7

The Spectral Theorem for Bounded Self-Adjoint Operators: Statements

In the present chapter, we will consider the spectral theorem for *bounded* self-adjoint operators, leaving a discussion of unbounded operators to Chaps. 9 and 10. The proofs of the main theorems (two different versions of the spectral theorem) are moderately long and are deferred to Chap. 8. After some elementary definitions and results in Sect. 7.1, we come to the main results in Sects. 7.2 and 7.3. Throughout the chapter, $\mathbf{H}$ will, as usual, denote a separable Hilbert space over $\mathbb{C}$.

7.1 Elementary Properties of Bounded Operators

As usual, we will let $\mathbf{H}$ denote a separable complex Hilbert space. Recall from Appendix A.3.4 that a linear operator A on $\mathbf{H}$ is said to be *bounded* if the *operator norm* of A,

$$\|A\| := \sup_{\psi\in\mathbf{H}\setminus\{0\}} \frac{\|A\psi\|}{\|\psi\|} \tag{7.1}$$

is finite. The space of bounded operators on $\mathbf{H}$ forms a Banach space under the operator norm, and we have the inequality

$$\|AB\| \leq \|A\|\,\|B\| \tag{7.2}$$

for all bounded operators A and B.

Definition 7.1 *The Banach space of bounded operators on* $\mathbf{H}$*, with respect to the operator norm (7.1), is denoted* $\mathcal{B}(\mathbf{H})$*.*

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Recall (Appendix A.4.3) that for any $A \in \mathcal{B}(\mathbf{H})$ there is a unique operator $A^* \in \mathcal{B}(\mathbf{H})$, called the adjoint of A, such that

$$\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle$$

for all $\phi, \psi \in \mathbf{H}$. An operator $A \in \mathcal{B}(\mathbf{H})$ is called *self-adjoint* if $A^* = A$. We say that $A \in \mathcal{B}(\mathbf{H})$ is *non-negative* if

$$\langle \psi, A\psi \rangle \geq 0 \tag{7.3}$$

for all $\psi \in \mathbf{H}$.

Proposition 7.2 *For all $A \in \mathcal{B}(\mathbf{H})$, we have*

$$\|A^*\| = \|A\|$$

and

$$\|A^*A\| = \|A\|^2 .$$

In particular, if A is self-adjoint, we have the useful result that $\left\|A^2\right\| = \|A\|^2$.

Proof. The operator norm of A can also be computed as

$$\|A\| = \sup_{\|\psi\|=1} \|A\psi\| .$$

Furthermore, for any vector $\phi \in \mathbf{H}$, $\|\phi\| = \sup_{\|\chi\|=1} |\langle \chi, \phi \rangle|$. (Inequality one direction is by the Cauchy–Schwarz inequality, and inequality the other direction is by taking χ to be a multiple of ϕ.) Thus,

$$\|A\| = \sup_{\|\phi\|=\|\psi\|=1} |\langle \phi, A\psi \rangle| .$$

From this, we get

$$\begin{aligned}
\|A^*\| &= \sup_{\|\phi\|=\|\psi\|=1} |\langle \phi, A^*\psi \rangle| \\
&= \sup_{\|\phi\|=\|\psi\|=1} |\langle A\phi, \psi \rangle| \\
&= \sup_{\|\phi\|=\|\psi\|=1} |\langle \psi, A\phi \rangle| \\
&= \|A\| .
\end{aligned}$$

Meanwhile, $\|A^*A\| \leq \|A^*\| \, \|A\| = \|A\|^2$. On the other hand,

$$\begin{aligned}
\|A^*A\| &= \sup_{\|\phi\|=\|\psi\|=1} |\langle \phi, A^*A\psi \rangle| \\
&= \sup_{\|\phi\|=\|\psi\|=1} |\langle A\phi, A\psi \rangle| \\
&\geq \sup_{\|\psi\|=1} |\langle A\psi, A\psi \rangle| \\
&= \|A\|^2 ,
\end{aligned}$$

which establishes the inequality in the other order. ■

We now record an elementary but very useful result.

Proposition 7.3 *For all* $A \in \mathcal{B}(\mathbf{H})$*, we have*

$$[\mathrm{Range}(A)]^{\perp} = \ker(A^*),$$

where for any $B \in \mathcal{B}(\mathbf{H})$*,* $\ker(B)$ *denotes the kernel of* B*.*

Proof. Suppose first that ψ belongs to $[\mathrm{Range}(A)]^{\perp}$. Then for all $\phi \in \mathbf{H}$, we have

$$0 = \langle \psi, A\phi \rangle = \langle A^*\psi, \phi \rangle . \tag{7.4}$$

This implies that $A^*\psi = 0$ and thus that $\psi \in \ker(A^*)$. Conversely, suppose $\psi \in \ker(A^*)$. Then for all $\phi \in \mathbf{H}$, (7.4) holds (reading the equation from right to left). This shows that ψ is orthogonal to every element of the form $A\phi$, meaning that $\psi \in [\mathrm{Range}(A)]^{\perp}$. ■

Next, we define the *spectrum* of a bounded operator, which plays the same role as the set of eigenvalues in the finite-dimensional case.

Definition 7.4 *For* $A \in \mathcal{B}(\mathbf{H})$*, the* ***resolvent set*** *of* A*, denoted* $\rho(A)$ *is the set of all* $\lambda \in \mathbb{C}$ *such that the operator* $(A - \lambda I)$ *has a bounded inverse. The* ***spectrum*** *of* A*, denoted by* $\sigma(A)$*, is the complement in* $\mathbb{C}$ *of the resolvent set. For* λ *in the resolvent set of* A*, the operator* $(A - \lambda I)^{-1}$ *is called the* ***resolvent*** *of* A *at* λ*.*

Saying that $(A - \lambda I)$ has a bounded inverse means that there exists a bounded operator B such that

$$(A - \lambda I)B = B(A - \lambda I) = I.$$

If A is bounded and $A - \lambda I$ is one-to-one and maps $\mathbf{H}$ onto $\mathbf{H}$, then it follows from the closed graph theorem (Theorem A.39) that the inverse map must be bounded. Thus, the resolvent set of A can alternatively be described as the set of $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is one-to-one and onto.

Proposition 7.5 *For all* $A \in \mathcal{B}(\mathbf{H})$*, the following results hold.*

1. *The spectrum* $\sigma(A)$ *of* A *is a closed, bounded, and nonempty subset of* $\mathbb{C}$*.*

2. *If* $|\lambda| > \|A\|$*, then* λ *is in the resolvent set of* A*.*

Lemma 7.6 *Suppose* $X \in \mathcal{B}(\mathbf{H})$ *satisfies* $\|X\| < 1$*. Then the operator* $I - X$ *is invertible, with the inverse given by the following convergent series in* $\mathcal{B}(\mathbf{H})$*:*

$$(I - X)^{-1} = I + X + X^2 + X^3 + \cdots \tag{7.5}$$

Proof. As a consequence of (7.2), we have $\|X^m\| \le \|X\|^m$. The (geometric) series on the right-hand side of (7.5) is therefore absolutely convergent and thus convergent in the Banach space $\mathcal{B}(\mathbf{H})$ (Appendix A.3.4). If we multiply this series on either side by $(I-X)$, everything will cancel except I, showing that the sum of the series is the inverse of $(I-X)$. ∎

Proof of Proposition 7.5. For any nonzero $\lambda \in \mathbb{C}$, consider the operator

$$A - \lambda I = -\lambda\left(I - \frac{A}{\lambda}\right).$$

If $|\lambda| > \|A\|$, then $\|A/\lambda\| < 1$, and $I - A/\lambda$ is invertible by the lemma. It then follows that $A - \lambda I$ is invertible, with

$$(A-\lambda I)^{-1} = -\frac{1}{\lambda}\left(I + \frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \cdots\right). \tag{7.6}$$

Thus, λ is in the resolvent set of A. This establishes Point 2 in the proposition and shows that $\sigma(A)$ is bounded.

Suppose now that $\lambda_0 \in \mathbb{C}$ is in the resolvent set of A. Then for another number $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} A - \lambda I &= A - \lambda_0 I - (\lambda - \lambda_0) I \\ &= (A - \lambda_0 I)\,(I - (\lambda - \lambda_0)\,(A - \lambda_0 I)^{-1}). \end{aligned} \tag{7.7}$$

Thus, if

$$|\lambda - \lambda_0| < \frac{1}{\|(A-\lambda_0 I)^{-1}\|},$$

both factors on the right-hand side of (7.7) will be invertible, so that $A - \lambda I$ is also invertible. Thus, the resolvent set of A is open and the spectrum is closed.

To show that $\sigma(A)$ is nonempty, note that $A - \lambda I$ may be computed as follows:

$$\begin{aligned} (A-\lambda I)^{-1} &= (I - (\lambda-\lambda_0)(A-\lambda_0 I)^{-1})^{-1}(A-\lambda_0 I)^{-1} \\ &= \left(\sum_{m=0}^{\infty} (\lambda-\lambda_0)^m ((A-\lambda_0 I)^{-1})^m\right)(A-\lambda_0 I)^{-1}. \end{aligned} \tag{7.8}$$

Thus, near any point λ_0 in the resolvent set of A, the resolvent $(A-\lambda I)^{-1}$ can be computed by the locally convergent series (7.8) in powers of $\lambda - \lambda_0$, with the coefficients of the series being elements of $\mathcal{B}(\mathbf{H})$. For any $\phi, \psi \in \mathbf{H}$, the map

$$\lambda \mapsto \left\langle \phi, (A-\lambda I)^{-1}\psi \right\rangle \tag{7.9}$$

will be given by a locally convergent power series with coefficients in $\mathbb{C}$, meaning that the function (7.9) is a holomorphic function on the resolvent set of A. Furthermore, from (7.6) we can see that $\left\|(A-\lambda I)^{-1}\right\|$ tends to zero as $|\lambda|$ tends to infinity, and so also does the right-hand side of (7.9).

If $\sigma(A)$ were the empty set, the function (7.9) would be holomorphic on all of $\mathbb{C}$ and tending to zero at infinity. By Liouville's theorem, the right-hand side of (7.9) would have to be identically zero for all ϕ and ψ, which would mean that $(A-\lambda I)^{-1}$ is the zero operator. But since $(A-\lambda I)(A-\lambda I)^{-1}=I$, the operator $(A-\lambda I)^{-1}$ cannot be zero. ∎

If $A\psi=\lambda\psi$ for some $\lambda\in\mathbb{C}$ and some nonzero $\psi\in\mathbf{H}$, then $(A-\lambda I)$ has a nonzero kernel and so λ is in the spectrum of A. Thus, any eigenvalue for A is contained in the spectrum of A. In the infinite-dimensional case, however, the converse is not true: A point in the spectrum may not be an eigenvalue for A. Nevertheless, for a bounded *self-adjoint* operator A, the spectrum of A may be described in a way that is not too far removed from what we have in the finite-dimensional case.

Proposition 7.7 *If $A\in\mathcal{B}(\mathbf{H})$ is self-adjoint, then the following results hold.*

1. *The spectrum of A is contained in the real line.*
2. *A number $\lambda\in\mathbb{R}$ belongs to the spectrum of A if and only if there exists a sequence ψ_n of nonzero vectors in $\mathbf{H}$ such that*

$$\lim_{n\to\infty}\frac{\|A\psi_n-\lambda\psi_n\|}{\|\psi_n\|}=0. \tag{7.10}$$

Condition 2 in the proposition says that $\lambda\in\mathbb{R}$ belongs to the spectrum if and only if λ is "almost an eigenvalue," meaning that there exists $\psi\neq 0$ for which $A\psi$ is equal to $\lambda\psi$ plus an error that is small compared to the size of ψ.

Lemma 7.8 *If $A\in\mathcal{B}(\mathbf{H})$ is self-adjoint, then for all $\lambda=a+ib\in\mathbb{C}$, we have*

$$\langle(A-\lambda I)\psi,(A-\lambda I)\psi\rangle\geq b^2\langle\psi,\psi\rangle. \tag{7.11}$$

Proof. We compute that

$$\begin{aligned}&\langle(A-(a+ib)I)\psi,(A-(a+ib)I)\psi\rangle\\&=\langle(A-aI)\psi,(A-aI)\psi\rangle+ib\langle\psi,(A-aI)\psi\rangle\\&-ib\langle(A-aI)\psi,\psi\rangle+b^2\langle\psi,\psi\rangle.\end{aligned} \tag{7.12}$$

Since A is self-adjoint, so is $A-aI$, from which we see that the second and third terms on the right-hand side of (7.12) cancel, leaving us with

$$\langle(A-\lambda I)\psi,(A-\lambda I)\psi\rangle=\langle(A-aI)\psi,(A-aI)\psi\rangle+b^2\langle\psi,\psi\rangle,$$

from which the desired inequality follows. ∎

Proof of Proposition 7.7. For Point 1, we need to show that any complex number $\lambda = a + ib$ with $b \neq 0$ belongs to the resolvent set of A. Since $b \neq 0$, (7.11) shows that $A - \lambda I$ is injective. Meanwhile, by Proposition 7.3, $\text{Range}(A - \lambda I)^{\perp} = \ker(A - \bar{\lambda} I)$. Since $\bar{\lambda}$ also has nonzero imaginary part, $A - \bar{\lambda} I$ is injective, and so the range of $A - \lambda I$ is dense in $\mathbf{H}$. To show that the range is all of $\mathbf{H}$, consider any $\phi \in \mathbf{H}$ and choose a sequence $\phi_n = (A - \lambda I)\psi_n$ in $\text{Range}(A - \lambda I)$ with $\phi_n \to \phi$. Applying (7.11) with ψ replaced by $\psi_n - \psi_m$ shows that $\langle \psi_n \rangle$ is a Cauchy sequence. Thus, $\psi_n \to \psi$ for some $\psi \in \mathbf{H}$. Since A is bounded,

$$(A - \lambda I)\psi = \lim_{n\to\infty} (A - \lambda I)\psi_n = \lim_{n\to\infty} \phi_n = \phi.$$

We conclude, then, that $A - \lambda I$ is one-to-one and onto. The inverse operator $(A - \lambda I)^{-1}$ is bounded, by (7.11) (or by the closed graph theorem).

For Point 2, assume there exists a sequence as in (7.10), and suppose that $A - \lambda I$ had an inverse. Letting $\phi_n = (A - \lambda I)\psi_n$, we have $\psi_n = (A - \lambda I)^{-1}\phi_n$ and so (7.10) says that

$$\lim_{n\to\infty} \frac{\|\phi_n\|}{\|(A - \lambda I)^{-1}\phi_n\|} = 0,$$

which shows that $(A - \lambda I)^{-1}$ is actually unbounded. Thus, $A - \lambda I$ cannot have a *bounded* inverse.

Conversely, if, for some $\lambda \in \mathbb{R}$, no such sequence exists, then there exists some $\varepsilon > 0$ such that

$$\|(A - \lambda I)\psi\| \geq \varepsilon \|\psi\| \tag{7.13}$$

for all $\psi \in \mathbf{H}$. Then $A - \lambda I$ is injective and Proposition 7.3 tells us that the range of the self-adjoint operator $A - \lambda I$ is dense in $\mathbf{H}$. Arguing as in the preceding paragraphs with (7.13) in place of (7.11), we can see that the range of $A - \lambda I$ is also closed, hence all of $\mathbf{H}$. This shows that $A - \lambda I$ has an inverse. ∎

Example 7.9 *Let* $\mathbf{H} = L^2([0,1])$ *and let* A *be the operator on* $\mathbf{H}$ *defined by*

$$(A\psi)(x) = x\psi(x).$$

Then this operator is bounded and self-adjoint, and its spectrum is given by

$$\sigma(A) = [0, 1].$$

As we have already noted in Sect. 6.1, the operator A does not have any (true) eigenvectors.

Proof. It is apparent that $\|A\psi\| \leq \|\psi\|$ and that $\langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle$ for all $\phi, \psi \in \mathbf{H}$, so that A is bounded and self-adjoint. Given $\lambda \in (0,1)$, consider the functions $\psi_n := 1_{[\lambda \lambda + 1/n]}$, which satisfy $\|\psi_n\|^2 = 1/n$. On the other hand, since $|x - \lambda| \leq 1/n$ on $[\lambda, \lambda + 1/n]$, we have

$$\|(A - \lambda I)\psi_n\|^2 \leq 1/n^3.$$

Thus, by Proposition 7.7, λ belongs to the spectrum of A. Since this holds for all $\lambda \in (0,1)$ and the spectrum of A is closed, $\sigma(A) \supset [0,1]$.

Meanwhile, if $\lambda \notin [0,1]$, then the function $1/(x-\lambda)$ is bounded on $[0,1]$, and so $A - \lambda I$ has a bounded inverse, consisting of multiplication by $1/(x-\lambda)$. Thus, $\sigma(A) = [0,1]$. ■

7.2 Spectral Theorem for Bounded Self-Adjoint Operators, I

7.2.1 Spectral Subspaces

Given a bounded (for now) self-adjoint operator A, we hope to associate with each Borel set $E \subset \sigma(A)$ a closed subspace V_E of $\mathbf{H}$, where we think intuitively that V_E is the closed span of the generalized eigenvectors for A with eigenvalues in E. [We could do this more generally for any $E \subset \mathbb{R}$, but we do not expect any contribution from $\mathbb{R}\backslash\sigma(A)$.] We would expect the collection of these subspaces to have the following properties.

1. $V_{\sigma(A)} = \mathbf{H}$ and $V_{\varnothing} = \{0\}$.

2. If E and F are disjoint, then $V_E \perp V_F$.

3. For any E and F, $V_{E\cap F} = V_E \cap V_F$.

4. If $E_1, E_2, \ldots$ are disjoint and $E = \cup_j E_j$, then
$$V_E = \bigoplus_j V_{E_j}.$$

5. For any E, V_E is invariant under A.

6. If $E \subset [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ and $\psi \in V_E$, then
$$\|(A - \lambda_0 I)\psi\| \leq \varepsilon \|\psi\|.$$

The condition $V_{\sigma(A)} = \mathbf{H}$ captures the idea that our generalized eigenvectors should span $\mathbf{H}$, while Property 2 captures the idea that our generalized eigenvectors should have some sort of orthogonality for distinct eigenvalues, even if they are not actually in the Hilbert space. In Property 4, there may be infinitely many of the E_j's, in which case, the direct sum is in the Hilbert space sense (Definition A.45). Properties 5 and 6 capture the idea that V_E is made up of generalized eigenvectors for A with eigenvalues in E.

7.2.2 Projection-Valued Measures

It is convenient to describe closed subspaces of a Hilbert space $\mathbf{H}$ in terms of the associated orthogonal projection operators. Recall (Proposition A.57) that, given a closed subspace V of $\mathbf{H}$, there exists a unique bounded operator P that equals the identity on V and equals zero on the orthogonal complement $V^\perp$ of V. This operator is called the orthogonal projection onto V and satisfies $P^2 = P$ and $P^* = P$. The following definition expresses the first four properties of our spectral subspaces—the ones that do not involve the operator A—in terms of the corresponding orthogonal projections. Since those properties are similar to those of a measure, we use the term *projection-valued measure*.

Definition 7.10 *Let X be a set and Ω a σ-algebra in X. A map $\mu : \Omega \to \mathcal{B}(\mathbf{H})$ is called a* ***projection-valued measure*** *if the following properties are satisfied.*

1. *For each $E \in \Omega$, $\mu(E)$ is an orthogonal projection.*
2. *$\mu(\varnothing) = 0$ and $\mu(X) = I$.*
3. *If $E_1, E_2, E_3, \ldots$ in Ω are disjoint, then for all $v \in \mathbf{H}$, we have*
$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) v = \sum_{j=1}^{\infty} \mu(E_j)v,$$
where the convergence of the sum is in the norm topology on $\mathbf{H}$.
4. *For all $E_1, E_2 \in \Omega$, we have $\mu(E_1 \cap E_2) = \mu(E_1)\mu(E_2)$.*

Note that if E_1 and E_2 are disjoint, then Properties 2 and 4 tell us that $\mu(E_1)\mu(E_2) = 0$, from which it follows (Exercise 10) that the range of $\mu(E_1)$ and the range of $\mu(E_2)$ are perpendicular. It is then not hard to verify that $\mu(E_1)\mu(E_2)$ is the projection onto the intersection of the ranges of $\mu(E_1)$ and $\mu(E_2)$ (Exercise 11). Thus, if we define, for each $E \in \Omega$, a closed subspace $V_E := \mathrm{Range}(\mu(E))$, then the collection of V_E's satisfy the first four properties that we anticipated for spectral subspaces.

In the next subsection, we will associate a projection-valued measure μ^A with each bounded self-adjoint operator A. In that case, the projection $\mu^A(E)$ will be thought of as a projection onto the spectral subspace corresponding to E. We are about to introduce the notion of operator-valued integration with respect to a projection-valued measure. In the case of the projection-valued measure μ^A associated with A, this operator-valued integral will be the *functional calculus* for A.

Observe that, for any projection-valued measure μ and $\psi \in \mathbf{H}$, we can form an ordinary (positive) real-valued measure μ_ψ by setting

$$\mu_\psi(E) = \langle \psi, \mu(E)\psi \rangle \tag{7.14}$$

for all $E \in \Omega$. This observation provides a link between integration with respect to a projection-valued measure and integration with respect to an ordinary measure.

Proposition 7.11 (Operator-Valued Integration) *Let Ω be a σ-algebra in a set X and let $\mu : \Omega \to \mathcal{B}(\mathbf{H})$ be a projection-valued measure. Then there exists a unique linear map, denoted $f \mapsto \int_\Omega f\ d\mu$, from the space of bounded, measurable, complex-valued functions on Ω into $\mathcal{B}(\mathbf{H})$ with the property that*

$$\left\langle \psi, \left(\int_X f\ d\mu \right) \psi \right\rangle = \int_X f\ d\mu_\psi \tag{7.15}$$

for all f and all $\psi \in \mathbf{H}$, where μ_ψ is given by (7.14). This integral has the following additional properties.

1. *For all $E \in \Omega$, we have*

$$\int_X 1_E\ d\mu = \mu(E).$$

   In particular, the integral of the constant function 1 is I.

2. *For all f, we have*

$$\left\| \int_X f\ d\mu \right\| \leq \sup_{\lambda \in X} |f(\lambda)|. \tag{7.16}$$

3. *Integration is multiplicative: For all f and g, we have*

$$\int_X fg\ d\mu = \left(\int_X f\ d\mu \right) \left(\int_X g\ d\mu \right). \tag{7.17}$$

4. *For all f, we have*

$$\int_X \bar{f}\ d\mu = \left(\int_X f\ d\mu \right)^*.$$

   In particular, if f is real-valued, then $\int_X f\ d\mu$ is self-adjoint.

By Property 1 and linearity, integration with respect to μ has the expected behavior on simple functions. It then follows from Property 2 that the integral of an arbitrary bounded measurable function f can be computed as follows. Take a sequence s_n of simple functions converging uniformly to f; the integral of f is then the limit, in the operator norm topology, of the integral of the s_n's.

Although the multiplicative property of the integral may seem surprising at first, observe that for any $E_1, E_2 \in \Omega$, Property 3 in Definition 7.10 tells us that

$$\left(\int_X 1_{E_1}\, d\mu\right)\left(\int_X 1_{E_2}\, d\mu\right) = \mu(E_1)\mu(E_2) = \mu(E_1 \cap E_2)$$
$$= \int_X 1_{E_1} \cdot 1_{E_2}\, d\mu.$$

Thus, multiplicativity of the integral at the level of indicator functions is built into the definition of a projection-valued measure.

If one wanted to make a real-valued measure for which the corresponding integral was multiplicative, then since $1_E \cdot 1_E = 1_E$, the integral of 1_E—namely, $\mu(E)$—would have to satisfy $\mu(E)^2 = \mu(E)$. This would mean that $\mu(E)$ is 0 or 1 for all E. For such measures, one would indeed obtain multiplicativity of the integral, but measures with this property are not very interesting. For operator-valued measures, we can have *interesting* examples where the integral is multiplicative, simply because there are many more idempotents (elements A with $A^2 = A$) in $\mathcal{B}(\mathbf{H})$ than in $\mathbb{R}$.

Proof of Proposition 7.11. Given a projection-valued measure μ and a bounded measurable function f on X, define a map $Q_f : \mathbf{H} \to \mathbb{C}$ by

$$Q_f(\psi) = \int_X f\, d\mu_\psi,$$

where μ_ψ is given by (7.14). If f is an indicator function, then $Q_f(\psi) = \langle \psi, \mu(E)\psi \rangle$ is a bounded quadratic form. (See Definition A.60.) It is straightforward to show, passing from indicator functions to simple functions and then to general functions, that for any bounded measurable f, Q_f is a bounded quadratic form, with

$$|Q_f(\psi)| \leq \left(\sup_{\lambda \in X} |f(\lambda)|\right) \|\psi\|^2 . \tag{7.18}$$

It then follows from Proposition A.63 that there is a unique bounded operator A_f such that

$$Q_f(\psi) = \langle \psi, A_f \psi \rangle$$

for all $\psi \in \mathbf{H}$. We set $\int_X f\, d\mu = A_f$. From the way A_f is defined, it satisfies (7.15). The uniqueness of the linear map $f \mapsto \int_X f\, d\mu$ follows from the uniqueness in Proposition A.63.

If $f = 1_E$, then $Q_f(\psi) = \mu_\psi(E) = \langle \psi, \mu(E)\psi \rangle$, in which case the unique associated operator A_f is $\mu(E)$. This establishes Property 1. Property 2 follows from (7.18).

For Property 3, we have already observed that multiplicativity of the integral, at the level of indicator functions, is built into the definition of a projection-valued measure. Since both sides of (7.17) are bilinear in (ϕ, ψ), we have (7.17) for simple functions. Using Property 2, we can then obtain (7.17) for all bounded measurable functions by taking limits.

Finally, if f is real valued, then $Q_f(\psi)$ will be real for all $\psi \in \mathbf{H}$. Thus, by Proposition A.63, the associated operator A_f will be self-adjoint. Property 4 then follows by linearity. ■

7.2.3 The Spectral Theorem

We are ready to state one version of the spectral theorem for bounded self-adjoint operators.

Theorem 7.12 (Spectral Theorem, First Form) *If $A \in \mathcal{B}(\mathbf{H})$ is self-adjoint, then there exists a unique projection-valued measure μ^A on the Borel σ-algebra in $\sigma(A)$, with values in projections on $\mathbf{H}$, such that*

$$\int_{\sigma(A)} \lambda \; d\mu^A(\lambda) = A. \tag{7.19}$$

Since the spectrum $\sigma(A)$ of A is bounded, the function $f(\lambda) := \lambda$ is bounded on $\sigma(A)$. The proof of this theorem is given in Chap. 8.

Definition 7.13 (Functional Calculus) *If $A \in \mathcal{B}(\mathbf{H})$ is self-adjoint and $f : \sigma(A) \to \mathbb{C}$ is a bounded measurable function, define an operator $f(A)$ by setting*

$$f(A) = \int_{\sigma(A)} f(\lambda) \; d\mu^A(\lambda),$$

where μ^A is the projection-valued measure in Theorem 7.12.

We may extend the projection-valued measure μ^A from $\sigma(A)$ to all of $\mathbb{R}$ by assigning measure 0 to $\mathbb{R} \setminus \sigma(A)$. Then, roughly speaking, $f(A)$ is the operator that is equal to $f(\lambda)I$ on the range of the projection operator $\mu^A([\lambda, \lambda + d\lambda))$.

Since the integral with respect to μ^A is multiplicative, it follows from (7.19) that if $f(\lambda) = \lambda^m$ for some positive integer m, then $f(A)$ is the mth power of A. Further, since the series $e^{a\lambda} = \sum_{m=0}^{\infty} (a\lambda)^m/m!$ converges uniformly on the compact set $\sigma(A)$, the operator e^{aA} (computed using the functional calculus for the function $f(\lambda) = e^{a\lambda}$) may be computed as a power series.

Definition 7.14 (Spectral Subspaces) *For $A \in \mathcal{B}(\mathbf{H})$, let μ^A be the associated projection-valued measure, extended to be a measure on $\mathbb{R}$ by setting $\mu^A(\mathbb{R} \setminus \sigma(A)) = 0$. Then for each Borel set $E \subset \mathbb{R}$, define the **spectral subspace** V_E of $\mathbf{H}$ by*

$$V_E = \mathrm{Range}(\mu^A(E)).$$

The definition of a projection-valued measure implies that these spectral subspaces satisfy the first four properties listed in Sect. 7.2.1. We now show that (7.19) implies the remaining two properties we anticipated for the spectral subspaces.

Proposition 7.15 *If $A \in \mathcal{B}(\mathbf{H})$ is self-adjoint, the spectral subspaces associated with A have the following properties.*

1. *Each spectral subspace V_E is invariant under A.*
2. *If $E \subset [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ then for all $\psi \in V_E$, we have*
$$\|(A - \lambda_0 I)\psi\| \leq \varepsilon \|\psi\| .$$
3. *The spectrum of $A|_{V_E}$ is contained in the closure of E.*
4. *If λ_0 is in the spectrum of A, then for every neighborhood U of λ_0, we have $V_U \neq \{0\}$, or, equivalently, $\mu(U) \neq 0$.*

Proof. For Point 1, observe that for any bounded measurable functions f and g on $\sigma(A)$, the operators $f(A)$ and $g(A)$ commute, since the product in either order is equal to the integral of the function $fg = gf$ with respect to μ^A. In particular, A, which is the integral of the function $f(\lambda) = \lambda$, commutes with $\mu^A(E)$, which is the integral of the function 1_E. Thus, given a vector $\mu^A(E)\phi$ in the range of $\mu^A(E)$, we have
$$A\mu^A(E)\phi = \mu^A(E)A\phi,$$
which is again in the range of $\mu^A(E)$, establishing the invariance of the spectral subspace.

For Point 2, suppose that $\psi \in V_E$, where $E \subset [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$. Then ψ is in the range of $\mu^A(E)$, and so
$$(A - \lambda_0 I)\psi = (A - \lambda_0 I)\mu^A(E)\psi.$$
But $\mu^A(E) = 1_E(A)$ and $A - \lambda_0 I = f(A)$, where $f(\lambda) = \lambda - \lambda_0$. By the multiplicativity of the integral, then,
$$(A - \lambda_0 I)\psi = (f 1_E)(A)\psi.$$
But $|f(\lambda)1_E(\lambda)| \leq \varepsilon$ and so by (7.16), the operator $(f1_E)(A)$ has norm at most ε.

For Point 3, if λ_0 is not in $\bar{E}$, then the function $g(\lambda) := 1_E(\lambda)(1/(\lambda - \lambda_0))$ is bounded. Thus, $g(A)$ is a bounded operator and
$$g(A)(A - \lambda_0 I) = (A - \lambda_0 I)g(A) = 1_E(A).$$
This shows that the restriction to V_E of $g(A)$ is the inverse of the restriction to V_E of A. Thus, λ_0 is not in the spectrum of $A|_{V_E}$.

For Point 4, fix $\lambda_0 \in \sigma(A)$ and suppose for some $\varepsilon > 0$, we have $\mu((\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)) = 0$. Consider, then, the bounded function f defined by
$$f(\lambda) = \begin{cases} \frac{1}{\lambda - \lambda_0} & |\lambda - \lambda_0| \geq \varepsilon \\ 0 & |\lambda - \lambda_0| < \varepsilon \end{cases} .$$

Since $f(\lambda)\cdot(\lambda-\lambda_0)$ equals 1 except on $(\lambda_0-\varepsilon,\lambda_0+\varepsilon)$, the equation $f(\lambda)\cdot(\lambda-\lambda_0)=1$ holds μ-almost everywhere. Thus, the integral of this function coincides with the integral of the constant function 1, which is I. Since the integral is multiplicative, we see that

$$f(A)(A-\lambda_0 I)=(A-\lambda_0 I)f(A)=I,$$

showing that the bounded operator $f(A)$ is the inverse of $(A-\lambda_0 I)$. This contradicts the assumption that $\lambda_0\in\sigma(A)$. ■

Proposition 7.16 *If $A\in\mathcal{B}(\mathbf{H})$ is self-adjoint and $B\in\mathcal{B}(\mathbf{H})$ commutes with A, the following results hold.*

1. *For all bounded measurable functions f on $\sigma(A)$, the operator $f(A)$ commutes with B.*
2. *Each spectral subspace for A is invariant under B.*

The proof of this proposition is deferred until Chap. 8. We conclude this section by fulfilling (at least for bounded self-adjoint operators) one of the goals of the spectral theorem, namely to give a probability measure describing the probabilities for measurements of a self-adjoint operator A in the state ψ.

Proposition 7.17 *Suppose $A\in\mathcal{B}(\mathbf{H})$ is self-adjoint and $\psi\in\mathbf{H}$ is a unit vector. Then there exists a unique probability measure μ_ψ^A on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}}\lambda^m\,d\mu_\psi^A(\lambda)=\langle\psi,A^m\psi\rangle$$

for all non-negative integers m.

We will prove a version of Proposition 7.17 for unbounded self-adjoint operators in Chap. 9. In the unbounded case, however, we will not obtain uniqueness of the probability measure, even if ψ is in the domain of A^m for all m. Even in the unbounded case, however, the spectral theorem provides a *canonical* choice of the probability measure.

Proof. We define a measure μ_ψ^A on $\sigma(A)$ as in Sect. 7.2.2 by

$$\mu_\psi^A(E)=\left\langle\psi,\mu^A(E)\psi\right\rangle.$$

The properties of integration with respect to μ^A then tell us that

$$\langle\psi,A^m\psi\rangle=\left\langle\psi,\left(\int_{\sigma(A)}\lambda^m\,d\mu^A(\lambda)\right)\psi\right\rangle=\int_{\sigma(A)}\lambda^m\,d\mu_\psi^A(\lambda).$$

We then extend μ_ψ^A to $\mathbb{R}$ by setting it equal to zero on $\mathbb{R}\backslash\sigma(A)$, establishing the existence of the desired probability measure on $\mathbb{R}$. Since

$$|\langle\psi,A^m\psi\rangle|\le\|\psi\|^2\,\|A^m\|\le\|\psi\|^2\,\|A\|^m,$$

the moments grow only exponentially with m. Thus, standard uniqueness results for the moment problem (e.g., Theorem 8.1 in Chap. 4 of [18]) give the uniqueness of μ_ψ^A. ■

7.3 Spectral Theorem for Bounded Self-Adjoint Operators, II

As we have already noted in Sect. 6.5, one version of the spectral theorem asserts that every self-adjoint operator is unitarily equivalent to a multiplication operator. In the case of a bounded self-adjoint operator A, on a separable Hilbert space $\mathbf{H}$, this result means that A is unitarily equivalent to the operator M_h on $L^2(X,\mu)$, where (X,μ) is a σ-finite measure space, h is a measurable, real-valued function, and M_h is the operator of multiplication by h:

$$(M_h\psi)(\lambda) = h(\lambda)\psi(\lambda).$$

Although the "multiplication operator" form of the spectral theorem (Theorem 7.20) has the advantage of being easy to state, there is an even better version involving the concept of a direct integral. It is straightforward to extend the notion of an L^2 space to an L^2 space with values in a Hilbert space $\mathbf{H}$. In a direct integral, we extend the concept one step further, by allowing the Hilbert space to depend on the point. We begin with a measure space (X,μ) and then have one Hilbert space $\mathbf{H}_\lambda$ for each λ in X. An element of the direct integral is a function s on X such that $s(\lambda)$ belongs to $\mathbf{H}_\lambda$ for each $\lambda \in X$. Given a *real-valued* measurable function h on X, it makes sense to multiply an element s of the direct integral by h.

The direct integral form of the spectral theorem says a bounded self-adjoint operator A is unitarily equivalent to a multiplication operator on a direct integral. By extending multiplication operators to the more general setting of direct integrals (instead of just ordinary L^2 spaces), we gain several benefits. First, the set X and the function h become canonical: The set X is simply the spectrum of A and the function h is simply $h(\lambda) = \lambda$. Second, the direct integral approach carries with it a notion of "generalized eigenvectors," since the space $\mathbf{H}_\lambda$ can be thought of as the space of generalized eigenvectors with eigenvalue λ. (The spaces $\mathbf{H}_\lambda$ are not, in general, contained in the direct integral Hilbert space. Thus, direct integrals give a rigorous meaning to the idea of "eigenvectors" that are not in the Hilbert space on which the operator acts.) Third, the direct integral approach gives a simple way to classify self-adjoint operators up to unitary equivalence: Two self-adjoint operators are unitarily equivalent if and only if their direct integral representations are equivalent in a natural sense (Proposition 7.24).

If one really wants the simplicity of the (ordinary) multiplication operator version of the spectral theorem, it is a simple matter to prove this result using precisely the same methods as in the proof of the direct integral version. (See Theorem 7.20.) Nevertheless, the direct integral version is, arguably, the most definitive version of the spectral theorem for a single self-adjoint operator.

We turn now to the definition of a direct integral. Suppose μ is a σ-finite measure on a σ-algebra Ω of sets in X. Suppose also that for each $\lambda \in X$, we have a separable Hilbert space $\mathbf{H}_\lambda$ with inner product $\langle \cdot, \cdot \rangle_\lambda$. We want to define the direct integral of the $\mathbf{H}_\lambda$'s with respect to μ. Elements of the direct integral will be *sections* s, meaning that s is a function on X with values in the union of the $\mathbf{H}_\lambda$'s, having the property that

$$s(\lambda) \in \mathbf{H}_\lambda$$

for each λ in X. We would like to define the norm of a section s by the formula

$$\|s\|^2 = \int_X \langle s(\lambda), s(\lambda) \rangle_\lambda \, d\mu(\lambda),$$

provided that the integral on the right-hand side is finite. The inner product of two sections s_1 and s_2 (with finite norm) should then be given by the formula

$$\langle s_1, s_2 \rangle := \int_X \langle s_1(\lambda), s_2(\lambda) \rangle_\lambda \, d\mu(\lambda).$$

The problem with this description of the norm and inner product on the direct integral is that we have not said anything about measurability. As things stand, it does not make sense to ask whether a section s is measurable, since the space in which $s(\lambda)$ takes its values is different for each λ. We must, therefore, introduce some additional structure that gives rise to a notion of measurability. (The measurability issue is a technicality that can be ignored on a first reading.)

One way to address the measurability issue is to choose a *simultaneous orthonormal basis* for each of the Hilbert spaces $\mathbf{H}_\lambda$. To deal with the possibility that different spaces can have different dimensions, we slightly modify the concept of an orthonormal basis. We say that a family $\{e_j\}$ of vectors is an orthonormal basis for a Hilbert space $\mathbf{H}$ if $\langle e_j, e_k \rangle = 0$ for $j \neq k$, the norm of each e_j is *either* 0 or 1, and the closure of the span of the e_j's is all of $\mathbf{H}$. This just means that we allow some of the vectors in our basis to be zero, with the nonzero vectors forming an orthonormal basis in the usual sense.

We now define a simultaneous orthonormal basis for a family $\{\mathbf{H}_\lambda\}$ of separable Hilbert spaces to be a collection $\{e_j(\cdot)\}_{j=1}^\infty$ of sections with the property that for each λ, $\{e_j(\lambda)\}_{j=1}^\infty$ is an orthonormal basis for $\mathbf{H}_\lambda$. Provided that the function $\lambda \mapsto \dim \mathbf{H}_\lambda$ is a measurable function from X into $[0, \infty]$, it is possible to choose a simultaneous orthonormal basis $\{e_j(\cdot)\}$ such that $\langle e_j(\lambda), e_k(\lambda) \rangle$ is measurable for all j and k. Having chosen a simultaneous orthonormal basis with this property, we *define* a section s to be measurable if the function

$$\lambda \mapsto \langle e_j(\lambda), s(\lambda)\rangle_\lambda$$

is a measurable complex-valued function for each j. Our assumption on the e_j's means that the e_j's themselves are measurable sections.

We refer to a choice of simultaneous orthonormal basis, chosen so that $\langle e_j(\lambda), e_k(\lambda)\rangle$ is measurable, as a *measurability structure* on the collection of $\mathbf{H}_\lambda$'s. Given two measurable sections s_1 and s_2, the function

$$\lambda \mapsto \langle s_1(\lambda), s_2(\lambda)\rangle_\lambda = \sum_{j=1}^{\infty} \langle s_1(\lambda), e_j(\lambda)\rangle_\lambda \langle e_j(\lambda), s_2(\lambda)\rangle_\lambda$$

is also measurable.

Definition 7.18 *Suppose the following structures are given: (1) a σ-finite measure space (X, Ω, μ), (2) a collection $\{\mathbf{H}_\lambda\}_{\lambda\in X}$ of separable Hilbert spaces for which the dimension function is measurable, and (3) a measurability structure on $\{\mathbf{H}_\lambda\}_{\lambda\in X}$. Then the* ***direct integral*** *of the $\mathbf{H}_\lambda$'s with respect to μ, denoted*

$$\int_X^{\oplus} \mathbf{H}_\lambda \; d\mu(\lambda),$$

is the space of equivalence classes of almost-everywhere-equal measurable sections s for which

$$\|s\|^2 := \int_X \langle s(\lambda), s(\lambda)\rangle_\lambda \;\; d\mu(\lambda) < \infty.$$

The inner product $\langle s_1, s_2\rangle$ of two such sections s_1 and s_2 is given by the formula

$$\langle s_1, s_2\rangle := \int_X \langle s_1(\lambda), s_2(\lambda)\rangle_\lambda \;\; d\mu(\lambda).$$

To see that the integral defining the inner product of two finite-norm sections is finite, note that $|\langle s_1(\lambda), s_2(\lambda)\rangle_\lambda| \leq \|s_1(\lambda)\|_\lambda \|s_2(\lambda)\|_\lambda$. By assumption, $\|s_j(\lambda)\|_\lambda$ is a square-integrable function of λ for $j = 1, 2$, and the product of two square-integrable functions is integrable. Thus, the integrand in the definition of $\langle s_1, s_2\rangle$ is also integrable. It is not hard to show, using an argument similar to the proof of completeness of L^2 spaces, that a direct integral of Hilbert spaces is a Hilbert space.

Let us think of two important special cases of the direct integral construction. First, if each of the $\mathbf{H}_\lambda$'s is simply $\mathbb{C}$, then the direct integral (with the obvious measurability structure) is simply $L^2(X, \mu)$. Second, suppose that $X = \{\lambda_1, \lambda_2, \ldots\}$ is countable, Ω is the σ-algebra of all subsets of X, and μ is the counting measure on X. Then the direct integral is the *Hilbert space direct sum* (Definition A.45).

Given a direct integral, suppose we have some $\lambda_0 \in X$ for which $\{\lambda_0\}$ is measurable and such that $c := \mu(\{\lambda_0\}) > 0$. Then we can embed $\mathbf{H}_{\lambda_0}$ isometrically into the direct integral by mapping each $\psi \in \mathbf{H}_{\lambda_0}$ to the section s given by

$$s(\lambda) = \begin{cases} \frac{1}{\sqrt{c}}\psi, & \lambda = \lambda_0 \\ 0, & \lambda \neq \lambda_0 \end{cases}.$$

Even if $\mu(\{\lambda_0\}) = 0$, we may still think that $\mathbf{H}_{\lambda_0}$ is a sort of "generalized subspace" of the direct integral.

Theorem 7.19 (Spectral Theorem, Second Form) *If $A \in \mathcal{B}(\mathbf{H})$ is self-adjoint, then there exists a σ-finite measure μ on $\sigma(A)$, a direct integral*

$$\int_{\sigma(A)}^{\oplus} \mathbf{H}_\lambda \; d\mu(\lambda),$$

and a unitary map U between $\mathbf{H}$ and the direct integral such that

$$\left[UAU^{-1}(s)\right](\lambda) = \lambda s(\lambda) \tag{7.20}$$

for all sections s in the direct integral.

The proof of Theorem 7.19 is given in the next chapter, along with the proof of our first version of the spectral theorem. In the meantime, let us think about what this version of the spectral theorem is saying. We may think that the unitary map U is an identification of our original Hilbert space $\mathbf{H}$ with a certain direct integral over the spectrum of A. Under this identification, the self-adjoint operator A becomes the operator of multiplication by λ, that is, the map sending the section $s(\lambda)$ to $\lambda s(\lambda)$. Roughly speaking, then, the operator A acts (under our identification) as λI on each space $\mathbf{H}_\lambda$. Thus, we may think of $\mathbf{H}_\lambda$ as being something like an "eigenspace" for A, for each element λ of the spectrum of A. Of course, unless $\mu(\{\lambda\}) > 0$, the Hilbert space $\mathbf{H}_\lambda$ is not actually contained in $\mathbf{H}$. Nevertheless, we may think of elements of a given $\mathbf{H}_\lambda$ as "generalized eigenvectors" for the operator A.

The direct integral formulation of the spectral theorem leads readily to a classification result for bounded self-adjoint operators. See Proposition 7.24 later in this section. Meanwhile, as we noted earlier in this section, the method of proof for Theorem 7.19 also yields a version of the spectral theorem involving multiplication operators on ordinary L^2 spaces.

Theorem 7.20 (Spectral Theorem, Multiplication Operator Form) *Suppose $A \in \mathcal{B}(\mathbf{H})$ is self-adjoint. Then there exists a σ-finite measure space (X, μ), a bounded, measurable, real-valued function h on X, and a unitary map $U : \mathbf{H} \to L^2(X, \mu)$ such that*

$$[UAU^{-1}(\psi)](\lambda) = h(\lambda)\psi(\lambda)$$

for all $\psi \in L^2(X, \mu)$.

We return now to a discussion of the direct integral version of the spectral theorem. This version gives a simple description of the functional calculus.

Proposition 7.21 *Suppose $A \in \mathcal{B}(\mathbf{H})$ is self-adjoint and U is a unitary map as in Theorem 7.19. Then for any bounded measurable function f on $\sigma(A)$, we have*

$$[Uf(A)U^{-1}(s)](\lambda) = f(\lambda)s(\lambda).$$

Thus, roughly speaking, $f(A)$ is defined to be $f(\lambda)I$ on each "generalized eigenspace" $\mathbf{H}_\lambda$. Proposition 7.21 follows directly from (7.20) if f is a polynomial; the result for continuous f then follows by taking uniform limits. The result for general f is then easily established by using the limiting arguments of Chap. 8, especially Exercise 3.

Let us now consider what sort of uniqueness there should be in the second version of the spectral theorem. There is a "trivial" source of nonuniqueness coming from the possibility that some of the $\mathbf{H}_\lambda$'s may have dimension 0. Let E_0 denote the set of λ for which $\dim \mathbf{H}_\lambda = 0$. Even if $\mu(E_0) > 0$, the set E_0 makes no contribution to the norm of a section, since every section is automatically zero on E_0. Thus, we may define a new measure $\tilde{\mu}$ by setting $\tilde{\mu}(E) = \mu(E \cap E_0^c)$, so that $\tilde{\mu}$ agrees with μ on E_0^c but is zero on E_0. Then the direct integrals of the $\mathbf{H}_\lambda$'s with respect to μ and with respect to $\tilde{\mu}$ are "indistinguishable." Thus, we can always modify a direct integral so as to assume that $\dim \mathbf{H}_\lambda > 0$ for almost every λ.

Meanwhile, unlike the projection-valued measure μ^A in Theorem 7.12, the measure μ in Theorem 7.19 is not unique, but only unique up to equivalence, where two σ-finite measures on a given measurable space are equivalent if they have precisely the same sets of measure zero. For a given measure μ, the Hilbert spaces $\mathbf{H}_\lambda$ are unique only up to unitary equivalence, meaning that only the *dimension* of the spaces is uniquely determined. Even the dimension of $\mathbf{H}_\lambda$ is uniquely determined only up to a set of μ-measure zero. As it turns out, the sources of nonuniqueness in this paragraph and the previous paragraph are all that exist.

Proposition 7.22 (Uniqueness in Theorem 7.19) *Suppose $A \in \mathcal{B}(\mathbf{H})$ is self-adjoint and consider two different direct integrals as in Theorem 7.19, one with measure $\mu^{(1)}$ and Hilbert spaces $\mathbf{H}_\lambda^{(1)}$ and the other with measure $\mu^{(2)}$ and Hilbert spaces $\mathbf{H}_\lambda^{(2)}$. If $\dim \mathbf{H}_\lambda^{(j)} > 0$ for $\mu^{(j)}$-almost every λ ($j = 1, 2$), then $\mu^{(1)}$ and $\mu^{(2)}$ are mutually absolutely continuous and*

$$\dim \mathbf{H}_\lambda^{(1)} = \dim \mathbf{H}_\lambda^{(2)}$$

for $\mu^{(j)}$-almost every λ ($j = 1, 2$).

See the end of the next chapter for a sketch of the proof of this uniqueness result.

Theorem 7.19 should be thought of as a refinement of our earlier form (Theorem 7.12) of the spectral theorem, in the sense that we can easily recover Theorem 7.12 from Theorem 7.19. In the setting of Theorem 7.19, and given a measurable set $E \subset \sigma(A)$, let V_E denote the space of (equivalence classes) of sections s that are supported on E, that is, for which $s(\lambda) = 0$ for μ-almost every λ in E^c. This is easily seen to be a closed subspace. Let P_E denote the orthogonal projection onto V_E, and define

$$\mu^A(E) = U^{-1} P_E U. \tag{7.21}$$

It is straightforward to check that μ^A is a projection-valued measure on $\sigma(A)$, with values in $\mathcal{B}(\mathbf{H})$, and that $\int_{\sigma(A)} \lambda \, d\mu^A(\lambda) = A$.

Note that both versions of the spectral theorem for A involve a measure, the first, denoted μ^A, being a projection-valued measure, and the second, denoted μ, being an ordinary measure with values in the non-negative real numbers. The following result shows the relationship between the two measures.

Proposition 7.23 *Suppose $A \in \mathcal{B}(\mathbf{H})$ is self-adjoint, μ^A is the projection-valued measure given by Theorem 7.12 and μ is a real-valued measure as in Theorem 7.19. If $\dim \mathbf{H}_\lambda > 0$ for μ-almost every λ, then for any Borel set $E \subset \sigma(A)$, $\mu^A(E) = 0$ if and only if $\mu(E) = 0$.*

Of course, the 0 in the expression $\mu^A(E) = 0$ is the zero *operator*, whereas the 0 in the expression $\mu(E) = 0$ is the *number* 0. Nevertheless, we may think of Proposition 7.23 as saying that μ^A and μ are equivalent in the usual measure-theoretic sense, having precisely the same sets of measure zero.

Proof. As we have remarked, given a direct integral as in Theorem 7.19, we can construct a projection-valued measure by means of (7.21), and this projection-valued measure satisfies $\int_{\sigma(A)} \lambda \, d\mu^A(\lambda) = A$. This projection-valued measure must coincide with the one in Theorem 7.12, by the uniqueness in that theorem.

Now, if $\mu(E) = 0$, then any section supported on E is zero almost everywhere and thus represents the zero element of the direct integral. In that case, $V_E = 0$ and so $\mu^A(E) = 0$ by (7.21). In the other direction, suppose $\mu(E) > 0$. Since μ is σ-finite, E will contain a measurable subset F such that $0 < \mu(F) < \infty$. Then let s be the section given by

$$s(\lambda) = \sum_{j=1}^{\infty} \frac{1}{2^j} e_j(\lambda)$$

for $\lambda \in F$ and $s(\lambda) = 0$ for $\lambda \in F^c$, where $\{e_j(\cdot)\}$ is our measurability structure for the direct integral. Then

$$\langle s(\lambda), e_j(\lambda) \rangle_\lambda = \frac{1}{2^j} \langle e_j(\lambda), e_j(\lambda) \rangle_\lambda \, 1_F(\lambda),$$

which is a measurable function of λ for all j, so that s is measurable. Since we assume that $\mathbf{H}_\lambda$ has nonzero dimension for μ-almost every λ, s will be nonzero almost everywhere on F and thus will have positive norm. The norm of s is finite because $\|s(\lambda)\| \leq 1$ and F has finite measure. Thus, $V_E \neq 0$ and $\mu^A(E) \neq 0$. ■

We say that self-adjoint operators A_1 and A_2 on Hilbert spaces $\mathbf{H}_1$ and $\mathbf{H}_2$ are *unitarily equivalent* if there exists a unitary map $U : \mathbf{H}_1 \to \mathbf{H}_2$ such that

$$A_2 = UA_1U^{-1}.$$

Using Proposition 7.22, we can give a classification of bounded self-adjoint operators on separable Hilbert spaces up to unitary equivalence. For a given bounded self-adjoint operator A, we call the function $\lambda \mapsto \dim \mathbf{H}_\lambda$ the *multiplicity function* for A. It is well defined (independent of the choice of direct integral decomposition) up to a set of measure zero. It turns out that bounded self-adjoint operators are characterized, up to unitary equivalence, by the spectrum of A as a set, the equivalence class of the measure μ in Theorem 7.19, and the multiplicity function.

Proposition 7.24 *Suppose A_1 and A_2 are bounded self-adjoint operators on separable Hilbert spaces $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively. Choose direct integral representations for A_1 and A_2 as in Theorem 7.19, with the associated measures μ_1 and μ_2 chosen so that $\dim \mathbf{H}_\lambda > 0$ for μ_j-almost every λ ($j = 1, 2$). Then A_1 and A_2 are unitarily equivalent if and only if the following conditions are satisfied.*

1. *$\sigma(A_1) = \sigma(A_2)$.*
2. *The measures μ_1 and μ_2 are mutually absolutely continuous.*
3. *The multiplicity functions of A_1 and A_2 coincide up to a set of measure zero.*

See Exercise 12 for a proof of this result.

7.4 Exercises

1. Suppose A and B are *commuting* linear operators on a nonzero finite-dimensional vector space.
   (a) Show that each eigenspace for A is invariant under B.
   (b) Show that A and B have at least one simultaneous eigenvector, that is, a nonzero vector v with $Av = \lambda v$ and $Bv = \mu v$, for some constants $\lambda, \mu \in \mathbb{C}$.

2. Suppose that $A \in \mathcal{B}(\mathbf{H})$ is *normal*, meaning that $AA^* = A^*A$. Suppose that for some $\psi \in \mathbf{H}$ and $\lambda \in \mathbb{C}$ we have $A\psi = \lambda\psi$. Show that $A^*\psi = \bar{\lambda}\psi$.

   Hint: Compute $\left\|(A^* - \bar{\lambda})\psi\right\|$.

3. Suppose a closed subspace V of $\mathbf{H}$ is invariant under a bounded operator A, meaning that $A\psi \in V$ for all $\psi \in V$. Show that the orthogonal complement $V^{\perp}$ of V is invariant under A^*.

4. (a) Suppose that $\mathbf{H}$ is a finite-dimensional Hilbert space over $\mathbb{C}$ and A is a normal linear operator on $\mathbf{H}$ in the sense of Exercise 2. Show that there exists an orthonormal basis for V consisting of simultaneous eigenvectors for A and A^*.

   Hint: Use Exercises 1 and 3.

   (b) Suppose A is a linear operator on a finite-dimensional Hilbert space $\mathbf{H}$ over $\mathbb{C}$ and suppose there exists an orthonormal basis for V consisting of eigenvectors of A. Show that A commutes with A^*.

5. Suppose $A \in \mathcal{B}(\mathbf{H})$ has an inverse A^{-1} in $\mathcal{B}(\mathbf{H})$. Show that $(A^{-1})^* A^* = A^*(A^{-1})^* = I$. Conclude that A^* is invertible and $(A^*)^{-1}{=}(A^{-1})^*$.

6. Suppose U is a unitary operator on $\mathbf{H}$ (Definition A.55). Show that the spectrum of U is contained in the unit circle.

   Hint: By writing $U - \lambda I$ as $(-\lambda)(I - U/\lambda)$ or as $U(I - \lambda U^{-1})$, show that any λ with $|\lambda| \neq 1$ is in the resolvent set of λ.

7. Suppose that $A \in \mathcal{B}(\mathbf{H})$ is self-adjoint and non-negative, that is, that A satisfies (7.3). Show that the spectrum of A is contained in the interval $[0, \infty)$.

   Note: Conversely, if $A \in \mathcal{B}(\mathbf{H})$ is self-adjoint and $\sigma(A) \subset [0, \infty)$, then A is non-negative. See Exercise 2 in Chap. 8.

8. Suppose $A \in \mathcal{B}(\mathbf{H})$ is invertible. Show that there exists $\varepsilon > 0$ such that for all $B \in \mathcal{B}(\mathbf{H})$ with $\|B - A\| < \varepsilon$, B is also invertible.

   Hint: Use a power series argument as in the proof of Proposition 7.5.

9. Assume $A \in \mathcal{B}(\mathbf{H})$ is self-adjoint.

   (a) Suppose $\lambda_0 \in \mathbb{C}$ is a point in the resolvent set of A. Show that

   $$\left\|(A - \lambda_0 I)^{-1}\right\| = \frac{1}{d(\lambda_0, \sigma(A))},$$

   where $d(\lambda_0, \sigma(A)) = \inf_{\lambda \in \sigma(A)} |\lambda - \lambda_0|$.
   Hint: Think of $(A - \lambda_0 I)^{-1}$ as a function of A in the sense of the functional calculus for A.

   (b) Given $\lambda_0 \in \mathbb{C}$, suppose that there exists some nonzero $\psi \in \mathbf{H}$ such that

   $$\|A\psi - \lambda_0 \psi\| < \varepsilon \|\psi\|.$$

   Show that there exists $\lambda \in \sigma(A)$ such that $|\lambda - \lambda_0| < \varepsilon$.

10. Suppose V_1 and V_2 are two closed subspaces of $\mathbf{H}$, with associated orthogonal projections P_1 and P_2. Show that V_1 and V_2 are orthogonal if and only if $P_1P_2 = 0$.

11. Suppose μ is a projection-valued measure on (X, Ω). Show that for any $E_1, E_2 \in \Omega$, $\mu(E_1)\mu(E_2)$ is the projection onto the closed subspace $\text{Range}(\mu(E_1)) \cap \text{Range}(\mu(E_2))$.

    Hint: Write E_1 as $E_1 = (E_1 \cap E_2) \cup (E_1 \backslash E_2)$ and use Exercise 10.

12. Prove Proposition 7.24.

    Hint: Use Proposition 7.22 and the Radon–Nikodym theorem (Theorem A.6).