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The Experimental Origins of Quantum Mechanics

Quantum mechanics, with its controversial probabilistic nature and curious blending of waves and particles, is a very strange theory. It was not invented because anyone thought this is the way the world *should* behave, but because various experiments showed that this is the way the world *does* behave, like it or not. Craig Hogan, director of the Fermilab Particle Astrophysics Center, put it this way:

> No theorist in his right mind would have invented quantum mechanics unless forced to by data.[1]

Although the first hint of quantum mechanics came in 1900 with Planck's solution to the problem of blackbody radiation, the full theory did not emerge until 1925–1926, with Heisenberg's matrix model, Schrödinger's wave model, and Born's statistical interpretation of the wave model.

1.1 Is Light a Wave or a Particle?

1.1.1 Newton Versus Huygens

Beginning in the late seventeenth century and continuing into the early eighteenth century, there was a vigorous debate in the scientific community

[1]Quoted in "Is Space Digital?" by Michael Moyer, *Scientific American*, February 2012, pp. 30–36.

B.C. Hall, *Quantum Theory for Mathematicians*, Graduate Texts in Mathematics 267, DOI 10.1007/978-1-4614-7116-5_1,

over the nature of light. One camp, following the views of Isaac Newton, claimed that light consisted of a group of particles or "corpuscles." The other camp, led by the Dutch physicist Christiaan Huygens, claimed that light was a wave. Newton argued that only a corpuscular theory could account for the observed tendency of light to travel in straight lines. Huygens and others, on the other hand, argued that a wave theory could explain numerous observed aspects of light, including the bending or "refraction" of light as it passes from one medium to another, as from air into water. Newton's reputation was such that his "corpuscular" theory remained the dominant one until the early nineteenth century.

1.1.2 The Ascendance of the Wave Theory of Light

In 1804, Thomas Young published two papers describing and explaining his double-slit experiment. In this experiment, sunlight passes through a small hole in a piece of cardboard and strikes another piece of cardboard containing two small holes. The light then strikes a third piece of cardboard, where the pattern of light may be observed. Young observed "fringes" or alternating regions of high and low intensity for the light. Young believed that light was a wave and he postulated that these fringes were the result of *interference* between the waves emanating from the two holes. Young drew an analogy between light and water, where in the case of water, interference is readily observed. If two circular waves of water cross each other, there will be some points where a peak of one wave matches up with a trough of another wave, resulting in *destructive interference*, that is, a partial cancellation between the two waves, resulting in a small amplitude of the combined wave at that point. At other points, on the other hand, a peak in one wave will line up with a peak in the other, or a trough with a trough. At such points, there is *constructive interference*, with the result that the amplitude of the combined wave is large at that point. The pattern of constructive and destructive interference will produce something like a checkerboard pattern of alternating regions of large and small amplitudes in the combined wave. The dimensions of each region will be roughly on the order of the wavelength of the individual waves.

Based on this analogy with water waves, Young was able to explain the interference fringes that he observed and to predict the wavelength that light must have in order for the specific patterns he observed to occur. Based on his observations, Young claimed that the wavelength of visible light ranged from about 1/36,000 in. (about 700 nm) at the red end of the spectrum to about 1/60,000 in. (about 425 nm) at the violet end of the spectrum, results that agree with modern measurements.

Figure 1.1 shows how circular waves emitted from two different points form an interference pattern. One should think of Young's second piece of cardboard as being at the top of the figure, with holes near the top left and

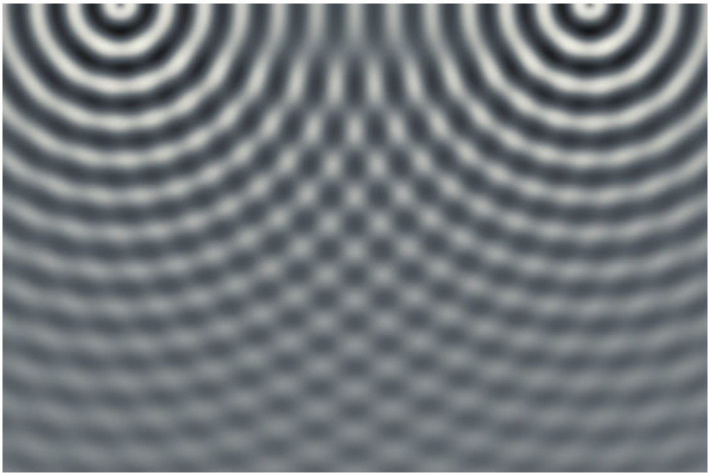

FIGURE 1.1. Interference of waves emitted from two slits.

top right of the figure. Figure 1.2 then plots the intensity (i.e., the square of the displacement) as a function of x, with y having the value corresponding to the bottom of Fig. 1.1.

Despite the convincing nature of Young's experiment, many proponents of the corpuscular theory of light remained unconvinced. In 1818, the French Academy of Sciences set up a competition for papers explaining the observed properties of light. One of the submissions was a paper by Augustin-Jean Fresnel in which he elaborated on Huygens's wave model of refraction. A supporter of the corpuscular theory of light, Siméon-Denis Poisson read Fresnel's submission and ridiculed it by pointing out that if that theory were true, light passing by an opaque disk would diffract around the edges of the disk to produce a bright spot in the center of the shadow of the disk, a prediction that Poisson considered absurd. Nevertheless, the head of the judging committee for the competition, François Arago, decided to put the issue to an experimental test and found that such a spot does in fact occur. Although this spot is often called "Arago's spot," or even, ironically, "Poisson's spot," Arago eventually realized that the spot had been observed 100 years earlier in separate experiments by Delisle and Maraldi.

Arago's observation of Poisson's spot led to widespread acceptance of the wave theory of light. This theory gained even greater acceptance in 1865, when James Clerk Maxwell put together what are today known as Maxwell's equations. Maxwell showed that his equations predicted that electromagnetic waves would propagate at a certain speed, which agreed with the observed speed of light. Maxwell thus concluded that light *is* simply an electromagnetic wave. From 1865 until the end of the nineteenth

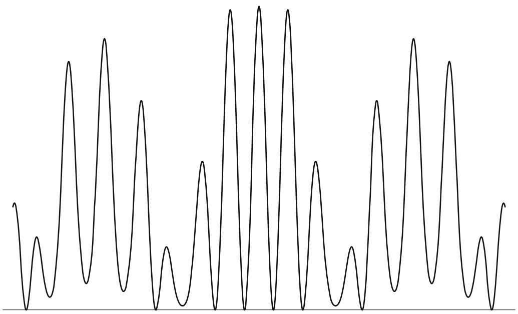

FIGURE 1.2. Intensity plot for a horizontal line across the bottom of Fig. 1.1
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century, the debate over the wave-versus-particle nature of light was considered to have been conclusively settled in favor of the wave theory.

1.1.3 Blackbody Radiation

In the early twentieth century, the wave theory of light began to experience new challenges. The first challenge came from the theory of *blackbody radiation.* In physics, a blackbody is an idealized object that perfectly absorbs all electromagnetic radiation that hits it. A blackbody can be approximated in the real world by an object with a highly absorbent surface such as "lamp black." The problem of blackbody radiation concerns the distribution of electromagnetic radiation in a cavity within a blackbody. Although the walls of the blackbody absorb the radiation that hits it, thermal vibrations of the atoms making up the walls cause the blackbody to emit electromagnetic radiation. (At normal temperatures, most of the radiation emitted would be in the infrared range.)

In the cavity, then, electromagnetic radiation is constantly absorbed and re-emitted until thermal equilibrium is reached, at which point the absorption and emission of radiation are perfectly balanced at each frequency. According to the "equipartition theorem" of (classical) statistical mechanics, the energy in any given mode of electromagnetic radiation should be exponentially distributed, with an average value equal to $k_B T$, where T is the temperature and k_B is Boltzmann's constant. (The temperature should be measured on a scale where absolute zero corresponds to $T = 0$.) The difficulty with this prediction is that the average amount of energy is *the same for every mode* (hence the term "equipartition"). Thus, once one adds up over all modes—of which there are infinitely many—the predicted amount of energy in the cavity is infinite. This strange prediction is referred to as the *ultraviolet catastrophe*, since the infinitude of the energy comes from the ultraviolet (high-frequency) end of the spectrum. This ultraviolet catastrophe does not seem to make physical sense and certainly does not match up with the observed energy spectrum within real-world blackbodies.

An alternative prediction of the blackbody energy spectrum was offered by Max Planck in a paper published in 1900. Planck postulated that the energy in the electromagnetic field at a given frequency ω should be "quantized," meaning that this energy should come only in integer multiples of a certain basic unit equal to $\hbar\omega$, where $\hbar$ is a constant, which we now call Planck's constant. Planck postulated that the energy would again be exponentially distributed, but only over integer multiples of $\hbar\omega$. At low frequencies, Planck's theory predicts essentially the same energy as in classical statistical mechanics. At high frequencies, namely at frequencies where $\hbar\omega$ is large compared to k_BT, Planck's theory predicts a rapid fall-off of the average energy (see Exercise 2 for details). Indeed, if we measure mass, distance, and time in units of grams, centimeters, and seconds, respectively, and we assign $\hbar$ the numerical value

$$\hbar = 1.054 \times 10^{-27},$$

then Planck's predictions match the experimentally observed blackbody spectrum.

Planck pictured the walls of the blackbody as being made up of independent oscillators of different frequencies, each of which is restricted to have energies of $\hbar\omega$. Although this picture was clearly not intended as a realistic physical explanation of the quantization of electromagnetic energy in blackbodies, it does suggest that Planck thought that energy quantization arose from properties of the walls of the cavity, rather than in intrinsic properties of the electromagnetic radiation. Einstein, on the other hand, in assessing Planck's model, argued that energy quantization was inherent in the radiation itself. In Einstein's picture, then, electromagnetic energy at a given frequency—whether in a blackbody cavity or not—comes in packets or *quanta* having energy proportional to the frequency. Each quantum of electromagnetic energy constitutes what we now call a *photon*, which we may think of as a particle of light. Thus, Planck's model of blackbody radiation began a rebirth of the particle theory of light.

It is worth mentioning, in passing, that in 1900, the same year in which Planck's paper on blackbody radiation appeared, Lord Kelvin gave a lecture that drew attention to another difficulty with the classical theory of statistical mechanics. Kelvin described two "clouds" over nineteenth-century physics at the dawn of the twentieth century. The first of these clouds concerned aether—a hypothetical medium through which electromagnetic radiation propagates—and the failure of Michelson and Morley to observe the motion of earth relative to the aether. Under this cloud lurked the theory of special relativity. The second of Kelvin's clouds concerned heat capacities in gases. The equipartition theorem of classical statistical mechanics made predictions for the ratio of heat capacity at constant pressure (c_p) and the heat capacity at constant volume (c_v). These predictions deviated substantially from the experimentally measured ratios. Under the second cloud lurked the theory of quantum mechanics, because the resolution of this discrepancy is similar to Planck's resolution of the blackbody problem. As in the case of blackbody radiation, quantum mechanics gives rise to a correction to the equipartition theorem, thus resulting in different predictions for the ratio of c_p to c_v, predictions that can be reconciled with the observed ratios.

1.1.4 The Photoelectric Effect

The year 1905 was Einstein's *annus mirabilis* (miraculous year), in which Einstein published four ground-breaking papers, two on the special theory of relativity and one each on Brownian motion and the photoelectric effect. It was for the photoelectric effect that Einstein won the Nobel Prize in physics in 1921. In the photoelectric effect, electromagnetic radiation striking a metal causes electrons to be emitted from the metal. Einstein found that as one increases the *intensity* of the incident light, the *number* of emitted electrons increases, but the *energy* of each electron does not change. This result is difficult to explain from the perspective of the wave theory of light. After all, if light is simply an electromagnetic wave, then increasing the intensity of the light amounts to increasing the strength of the electric and magnetic fields involved. Increasing the strength of the fields, in turn, ought to increase the amount of energy transferred to the electrons.

Einstein's results, on the other hand, are readily explained from a particle theory of light. Suppose light is actually a stream of particles (photons) with the energy of each particle determined by its frequency. Then increasing the intensity of light at a given frequency simply increases the number of photons and does not affect the energy of each photon. If each photon has a certain likelihood of hitting an electron and causing it to escape from the metal, then the energy of the escaping electron will be determined by the *frequency* of the incident light and not by the *intensity* of that light. The photoelectric effect, then, provided another compelling reason for believing that light can behave in a particlelike manner.

1.1.5 The Double-Slit Experiment, Revisited

Although the work of Planck and Einstein suggests that there is a particlelike aspect to light, there is certainly also a wavelike aspect to light, as shown by Young, Arago, and Maxwell, among others. Thus, somehow, light must in some situations behave like a wave and in some situations like a particle, a phenomenon known as "wave–particle duality." William Lawrence Bragg described the situation thus:

> God runs electromagnetics on Monday, Wednesday, and Friday by the wave theory, and the devil runs them by quantum theory on Tuesday, Thursday, and Saturday.

(Apparently Sunday, being a day of rest, did not need to be accounted for.)

In particular, we have already seen that Young's double-slit experiment in the early nineteenth century was one important piece of evidence in favor of the wave theory of light. If light is really made up of particles, as blackbody radiation and the photoelectric effect suggest, one must give a particle-based explanation of the double-slit experiment. J.J. Thomson suggested in 1907 that the patterns of light seen in the double-slit experiment could be the result of different photons somehow interfering with one another. Thomson thus suggested that if the intensity of light were sufficiently reduced, the photons in the light would become widely separated and the interference pattern might disappear. In 1909, Geoffrey Ingram Taylor set out to test this suggestion and found that even when the intensity of light was drastically reduced (to the point that it took three *months* for one of the images to form), the interference pattern remained the same.

Since Taylor's results suggest that interference remains even when the photons are widely separated, the photons are not interfering with one another. Rather, as Paul Dirac put it in Chap. 1 of [6], "Each photon then interferes only with itself." To state this in a different way, since there is no interference when there is only one slit, Taylor's results suggest that each individual photon passes through *both* slits. By the early 1960s, it became possible to perform double-slit experiments with electrons instead of photons, yielding even more dramatic confirmations of the strange behavior of matter in the quantum realm. (See Sect. 1.2.4.)

1.2 Is an Electron a Wave or a Particle?

In the early part of the twentieth century, the atomic theory of matter became firmly established. (Einstein's 1905 paper on Brownian motion was an important confirmation of the theory and provided the first calculation of atomic masses in everyday units.) Experiments performed in 1909 by Hans Geiger and Ernest Marsden, under the direction of Ernest Rutherford, led Rutherford to put forward in 1911 a picture of atoms in which a small nucleus contains most of the mass of the atom. In Rutherford's model, each atom has a positively charged nucleus with charge nq, where n is a positive integer (the *atomic number*) and q is the basic unit of charge first observed in Millikan's famous oil-drop experiment. Surrounding the nucleus is a cloud of n electrons, each having negative charge $-q$. When atoms bind into molecules, some of the electrons of one atom may be shared with another atom to form a bond between the atoms. This picture of atoms and their binding led to the modern theory of chemistry.

Basic to the atomic theory is that electrons are particles; indeed, the number of electrons per atom is supposed to be the atomic number. Nevertheless, it did not take long after the atomic theory of matter was confirmed before wavelike properties of electrons began to be observed. The situation, then, is the reverse of that with light. While light was long thought to be a wave (at least from the publication of Maxwell's equations in 1865 until Planck's work in 1900) and was only later seen to have particlelike behavior, electrons were initially thought to be particles and were only later seen to have wavelike properties. In the end, however, both light and electrons have both wavelike and particlelike properties.

1.2.1 The Spectrum of Hydrogen

If electricity is passed through a tube containing hydrogen gas, the gas will emit light. If that light is separated into different frequencies by means of a prism, bands will become apparent, indicating that the light is not a continuous mix of many different frequencies, but rather consists only of a discrete family of frequencies. In view of the photonic theory of light, the energy in each photon is proportional to its frequency. Thus, each observed frequency corresponds to a certain amount of energy being transferred from a hydrogen atom to the electromagnetic field.

Now, a hydrogen atom consists of a single proton surrounded by a single electron. Since the proton is much more massive than the electron, one can picture the proton as being stationary, with the electron orbiting it. The idea, then, is that the current being passed through the gas causes some of the electrons to move to a higher-energy state. Eventually, that electron will return to a lower-energy state, emitting a photon in the process. In this way, by observing the energies (or, equivalently, the frequencies) of the emitted photons, one can work backwards to the change in energy of the electron.

The curious thing about the state of affairs in the preceding paragraph is that the energies of the emitted photons—and hence, also, the energies of the electron—come only in a discrete family of possible values. Based on the observed frequencies, Johannes Rydberg concluded in 1888 that the possible energies of the electron were of the form

$$E_n = -\frac{R}{n^2}. \tag{1.1}$$

Here, R is the "Rydberg constant," given (in "Gaussian units") by

$$R = \frac{m_e Q^4}{2\hbar^2},$$

where Q is the charge of the electron and m_e is the mass of the electron. (Technically, m_e should be replaced by the reduced mass μ of the proton–electron system; that is, $\mu = m_e m_p/(m_e + m_p)$, where m_p is the mass of the proton. However, since the proton mass is much greater than the electron mass, μ is almost the same as m_e and we will neglect the difference between the two.) The energies in (1.1) agree with experiment, in that all the observed frequencies in hydrogen are (at least to the precision available at the time of Rydberg) of the form

$$\omega = \frac{1}{\hbar}\left(E_n - E_m\right), \tag{1.2}$$

for some $n > m$. It should be noted that Johann Balmer had already observed in 1885 frequencies of the same form, but only in the case $m = 2$, and that Balmer's work influenced Rydberg.

The frequencies in (1.2) are known as the *spectrum* of hydrogen. Balmer and Rydberg were merely attempting to find a simple formula that would match the observed frequencies in hydrogen. Neither of them had a theoretical explanation for *why* only these particular frequencies occur. Such an explanation would have to wait until the beginnings of quantum theory in the twentieth century.

1.2.2 The Bohr–de Broglie Model of the Hydrogen Atom

In 1913, Niels Bohr introduced a model of the hydrogen atom that attempted to explain the observed spectrum of hydrogen. Bohr pictured the hydrogen atom as consisting of an electron orbiting a positively charged nucleus, in much the same way that a planet orbits the sun. Classically, the force exerted on the electron by the proton follows the *inverse square law* of the form

$$F = \frac{Q^2}{r^2}, \tag{1.3}$$

where Q is the charge of the electron, in appropriate units.

If the electron is in a circular orbit, its trajectory in the plane of the orbit will take the form

$$(x(t), y(t)) = (r\cos(\omega t), r\sin(\omega t)).$$

If we take the second derivative with respect to time to obtain the acceleration vector $\mathbf{a}$, we obtain

$$\mathbf{a}(t) = (-\omega^2 r\cos(\omega t), -\omega^2 r\sin(\omega t)),$$

so that the magnitude of the acceleration vector is $\omega^2 r$. Newton's second law, $F = ma$, then requires that

$$m_e\omega^2 r = \frac{e^2}{r^2},$$

so that

$$\omega = \sqrt{\frac{Q^2}{m_e r^3}}.$$

From the formula for the frequency, we can calculate that the momentum (mass times velocity) has magnitude

$$p = \sqrt{\frac{m_e Q^2}{r}}. \tag{1.4}$$

We can also calculate the angular momentum J, which for a circular orbit is just the momentum times the distance from the nucleus, as

$$J = \sqrt{m_e Q^2 r}.$$

Bohr postulated that the electron obeys classical mechanics, *except* that its angular momentum is "quantized." Specifically, in Bohr's model, the angular momentum is required to be an integer multiple of $\hbar$ (Planck's constant). Setting J equal to $n\hbar$ yields

$$r_n = \frac{n^2 \hbar^2}{m_e Q^2}. \tag{1.5}$$

If one calculates the energy of an orbit with radius r_n, one finds (Exercise 3) that it agrees precisely with the Rydberg energies in (1.1). Bohr further postulated that an electron could move from one allowed state to another, emitting a packet of light in the process with frequency given by (1.2).

Bohr did not explain *why* the angular momentum of an electron is quantized, nor how it moved from one allowed orbit to another. As such, his theory of atomic behavior was clearly not complete; it belongs to the "old quantum mechanics" that was superseded by the matrix model of Heisenberg and the wave model of Schrödinger. Nevertheless, Bohr's model was an important step in the process of understanding the behavior of atoms, and Bohr was awarded the 1922 Nobel Prize in physics for his work. Some remnant of Bohr's approach survives in modern quantum theory, in the WKB approximation (Chap. 15), where the Bohr–Sommerfeld condition gives an approximation to the energy levels of a one-dimensional quantum system.

In 1924, Louis de Broglie reinterpreted Bohr's condition on the angular momentum as a wave condition. The *de Broglie hypothesis* is that an electron can be described by a wave, where the spatial frequency k of the wave is related to the momentum of the electron by the relation

$$p = \hbar k. \tag{1.6}$$

Here, "frequency" is defined so that the frequency of the function $\cos(kx)$ is k. This is "angular" frequency, which differs by a factor of 2π from the cycles-per-unit-distance frequency. Thus, the period associated with a given frequency k is $2\pi/k$.

In de Broglie's approach, we are supposed to imagine a wave superimposed on the classical trajectory of the electron, with the quantization

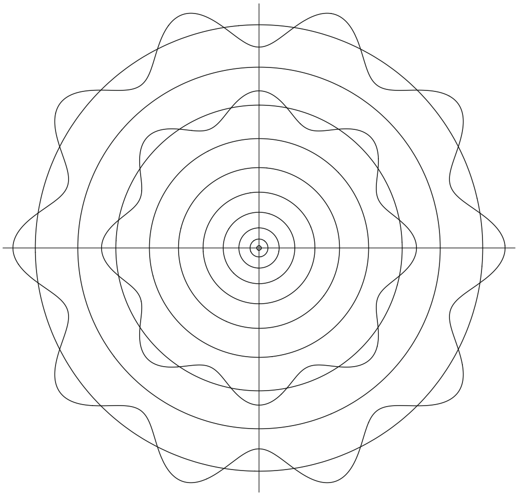

FIGURE 1.3. The Bohr radii for $n = 1$ to $n = 10$, with de Broglie waves superimposed for $n = 8$ and $n = 10$.

condition now being that the wave should match up with itself when going all the way around the orbit. This condition means that the orbit should consist of an integer number of periods of the wave:

$$2\pi r = n\frac{2\pi}{k}.$$

Using (1.6) along with the expression (1.4) for p, we obtain

$$2\pi r = n2\pi\frac{\hbar}{p} = 2\pi n\hbar\sqrt{\frac{r}{m_e Q^2}}.$$

Solving this equation for r gives precisely the Bohr radii in (1.5).

Thus, de Broglie's wave hypothesis gives an alternative to Bohr's quantization of angular momentum as an explanation of the allowed energies of hydrogen. Of course, if one accepts de Broglie's wave hypothesis for electrons, one would expect to see wavelike behavior of electrons not just in the hydrogen atom, but in other situations as well, an expectation that would soon be fulfilled. Figure 1.3 shows the first 10 Bohr radii. For the 8th and 10th radii, the de Broglie wave is shown superimposed onto the orbit.

1.2.3 Electron Diffraction

In 1925, Clinton Davisson and Lester Germer were studying properties of nickel by bombarding a thin film of nickel with low-energy electrons. As a result of a problem with their equipment, the nickel was accidentally heated to a very high temperature. When the nickel cooled, it formed into large crystalline pieces, rather than the small crystals in the original sample. After this recrystallization, Davisson and Germer observed peaks in the pattern of electrons reflecting off of the nickel sample that had not been present when using the original sample. They were at a loss to explain this pattern until, in 1926, Davisson learned of the *de Broglie hypothesis* and suspected that they were observing the wavelike behavior of electrons that de Broglie had predicted.

After this realization, Davisson and Germer began to look systematically for wavelike peaks in their experiments. Specifically, they attempted to show that the pattern of angles at which the electrons reflected matched the patterns one sees in x-ray diffraction. After numerous additional measurements, they were able to show a very close correspondence between the pattern of electrons and the patterns seen in x-ray diffraction. Since x-rays were by this time known to be waves of electromagnetic radiation, the Davisson–Germer experiment was a strong confirmation of de Broglie's wave picture of electrons. Davisson and Germer published their results in two papers in 1927, and Davisson shared the 1937 Nobel Prize in physics with George Paget, who had observed electron diffraction shortly after Davisson and Germer.

1.2.4 The Double-Slit Experiment with Electrons

Although quantum theory clearly predicts that electrons passing through a double slit will experience interference similar to that observed in light, it was not until Clauss Jönsson's work in 1961 that this prediction was confirmed experimentally. The main difficulty is the much smaller wavelength for electrons of reasonable energy than for visible light. Jönsson's electrons, for example, had a de Broglie wavelength of 5 nm, as compared to a wavelength of roughly 500 nm for visible light (depending on the color).

In results published in 1989, a team led by Akira Tonomura at Hitachi performed a double-slit experiment in which they were able to record the results *one electron at a time*. (Similar but less definitive experiments were carried out by Pier Giorgio Merli, GianFranco Missiroli and Giulio Pozzi in Bologna in 1974 and published in the *American Journal of Physics* in 1976.) In the Hitachi experiment, each electron passes through the slits and then strikes a screen, causing a small spot of light to appear. The location of this spot is then recorded for each electron, one at a time. The key point is that each individual electron strikes the screen at a *single point*. That is to say, *individual* electrons are not smeared out across the screen in a wavelike pattern, but rather behave like point particles, in that the observed location of the electron is indeed a point. Each electron, however, strikes the screen at a different point, and once a large number of the electrons have struck and their locations have been recorded, an interference pattern emerges.

It is not the variability of the locations of the electrons that is surprising, since this could be accounted for by small variations in the way the electrons

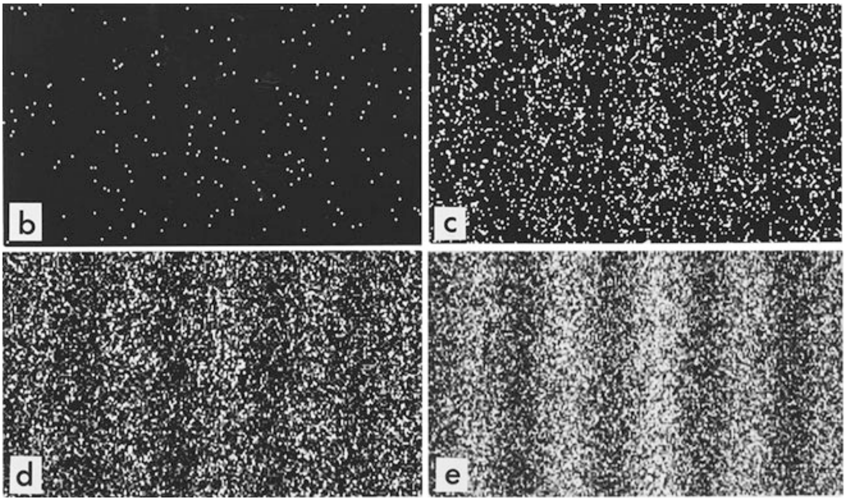

FIGURE 1.4. Four images from the 1989 experiment at Hitachi showing the impact of individual electrons gradually building up to form an interference pattern. Image by Akira Tonomura and Wikimedia Commons user Belsazar. File is licensed under the Creative Commons Attribution-Share Alike 3.0 Unported license.

are shot toward the slits. Rather, it is the distinctive *interference* pattern that is surprising, with rapid variations in the pattern of electron strikes over short distances, including regions where almost no electron strikes occur. (Compare Fig. 1.4 to Fig. 1.2.) Note also that in the experiment, the electrons are widely separated, so that there is never more than one electron in the apparatus at any one time. Thus, the electrons cannot interfere with one another; rather, *each electron interferes with itself.* Figure 1.4 shows results from the Hitachi experiment, with the number of observed electrons increasing from about 150 in the first image to 160,000 in the last image.

1.3 Schrödinger and Heisenberg

In 1925, Werner Heisenberg proposed a model of quantum mechanics based on treating the position and momentum of the particle as, essentially, matrices of size $\infty \times \infty$. Actually, Heisenberg himself was not familiar with the theory of matrices, which was not a standard part of the mathematical education of physicists at the time. Nevertheless, he had quantities of the form x_{jk} and p_{jk} (where j and k each vary over all integers), which we can recognize as matrices, as well as expressions such as $\sum_l x_{jl}p_{lk}$, which we can recognize as a matrix product. After Heisenberg explained his theory to Max Born, Born recognized the connection of Heisenberg's formulas to matrix theory and made the matrix point of view explicit, in a paper coauthored by Born and his assistant, Pascual Jordan. Born, Heisenberg, and Jordan then all published a paper together elaborating upon their theory. The papers of Heisenberg, of Born and Jordan, and of Born, Heisenberg, and Jordan all appeared in 1925. Heisenberg received the 1932 Nobel Prize in physics (actually awarded in 1933) for his work. Born's exclusion from this prize was controversial, and may have been influenced by Jordan's connections with the Nazi party in Germany. (Heisenberg's own work for the Nazis during World War II was also a source of much controversy after the war.) In any case, Born was awarded the Nobel Prize in physics in 1954 for his work on the statistical interpretation of quantum mechanics (Sect. 1.4).

Meanwhile, in 1926, Erwin Schrödinger published four remarkable papers in which he proposed a wave theory of quantum mechanics, along the lines of the *de Broglie hypothesis*. In these papers, Schrödinger described how the waves evolve over time and showed that the energy levels of, for example, the hydrogen atom could be understood as *eigenvalues* of a certain operator. (See Chap. 18 for the computation for hydrogen.) Schrödinger also showed that the Heisenberg–Born–Jordan matrix model could be incorporated into the wave theory, thus showing that the matrix theory and the wave theory were equivalent (see Sect. 3.8). This book describes the mathematical structure of quantum mechanics in essentially the form proposed by Schrödinger in 1926. Schrödinger shared the 1933 Nobel Prize in physics with Paul Dirac.

1.4 A Matter of Interpretation

Although Schrödinger's 1926 papers gave the correct mathematical description of quantum mechanics (as it is generally accepted today), he did not provide a widely accepted *interpretation* of the theory. That task fell to Born, who in a 1926 paper proposed that the "wave function" (as the wave appearing in the Schrödinger equation is generally called) should be interpreted statistically, that is, as determining the *probabilities* for observations of the system. Over time, Born's statistical approach developed into the *Copenhagen interpretation* of quantum mechanics. Under this interpretation, the wave function ψ of the system is not directly observable. Rather, ψ merely determines the probability of observing a particular result.

In particular, if ψ is properly normalized, then the quantity $|\psi(x)|^2$ is the probability distribution for the position of the particle. Even if ψ itself is spread out over a large region in space, any measurement of the position of the particle will show that the particle is located at a *single point*, just as we see for the electrons in the two-slit experiment in Fig. 1.4. Thus, a measurement of a particle's position does not show the particle "smeared out" over a large region of space, even if the wave function ψ *is* smeared out over a large region.

Consider, for example, how Born's interpretation of the Schrödinger equation would play out in the context of the Hitachi double-slit experiment depicted in Fig. 1.4. Born would say that each electron has a wave function that evolves in time according to the Schrödinger equation (an equation of wave type). Each particle's wave function, then, will propagate through the slits in a manner similar to that pictured in Fig. 1.1. If there is a screen at the bottom of Fig. 1.1, then the electron will hit the screen at a single point, even though the wave function is very spread out. The wave function does not determine where the particle hits the screen; it merely determines the probabilities for where the particle hits the screen. If a whole sequence of electrons passes through the slits, one after the other, over time a probability distribution will emerge, determined by the square of the magnitude of the wave function, which is shown in Fig. 1.2. Thus, the probability distribution of electrons, as seen from a large number of electrons as in Fig. 1.4, shows wavelike interference patterns, even though each individual electron strikes the screen at a single point.

It is essential to the theory that the wave function $\psi(x)$ itself is not the probability density for the location of the particle. Rather, the probability density is $|\psi(x)|^2$. The difference is crucial, because probability densities are intrinsically positive and thus do not exhibit destructive interference. The wave function itself, however, is complex-valued, and the real and imaginary parts of the wave function take on both positive and negative values, which can interfere constructively or destructively. The part of the wave function passing through the first slit, for example, can interfere with the part of the wave function passing through the second slit. Only *after* this interference has taken place do we take the magnitude squared of the wave function to obtain the probability distribution, which will, therefore, show the sorts of peaks and valleys we see in Fig. 1.2.

Born's introduction of a probabilistic element into the interpretation of quantum mechanics was—and to some extent still is—controversial. Einstein, for example, is often quoted as saying something along the lines of, "God does not play at dice with the universe." Einstein expressed the same sentiment in various ways over the years. His earliest known statement to this effect was in a letter to Born in December 1926, in which he said,

> Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the "old one." I, at any rate, am convinced that *He* does not throw dice.

Many other physicists and philosophers have questioned the probabilistic interpretation of quantum mechanics, and have sought alternatives, such as "hidden variable" theories. Nevertheless, the Copenhagen interpretation of quantum mechanics, essentially as proposed by Born in 1926, remains the standard one. This book resolutely avoids all controversies surrounding the interpretation of quantum mechanics. Chapter 3, for example, presents the standard statistical interpretation of the theory without question. The book may nevertheless be of use to the more philosophically minded reader, in that one must learn something of quantum mechanics before delving into the (often highly technical) discussions about its interpretation.

1.5 Exercises

1. Beginning with the formula for the sum of a geometric series, use differentiation to obtain the identity

$$\sum_{n=0}^{\infty} ne^{-An} = \frac{e^{-A}}{(1-e^{-A})^2}.$$

2. In Planck's model of blackbody radiation, the energy in a given frequency ω of electromagnetic radiation is distributed randomly over all numbers of the form $n\hbar\omega$, where $n = 0, 1, 2, \ldots$. Specifically, the likelihood of finding energy $n\hbar\omega$ is postulated to be

$$p(E = n\hbar\omega) = \frac{1}{Z} e^{-\beta n\hbar\omega},$$
$$Z = \frac{1}{1-e^{-\beta\hbar\omega}}$$

where Z is a normalization constant, which is chosen so that the sum over n of the probabilities is 1. Here $\beta = 1/(k_B T)$, where T is the temperature and k_B is Boltzmann's constant. The *expected value* of the energy, denoted $\langle E \rangle$, is defined to be

$$\langle E \rangle = \frac{1}{Z} \sum_{n=0}^{\infty} (n\hbar\omega) e^{-\beta n\hbar\omega}.$$

(a) Using Exercise 1, show that

$$\langle E \rangle = \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}.$$

(b) Show that $\langle E \rangle$ behaves like $1/\beta = k_B T$ for small ω, but that $\langle E \rangle$ decays exponentially as ω tends to infinity.

Note: In applying the above calculation to blackbody radiation, one must also take into account the number of modes having frequency in a given range, say between ω_0 and $\omega_0 + \varepsilon$. The exact number of such frequencies depends on the shape of the cavity, but according to Weyl's law, this number will be approximately proportional to $\varepsilon\omega_0^2$ for large values of ω_0. Thus, the amount of energy per unit of frequency is

$$C\frac{\hbar\omega^3}{e^{\beta\hbar\omega} - 1}, \tag{1.7}$$

where C is a constant involving the volume of the cavity and the speed of light. The relation (1.7) is known as Planck's law.

3. In classical mechanics, the kinetic energy of an electron is $m_e v^2/2$, where v is the magnitude of the velocity. Meanwhile, the potential energy associated with the force law (1.3) is $V(r) = -Q^2/r$, since $dV/dr = F$. Show that if the particle is moving in a circular orbit with radius r_n given by (1.5), then the total energy (kinetic plus potential) of the particle is E_n, as given in (1.1).