, where V is a finite-dimensional Hilbert space over \mathbb{C} . A subspace W of V is said to be **invariant** under Π if for each $A \in G$, W is invariant under U for every $U \in \mathfrak{U}(V)$ such that $[U] = \Pi(A)$. A projective unitary representation (Π, V) is **irreducible** if the only invariant subspaces are $\{0\}$ and V .

Given an ordinary unitary representation, $\Sigma : G \rightarrow \mathfrak{U}(V)$, we can always form a projective representation, $\Pi : G \rightarrow \text{PU}(V)$, simply by setting $\Pi = Q \circ \Sigma$. Not every projective representation, however, arises in this fashion. Thus, considering projective representations gives us more flexibility than considering ordinary unitary representations.

Proposition 16.46 Let $\Pi : G \rightarrow \text{PU}(V)$ be a finite-dimensional projective unitary representation of a matrix Lie group G , and let $\pi : \mathfrak{g} \rightarrow \mathfrak{pu}(V)$ be the associated Lie algebra homomorphism. Then there exists a Lie algebra homomorphism $\sigma : \mathfrak{g} \rightarrow \mathfrak{u}(V)$ such that $\pi(X) = q(\sigma(X))$ for all $X \in \mathfrak{g}$. It is possible to choose σ so that $\text{trace}(\sigma(X)) = 0$ for all $X \in \mathfrak{g}$, and σ is unique if we require this condition.

That is to say, every finite-dimensional projective representation can be “de-projectivized” at the Lie algebra level. In general, σ is not unique, because there may be σ 's for which $\text{trace}(\sigma(X))$ is nonzero for some X . On the other hand, if \mathfrak{g} has the property that every $X \in \mathfrak{g}$ is a linear combination of commutators—which is true if $\mathfrak{g} = \mathfrak{so}(3)$ —then σ is unique. See Exercise 15.

Proof. Recall that $\mathfrak{pu}(V) \cong \mathfrak{u}(V)/\{iaI\}$. That is, for each $X \in \mathfrak{g}$, $\pi(X)$ denotes a whole family of operator that differ by adding iaI . If $Y \in \mathfrak{u}(n)$ is any representative of $\pi(X)$, then since $Y^* = -Y$, the trace of Y will be pure imaginary. Thus, there is a unique pure-imaginary constant $c =$

$-\text{trace}(Y)/\dim V$ such that the trace of $Y + cI$ is zero. Let us then set $\sigma(X) = Y + cI$. Since π is a Lie algebra homomorphism, $\sigma([X, Y])$ will equal $[\sigma(X), \sigma(Y)] + iaI$, for some $a \in \mathbb{R}$. Since $\text{trace}(\sigma([X, Y])) = 0$ by construction and since the commutator of any two matrices has trace zero, we see that actually $a = 0$. Thus, a σ as in the proposition exists, and it is unique if we require that $\sigma(X)$ have trace zero. ■

Theorem 16.47 *Suppose G is a matrix Lie group and \tilde{G} is a universal cover of G , with covering map Φ . Then the following hold.*

1. *Let $\Pi : G \rightarrow \text{PU}(V)$ be a finite-dimensional projective unitary representation of G . Then there is an ordinary unitary representation $\Sigma : \tilde{G} \rightarrow \text{U}(V)$ of \tilde{G} such that $\Pi \circ \Phi = Q \circ \Sigma$. Any such Σ is irreducible if and only if Π is irreducible. It is possible to choose Σ so that $\det(\Sigma(A)) = 1$ for all $A \in \tilde{G}$, and Σ is unique if we require this condition.*
2. *Let Σ be a finite-dimensional irreducible unitary representation of \tilde{G} . Then the kernel of the associated projective unitary representation $Q \circ \Sigma$ contains the kernel of the covering map Φ . Thus, $Q \circ \Sigma$ factors through G and gives rise to a projective unitary representation of G .*

In the finite-dimensional case, then, there is a one-to-one correspondence between irreducible projective unitary representations of G and irreducible, determinant-one ordinary unitary representations of \tilde{G} . Point 1 of the theorem means that any finite-dimensional projective unitary representation of the group G can be “de-projectivized” at the expense of passing to the universal cover \tilde{G} of G .

Note that Theorem 16.47 applies only to *finite-dimensional* projective unitary representations. Example 16.56 will provide an infinite-dimensional example in which Point 1 of the theorem fails.

Proof. If \mathfrak{g} is the Lie algebra of G , Proposition 16.46 tells us that we can find an ordinary representation $\sigma : \mathfrak{g} \rightarrow \mathfrak{u}(V)$ such that $q \circ \sigma = \pi$. We then define a representation $\tilde{\sigma} : \tilde{\mathfrak{g}} \rightarrow \mathfrak{u}(V)$ of the Lie algebra $\tilde{\mathfrak{g}}$ of \tilde{G} by setting $\tilde{\sigma}(X) = \sigma(\phi(X))$, $X \in \tilde{\mathfrak{g}}$. Since \tilde{G} is simply connected, we can then find a unique representation $\Sigma : \tilde{G} \rightarrow \text{U}(V)$ such that $\Sigma(e^X) = e^{\tilde{\sigma}(X)}$ for all $X \in \tilde{\mathfrak{g}}$. Since

$$q \circ \tilde{\sigma} = q \circ \sigma \circ \phi = \pi \circ \phi,$$

it follows that $Q \circ \Sigma = \Pi \circ \Phi$. Furthermore, if Σ maps into $\text{SU}(V)$, $\sigma = \tilde{\sigma} \circ \phi^{-1}$ maps into $\mathfrak{su}(n)$. This condition uniquely determines σ and thus also $\tilde{\sigma}$ and Σ , establishing Point 1 of the theorem.

For Point 2, observe that $\ker \Phi$ is a discrete normal subgroup of \tilde{G} , which is therefore central (Exercises 1 and 12). Thus, for all $A \in \ker \Phi$, we have

$$\Sigma(A)\Sigma(B) = \Sigma(AB) = \Sigma(BA) = \Sigma(B)\Sigma(A)$$

for all $B \in \tilde{G}$. That is to say, $\Sigma(A)$ is an intertwining map of V to itself. Since V is also irreducible as a representation of \tilde{G} , Schur's lemma tells us that $\Sigma(A) = cI$, where $|c| = 1$ because $\Sigma(A) \in U(V)$. Thus, A is in the kernel of the associated projective representation $Q \circ \Sigma$. ■

16.8 New Representations from Old

In this section, we consider three basic mechanisms for combining representations to produce new representations: direct sums, tensor products, and duals. This section assumes familiarity with these notions at the level of vector spaces; a brief review is provided in Appendix A.1.

Definition 16.48 Suppose (Π_1, V_1) and (Π_2, V_2) are representations of a matrix Lie group G . The **direct sum** of these two representations is the representation $\Pi_1 \oplus \Pi_2 : G \rightarrow \text{GL}(V_1 \oplus V_2)$ given by

$$(\Pi_1 \oplus \Pi_2)(A) = \Pi_1(A) \oplus \Pi_2(A).$$

The **tensor product** of Π_1 and Π_2 is the representation $\Pi_1 \otimes \Pi_2 : G \rightarrow \text{GL}(V_1 \otimes V_2)$ given by

$$(\Pi_1 \otimes \Pi_2)(A) = \Pi_1(A) \otimes \Pi_2(A).$$

Finally, the **dual** of Π_1 is the representation $\Pi_1^{tr} : G \rightarrow \text{GL}(V^*)$ given by

$$\Pi_1^{tr}(A) = \Pi_1(A^{-1})^{tr} = (\Pi_1(A)^{tr})^{-1}.$$

Similarly, the direct sum, tensor product, and dual of Lie algebra representations can be defined by

$$\begin{aligned} (\pi_1 \oplus \pi_2)(X) &= \pi_1(X) \oplus \pi_2(X) \\ (\pi_1 \otimes \pi_2)(X) &= \pi_1(X) \otimes I + I \otimes \pi_2(X) \\ \pi_1^{tr}(X) &= -\pi_1(X)^{tr}. \end{aligned}$$

It is important to note the differences in formulas between the group and the Lie algebra in the case of tensor products and dual representations. It is easy to motivate the definitions for the Lie algebra: If G acts on $V_1 \otimes V_2$ by $\Pi_1(A) \otimes \Pi_2(A)$, then the associated Lie algebra action will be given by

$$\left. \frac{d}{dt} \Pi_1(e^{tX}) \otimes \Pi_2(e^{tX}) \right|_{t=0} = \pi_1(X) \otimes I + I \otimes \pi_2(X).$$

Of course, we continue to use this last formula for tensor products of Lie algebra representations, even if there is no associated group representations.

Remark 16.49 If (Π_1, V_1) and (Π_2, V_2) are representations of a group G , it is possible to view $V_1 \otimes V_2$ as a representation of the direct product group $G \times G$, by setting

$$(\Pi_1 \otimes \Pi_2)(A, B) = \Pi_1(A) \otimes \Pi_2(B).$$

Similarly, if (π_1, V_1) and (π_2, V_2) are representations of a Lie algebra \mathfrak{g} , it is possible to view $V_1 \otimes V_2$ as a representation of $\mathfrak{g} \oplus \mathfrak{g}$ by setting

$$(\pi_1 \otimes \pi_2)(X, Y) = \pi_1(X) \otimes I + I \otimes \pi_2(Y).$$

Nevertheless, it is, in most cases, more natural to view $V_1 \otimes V_2$ as a representation of G itself, rather than of $G \times G$. Even if V_1 and V_2 are irreducible representations of G , the space $V_1 \otimes V_2$ will in most cases fail to be irreducible as a representation of G . If, for example, we take $V_1 = V_2 = V$, then the space of symmetric tensors inside $V \otimes V$ will form a nontrivial invariant subspace, unless $\dim V = 1$. An important problem in representation theory is to decompose $V_1 \otimes V_2$ as a direct sum of irreducible representations, where V_1 and V_2 are irreducible representations of a fixed group or Lie algebra. In the case of the Lie algebra $\mathfrak{su}(2)$, this decomposition is discussed in Sect. 17.9.

Definition 16.50 A finite-dimensional representation of a group or Lie algebra is said to be **completely reducible** if it is isomorphic to a direct sum of irreducible representations.

Proposition 16.51 Every finite-dimensional unitary representation of a group or Lie algebra is completely reducible.

Proof. Suppose (Π, V) is a unitary representation of a matrix Lie group G . If W is a subspace of V invariant under each $\Pi(A)$, then W^\perp is invariant under each $\Pi(A)^*$, as the reader may easily verify. But since Π is unitary,

$$\Pi(A)^* = \Pi(A)^{-1} = \Pi(A^{-1}).$$

Thus, W^\perp is invariant under $\Pi(A^{-1})$ for all $A \in G$, hence under $\Pi(A)$ for all $A \in G$. We conclude that, in the unitary case, the orthogonal complement of an invariant subspace is always invariant.

If V is irreducible, there is nothing to prove. If not, we pick a nontrivial invariant subspace W and decompose V as $W \oplus W^\perp$. The restriction of Π to W or to W^\perp is again a unitary representation, so we can repeat this procedure for each of these subspaces. Since V is finite dimensional, the process must eventually terminate, yielding an orthogonal decomposition of V as a direct sum of irreducible invariant subspaces.

If we consider a unitary representation π of a Lie algebra \mathfrak{g} , we have the same argument, but with the identity $\Pi(A)^* = \Pi(A^{-1})$ replaced by $\pi(X)^* = -\pi(X)$. ■

Proposition 16.52 *Suppose K is a compact matrix Lie group. For any finite-dimensional representation (Π, V) of K , there exists an inner product on V such that $\Pi(A)$ is unitary for all $A \in G$. In particular, every finite-dimensional representation of K is completely reducible.*

See Proposition 4.36 in [21].

16.9 Infinite-Dimensional Unitary Representations

For the applications we have in mind, we need to consider representations that are infinite dimensional. The theory of such representations is inevitably more complicated than that of finite-dimensional representations. For our purposes, it suffices to consider the nicest sort of infinite-dimensional representations—unitary representations in a Hilbert space.

16.9.1 Ordinary Unitary Representations

We begin by considering ordinary representations and then turn to projective representations.

Definition 16.53 *Suppose G is a matrix Lie group. Then a **unitary representation** of G is a strongly continuous homomorphism $\Pi : G \rightarrow \mathbf{U}(\mathbf{H})$, where \mathbf{H} is a separable Hilbert space and $\mathbf{U}(\mathbf{H})$ is the group of unitary operators on \mathbf{H} . Here, strong continuity of Π means that if a sequence A_m in G converges to $A \in G$, then*

$$\lim_{m \rightarrow \infty} \|\Pi(A_m)\psi - \Pi(A)\psi\| = 0$$

for all $\psi \in \mathbf{H}$.

We can attempt to associate to a unitary representation Π of G some sort of representation π of the Lie algebra \mathfrak{g} of G , by imitating the construction in Theorem 16.23. For any $X \in \mathfrak{g}$, the map $t \mapsto \Pi(e^{tX})$ is a strongly continuous one-parameter unitary group. Thus, Stone's theorem (Theorem 10.15) tells us that there exists a unique self-adjoint operator A such that $\Pi(e^{tX}) = e^{itA}$ for all $t \in \mathbb{R}$. If we let $\pi(X)$ denote the skew-self-adjoint operator iA , we will have

$$\Pi(e^{tX}) = e^{t\pi(X)}. \tag{16.5}$$

The operators $\pi(X)$, $X \in \mathfrak{g}$, are in general unbounded and defined only on a dense subspace of \mathbf{H} . Nevertheless, it can be shown (see, e.g., [43]) that there exists a dense subspace V of \mathbf{H} contained in the domain of each $\pi(X)$ and that is invariant under each $\pi(X)$, and on which we have $\pi([X, Y]) = [\pi(X), \pi(Y)]$. In the case of the particular representation that we will consider in the next chapter, we can avoid these difficulties by looking at finite-dimensional invariant subspaces.

Proposition 16.54 *Suppose G is a matrix Lie group and $\Pi : G \rightarrow \mathbf{U}(\mathbf{H})$ is a unitary representation of G . For each $X \in \mathfrak{g}$, let $\pi(X)$ denote the operator in (16.5). Suppose $V \subset \mathbf{H}$ is a finite-dimensional subspace of \mathbf{H} such that $\Pi(A)$ maps V into V , for all $A \in G$. Then for all $X \in \mathfrak{g}$, $V \subset \text{Dom}(\pi(X))$, $\pi(X)$ maps V into V , and we have*

$$\pi([X, Y])v = [\pi(X), \pi(Y)]v \tag{16.6}$$

for all $v \in V$.

In the other direction, suppose G is connected and suppose V is any finite-dimensional subspace of \mathbf{H} such that for all $X \in \mathfrak{g}$, $V \subset \text{Dom}(\pi(X))$ and $\pi(X)$ maps V into V . Then $\Pi(A)$ also maps V into V , for all $A \in G$.

Proof. Since V is invariant under both $\Pi(A)$ and $\Pi(A)^* = \Pi(A^{-1})$, the restriction to V of each $\Pi(A)$ is unitary. The operators $\Pi(A)|_V$ form a finite-dimensional unitary representation of G that is strongly continuous and thus continuous. (In the finite-dimensional case, all reasonable notions of continuity for representations coincide.) For each $X \in \mathfrak{g}$, Theorem 16.18 tells us that there is an operator \tilde{X} on V such that

$$\Pi(e^{tX})|_V = e^{t\tilde{X}}.$$

Thus, for any $v \in V$, we have

$$\lim_{t \rightarrow 0} \frac{\Pi(e^{tX})v - v}{t} = \lim_{t \rightarrow 0} \frac{e^{t\tilde{X}}v - v}{t} = \tilde{X}v.$$

This calculation shows that v is in the domain of the infinitesimal generator $\pi(X)$ of the unitary group $\Pi(e^{tX})$, and that $\pi(X)v = \tilde{X}v$. Since the operators \tilde{X} , $X \in \mathfrak{g}$, form a representation of \mathfrak{g} , we have the relation (16.6).

In the other direction, if V is invariant under $\pi(X)$, the restriction of $\pi(X)$ to V is automatically bounded. Thus, there is a constant C such that

$$\|\pi(X)^m v\| \leq C^m \|v\| \tag{16.7}$$

for all $v \in V$. If we use the direct-integral form of the spectral theorem for the self-adjoint operator $A := -i\pi(X)$, it is easy to see that (16.7) can only hold if v , viewed as an element of the direct integral, is supported on a bounded interval inside the spectrum of A . Since the power series of the function $\lambda \mapsto e^{t\lambda}$ converges to $e^{t\lambda}$ uniformly on any finite interval, we will have

$$\Pi(e^{tX})v = e^{itA}v = \sum_{m=0}^{\infty} \frac{t^m \pi(X)^m}{m!} v.$$

Each term in the above power series belongs to V , which is finite dimensional and thus closed. We conclude that $\Pi(e^{tX})v$ belongs to V for all $X \in \mathfrak{g}$. Since G is connected, each element of G is a product of exponentials of Lie algebra elements, and we have the claim. ■

16.9.2 Projective Unitary Representations

Given a Hilbert space \mathbf{H} , let $S^{\mathbf{H}}$ denote the unit sphere in \mathbf{H} , that is, the set of vectors with norm 1. Let $P\mathbf{H}$ be the quotient space $(S^{\mathbf{H}})/\sim$, where “ \sim ” denotes the equivalence relation in which $u \sim v$ if and only if $u = e^{i\theta}v$ for some $\theta \in \mathbb{R}$. The quotient map $q : S^{\mathbf{H}} \rightarrow P\mathbf{H}$ induces a topology on $P\mathbf{H}$ in which a set $U \subset P\mathbf{H}$ is open if and only if $q^{-1}(U)$ is open as a subset of the metric space $S^{\mathbf{H}} \subset \mathbf{H}$.

As in the finite-dimensional case, we can form the quotient group

$$\text{PU}(\mathbf{H}) := \text{U}(\mathbf{H})/\{e^{i\theta}I\}.$$

The action of $\text{U}(\mathbf{H})$ on $S^{\mathbf{H}}$ descends to a well-defined action of $\text{PU}(\mathbf{H})$ on $P\mathbf{H}$.

Definition 16.55 *A projective unitary representation of a matrix Lie group G is a homomorphism $\Pi : G \rightarrow \text{PU}(\mathbf{H})$, for some Hilbert space \mathbf{H} , with the property that if a sequence A_m in G converges to A in G , then*

$$\Pi(A_m)x \rightarrow \Pi(A)x$$

for all $x \in P\mathbf{H}$.

Recall that in the finite-dimensional case, every projective unitary representation of G can be “de-projectivized” at the expense of possibly having to pass to the universal cover \tilde{G} of G (Theorem 16.47). The de-projectivization proceeds by passing to the Lie algebra, choosing the trace-zero representative of each equivalence class, and then exponentiating back to the universal cover of the original group. This approach does not work in the infinite-dimensional case. After all, even assuming we can construct a Lie algebra homomorphism $\pi(X)$ for each $X \in \mathfrak{g}$, the representatives of $\pi(X)$ are typically *unbounded* operators on \mathbf{H} , for which the notion of trace does not make sense. This difficulty is not just a technicality; the corresponding result in the infinite-dimensional case is false, as we will now see.

Example 16.56 For all $(a, b) \in \mathbb{R}^2$, define an operator $T_{(a,b)}$ on $L^2(\mathbb{R})$ by

$$(T_{(a,b)}\psi)(x) = e^{iax}\psi(x - b).$$

Then $T_{(a,b)}$ is unitary for all $(a, b) \in \mathbb{R}^2$ and we have

$$\begin{aligned} (T_{(a,b)}T_{(a',b')}\psi)(x) &= e^{iax}e^{ia'(x-b)}\psi(x - (b + b')) \\ &= e^{-ia'b} (T_{(a+a',b+b')}\psi)(x). \end{aligned} \tag{16.8}$$

The map $(a, b) \mapsto [T_{(a,b)}]$ is a homomorphism of \mathbb{R}^2 into $\text{PU}(L^2(\mathbb{R}))$, and this homomorphism is continuous in the sense of Definition 16.55. There does not, however, exist any homomorphism $S : \mathbb{R}^2 \rightarrow \text{U}(L^2(\mathbb{R}))$ such that $[S_{(a,b)}] = [T_{(a,b)}]$ for all $(a, b) \in \mathbb{R}^2$.

Thus, even though \mathbb{R}^2 is simply connected (and thus its own universal cover), there is no way to de-projectivize the projective unitary representation $(a, b) \mapsto [T_{(a,b)}]$ of \mathbb{R}^2 .

Proof. The map $(a, b) \rightarrow T_{(a,b)}$ is easily seen to be strongly continuous, and thus the map $(a, b) \mapsto [T_{(a,b)}]$ is continuous in the sense of Definition 16.55. If a homomorphism S with the indicated properties existed, then there would be constants $\theta_{a,b}$ such that $S_{(a,b)} = e^{i\theta_{a,b}} T_{(a,b)}$. But then since S is a homomorphism from the commutative group \mathbb{R}^2 into $U(L^2(\mathbb{R}))$, the operator $S_{(a,b)}$ would have to commute with $S_{(a',b')}$ for all (a, b) and (a', b') . But then the operators $T_{(a,b)}$ and $T_{(a',b')}$, being constant multiples of commuting operators, would need to commute as well. But this is not the case; for example, $T_{(a,0)}$ does not commute with $T_{(0,b')}$, as is easily verified using (16.8). ■

Despite the negative result in Example 16.56, there is a positive result in this direction: If G is connected and “semi-simple,” every projective unitary representation of G can be de-projectivized after passing to the universal cover. Here, a Lie algebra \mathfrak{g} is said to be *simple* if \mathfrak{g} has no nontrivial ideals and $\dim \mathfrak{g} \geq 2$. A Lie algebra is said to be *semi-simple* if it is a direct sum of simple algebras. Finally, a Lie group G is said to be *semi-simple* if the Lie algebra \mathfrak{g} of G is semi-simple.

For any connected Lie group G , a projective unitary representation Π of G can be de-projectivized by passing to a *one-dimensional central extension*. A one-dimensional central extension of G is a Lie group G' together with a surjective homomorphism $\Phi : G' \rightarrow G$ such that the kernel of Φ is one-dimensional and contained in the center of G' . See the article [1] of V. Bargmann for more information about these issues.

16.10 Exercises

1. Suppose that G is a connected matrix Lie group and that N is a discrete normal subgroup of G , meaning that there is some neighborhood U of I in G such that $U \cap N = \{I\}$. Show that N is contained in the center of G .

Hint: Consider the quantity gng^{-1} for $g \in G$ and $n \in N$.

2. (a) Suppose two elements U and V of $SU(2)$ commute. Show that each eigenspace for U is invariant under V and vice versa.
(b) Show that if U is in the center of $SU(2)$, then $U = I$ or $U = -I$.
3. Define the *Hilbert–Schmidt norm* of a matrix $X \in M_n(\mathbb{C})$ by the formula

$$\|X\|_{\text{HS}}^2 = \sum_{j,k=1}^n |X_{jk}|^2.$$

Using the Cauchy–Schwarz inequality, show that

$$\|XY\|_{\text{HS}} \leq \|X\|_{\text{HS}} \|Y\|_{\text{HS}} \quad (16.9)$$

for all $X, Y \in M_n(\mathbb{C})$.

4. Using term-by-term differentiation of power series, show that for all $X \in M_n(\mathbb{C})$ and all $1 \leq j, k \leq n$, we have

$$\left. \frac{d}{dt} \left[(e^{tX})_{jk} \right] \right|_{t=0} = X_{jk}.$$

5. Verify Property 4 of Theorem 16.15. This should be easy in the case that X is diagonalizable. In the general case, either use the Jordan canonical form or appeal to the fact that diagonalizable matrices are dense in $M_n(\mathbb{C})$.
6. Suppose X and Y are commuting $n \times n$ matrices. Show that

$$e^X e^Y = e^{X+Y}.$$

This is Property 5 of Theorem 16.15.

Hint: Multiply together the power series for e^X and e^Y and then group terms where the total power of X and Y is n .

7. For $A \in M_n(\mathbb{C})$, define the *logarithm* of A by the power series

$$\log A = A - I - \frac{(A - I)^2}{2} + \frac{(A - I)^3}{3} - \dots$$

whenever this series converges. Assume the following result: If A is sufficiently close to I , then $\log A$ is defined and $\exp(\log A) = A$. [This can be seen easily when A is diagonalizable, and the set of diagonalizable matrices is dense in $M_n(\mathbb{C})$.]

- (a) Show that there exists a constant C such that for all A with $\|A - I\| < 1/2$ we have

$$\|\log A - (A - I)\| \leq C \|A - I\|^2.$$

- (b) Show that for all $X, Y \in M_n(\mathbb{C})$ we have

$$\log \left(e^{X/m} e^{Y/m} \right) = \frac{X}{m} + \frac{Y}{m} + O \left(\frac{1}{m^2} \right). \quad (16.10)$$

Note that $e^{X/m} e^{Y/m}$ tends to I as m tends to infinity, so that the left-hand side of (16.10) is defined for all sufficiently large m .

- (c) Prove the Lie Product Formula.

8. (a) Show that for all $X, Y \in M_n(\mathbb{C})$,

$$\left\| \frac{d}{dt}(X + tY)^m \Big|_{t=0} \right\| \leq m \|X\|^{m-1} \|Y\|.$$

- (b) Show that the map $X \mapsto e^{tX}$ is a continuously differentiable map of $M_n(\mathbb{C}) \cong \mathbb{R}^{2n^2}$ to itself.

- (c) Using Exercise 4, show that the differential of the map $X \mapsto e^X$ at $X = 0$ is the identity map of $M_n(\mathbb{C})$ to itself. (Recall that the differential of smooth map of \mathbb{R}^j to \mathbb{R}^k , evaluated at a point in \mathbb{R}^j , is a linear map of \mathbb{R}^j to \mathbb{R}^k .)

9. Suppose \mathfrak{g} is a Lie algebra and \mathfrak{h} is an ideal in \mathfrak{g} . Let $\mathfrak{g}/\mathfrak{h}$ denote the vector space quotient of \mathfrak{g} by \mathfrak{h} . Show that the bracket on \mathfrak{g} descends unambiguously to a bilinear map on $\mathfrak{g}/\mathfrak{h}$, and that $\mathfrak{g}/\mathfrak{h}$ forms a Lie algebra under this map.

10. Suppose that G_1, G_2 , and G_3 are matrix Lie groups with Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$, and \mathfrak{g}_3 , respectively. Suppose that $\Phi : G_1 \rightarrow G_2$ and $\Psi : G_2 \rightarrow G_3$ are Lie group homomorphisms with associated Lie algebra homomorphisms ϕ and ψ , respectively. Show that the Lie algebra homomorphism associated to $\Psi \circ \Phi : G_1 \rightarrow G_3$ is $\psi \circ \phi$.

11. Show that isomorphic matrix Lie groups have isomorphic Lie algebras.

12. Suppose G_1 and G_2 are matrix Lie groups with Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 , respectively. Suppose $\Phi : G_1 \rightarrow G_2$ is a Lie group homomorphism with the property that the associated Lie algebra homomorphism $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is injective. Show that there exists a neighborhood U of the identity in G_1 such that $U \cap \ker \Phi = \{I\}$.

Hint: Use Theorem 16.25.

13. (a) Show that every $R \in \text{SO}(3)$ has an eigenvalue of 1.
 (b) Show that every $R \in \text{SO}(3)$ is conjugate in $\text{SO}(3)$ to matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$$

for some $\theta \in \mathbb{R}$.

- (c) Show that the exponential map from $\mathfrak{so}(3)$ to $\text{SO}(3)$ is surjective.
 (d) Show that $\text{SO}(3)$ is connected.

14. Show that the center of $\text{SO}(3)$ is trivial.

Hint: Use Part (a) of Exercise 13.

15. Given a Lie algebra \mathfrak{g} , let $[\mathfrak{g}, \mathfrak{g}]$ denote the space of linear combinations of commutators, that is, the space spanned by elements of the form $[X, Y]$ with $X, Y \in \mathfrak{g}$.
- Show that $[\mathfrak{g}, \mathfrak{g}]$ is an ideal in \mathfrak{g} and that the quotient $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is commutative. (The ideal $[\mathfrak{g}, \mathfrak{g}]$ is called the *commutator ideal* of \mathfrak{g} .)
 - If $\mathfrak{g} = \mathfrak{so}(3)$, show that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.
 - If $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is any finite-dimensional representation of \mathfrak{g} , show that $\pi([\mathfrak{g}, \mathfrak{g}])$ is contained in $\mathfrak{sl}(V)$, the space of endomorphisms of V with trace zero.
16. (a) Show that the Lie algebra $\mathfrak{pu}(n) \cong \mathfrak{u}(n)/\{ia\mathbb{R}\}$ is isomorphic to the Lie algebra $\mathfrak{su}(n)$.
- (b) Let $\{e^{2\pi ik/n}I\}$ denote the group of matrices that are of the form of an n th root of unity times the identity. Show that the group $\mathfrak{PU}(n)$ is isomorphic to $\mathfrak{SU}(n)/\{e^{2\pi ik/n}I\}$.
17. Suppose that G is a matrix Lie group with Lie algebra \mathfrak{g} and that A is an element of G . Show that the operation of left multiplication by A^{-1} is a diffeomorphism of $M_n(\mathbb{C})$. Now show that there exist neighborhoods U of 0 in $M_n(\mathbb{C})$ and V of A in $M_n(\mathbb{C})$ such that the map $X \mapsto Ae^X$ maps U diffeomorphically onto V and such that for $X \in U$, we have $X \in \mathfrak{g}$ if and only if $Ae^X \in V$. (Use Theorem 16.25.)
18. Suppose that $Z \in M_n(\mathbb{C})$ has the property that $ZX = XZ$ for all $X \in M_n(\mathbb{C})$. Show that $Z = cI$ for some $c \in \mathbb{C}$.
19. Suppose (Π, \mathbf{H}) is a unitary representation of a matrix Lie group G , and suppose V_1 and V_2 are finite-dimensional irreducible invariant subspaces of \mathbf{H} . Show that if V_1 and V_2 are not isomorphic as representations of G , then V_1 is orthogonal to V_2 inside \mathbf{H} .
- Hint:* Show that the orthogonal projection of \mathbf{H} onto V_1 or V_2 is an intertwining map, and use Schur's lemma.