Chapter 14
Theory Contraction and Base Contraction Unified

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General Introduction

The AGM (Alchourrón-Gärdenfors-Makinson) model of belief change has acquired the status of a standard model. In that model, a belief state is represented by a set of sentences that is closed under logical consequence, the *belief set* or theory. (Alchourrón et al. 1985); (Gärdenfors 1988) Among the major rivals are models in which a belief state is represented by a *belief base*, a set that is not (except in a limiting case) closed under consequence. (Fuhrmann 1991); (Hansson 1989; 1991; 1992; 1993); (Nebel 1992) Obviously, the logical closure of a belief base is a belief set, and each non-trivial belief set can be represented by several different belief bases. It has been argued that different belief bases for one and the same belief set represent different ways of holding the same beliefs. Roughly, the elements of the belief base represent "basic", or independently grounded beliefs, in contrast to the "merely derived" beliefs that form the rest of the belief set. (Fuhrmann 1991, pp. 183–184); (Hansson 1989; 1992)

As has been accurately pointed out by Fuhrmann (1991), operations on a belief base generate operations on the corresponding belief set. In particular, if a belief base for a belief set **K** is contracted by a sentence α, then the logical closure of the contracted belief base is a belief set from which α has been contracted.

The purpose of this paper is to characterize the operators of contraction on a belief set **K** that can be obtained by assigning to it (1) a belief base, and (2) an operator of partial meet contraction for that belief base. The section

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"Partial meet contraction" contains introductory material. In the section "The new postulates", the postulates that will be used for the characterizations are introduced, and in the section "Axiomatic characterizations" axiomatic characterizations of various types of base-generated theory contractions are given. The section "Proofs" provides proofs of the results reported in the section "Axiomatic characterizations".

Partial Meet Contraction

We will assume a language L that is closed under truth-functional operations and a consequence operator Cn for L. Cn satisfies the standard conditions for a consequence operator, namely inclusion ($A \subseteq Cn(A)$), monotony (if $A \subseteq B$, then $Cn(A) \subseteq Cn(B)$), and iteration ($Cn(A) = Cn(Cn(A))$). Furthermore, it satisfies the properties of supraclassicality (if α follows by classical truth-functional logic from A, then $\alpha \in Cn(A)$), deduction (if $\beta \in Cn(A \cup \{\alpha\})$, then $(\alpha \to \beta) \in Cn(A)$) and compactness (if $\alpha \in Cn(A)$, then $\alpha \in Cn(A')$ for some finite set $A' \subseteq A$). $A \vdash \alpha$ will be used as an alternative notation for $\alpha \in Cn(A)$.

A *belief set* is a subset $\mathbf{K}$ of the language such that $\mathbf{K} = Cn(\mathbf{K})$. An operator of *contraction* for $\mathbf{K}$ is an operator $\div$ from $\mathcal{P}(L) \times L$ to $\mathcal{P}(L)$ such that for all sentences α, $\mathbf{K} \div \alpha \subseteq \mathbf{K}$, and if $\alpha \notin Cn(\varnothing)$, then $\alpha \notin Cn(\mathbf{K} \div \alpha)$. A particularly interesting type of contraction is *partial meet contraction*, which was introduced by Alchourrón, Gärdenfors, and Makinson (1985). (Gärdenfors 1984) It is defined by the identity:

$$\mathbf{K} \div \alpha = \bigcap \gamma(\mathbf{K} \perp \alpha),$$

where $\mathbf{K} \perp \alpha$ denotes the set of inclusion-maximal subsets of $\mathbf{K}$ that do not have α as a logical consequence. γ is a *selection function*, such that $\gamma(\mathbf{K} \perp \alpha)$ is a non-empty subset of $\mathbf{K} \perp \alpha$, unless the latter is empty, in which case $\gamma(\mathbf{K} \perp \alpha) = \{\mathbf{K}\}$.

Let γ be a selection function for $\mathbf{K}$. Then the *completion* of γ is the function γ^* such that $\gamma^*(\mathbf{K} \perp \alpha) = \{X \in \mathbf{K} \perp \alpha \mid \bigcap \gamma(\mathbf{K} \perp \alpha) \subseteq X\}$, unless $\mathbf{K} \perp \alpha$ is empty, in which case $\gamma^*(\mathbf{K} \perp \alpha) = \{\mathbf{K}\}$. γ^* is a selection function for $\mathbf{K}$, and gives rise to the same operator of partial meet contraction as γ. (Alchourrón et al. 1985, p. 519) It is in some contexts convenient to make the technical assumption that a selection function is *completed*, i.e. identical to its own completion.

Full meet contraction is the limiting case when $\gamma(\mathbf{K} \perp \alpha) = \mathbf{K} \perp \alpha$ for all non-empty $\mathbf{K} \perp \alpha$. In the other limiting case, when $\gamma(\mathbf{K} \perp \alpha)$ is a singleton for all α, $\div$ is an operator of *maxichoice* contraction. An operator of partial meet contraction is *relational* if and only if it is based on a selection function γ for which there is a relation $\underline{\ll}$ such that for all non-empty $\mathbf{K} \perp \alpha$ we have

$$\gamma(\mathbf{K} \perp \alpha) = \left\{\mathbf{K}' \in \mathbf{K} \perp \alpha \;\middle|\; \mathbf{K}'' \underline{\ll} \mathbf{K}' \text{ for all } \mathbf{K}'' \in \mathbf{K} \perp \alpha\right\}$$

If this condition holds for a transitive relation $\underline{\ll}$, then the operator is *transitively relational*.

Partial meet contraction derives much of its attractiveness from a representation theorem by Alchourrón et al. (1985): An operation $\div$ on a logically closed set $\mathbf{K}$ is a partial meet contraction if and only if it satisfies the following six postulates, the basic *Gärdenfors postulates*:

(G÷1) $\mathbf{K} \div \alpha$ is a theory if $\mathbf{K}$ is a theory (*closure*)
(G÷2) $\mathbf{K} \div \alpha \subseteq \mathbf{K}$ (*inclusion*)
(G÷3) If $\alpha \notin Cn(\mathbf{K})$ then $\mathbf{K} \div \alpha = \mathbf{K}$ (*vacuity*)
(G÷4) If $\alpha \notin Cn(\varnothing)$ then $\alpha \notin Cn(\mathbf{K} \div \alpha)$ (*success*)
(G÷5) If $Cn(\alpha) = Cn(\beta)$ then $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$ (*preservation*)
(G÷6) $\mathbf{K} \subseteq Cn((\mathbf{K} \div \alpha) \cup \{\alpha\})$ whenever $\mathbf{K}$ is a theory (*recovery*).

Furthermore, an operator of partial meet contraction on a logically closed set is transitively relational if and only if it also satisfies:

(G÷7) $(\mathbf{K} \div \alpha) \cap (\mathbf{K} \div \beta) \subseteq \mathbf{K} \div (\alpha \& \beta)$ (*intersection*)
(G÷8) If $\alpha \notin \mathbf{K} \div (\alpha \& \beta)$ then $\mathbf{K} \div (\alpha \& \beta) \subseteq \mathbf{K} \div \alpha$ (*conjunction*).

A *belief base* for $\mathbf{K}$ is a set $\boldsymbol{B}$ such that $\mathbf{K} = Cn(\boldsymbol{B})$. Partial meet contraction for belief bases is defined in the same way as for belief sets, i.e., by the identity $\boldsymbol{B} \div \alpha = \bigcap \gamma(\boldsymbol{B} \perp \alpha)$. Full meet, maxichoice, relational, and transitively relational contraction is also defined in the same way as for belief sets. Furthermore, an operator of partial meet contraction is *transitively, maximizingly relational* (*TMR*) if and only if it is relational by a transitive relation $\underline{\ll}$ such that, for its strict counterpart $\ll$, if $A \subset B$, then $A \ll B$.[1]

Contractions on belief bases may be studied in their own right.[2] In this paper, however, they will be treated as means for contractions on belief sets.

Definition An operation $\div$ on a belief set $\mathbf{K}$ is *generated by* a partial meet base contraction if and only if there is a belief base $\boldsymbol{B}$ for $\mathbf{K}$ and an operator $\sim_\gamma$ of partial meet contraction for $\boldsymbol{B}$ such that $\mathbf{K} \div \alpha = Cn\left(\boldsymbol{B} \sim_\gamma \alpha\right)$ for all $\alpha \in L$.

We will see directly that if an operation on a belief set is generated by some partial meet base contraction, then it satisfies the first five of the basic Gärdenfors postulates, (G÷1)–(G÷5), but it does not, in general, satisfy recovery (G÷6).[3]

[1] The maximizing property may be interpreted as saying that all elements of the belief base have positive epistemic value. This property might at first hand seem superfluous. If $K' \subset K''$, then $K' \in K \perp \alpha$ and $K'' \in K \perp \alpha$ cannot both be true, so that K' and K'' cannot be candidates between which the selection function has to choose. However, when more than one contraction is taken into account, the property need not be superfluous. If $K_1 \subset K_2$, and K_3 is neither a subset nor a superset of either K_1 or K_2, then $\gamma(\{K_1,K_3\}) = \{K_1\}$ and $\gamma(\{K_2,K_3\}) = \{K_3\}$ may both hold for a transitively relational selection function γ, but not if it is in addition required to be maximizing.

[2] See (Hansson 1993) for some results, including an axiomatic characterization of the partial meet contractions on a belief base.

[3] In Makinson's (1987) terminology, an operation that satisfies (G÷1)-(G÷5) but not necessarily (G÷6) is a "withdrawal". However, I will use "contraction" in the wide sense indicated above, thus including Makinson's withdrawals.

Recovery is the most controversial of the six postulates, and has been seriously questioned by several authors. See (Hansson 1991); (Makinson 1987); (Niederée 1991).

The New Postulates

It has often been remarked that the only realistic belief sets are those that have a finite representation. (Gärdenfors and Makinson 1988) We are going to assume that both the original belief set and the belief sets that are obtained through contraction have finite representations (belief bases); thus for every α, there is some finite set A such that $\mathbf{K} \div \alpha = Cn(A)$. Furthermore, we will assume that although there may be infinitely many sentences by which the belief set can be contracted, there is only a finite number of belief sets that can be obtained through contraction, i.e. $\{\mathbf{K}' \mid (\exists\alpha)(\mathbf{K}' = \mathbf{K} \div \alpha)\}$ is finite. These two finiteness properties can be combined into the following:

There is a finite set A such that for every α, there is some $A' \subseteq A$ such that $\mathbf{K} \div \alpha = Cn(A')$. (*finitude*)

In the presence of vacuity (G÷3), *finitude* implies that there is some finite set A such that $\mathbf{K} = Cn(A)$.

If $\div$ is a contraction operator for $\mathbf{K}$, and α is not a logical theorem, then $\mathbf{K} \div \alpha$ does not contain α. However, to contract $\mathbf{K}$ by α is not the only way to exclude α from the belief set. Typically, there are several β distinct from α such that $\alpha \notin \mathbf{K} \div \beta$. This must be the case if α logically implies β, and it can be true in other cases as well. A contraction $\mathbf{K} \div \beta$ such that $\alpha \notin \mathbf{K} \div \beta$ will be called an α-*removal*.

Two different beliefs may have exactly the same justification(s). As an example, I believe that either Paris or Oslo is the capital of France (α). I also believe that either Paris or Stockholm is the capital of France (β). Both these beliefs are entirely based on my belief that Paris is the capital of France. Therefore, a contraction by some sentence δ removes α if and only if it removes β (namely if and only if it removes the common justification of these two beliefs). There is no contraction by which I can retract α without retracting β or vice versa. It is not unreasonable to require that if two beliefs in a belief set stand or fall together in this way, then their contractions are identical. In other words, *if all* α*-removals are* β*-removals and vice versa, then* $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$. In the formal language:

If $\mathbf{K} \div \delta \vdash \alpha$ iff $\mathbf{K} \div \delta \vdash \beta$ for all δ, then $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$. (*symmetry*)

An α-removal $\mathbf{K} \div \beta$ will be called a *preservative* α*-removal* if and only if $\mathbf{K} \div \alpha \subseteq \mathbf{K} \div \beta$, and a *strictly preservative* α*-removal* if and only if $\mathbf{K} \div \alpha \subset \mathbf{K} \div \beta$. A strictly preservative α-removal is an operation that removes α, and does this in a more economical way than what is done by the contraction by α.

Often, a belief can be removed more economically if more specified information is obtained. As an example, I believe that Albert Schweitzer was a German Missionary (α). Let α_1 denote that he was a German and α_2 that he was a Missionary, so that $\alpha \equiv \alpha_1 \& \alpha_2$. If I have to contract my belief set by α, then the contracted belief set will contain neither α_1 nor α_2. Admittedly it would be logically sufficient to withdraw one of them. However, they are both equally entrenched, so that I do not know which to choose in preference over the other. Therefore, both will have to go. On the other hand, if I have to contract my belief set by α_1, then I have no reason to let go of α_2.[4] To contract by α_1 is, given the structure of my belief state, a more specified way to remove α. Thus we may expect that $\mathbf{K} \div \alpha \subset \mathbf{K} \div \alpha_1$, so that $\mathbf{K} \div \alpha_1$ is a strictly preservative α-removal.

Let δ denote that Albert Schweitzer was a Swede, and let us consider the contraction of $\mathbf{K}$ by $\alpha_1 \vee \delta$, "Albert Schweitzer was a German or a Swede." Since I believe in $\alpha_1 \vee \delta$ only as a consequence of my belief in α_1, I can only retract $\alpha_1 \vee \delta$ by retracting α_1. Therefore, $\mathbf{K} \div (\alpha_1 \vee \delta)$ is not a proper superset of $\mathbf{K} \div \alpha_1$, i.e., it is not a more conservative α-withdrawal than $\mathbf{K} \div \alpha_1$. Indeed, the way my beliefs are structured, α_1 cannot be further subdivided in the way that α was subdivided into α_1 and α_2. There is no part of α_1 that stands on its own and can be retracted from $\mathbf{K}$ without the rest of α_1 being lost as well. In this sense, no α-removal can be more conservative than $\mathbf{K} \div \alpha_1$.

More generally, $\mathbf{K} \div \beta$ is a *maximally preservative* α-*removal* if and only if it is a preservative α-removal and there is no α-removal $\mathbf{K} \div \delta$ such that $\mathbf{K} \div \beta \subset \mathbf{K} \div \delta$. Intuitively, to perform a maximally preservative α-removal is to make the belief set not imply α, making use of information that is sufficiently specified to allow one to remove a part of α so small that no smaller part of it can be removed alone.

Contraction should be conservative in the sense that every element of $\mathbf{K}$ is retained in $\mathbf{K} \div \alpha$ unless there is some good reason to exclude it. As was noted in (Hansson 1991), (G÷1)–(G÷5) do not ensure this property, since they are satisfied by the operation $\div$ such that if $\alpha \notin Cn(\mathbf{K})$, then $\mathbf{K} \div \alpha = \mathbf{K}$, and otherwise $\mathbf{K} \div \alpha = \mathbf{K} \cap Cn(\varnothing)$. The same applies if *symmetry* is added to the list of postulates. We need some further postulate to prevent elements of $\mathbf{K}$ from being lost for no reason in operations of contraction.

One way to achieve this is to require that what is lost in the contraction by α must be incompatible with some reasonable way to remove α. In other words, if a unit of belief is lost in contraction by α, then there should be at least one preservative α-removal to which the lost unit cannot be added without α being implied. However, it would be too far-reaching to require this to hold for units of belief that cannot stand on their own (such as $\alpha_1 \vee \delta$ in our Albert Schweizer example). Such a unit of belief can be lost merely due to loss of the larger, self-sustained unit(s) of which it is a part. Thus, our principle of conservativity should refer only to units of belief that *can* stand on their own.

In order to formulate a conservativity principle, we therefore need a formal criterion that excludes non-self-sustained units of belief. In our example, there can be no sentence β such that $\mathbf{K} \div \beta$ consists exactly of $\alpha_1 \vee \delta$ and its logical consequences, since $\alpha_1 \vee \delta$ cannot stand on its own without α_1. Admittedly, this is a

[4]It is here assumed that $\alpha_1 \equiv \alpha_2$ is less entrenched than α_1 and α_2.

weak criterion, and it can be strengthened in various ways.[5] However, it turns out to be sufficient for our present purposes, and it will therefore be used in our postulate of conservativity:

If $\mathbf{K} \div \beta \not\subseteq \mathbf{K} \div \alpha$, then there is some δ such that $\mathbf{K} \div \alpha \subseteq \mathbf{K} \div \delta \nvdash \alpha$ and $(\mathbf{K} \div \beta) \cup (\mathbf{K} \div \delta) \vdash \alpha$. (*conservativity*)

An obvious way to strengthen *conservativity* is to require that if a unit of belief is lost in contraction by α, then α will be implied if it is added to $\mathbf{K} \div \alpha$ (and not merely if it is added to some preservative α-removal):

If $\mathbf{K} \div \beta \not\subseteq \mathbf{K} \div \alpha$ then $\mathbf{K} \div \alpha \nvdash \alpha$ and $(\mathbf{K} \div \beta) \cup (\mathbf{K} \div \alpha) \vdash \alpha$. (*strong conservativity*)

Strong conservativity is much less plausible than *conservativity*. This can be seen from our Albert Schweitzer example. In that example, it may reasonably be assumed that $\alpha_1 \notin \mathbf{K} \div (\alpha_1\&\alpha_2)$, $\alpha_2 \notin \mathbf{K} \div (\alpha_1\&\alpha_2)$, $\alpha_1 \in \mathbf{K} \div \alpha_2$, $\alpha_2 \in \mathbf{K} \div \alpha_1$, $\mathbf{K} \div (\alpha_1\&\alpha_2) \subseteq \mathbf{K} \div \alpha_1$, and $\mathbf{K} \div (\alpha_1\&\alpha_2) \subseteq \mathbf{K} \div \alpha_2$. However, this is incompatible with *strong conservativity*. Since $\mathbf{K} \div \alpha_1 \not\subseteq \mathbf{K} \div (\alpha_1\&\alpha_2)$, this postulate requires that $\mathbf{K} \div \alpha_1 \cup \mathbf{K} \div (\alpha_1\&\alpha_2) \vdash (\alpha_1\&\alpha_2)$, contrary to our assumptions for this case. More generally, *strong conservativity* is implausible since it precludes the removal of two or more sentences (in this case α_1 and α_2), when it would have been logically sufficient to remove only one of them. Such epistemic behaviour is rational enough when the beliefs in question are equally entrenched, or have equal epistemic utility.

The concepts of epistemic entrenchment and epistemic utility refer to extra-logical reasons that one may have for preferring one way to remove a sentence α rather than another. It is conceivable for an epistemic agent to make no use of such extra-logical information. Such an agent is indecisive in the sense of not being able to make a choice among different ways to remove a belief, if these are on an equal footing from a logical point of view. Formally, this is close to a reversal of *conservativity*: If a (self-sustained) unit of belief conflicts with some way to remove α from $\mathbf{K}$, then it is not a part of $\mathbf{K} \div \alpha$:

If there is some δ such that $\mathbf{K} \div \delta \nvdash \alpha$ and $(\mathbf{K} \div \beta) \cup (\mathbf{K} \div \delta) \vdash \alpha$, then $\mathbf{K} \div \beta \not\subseteq \mathbf{K} \div \alpha$. (*indecisiveness*)

Non-logical considerations play an important role in actual human epistemic behaviour. Arguably, it would in many cases be irrational not to let them do so. Therefore, *indecisiveness* is not a plausible general property of rational belief change.

[5]Two such strengthened versions should be mentioned: (1) $\div$ can be extended to contraction by sets (multiple contraction), such that if $A \cap Cn(\varnothing) = \varnothing$, then $\mathbf{K}\div A$ is a logically closed subset of $\mathbf{K}$ that does not imply any element of A. It a unit of belief cannot stand on its own, then it should not be equal to $\mathbf{K}\div A$ for any set A. (2) Iterated contraction can be used for the same purpose: If a unit of belief cannot stand on its own, then it should not be equal to $\mathbf{K}\div\beta_1\div\beta_2 \ldots \div\beta_n$ for any series $\beta_1 \ldots \beta_n$ of sentences.

The next postulate will again be concerned with maximally preservative removals, and thus with the smallest units of belief that can be removed from a belief set. In an orderly and coherent belief state, one would expect the identities and the relations of entrenchment of these units to be constant, i.e. they should be the same independently of what sentence we are contracting by.

As an illustration, let us extend the Albert Schweitzer example. Let α_1 denote that Schweitzer was a German, α_2 that he was a missionary and α_3 that he was a physician. Let us assume that $\mathbf{K} \div \alpha_1$ is a maximally preservative $\alpha_1 \& \alpha_2 \& \alpha_3$-removal, i.e. a maximally economical way to remove $\alpha_1 \& \alpha_2 \& \alpha_3$ from the belief set. Since $\mathbf{K} \div \alpha_1$ is also an $\alpha_1 \& \alpha_2$-removal, and since the $\alpha_1 \& \alpha_2$-removals are a subset of the $\alpha_1 \& \alpha_2 \& \alpha_3$-removals, it should also be maximally preservative among these. Furthermore, if $\mathbf{K} \div \alpha_2$ is equally economical as $\mathbf{K} \div \alpha_1$ in the context of removing $\alpha_1 \& \alpha_2$ (i.e., if it is also a maximaly preservative $\alpha_1 \& \alpha_2$-removal), then it should also be equally economical as $\mathbf{K} \div \alpha_1$ in the context of removing $\alpha_1 \& \alpha_2 \& \alpha_3$ (i.e., it should also be a maximally preservative $\alpha_1 \& \alpha_2 \& \alpha_3$-removal). In general:

If $\vdash \alpha \rightarrow \beta$ and the set of β-removals that are also maximally preservative α-removals is non-empty, then it coincides with the set of maximally preservative β-removals. (*regularity*)

It may be convenient to divide *regularity* into two parts[6]:

If $\vdash \alpha \rightarrow \beta$ and some maximally preservative α-removal is also a β-removal, then all maximally preservative β-removals are maximally preservative α-removals. (*regularity 1*)

If $\vdash \alpha \rightarrow \beta$ and $\mathbf{K} \div \delta$ is both a β-removal and a maximally preservative α-removal, then it is a maximally preservative β-removal. (*regularity 2*)

Clearly, *regularity* holds if and only if both *regularity 1* and *regularity 2* hold.

It does *not* follow from *regularity* that if, in our example, $\mathbf{K} \div (\alpha_1 \& \alpha_2 \& \alpha_3)$ is an $\alpha_1 \& \alpha_2$-removal, then $\mathbf{K} \div (\alpha_1 \& \alpha_2 \& \alpha_3) = \mathbf{K} \div (\alpha_1 \& \alpha_2)$. To see why this would not be plausible as a general principle, let us modify the example and assume that α_2 and α_3 are equally entrenched, whereas α_1 is more entrenched than both α_2 and α_3. Then we should expect $\alpha_3 \notin \mathbf{K} \div (\alpha_1 \& \alpha_2 \& \alpha_3)$ but $\alpha_3 \in \mathbf{K} \div (\alpha_1 \& \alpha_2)$. However, as a limiting case, we can imagine an epistemic agent who is never indifferent between different ways to remove a belief from her belief set. Such an agent will always, when contracting by α, remove one of the smallest removable parts of α. Thus, the contraction by α will itself be the only maximally preservative α-removal. In our example, when contracting by $\alpha_1 \& \alpha_2 \& \alpha_3$, this agent will remove exactly one of α_1, α_2, and α_3, and when contracting by $\alpha_1 \& \alpha_2$, she will remove exactly one of α_1 and α_2. Assuming that the relative entrenchment of her beliefs is context-independent, if she removes α_1 when contracting by $\alpha_1 \& \alpha_2 \& \alpha_3$, then α_1 is

[6] *Regularity 1* is closely related to Amartya Sen's β property for rational choice behaviour, and *regularity 2* to his α property. (Sen 1970).

also the removed unit of belief when α_1 & α_2 is contracted. In general, her epistemic behaviour should satisfy the following postulate:

If $\vdash \alpha \rightarrow \beta$ and $\mathbf{K} \div \alpha \nvdash \beta$, then $\mathbf{K} \div \beta = \mathbf{K} \div \alpha$. (*hyperregularity*)

Hyperregularity implies that for all α and β, $\mathbf{K} \div (\alpha\&\beta) = \mathbf{K} \div \alpha$ or $\mathbf{K} \div (\alpha\&\beta) = \mathbf{K} \div \beta$. This condition has also been called "decomposition" (Alchourrón et al. 1985, p. 525) As was noted by Gärdenfors (1988, p. 66), it is too strong a principle. Thus, *hyperregularity* is a limiting case of some interest, but not a plausible criterion of rational belief change. The same applies to *strong conservativity* and *indecisiveness*, whereas *symmetry*, *conservativity*, and *regularity* are proposed as reasonable postulates for rational belief change.

Axiomatic Characterizations

Symmetry and *conservativity* are sufficient to characterize, together with (G÷1)-(G÷5) and *finitude*, the contractions of belief sets that are generated by partial meet base contraction. Since these are all fairly plausible postulates, this result adds to the credibility of theory contraction through partial meet base contraction.

Theorem 14.1 An operation ÷ on a belief set **K** is generated by partial meet contraction of a finite base for **K** iff ÷ satisfies (G÷1), (G÷2), (G÷3), (G÷4), (G÷5), *finitude*, *symmetry* and *conservativity*.

If *conservativity* is strengthened to the (much less plausible) postulate of *strong conservativity*, then a characterization is obtained of operations that are generated by maxichoice contractions of finite bases.

Theorem 14.2 An operation ÷ on a belief set **K** is generated by maxichoice partial meet contraction of a finite base for **K** iff ÷ satisfies (G÷1), (G÷2), (G÷3), (G÷4), (G÷5), *finitude*, *symmetry*, and *strong conservativity*.

Indecisiveness, in combination with *conservativity*, is a characteristic postulate for operations based on full meet base contraction.

Theorem 14.3 An operation ÷ on a belief set **K** is generated by full meet contraction of a finite base for **K** iff ÷ satisfies (G÷1), (G÷2), (G÷3), (G÷4), (G÷5), *finitude*, *conservativity*, and *indecisiveness*.

Regularity ensures that the partial meet base contraction that generates ÷ is transitively, maximizingly relational (cf. the section "Partial meet contraction").

Theorem 14.4 An operation ÷ on a belief set **K** is generated by transitively, maximizingly relational partial meet contraction of a finite base for **K**, by a completed selection function,[7] iff ÷ satisfies (G÷1), (G÷2), (G÷3), (G÷4), (G÷5), *finitude*, *symmetry*, *conservativity*, and *regularity*.

[7]The completeness of γ is used in the proof. I do not know if it can be dispensed with.

For maxichoice operations that are transitively, maximizingly relational, the following axiomatic characterization has been obtained:

Theorem 14.5 An operation ÷ on a belief set **K** is generated by a transitively, maximizingly relational maxichoice contraction of a finite base for **K** iff ÷ satisfies (G÷1), (G÷2), (G÷3), (G÷4), (G÷5), *finitude*, *symmetry*, *strong conservativity*, and *hyperregularity*.

Some of the postulates used in Theorems 14.2, 14.3, and 14.5 were shown in the section "The new postulates" to be quite implausible. Indeed, maxichoice and full meet contraction are of interest only as limiting cases. In contrast, Theorems 14.1 and 14.4 only employ fairly plausible postulates of rational belief change. It is proposed that the classes of base-generated contractions of belief sets that are characterized in these theorems represent reasonable types of belief contraction.

Two further important properties have been obtained for the class of operations that were referred to in Theorem 14.4, namely contractions of belief sets that are generated by transitively, maximizingly relational partial meet base contractions:

Theorem 14.6 Let the operation ÷ on the belief set **K** be generated by some transitively, maximizingly relational partial meet contraction of a finite base for **K**. Then:

(1) If $\mathbf{K} \div \delta \subseteq (\mathbf{K} \div \alpha) \cap (\mathbf{K} \div \beta)$, then $\mathbf{K} \div \delta \subseteq \mathbf{K} \div (\alpha \& \beta)$. (*weak intersection*)
(2) If $\alpha \notin \mathbf{K} \div (\alpha \& \beta)$, then $\mathbf{K} \div (\alpha \& \beta) \subseteq \mathbf{K} \div \alpha$. (*conjunction*)

Weak intersection is a weaker form of Gärdenfors's (G÷7) postulate, namely that "the beliefs that are both in $\mathbf{K} \div \alpha$ and $\mathbf{K} \div \beta$ are also in $\mathbf{K} \div (\alpha \& \beta)$" (Gärdenfors 1988, p. 64).[8] It differs from Gärdenfors's original postulate in being restricted to beliefs that are self-sustained in the sense that was accounted for in section "The new postulates". To see that this is a reasonable restriction, let α and β be self-sustained beliefs that have the same degree of entrenchment, and such that $\alpha \vee \beta$ is believed only as a logical consequence of α and β. For a plausible practical example, let α denote that Algiers is a capital and β that Bern is a capital. I have both these beliefs, and they are equally entrenched. If I contract my belief set by α, then β is unperturbed, so that $\beta \in \mathbf{K} \div \alpha$, and as a consequence of that, $\alpha \vee \beta \in \mathbf{K} \div \alpha$. For symmetrical reasons, $\alpha \vee \beta \in \mathbf{K} \div \beta$. However, if I contract my belief set by $\alpha \& \beta$, then since α and β are equally entrenched I cannot choose between them, so that they must both go. Since neither α nor β is in $\mathbf{K} \div (\alpha \& \beta)$, and $\alpha \vee \beta$ was believed only as a consequence of (each of) these two beliefs, $\alpha \vee \beta$ will be lost as well. Thus, $\alpha \vee \beta \in (\mathbf{K} \div \alpha) \cap (\mathbf{K} \div \beta)$ but $\alpha \vee \beta \notin \mathbf{K} \div (\alpha \& \beta)$, contrary to (G÷7) but in accordance with *weak intersection*.

Conjunction is Gärdenfors's (G÷8) postulate. To motivate it, the Algiers and Bern example may again be used. In that case, to remove α is a way to remove a specified part of $\alpha \& \beta$. In general, the removal of a part of a certain belief is at least

[8]The formulas of the quotation have been adapted to the notational convention used here.

as economical (conservative, retentive of the original belief set) as the removal of that belief in in entirely. Therefore, if $\mathbf{K} \div (\alpha \& \beta)$ is an α-removal, then $\mathbf{K} \div \alpha$ should be at least as economical as $\mathbf{K} \div (\alpha \& \beta)$.

In conclusion, with base-generated theory contraction we can avoid the problematic *recovery* postulate (G÷6) of the AGM framework, but still have the plausible basic postulates (G÷1)-(G÷5) and the additional postulates (G÷7) (in a weakened but credibilized form) and (G÷8) (in its original form).[9]

Proofs

A set A is *finite-based* iff there is some finite set A' such that $Cn(A') = Cn(A)$. For any non-empty, finite-based set A, $\& A$ denotes the conjunction of all elements of some finite base of A. For any finite, non-empty set A, $\mathcal{V}(A)$ denotes the disjunction of the elements of A.

The following lemmas will be needed for the proofs:

Lemma 14.1 $\boldsymbol{B} \perp (\alpha \& \beta) \subseteq \boldsymbol{B} \perp \alpha \cup \boldsymbol{B} \perp \beta$.

Proof of Lemma 14.1 Let $W \in \boldsymbol{B} \perp (\alpha \& \beta)$. Then either $W \nvdash \alpha$ or $W \nvdash \beta$. It follows from $W \nvdash \alpha$ and $W \in \boldsymbol{B} \perp (\alpha \& \beta)$ that $W \in \boldsymbol{B} \perp \alpha$, and in the same way from $W \nvdash \beta$ and $W \in \boldsymbol{B} \perp (\alpha \& \beta)$ that $W \in \boldsymbol{B} \perp \beta$.

Lemma 14.2 If $X \not\subseteq Y \not\subseteq X$ for all $X \in \boldsymbol{B} \perp \alpha$ and $Y \in \boldsymbol{B} \perp \beta$, then $\boldsymbol{B} \perp (\alpha \& \beta) = \boldsymbol{B} \perp \alpha \cup \boldsymbol{B} \perp \beta$.

We will mostly use a special case of the lemma. Namely if $\{X\} = \boldsymbol{B} \perp \alpha$ and $\{Y\} = \boldsymbol{B} \perp \beta$, and $X \not\subseteq Y \not\subseteq X$, then $\{X, Y\} = \boldsymbol{B} \perp (\alpha \& \beta)$.

Proof of Lemma 14.2 One direction follows from *lemma* 14.1. For the other direction, let $X \in \boldsymbol{B} \perp \alpha$. Then $X \nvdash (\alpha \& \beta)$. In order to prove that $X \in \boldsymbol{B} \perp (\alpha \& \beta)$, suppose to the contrary that there is some W such that $X \subset W \subseteq \boldsymbol{B}$ and $W \nvdash (\alpha \& \beta)$. From $X \subset W$ it follows that $W \vdash \alpha$. With $W \nvdash (\alpha \& \beta)$ this yields $W \nvdash \beta$, from which it follows that $W \subseteq Y$ for some $Y \in \boldsymbol{B} \perp \beta$. We therefore have $X \subset W \subseteq Y$, contradicting the assumption that $X \not\subseteq Y \not\subseteq X$. We may conclude that $X \in \boldsymbol{B} \perp (\alpha \& \beta)$.

In the same way it follows that if $Y \in \boldsymbol{B} \perp \beta$ then $Y \in \boldsymbol{B} \perp (\alpha \& \beta)$.

Lemma 14.3 If $X \in \boldsymbol{B} \perp \alpha$ and $\boldsymbol{B}$ is finite, then there is some β such that $\vdash \alpha \rightarrow \beta$ and $\{X\} = \boldsymbol{B} \perp \beta$.

Proof of Lemma 14.3 If $X = \boldsymbol{B}$, then let $\beta = \alpha$. Otherwise, let $\boldsymbol{B} \backslash X = \{\xi_1, \dots \xi_n\}$, and let β be $\alpha \vee \xi_1 \vee \dots \vee \xi_n$. First suppose that $X \vdash \alpha \vee \xi_1 \vee \dots \vee \xi_n$. It follows from $X \in \boldsymbol{B} \perp \alpha$ that that $X \vdash \xi_k \rightarrow \alpha$ for all $\xi_k \in \boldsymbol{B} \backslash X$. We therefore have $X \vdash$

[9] I do not know if *weak intersection* and *conjunction* can replace *regularity* in theorem 14.4.

α, contrary to the conditions. It may be concluded that $X \nvdash \alpha \vee \xi_1 \vee \cdots \vee \xi_n$. Since $X \cup \{\xi_k\} \vdash \alpha$, and thus $X \cup \{\xi_k\} \vdash \alpha \vee \xi_1 \vee \cdots \vee \xi_n$ holds for all $\xi_k \in \boldsymbol{B}\backslash X$, it follows that $X \in \boldsymbol{B} \perp (\alpha \vee \xi_1 \vee \cdots \vee \xi_n)$.

Next, let $Z \in \boldsymbol{B} \perp (\alpha \vee \xi_1 \vee \cdots \vee \xi_n)$. Since all elements of $\boldsymbol{B}\backslash X$ imply $\alpha \vee \xi_1 \vee \cdots \vee \xi_n$, we have $Z \subseteq X$. From this and $X \in \boldsymbol{B} \perp (\alpha \vee \xi_1 \vee \cdots \vee \xi_n)$ it follows that $\{X\} = \boldsymbol{B} \perp (\alpha \vee \xi_1 \vee \cdots \vee \xi_n)$.

Lemma 14.4 Let $\boldsymbol{B}$ be a finite set. If $\boldsymbol{B} \neq Z \in \boldsymbol{B} \perp \alpha$ for some α, then $\{Z\} = \boldsymbol{B} \perp \mathcal{V}(\boldsymbol{B}\backslash Z)$. ($\mathcal{V}(A)$ is the disjunction of all elements of A.)

Proof of Lemma 14.4 Let $\boldsymbol{B}\backslash Z = \{\xi_1, \ldots \xi_n\}$. It follows by $Z \in \boldsymbol{B} \perp \alpha$ that for each $\xi_k, Z \vdash \xi_k \rightarrow \alpha$. It follows from this and $Z \nvdash \alpha$ that $Z \nvdash (\xi_1 \vee \cdots \vee \xi_n)$ Since every Z' such that $Z \subset Z' \subseteq \boldsymbol{B}$ contains one of $\xi_1, \ldots \xi_n$, we can conclude that $Z \in \boldsymbol{B} \perp (\xi_1 \vee \cdots \vee \xi_n)$.

Next, suppose that $W \in \boldsymbol{B} \perp (\xi_1 \vee \cdots \vee \xi_n)$. Since all elements of $\boldsymbol{B}\backslash Z$ imply $\xi_1 \vee \cdots \vee \xi_n$, $W \cap (\boldsymbol{B}\backslash Z) = \varnothing$, i.e. $W \subseteq Z$. From this and $Z \in \boldsymbol{B} \perp (\xi_1 \vee \cdots \vee \xi_n)$ it follows that $W = Z$. We may conclude that $\{Z\} = \boldsymbol{B} \perp (\xi_1 \vee \cdots \vee \xi_n)$.

Lemma 14.5 Let $\boldsymbol{B}$ be a finite belief base and $\boldsymbol{B}''$ its closure under conjunction. ($\boldsymbol{B}''$ consists of the sentences that are elements of $\boldsymbol{B}$ or conjunctions of elements of $\boldsymbol{B}$.) If an operation $\div$ on $Cn(\boldsymbol{B})$ is generated by a partial meet contraction on $\boldsymbol{B}$, then it is generated by a partial meet contraction on $\boldsymbol{B}''$.

Definitions for the Proof of Lemma 14.5

1. Let A and $\boldsymbol{B}$ be sets of sentences. Then A is $\boldsymbol{B}$-*closed* iff $Cn(A) \cap \mathbf{B} \subseteq A$.
2. $\Re(\boldsymbol{B})$ is the set of $\boldsymbol{B}$-closed subsets of $\boldsymbol{B}$ (Hansson 1991).

Proof of Lemma 14.5 Let f be the function such that for each element A of $\Re(\boldsymbol{B})$, $f(A)$ is the closure under conjunction of A.

We first need to show that f is a one-to-one correspondence. Suppose that it is not. Then there are two elements A and A' of $\Re(\boldsymbol{B})$ such that $A \neq A'$ and $f(A) = f(A')$. We may, without loss of generality, assume that there is then some α such that $\alpha \in A$ and $\alpha \notin A'$. If follows from $\alpha \in A$ and $Cn(A) = Cn(f(A)) = Cn(f(A')) = Cn(A')$ that $\alpha \in Cn(A')$. Since A' is $\boldsymbol{B}$-closed, it follows from $\alpha \in Cn(A')$ and $\alpha \in \boldsymbol{B}$ that $\alpha \in A'$. We may conclude from this contradiction that f is a one-to-one correspondence.

In order to prove the lemma, suppose that the operation $\div$ on $Cn(\boldsymbol{B})$ is based on the partial meet contraction $\sim_\gamma$ on $\boldsymbol{B}$. Let γ'' be the function such that:

(1) If $\boldsymbol{B}'' \perp \alpha \neq \varnothing$, then $\gamma''\left(\boldsymbol{B}'' \perp \alpha\right) = \left\{X \in \boldsymbol{B}'' \perp \alpha \mid f^{-1}(X) \in \gamma(\boldsymbol{B} \perp \alpha)\right\}$
(2) If $\boldsymbol{B}'' \perp \alpha = \varnothing$, then $\gamma''\left(\boldsymbol{B}'' \perp \alpha\right) = \{\boldsymbol{B}''\}$.

We need to prove (1) that γ'' is a selection function for $\boldsymbol{B}''$, and (2) that for all α, $\mathbf{K} \div \alpha = Cn\left(\bigcap \gamma''\left(\boldsymbol{B}'' \perp \alpha\right)\right)$.

Part 1: In order to prove that γ'' is a selection function for $\boldsymbol{B}''$ we have to show that if $\boldsymbol{B}'' \perp \alpha \neq \varnothing$, then $\gamma''\left(\boldsymbol{B}'' \perp \alpha\right) \neq \varnothing$. Let $\boldsymbol{B}'' \perp \alpha \neq \varnothing$. Then $\boldsymbol{B} \perp \alpha \neq \varnothing$, from which follows that $\gamma(\boldsymbol{B} \perp \alpha)$ is nonempty. Let $X \in \gamma(\boldsymbol{B} \perp \alpha)$. It follows from $Cn(f(X)) = Cn(X)$ that $f(X) \nvdash \alpha$.

Suppose that there is some $Y \subseteq \boldsymbol{B}''$ such that $f(X) \subset Y \nvdash \alpha$. There is then a set Y with this property that is closed under conjunction. It follows that $X \subset f^{-1}(Y) \nvdash \alpha$ and $f^{-1}(Y) \subseteq \boldsymbol{B}$, contrary to $X \in \boldsymbol{B} \perp \alpha$. We may conclude from this contradiction that there is no $Y \subseteq \boldsymbol{B}''$ such that $f(X) \subset Y \nvdash \alpha$. Since $f(X) \nvdash \alpha$ it follows that $f(X) \in \boldsymbol{B}'' \perp \alpha$. Since $f^{-1}(f(X)) = X \in \gamma(\boldsymbol{B} \perp \alpha)$, it follows from the construction of γ'' that $\gamma''\left(\boldsymbol{B}'' \perp \alpha\right) \neq \varnothing$.

Part 2: Since, by the assumptions, $(Cn(\boldsymbol{B})) \div \alpha = Cn(\bigcap \gamma(\boldsymbol{B} \perp \alpha))$, it is sufficient to prove that $Cn\left(\bigcap \gamma''\left(\boldsymbol{B}'' \perp \alpha\right)\right) = Cn(\bigcap \gamma(\boldsymbol{B} \perp \alpha))$. Since $\boldsymbol{B}'' \perp \alpha \neq \varnothing$ if and only if $\boldsymbol{B} \perp \alpha \neq \varnothing$, and $Cn(\boldsymbol{B}'') = Cn(\boldsymbol{B})$, only the case when $\boldsymbol{B}'' \perp \alpha \neq \varnothing$ requires further consideration.

For one direction, let $\delta \in \bigcap \gamma(\boldsymbol{B} \perp \alpha)$. Then $\delta \in Z$ for every $Z \in \gamma(\boldsymbol{B} \perp \alpha)$. By the construction of γ'', $\delta \in Z''$ for every $Z'' \in \gamma''\left(\boldsymbol{B}'' \perp \alpha\right)$. It follows that $\delta \in \bigcap \gamma''\left(\boldsymbol{B}'' \perp \alpha\right)$. Thus, $\bigcap \gamma(\boldsymbol{B} \perp \alpha) \subseteq \bigcap \gamma''\left(\boldsymbol{B}'' \perp \alpha\right)$; from which $Cn(\bigcap \gamma(\boldsymbol{B} \perp \alpha)) \subseteq Cn\left(\bigcap \gamma''\left(\boldsymbol{B}'' \perp \alpha\right)\right)$ can be concluded.

For the other direction, suppose that $\varepsilon \in \bigcap \gamma''\left(\boldsymbol{B}'' \perp \alpha\right)$. It follows from $\varepsilon \in \boldsymbol{B}''$ that there are elements $\varepsilon_1, \ldots \varepsilon_n$ of $\boldsymbol{B}$ such that $\varepsilon \equiv \varepsilon_1 \& \ldots \& \varepsilon_n$. Let $W \in \gamma(\boldsymbol{B} \perp \alpha)$. By the construction of γ'', $f(W) \in \gamma''\left(\boldsymbol{B}'' \perp \alpha\right)$. It follows from $f(W) \in \boldsymbol{B}'' \perp \alpha$ that $f(\mathrm{W})$ is $\boldsymbol{B}''$-closed. We may conclude from $\{\varepsilon_1, \ldots \& \varepsilon_n\} \subseteq \boldsymbol{B}''$, $\varepsilon \in f(W)$ and the $\boldsymbol{B}''$-closure of $f(\mathrm{W})$ that $\{\varepsilon_1, \ldots \& \varepsilon_n\} \subseteq f(W)$.

It follows from $W \in \boldsymbol{B} \perp \alpha$ that W is $\boldsymbol{B}$-closed. Since $Cn(W) = Cn(f(W))$ and $\{\varepsilon_1, \ldots \& \varepsilon_n\} \subseteq \boldsymbol{B}$, we may conclude from $\{\varepsilon_1, \ldots \& \varepsilon_n\} \subseteq f(W)$ that $\{\varepsilon_1, \ldots \& \varepsilon_n\} \subseteq W$. Since this holds for all $W \in \gamma(\boldsymbol{B} \perp \alpha)$, we have $\{\varepsilon_1, \ldots \& \varepsilon_n\} \subseteq \bigcap \gamma(\boldsymbol{B} \perp \alpha)$. Thus, $\varepsilon \in Cn(\bigcap \gamma(\boldsymbol{B} \perp \alpha))$. We have proved that $\bigcap \gamma''\left(\boldsymbol{B}'' \perp \alpha\right) \subseteq Cn(\bigcap \gamma(\boldsymbol{B} \perp \alpha))$, from which $Cn\left(\bigcap \gamma''\left(\boldsymbol{B}'' \perp \alpha\right)\right) \subseteq Cn(\bigcap \gamma(\boldsymbol{B} \perp \alpha))$ follows as desired.

Lemma 14.6 Let the operation $\div$ on the belief set $\mathbf{K}$ be generated by the partial meet contraction $\sim_\gamma$ on the finite base $\boldsymbol{B}$ for $\mathbf{K}$. Then $\mathbf{K} \div \beta$ is a maximally preservative α-removal iff $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \subseteq \boldsymbol{B} \cap (\mathbf{K} \div \beta) \in \boldsymbol{B} \perp \alpha$.

Proof of Lemma 14.6 For the non-trivial direction, suppose that $\mathbf{K} \div \beta$ is a maximally preservative α-removal and that $\boldsymbol{B} \cap (\mathbf{K} \div \beta)$ is not an element of $\boldsymbol{B} \perp \alpha$. Then there must be some $X \subseteq \boldsymbol{B}$ such that $\boldsymbol{B} \cap (\mathbf{K} \div \beta) \subset X \in \boldsymbol{B} \perp \alpha$ and that there is no δ for which $Cn(X) = \mathbf{K} \div \delta$. However, this is impossible since by *lemma* 14.3 there is some δ such that $\{X\} = \boldsymbol{B} \perp \delta$, and by the definition of partial meet contraction $\boldsymbol{B} \sim_\gamma \delta = X$, so that $\mathbf{K} \div \delta = Cn(X)$.

Lemma 14.7 Let $\div$ be an operator on $\mathbf{K}$ that satisfies *closure* (G$\div$1) and *finitude*, and let $\boldsymbol{B} = \{\& X \mid (\exists \alpha)(X = \mathbf{K} \div \alpha)\}$. Then $Cn(\boldsymbol{B} \cap (\mathbf{K} \div \alpha)) = \mathbf{K} \div \alpha$.

Proof of Lemma 14.7 By the construction, $\&(\mathbf{K} \div \alpha) \in \boldsymbol{B}$. By *closure* (G$\div$1), $\&(\mathbf{K} \div \alpha) \in \mathbf{K} \div \alpha$. It follows that $\&(\mathbf{K} \div \alpha) \in (\boldsymbol{B} \cap (\mathbf{K} \div \alpha))$, so that

$$Cn(\{\&(\mathbf{K} \div \alpha)\}) \subseteq Cn(\boldsymbol{B} \cap (\mathbf{K} \div \alpha)).$$

We also have $\mathbf{K} \div \alpha \subseteq Cn(\{\&(\mathbf{K} \div \alpha)\})$, so that $\mathbf{K} \div \alpha \subseteq Cn(\boldsymbol{B} \cap (\mathbf{K} \div \alpha))$. The other direction follows directly from *closure* (G$\div$1).

Lemma 14.8 Let $\div$ be an operator on the belief set $\mathbf{K}$ that satisfies *closure* (G$\div$1), *success* (G$\div$4), *finitude* and *conservativity*, and let

$$\boldsymbol{B} = \{\&X \mid (\exists\alpha)(X = \mathbf{K} \div \alpha)\}.$$

Then:

$$\text{If } \{X\} = \boldsymbol{B} \perp \delta, \text{ then } X = \boldsymbol{B} \cap (\mathbf{K} \div \delta).$$

Proof of Lemma 14.8 Let $\{X\} = \boldsymbol{B} \perp \delta$. It follows by *success* (G$\div$4) that $\boldsymbol{B} \cap (\mathbf{K} \div \delta) \subseteq X$. Suppose that $X \nsubseteq \boldsymbol{B} \cap (\mathbf{K} \div \delta)$. Then, by the construction of $\boldsymbol{B}$, there is some ϕ such that $\&(\mathbf{K} \div \phi) \in X$ and $\&(\mathbf{K} \div \phi) \notin \boldsymbol{B} \cap (\mathbf{K} \div \delta)$. By *closure* (G$\div$1), $(\mathbf{K} \div \phi) \nsubseteq (\mathbf{K} \div \delta)$.

It follows by *conservativity* that there is some ψ such that $\mathbf{K} \div \delta \subseteq \mathbf{K} \div \psi \nvdash \delta$ and $(\mathbf{K} \div \phi) \cup (\mathbf{K} \div \psi) \vdash \delta$. However, it follows from $\mathbf{K} \div \psi \nvdash \delta$ and $\{X\} = \boldsymbol{B} \perp \delta$ that $\boldsymbol{B} \cap (\mathbf{K} \div \psi) \subseteq X$. Since both $\boldsymbol{B} \cap (\mathbf{K} \div \phi)$ and $\boldsymbol{B} \cap (\mathbf{K} \div \psi)$ are subsets of X, it follows by *lemma* 14.7 that $(\mathbf{K} \div \phi) \cup (\mathbf{K} \div \psi) \nvdash \delta$. We may conclude from this contradiction that $X = \boldsymbol{B} \cap (\mathbf{K} \div \delta)$.

Lemma 14.9 Let $\div$ be an operation on the belief set $\mathbf{K}$ that satisfies *closure* (G$\div$1), *success* (G$\div$4), *finitude*, *symmetry*, *conservativity* and *regularity 2*. Let $\boldsymbol{B} = \{\&X \mid (\exists\alpha)(X = \mathbf{K} \div \alpha)\}$. Then, if $\{X, Y\} \subseteq \boldsymbol{B} \perp \beta$ and $\mathbf{K} \div \beta \subseteq Cn(X)$, there is some β' such that $\{X, Y\} = \boldsymbol{B} \perp \beta'$ and $\mathbf{K} \div \beta' \subseteq Cn(X)$.

Proof of Lemma 14.9 Let $\boldsymbol{B} \perp \beta = \{X, Y, Z_1, \dots Z_n\}$, and suppose that

$$\mathbf{K} \div \beta \subseteq Cn(X).$$

Then $\boldsymbol{B} \cap (\mathbf{K} \div \beta) \subseteq \boldsymbol{B} \cap Cn(X) = X$.

It follows from $X \in \boldsymbol{B} \perp \beta$, by *lemma* 14.3, that $\{X\} = \boldsymbol{B} \perp \delta$ for some δ such that $\vdash \beta \rightarrow \delta$. By *lemma* 14.8, $X = \boldsymbol{B} \cap (\mathbf{K} \div \delta)$, so that by *closure* (G$\div$1) and *lemma* 14.7, $\mathbf{K} \div \delta = Cn(X)$. Similarly, there are ε and $\zeta_1, \dots \zeta_n$ such that $\{Y\} = \boldsymbol{B} \perp \varepsilon$ and $\mathbf{K} \div \varepsilon = Cn(Y)$, and that $\{Z_k\} = \boldsymbol{B} \perp \zeta_k$ and $\mathbf{K} \div \zeta_k = Cn\,(Z_k)$ for all k.

By repeated applications of *lemma* 14.2, $\boldsymbol{B} \perp \beta = \boldsymbol{B} \perp (\delta\&\varepsilon\&\zeta_1\&\dots\&\zeta_n)$. Thus, for all ϕ, $\boldsymbol{B} \cap (\mathbf{K} \div \phi) \vdash \beta$ iff $\boldsymbol{B} \cap (\mathbf{K} \div \phi) \vdash \delta\&\varepsilon\&\zeta_1\&\dots\&\zeta_n$. By *lemma* 14.7, $\mathbf{K} \div \phi \vdash \beta$ iff $\mathbf{K} \div \phi \vdash \delta\&\varepsilon\&\zeta_1\&\dots\&\zeta_n$. By *symmetry*, $\mathbf{K} \div \beta = \mathbf{K} \div (\delta\&\varepsilon\&\zeta_1\&\dots\&\zeta_n)$.

We have assumed that $\mathbf{K} \div \beta \subseteq Cn(X)$, i.e., $\mathbf{K} \div (\delta\&\varepsilon\&\zeta_1\&\dots\&\zeta_n) \subseteq \mathbf{K} \div \delta$. Suppose that $\mathbf{K} \div \delta$ is not a maximally preservative $\delta \,\&\, \varepsilon \,\&\, \zeta_1 \,\&\, \dots \,\&\zeta_n$-removal. Then there is some ϕ such that $\mathbf{K} \div \delta \subset \mathbf{K} \div \phi \nvdash \delta\&\varepsilon\&\zeta_1\&\dots\&\zeta_n$. By *lemma* 14.7, $Cn\,(\boldsymbol{B} \cap (\mathbf{K} \div \delta)) \subset Cn\,(\boldsymbol{B} \cap (\mathbf{K} \div \phi))$, so that

$$\boldsymbol{B} \cap (\mathbf{K} \div \delta)) \subset \boldsymbol{B} \cap (\mathbf{K} \div \phi) \nvdash \delta\&\varepsilon\&\zeta_1\&\dots\&\zeta_n,$$

contrary to $\boldsymbol{B} \cap (\mathbf{K} \div \delta) \in \boldsymbol{B} \perp (\delta\&\varepsilon\&\zeta_1\&\dots\&\zeta_n)$. We may conclude that $\mathbf{K}\div\delta$ is a maximally preservative $\delta \,\&\, \varepsilon \,\&\, \zeta_1 \,\&\, \dots \,\&\zeta_n$-removal. By *success* (G$\div$4), it is also a $\delta\&\varepsilon$-removal. It follows by *regularity 2* that $\mathbf{K} \div \delta$ is a maximally preservative

$\delta\&\varepsilon$-removal, so that $\mathbf{K} \div (\delta \div \varepsilon) \subseteq \mathbf{K} \div \delta$. By *lemma* 14.2, $\boldsymbol{B} \perp (\delta\&\varepsilon) = \{X, Y\}$. Since $Cn(X) = \mathbf{K} \div \delta$ we therefore have $\{X, Y\} = \boldsymbol{B} \perp (\delta\&\varepsilon)$ and $\mathbf{K} \div (\delta\&\varepsilon) \subseteq \mathbf{K} \div \delta = Cn(X)$ as desired.

Proof of Theorem 14.1, Left-to-Right Let $\div$ be an operation on $\mathbf{K}$ that is generated by an operator $\sim_\gamma$ of partial meet contraction on a finite belief base $\boldsymbol{B}$ for $\mathbf{K}$. It should be obvious that $\div$ satifies (G$\div$1)-(G$\div$5) and *finitude*.

Symmetry: We use the converse form of *symmetry*. Suppose that $\mathbf{K} \div \alpha \neq \mathbf{K} \div \beta$, i.e., that $Cn\left(\bigcap \gamma(\boldsymbol{B} \perp \alpha)\right) \neq Cn\left(\bigcap \gamma(\boldsymbol{B} \perp \beta)\right)$. Then $\boldsymbol{B} \perp \alpha \neq \boldsymbol{B} \perp \beta$. Without loss of generality we may assume that there is some $X \in \boldsymbol{B} \perp \alpha$ such that $X \notin \boldsymbol{B} \perp \beta$. There are two cases:

Case 1, $X \vdash \beta$: By *lemma* 14.3 there is some δ such that $\{X\} = \boldsymbol{B} \perp \delta$. By the definition of partial meet contraction, $\boldsymbol{B}{\sim_\gamma}\delta = X$, so that $\mathbf{K} \div \delta = Cn(X)$. It follows that $\mathbf{K} \div \delta \nvdash \alpha$ and $\mathbf{K} \div \delta \vdash \beta$.

Case 2, $X \nvdash \beta$: Then there is some X' such that $X \subset X' \in \boldsymbol{B} \perp \beta$. *Lemma* 14.3 can be used in the same way as in case 1 to show that there is some δ such that $\mathbf{K} \div \delta = Cn\,(X')$. It follows that $\mathbf{K} \div \delta \nvdash \beta$ and $\mathbf{K} \div \delta \vdash \alpha$.

Conservativity: By *lemma* 14.5, we may assume that $\boldsymbol{B}$ is closed under conjunction.

Suppose that $\mathbf{K} \div \beta \nsubseteq \mathbf{K} \div \alpha$. Then $\&\,(\mathbf{K} \div \beta) \notin \boldsymbol{B}{\sim_\gamma}\alpha$ It follows by the definition of partial meet contraction that there is some $X \in \gamma(\boldsymbol{B} \perp \alpha)$ such that $\&\,(\mathbf{K} \div \beta) \notin X$. By *lemma* 14.3, there is some δ such that $\{X\} = \boldsymbol{B} \perp \delta$. By the definition of partial meet contraction, $X = \boldsymbol{B}{\sim_\gamma}\delta$. Since $\boldsymbol{B}$ is, by assumption, closed under conjunction, it follows from $\boldsymbol{B}{\sim_\gamma}\beta \subseteq \boldsymbol{B}$ that $\&\left(\boldsymbol{B}{\sim_\gamma}\beta\right) \in \boldsymbol{B}$. We also have $\&\,(\mathbf{K} \div \beta) = \&\left(\boldsymbol{B}{\sim_\gamma}\beta\right)$, and it therefore follows from $\&\,(\mathbf{K} \div \beta) \notin \boldsymbol{B}{\sim_\gamma}\delta \in \boldsymbol{B} \perp \alpha$ that $\left(\boldsymbol{B}{\sim_\gamma}\delta\right) \cup \{\&\,(\mathbf{K} \div \beta)\} \vdash \alpha$, from which $(\mathbf{K} \div \delta) \cup (\mathbf{K} \div \beta) \vdash \alpha$ can be concluded. It follows from $\boldsymbol{B}{\sim_\gamma}\delta \in \gamma\,(\boldsymbol{B} \perp \alpha)$ that $\boldsymbol{B}{\sim_\gamma}\alpha \subseteq \boldsymbol{B}{\sim_\gamma}\delta \nvdash \alpha$, from which we may conclude that $\mathbf{K} \div \alpha \subseteq \mathbf{K} \div \delta \nvdash \alpha$.

Proof of Theorem 14.1, Right-to-Left Let $\boldsymbol{B} = \{\&X|\,(\exists\alpha)(X = \mathbf{K} \div \alpha)\}$, and let γ be is defined as follows:

(1) If $\boldsymbol{B} \perp \alpha \neq \varnothing$, then $\gamma(\boldsymbol{B} \perp \alpha) = \{X \in \boldsymbol{B} \perp \alpha|\,(\mathbf{K} \div \alpha) \subseteq Cn(X)\}$.
(2) If $\boldsymbol{B} \perp \alpha = \varnothing$, then $\gamma(\boldsymbol{B} \perp \alpha) = \{\boldsymbol{B}\}$

We need to show (1) that $\boldsymbol{B}$ is a finite base for $\mathbf{K}$, (2) that γ is a function, (3) that γ is a selection function for $\boldsymbol{B}$, and (4) that for all α: $\mathbf{K} \div \alpha = Cn\left(\bigcap \gamma(\boldsymbol{B} \perp \alpha)\right)$.

Part 1: It follows from *vacuity* (G$\div$3) and *finitude* that $\&\mathbf{K} \in \boldsymbol{B}$. Thus, $\mathbf{K} \subseteq Cn\,(\boldsymbol{B})$. It follows from *inclusion* (G$\div$2) that $\boldsymbol{B} \subseteq \mathbf{K}$. Thus, $Cn(\boldsymbol{B}) = \mathbf{K}$. By *finitude*, $\boldsymbol{B}$ is finite.

Part 2: Suppose that γ is not a function over the given domain. Then there are α and β such that $\boldsymbol{B} \perp \alpha = \boldsymbol{B} \perp \beta$ and

$$\{X \in \boldsymbol{B} \perp \alpha|\,(\mathbf{K} \div \alpha) \subseteq Cn(X)\} \neq \{X \in \boldsymbol{B} \perp \alpha|\,(\mathbf{K} \div \beta) \subseteq Cn(X)\}\,.$$

It follows that $\mathbf{K} \div \alpha \neq \mathbf{K} \div \beta$. However, from $\boldsymbol{B} \perp \alpha = \boldsymbol{B} \perp \beta$ it follows that for all δ, $\boldsymbol{B} \cap (\mathbf{K} \div \delta) \vdash \alpha$ iff $\boldsymbol{B} \cap (\mathbf{K} \div \delta) \vdash \beta$. By *lemma* 14.7, $Cn\,(\boldsymbol{B} \cap (\mathbf{K} \div \delta)) = \mathbf{K} \div \delta$.

Thus, $\mathbf{K} \div \delta \vdash \alpha$ iff $\mathbf{K} \div \delta \vdash \beta$. By *symmetry*, $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$, contrary to what was just shown. This contradiction concludes part (2) of the proof.

Part 3: In order to prove that γ is a selection function for $\boldsymbol{B}$, it remains to be shown that if $\boldsymbol{B} \perp \alpha$ is non-empty, then so is $\gamma(\boldsymbol{B} \perp \alpha)$. If $\boldsymbol{B} \perp \alpha$ is non-empty, then α is not a logical truth. By *success* (G÷4), $\mathbf{K} \div \alpha \nvdash \alpha$. Thus $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \nvdash \alpha$, so that there is some X with $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \subseteq X \in \boldsymbol{B} \perp \alpha$. By *lemma* 14.7, $Cn(\boldsymbol{B} \cap (\mathbf{K} \div \alpha)) = \mathbf{K} \div \alpha$. It follows that $\mathbf{K} \div \alpha \subseteq Cn(X)$. Then by the definition of γ, $\gamma(\boldsymbol{B} \perp \alpha)$ is nonempty.

Part 4: If α is a logical theorem, then let $\beta \notin \mathbf{K}$. It follows by *vacuity* (G÷3) that $\mathbf{K} \div \beta = \mathbf{K}$. By *conservativity*, if $\mathbf{K} \div \beta \nsubseteq \mathbf{K} \div \alpha$, then there is some δ such that $\mathbf{K} \div \delta \nvdash \alpha$. By *closure* (G÷1), this is impossible. Thus $\mathbf{K} \div \beta \subseteq \mathbf{K} \div \alpha$, i.e. $\mathbf{K} \subseteq \mathbf{K} \div \alpha$. With *inclusion* (G÷2), this yields $\mathbf{K} = \mathbf{K} \div \alpha$. By the definition of partial meet contraction, $\bigcap \gamma(\boldsymbol{B} \perp \alpha) = \boldsymbol{B}$. Using the result of part 1 of the present proof, we obtain $Cn(\bigcap \gamma(\boldsymbol{B} \perp \alpha)) = Cn(\boldsymbol{B}) = \mathbf{K} = \mathbf{K} \div \alpha$.

If α is not a logical theorem, then we use the construction of $\boldsymbol{B}$ to obtain $\&(\mathbf{K} \div \alpha) \in \boldsymbol{B}$. By *closure* (G÷1), $\&(\mathbf{K} \div \alpha) \in \mathbf{K} \div \alpha$. The construction of γ yields $\&(\mathbf{K} \div \alpha) \in \bigcap \gamma(\boldsymbol{B} \perp \alpha)$. It follows that $\mathbf{K} \div \alpha \subseteq Cn(\bigcap \gamma(\boldsymbol{B} \perp \alpha))$.

For the other direction, suppose that $\varepsilon \notin \boldsymbol{B} \cap (\mathbf{K} \div \alpha)$. If there is no β such that $\varepsilon = \&(\mathbf{K} \div \beta)$, then $\varepsilon \notin \boldsymbol{B}$ so that $\varepsilon \notin \bigcap \gamma(\boldsymbol{B} \perp \alpha)$. If $\varepsilon = \&(\mathbf{K} \div \beta)$ for some β, then it follows from by *closure* (G÷1) from $\&(\mathbf{K} \div \beta) \notin \boldsymbol{B} \cap (\mathbf{K} \div \alpha)$ that $\mathbf{K} \div \beta \nsubseteq \mathbf{K} \div \alpha$. By *conservativity* there is some δ such that $\mathbf{K} \div \alpha \subseteq \mathbf{K} \div \delta \nvdash \alpha$ and $(\mathbf{K} \div \beta) \cup (\mathbf{K} \div \delta) \vdash \alpha$. By *lemma* 14.7, $Cn(\boldsymbol{B} \cap (\mathbf{K} \div \delta)) = \mathbf{K} \div \delta$, so that $(\boldsymbol{B} \cap (\mathbf{K} \div \delta)) \cup \{\&(\mathbf{K} \div \beta)\} \vdash \alpha$. It follows from this and $(\boldsymbol{B} \cap (\mathbf{K} \div \alpha)) \subseteq (\boldsymbol{B} \cap (\mathbf{K} \div \delta)) \nvdash \alpha$ that there is some Y such that $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \subseteq \boldsymbol{B} \cap (\mathbf{K} \div \delta) \subseteq Y \in \boldsymbol{B} \perp \alpha$ and $\&(\mathbf{K} \div \beta) \notin Y$. By *lemma* 14.7 and the definition of γ, $Y \in \gamma(\boldsymbol{B} \perp \alpha)$. Since $\&(\mathbf{K} \div \beta) \notin Y$, we have $\&(\mathbf{K} \div \beta) \notin \bigcap \gamma(\boldsymbol{B} \perp \alpha)$, i.e. $\varepsilon \notin \bigcap \gamma(\boldsymbol{B} \perp \alpha)$.

Thus if $\varepsilon \notin \boldsymbol{B} \cap (\mathbf{K} \div \alpha)$, then $\varepsilon \notin \bigcap \gamma(\boldsymbol{B} \perp \alpha)$. We may conclude that $\bigcap \gamma(\boldsymbol{B} \perp \alpha) \subseteq \boldsymbol{B} \cap (\mathbf{K} \div \alpha)$. It follows by *lemma* 14.7 that $Cn(\bigcap \gamma(\boldsymbol{B} \perp \alpha)) \subseteq \mathbf{K} \div \alpha$.

Proof of Theorem 14.2, Left-to-Right Let $\div$ be an operation on $\mathbf{K}$ that is generated by an operator $\sim_\gamma$ of maxichoice partial meet contraction on a finite belief base $\boldsymbol{B}$ for $\mathbf{K}$. We can make use of the corresponding part of the proof of Theorem 14.1, so that it only remains to prove that *strong conservativity* holds.

By *lemma* 14.5, we may assume that $\boldsymbol{B}$ is closed under conjunction. Suppose that $\mathbf{K} \div \beta \nsubseteq \mathbf{K} \div \alpha$. Then $\boldsymbol{B} \sim_\gamma \beta \nsubseteq \boldsymbol{B} \sim_\gamma \alpha$. It follows that α is not a logical theorem, so that $\boldsymbol{B} \sim_\gamma \alpha \nvdash \alpha$, i.e. $\mathbf{K} \div \alpha \nvdash \alpha$.

Since $\boldsymbol{B}$ is closed under conjunction, $\&(\boldsymbol{B} \sim_\gamma \beta) \in \boldsymbol{B}$. Since $\sim_\gamma$ is maxichoice, $\boldsymbol{B} \sim_\gamma \alpha \in \boldsymbol{B} \perp \alpha$. Since $\&\left(\boldsymbol{B} \sim_\gamma \beta\right) \notin \boldsymbol{B} \sim_\gamma \alpha$, we may conclude that $\boldsymbol{B} \sim_\gamma \alpha \cup \left\{\&\left(\boldsymbol{B} \sim_\gamma \beta\right)\right\} \vdash \alpha$, from which it follows that $\mathbf{K} \div \beta \cup \mathbf{K} \div \alpha \vdash \alpha$.

Proof of Theorem 14.2, Right-to-Left $\boldsymbol{B}$ and γ are constructed in the same way as in the proof of Theorem 14.1. We have to prove: (1) that $\boldsymbol{B}$ is a finite base for $\mathbf{K}$, (2) that γ is a function, (3) that γ is a selection function for $\boldsymbol{B}$, (4) that for all α: $\mathbf{K} \div \alpha = Cn(\bigcap \gamma(\boldsymbol{B} \perp \alpha))$, and (5) that γ is maxichoice. Parts 1–3 coincide with the corresponding parts of the proof of Theorem 14.1. Since *strong conservativity* implies *conservativity*, the proof of part 4 of Theorem 14.1 is also a proof of part 4 of the present proof.

Part 5: Let $\delta \in \boldsymbol{B} \backslash (\boldsymbol{B} \sim_\gamma \alpha)$. By the construction of $\boldsymbol{B}$, $\delta = \&(\mathbf{K} \div \beta)$ for some β. Since $\&(\mathbf{K} \div \beta) \in \boldsymbol{B}$ and $\boldsymbol{B} \sim_\gamma \alpha$ is $\boldsymbol{B}$-closed (cf. the definition for *lemma* 14.5), it follows from $\& (\mathbf{K} \div \beta) \notin \bigcap \gamma (\boldsymbol{B} \perp \alpha)$ that $\& (\mathbf{K} \div \beta) \notin Cn (\bigcap \gamma (\boldsymbol{B} \perp \alpha))$, thus by the result of part 4, $\&(\mathbf{K} \div \beta) \notin \mathbf{K} \div \alpha$. By *closure* (G÷1), $\mathbf{K} \div \beta \not\subseteq \mathbf{K} \div \alpha$. By *strong conservativity*, $\mathbf{K} \div \beta \cup \mathbf{K} \div \alpha \vdash \alpha$. Thus, $\boldsymbol{B} \sim_\gamma \alpha \cup \{\&(\mathbf{K} \div \beta)\} \vdash \alpha$, i.e., $\boldsymbol{B} \sim_\gamma \alpha \cup \{\delta\} \vdash \alpha$. Since this holds for all $\delta \in \boldsymbol{B} \backslash (\boldsymbol{B} \sim_\gamma \alpha)$, we can conclude that $\boldsymbol{B} \sim_\gamma \alpha \in \boldsymbol{B} \perp \alpha$.

Proof of Theorem 14.3, Left-to-Right Let ÷ be the operation on **K** that is generated by the operator ∼ of full meet contraction on a finite belief base $\boldsymbol{B}$ for **K**. Making use of the corresponding part of the proof of Theorem 14.1, it only remains for us to prove that *indecisiveness* holds. Just as in Theorem 14.1 we may, due to *lemma* 14.5, assume that $\boldsymbol{B}$ is closed under conjunction.

Suppose that $\mathbf{K} \div \delta \nvdash \alpha$ and $(\mathbf{K} \div \beta) \cup (\mathbf{K} \div \delta) \vdash \alpha$. Then $\boldsymbol{B} \sim \delta \nvdash \alpha$ and $(\boldsymbol{B} \sim \beta) \cup (\boldsymbol{B} \sim \delta) \vdash \alpha$. Since $\boldsymbol{B} \sim \beta$ and $\boldsymbol{B} \sim \delta$ are both subsets of $\boldsymbol{B}$ there is some subset X of $\boldsymbol{B}$ such that $\boldsymbol{B} \sim \delta \subseteq X \in \boldsymbol{B} \perp \alpha$ and $\boldsymbol{B} \sim \beta \not\subseteq X$. By the definition of full meet contraction, $\boldsymbol{B} \sim \alpha \subseteq X$. Suppose that $\boldsymbol{B} \sim \beta \subseteq \boldsymbol{B} \sim \alpha$. It would then follow from $(\boldsymbol{B} \sim \beta) \cup (\boldsymbol{B} \sim \delta) \vdash \alpha$, $\boldsymbol{B} \sim \delta \subseteq X$ and $\boldsymbol{B} \sim \alpha \subseteq X$ that $X \vdash \alpha$, contrary to $X \in \boldsymbol{B} \perp \alpha$. We may conclude that $\boldsymbol{B} \sim \beta \not\subseteq \boldsymbol{B} \sim \alpha$.

Since all elements of $\boldsymbol{B} \perp \alpha$ are $\boldsymbol{B}$-closed (cf. the definition for *lemma* 14.5), their intersection $\boldsymbol{B} \sim \alpha$ is also $\boldsymbol{B}$-closed. Similarly, $\boldsymbol{B} \sim \beta$ is $\boldsymbol{B}$-closed. It therefore follows from $\boldsymbol{B} \sim \beta \not\subseteq \boldsymbol{B} \sim \alpha$. that $Cn(\boldsymbol{B} \sim \beta) \not\subseteq Cn(\boldsymbol{B} \sim \alpha)$, i.e. $\mathbf{K} \div \beta \not\subseteq \mathbf{K} \div \alpha$.

Proof of Theorem 14.3, Right-to-Left Let $\boldsymbol{B} = \{\&X | (\exists \alpha)(X = \mathbf{K} \div \alpha)\}$. We need to show that (1) $\boldsymbol{B}$ is a finite base for **K**, (2) for all α, $Cn (\bigcap (\boldsymbol{B} \perp \alpha)) \subseteq \mathbf{K} \div \alpha$, and (3) for all α, $\mathbf{K} \div \alpha \subseteq Cn (\bigcap (\boldsymbol{B} \perp \alpha))$. The proof of part 1 coincides with that of part 1 of Theorem 14.1.

Part 2: We are going to prove that $\bigcap (\boldsymbol{B} \perp \alpha) \subseteq \mathbf{K} \div \alpha$. Let $\zeta \notin \mathbf{K} \div \alpha$. If there is no β such that $\zeta = \mathbf{K} \div \beta$, then it follows by the construction of $\boldsymbol{B}$ that $\zeta \notin \bigcap (\boldsymbol{B} \perp \alpha)$. In the principal case, $\zeta = \& (\mathbf{K} \div \beta)$.

It follows from $\& (\mathbf{K} \div \beta) \notin \mathbf{K} \div \alpha$ and *closure* (G÷1) that $\mathbf{K} \div \beta \not\subseteq \mathbf{K} \div \alpha$. By *conservativity*, there is some δ such that $\mathbf{K} \div \alpha \subseteq \mathbf{K} \div \delta \nvdash \alpha$ and $(\mathbf{K} \div \beta) \cup (\mathbf{K} \div \delta) \vdash \alpha$. By *lemma* 14.7, $(\boldsymbol{B} \cap (\mathbf{K} \div \beta)) \cup (\boldsymbol{B} \cap (\mathbf{K} \div \delta)) \vdash \alpha$. Thus, there is some X such that $\& (\boldsymbol{B} \cap (\mathbf{K} \div \beta)) \notin X \in \boldsymbol{B} \perp \alpha$, i.e. $\& (\mathbf{K} \div \beta) \notin X \in \boldsymbol{B} \perp \alpha$. It follows that $\zeta = \& (\mathbf{K} \div \beta) \notin \bigcap (\boldsymbol{B} \perp \alpha)$.

Thus, if $\zeta \notin \mathbf{K} \div \alpha$, then $\zeta \notin \bigcap (\boldsymbol{B} \perp \alpha)$. i.e. $\bigcap (\boldsymbol{B} \perp \alpha) \subseteq \mathbf{K} \div \alpha$. By *closure* (G÷1), we can conclude that $Cn (\bigcap (\boldsymbol{B} \perp \alpha)) \subseteq \mathbf{K} \div \alpha$.

Part 3: We are going to show that $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \subseteq \bigcap (\boldsymbol{B} \perp \alpha)$. Let $\delta \notin \bigcap (\boldsymbol{B} \perp \alpha)$. If there is no β such that $\delta = \& (\mathbf{K} \div \beta)$, then clearly $\delta \notin \boldsymbol{B} \cap (\mathbf{K} \div \alpha)$. In the principal case, let $\delta = \& (\mathbf{K} \div \beta)$. It follows from $\& (\mathbf{K} \div \beta) \notin \bigcap (\boldsymbol{B} \perp \alpha)$ that there is some X such that $\& (\mathbf{K} \div \beta) \notin X \in \boldsymbol{B} \perp \alpha$.

By *lemma* 14.3, $\{X\} = \boldsymbol{B} \perp \phi$ for some ϕ. By *lemma* 14.8, $X = \boldsymbol{B} \cap (\mathbf{K} \div \phi)$. By *lemma* 14.7, $Cn(X) = \mathbf{K} \div \phi$. Therefore, it follows from $\& (\mathbf{K} \div \beta) \notin \boldsymbol{B} \cap (\mathbf{K} \div \phi) \in \boldsymbol{B} \perp \alpha$ that $\mathbf{K} \div \phi \nvdash \alpha$ and $(\mathbf{K} \div \beta) \cup (\mathbf{K} \div \phi) \vdash \alpha$. By *indecisiveness*,

$\mathbf{K}{\div}\beta \nsubseteq \mathbf{K}{\div}\alpha$. By *closure* (G÷1), $\&(\mathbf{K} \div \beta) \notin \mathbf{K} \div \alpha$, so that $\delta = \&(\mathbf{K} \div \beta) \notin \boldsymbol{B} \cap (\mathbf{K} \div \alpha)$.

We may conclude from this that $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \subseteq \bigcap(\boldsymbol{B} \perp \alpha)$. By *lemma* 14.7, $\mathbf{K} \div \alpha \subseteq Cn\,(\bigcap(\boldsymbol{B} \perp \alpha))$.

Proof of Theorem 14.4, Left-to-Right Let γ be a complete selection function such that $\sim_\gamma$ is an operator of transitively, maximizingly, relational partial meet contraction on a finite belief base $\boldsymbol{B}$ for $\mathbf{K}$, and that $\sim_\gamma$ generates the operation ÷ on $\mathbf{K}$. We can make use of the corresponding part of the proof of Theorem 14.1, so that it only remains to prove that *regularity 1* and *regularity 2* hold.

Regularity 1: Suppose that $\vdash \alpha \to \beta$ and that there is some ζ such that $\mathbf{K} \div \zeta$ is both a β-removal and a maximally preservative α-removal. Let $Z = \boldsymbol{B} \cap (\mathbf{K} \div \zeta)$. By *lemma* 14.6, $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \subseteq Z \in \boldsymbol{B} \perp \alpha$.

We are first going to show that $\boldsymbol{B} \cap (\mathbf{K} \div a) = \boldsymbol{B}{\sim_\gamma}\alpha$. It follows from $\mathbf{K} \div \alpha = Cn\left(\boldsymbol{B}{\sim_\gamma}\alpha\right)$ that $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) = \boldsymbol{B} \cap Cn\left(\boldsymbol{B}{\sim_\gamma}\alpha\right)$. Since each element of $\gamma\,(\boldsymbol{B} \perp \alpha)$ is $\boldsymbol{B}$-closed (cf. the definition for *lemma* 14.5), so is $\bigcap \gamma\,(\boldsymbol{B} \perp \alpha) = \boldsymbol{B}{\sim_\gamma}\alpha$, thus $\boldsymbol{B} \cap Cn\left(\boldsymbol{B}{\sim_\gamma}\alpha\right) = \boldsymbol{B}{\sim_\gamma}\alpha$, i.e. $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) = \boldsymbol{B}{\sim_\gamma}\alpha$.

We now have $\boldsymbol{B}{\sim_\gamma}\alpha \subseteq Z \in \boldsymbol{B} \perp \alpha$ so that, by the completeness of γ, $Z \in \gamma\,(\boldsymbol{B} \perp \alpha)$. It follows from $Z \nvdash \beta$ and $\vdash \alpha \to \beta$ that $Z \in \boldsymbol{B} \perp \beta$.

Next, we are going to show that $\gamma\,(\boldsymbol{B} \perp \beta) \subseteq \gamma\,(\boldsymbol{B} \perp \alpha)$

Let $X \in \gamma\,(\boldsymbol{B} \perp \beta)$. Suppose that $X \notin \gamma\,(\boldsymbol{B} \perp \alpha)$. If $X \in \boldsymbol{B} \perp \alpha$, then $Z \in \gamma\,(\boldsymbol{B} \perp \alpha)$ yields $X \ll Z$. If $X \notin \boldsymbol{B} \perp \alpha$, then there is some X' such that $X \subset X' \in \boldsymbol{B} \perp \alpha$. It follows by the maximizing property that $X \ll X'$ and by $Z \in \gamma(\boldsymbol{B}\perp\alpha)$ that $X' \underline{\ll} Z$. Transitivity yields $X \ll Z$, in this case as well.

From $X \ll Z$ and $Z \in \boldsymbol{B} \perp \beta$ it follows that $X \notin \gamma\,(\boldsymbol{B} \perp \beta)$. From this contradiction we may conclude that if $X \in \gamma\,(\boldsymbol{B} \perp \beta)$ then $X \in \gamma\,(\boldsymbol{B} \perp \alpha)$, i.e., that $\gamma\,(\boldsymbol{B} \perp \beta) \subseteq \gamma\,(\boldsymbol{B} \perp \alpha)$.

Now let $\mathbf{K}{\div}\delta$ be a maximally preservative β-removal. By *lemma* 14.6,

$$\boldsymbol{B} \cap (\mathbf{K} \div \beta) \subseteq \boldsymbol{B} \cap (\mathbf{K} \div \delta) \in \boldsymbol{B} \perp \beta.$$

Since $\boldsymbol{B} \cap (\mathbf{K} \div \beta) = \boldsymbol{B}{\sim_\gamma}\beta$ (that follows in the same way as $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) = \boldsymbol{B}{\sim_\gamma}\alpha$), we have $\boldsymbol{B}{\sim_\gamma}\beta \subseteq \boldsymbol{B} \cap (\mathbf{K} \div \delta) \in \boldsymbol{B} \perp \beta$ so that, by the completeness of γ, $\boldsymbol{B} \cap (\mathbf{K} \div \delta) \in \gamma\,(\boldsymbol{B} \perp \beta)$. We have just shown that $\gamma\,(\boldsymbol{B} \perp \beta) \subseteq \gamma\,(\boldsymbol{B} \perp \alpha)$, and may conclude that $\boldsymbol{B} \cap (\mathbf{K} \div \delta) \in \gamma\,(\boldsymbol{B} \perp \alpha)$.

It also follows from $\gamma\,(\boldsymbol{B} \perp \beta) \subseteq \gamma\,(\boldsymbol{B} \perp \alpha)$ that $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \subseteq \boldsymbol{B} \cap (\mathbf{K} \div \beta)$. From this and $\boldsymbol{B}\cap(\mathbf{K} \div \beta) \subseteq \boldsymbol{B}\cap(\mathbf{K} \div \delta)$ it follows that $\boldsymbol{B}\cap(\mathbf{K} \div \alpha) \subseteq \boldsymbol{B}\cap(\mathbf{K} \div \delta)$. We therefore have $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \subseteq \boldsymbol{B} \cap (\mathbf{K} \div \delta) \in \boldsymbol{B} \perp \alpha$. By *lemma* 14.6, $\mathbf{K}{\div}\delta$ is a maximally preservative α-removal.

Regularity 2: If β is a logical theorem, then $\mathbf{K}{\div}\alpha \vdash \beta$ so that *regularity 2* holds vacuously. For the principal case, suppose to the contrary that $\vdash \alpha \to \beta$, that $\mathbf{K}{\div}\delta$ is a maximally preservative α-removal, and that it is a β-removal but not a maximally preservative β-removal. It follows by *lemma* 14.6 that $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \subseteq \boldsymbol{B} \cap (\mathbf{K} \div \delta) \in \boldsymbol{B} \perp \alpha$. By $\boldsymbol{B}{\sim_\gamma}\alpha = \boldsymbol{B} \cap (\mathbf{K} \div \alpha)$ (cf. the proof for *regularity 1*) and the completeness of γ we have $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \subseteq \boldsymbol{B} \cap (\mathbf{K} \div \delta) \in \gamma\,(\boldsymbol{B} \perp \alpha)$. It follows from $\boldsymbol{B} \cap (\mathbf{K} \div \delta) \in B \perp \alpha$, $\boldsymbol{B} \cap (\mathbf{K} \div \delta) \nvdash \beta$ and $\vdash \alpha \rightarrow \beta$ that $\boldsymbol{B} \cap (\mathbf{K} \div \delta) \in \boldsymbol{B} \perp \beta$.

Suppose that $\boldsymbol{B} \cap (\mathbf{K} \div \delta) \in \gamma\,(\boldsymbol{B} \perp \beta)$. Then $\boldsymbol{B} \sim_{\gamma} \beta \subseteq \boldsymbol{B} \cap (\mathbf{K} \div \delta)$, i.e. $\boldsymbol{B} \cap (\mathbf{K} \div \beta) \subseteq \boldsymbol{B} \cap (\mathbf{K} \div \delta)$. By *lemma* 14.6, $\boldsymbol{B} \cap (\mathbf{K} \div \beta) \subseteq \boldsymbol{B} \cap (\mathbf{K} \div \delta) \in \boldsymbol{B} \perp \beta$ contradicts the condition that $\mathbf{K} \div \delta$ is not a maximally preservative β-removal. We may conclude that $\boldsymbol{B} \cap (\mathbf{K} \div \delta) \notin \gamma\,(\boldsymbol{B} \perp \beta)$. It follows that there is some $Z \in \boldsymbol{B} \perp \beta$ such that $Z \underline{\ll} \boldsymbol{B} \cap (\mathbf{K} \div \delta)$ does not hold.

If $Z \in \boldsymbol{B} \perp \alpha$, then $\boldsymbol{B} \cap (\mathbf{K} \div \delta) \in \gamma\,(\boldsymbol{B} \perp \alpha)$ yields $Z \underline{\ll} \boldsymbol{B} \cap (\mathbf{K} \div \delta)$. If $Z \notin \boldsymbol{B} \perp \alpha$, then there is some Z' such that $Z \subset Z' \in \boldsymbol{B} \perp \alpha$. The maximizing property yields $Z \ll Z'$. It follows from $\boldsymbol{B} \cap (\mathbf{K} \div \delta) \in \gamma\,(\boldsymbol{B} \perp \alpha)$ that $Z' \underline{\ll} \boldsymbol{B} \cap (\mathbf{K} \div \delta)$. By transitivity, $Z \ll \boldsymbol{B} \cap (\mathbf{K} \div \delta)$, again contradicting what we have just shown. This contradiction concludes the proof.

Proof of Theorem 14.4, Right-to-Left Let $\boldsymbol{B} = \{\&X \mid (\exists\alpha)(X = \mathbf{K} \div \alpha)\}$. Let $\underline{\ll}$ be the relation such that $Y \underline{\ll} X$ iff either $Y \subset X$ or there is some β such that $\{X, Y\} \subset \boldsymbol{B} \perp \beta$ and $\mathbf{K} \div \beta \subseteq Cn(X)$. Furthermore, let the function γ be defined as follows:

(1) If $\boldsymbol{B} \perp \alpha \neq \varnothing$, then $\gamma\,(\boldsymbol{B} \perp \alpha) = \{\boldsymbol{B}' \in \boldsymbol{B} \perp \alpha \mid \boldsymbol{B}'' \underline{\ll} \boldsymbol{B}'$ for all $\boldsymbol{B}'' \in \boldsymbol{B} \perp \alpha\}$.
(2) Otherwise, $\gamma\,(\boldsymbol{B} \perp \alpha) = \{\boldsymbol{B}\}$.

We need to show that (1) $\boldsymbol{B}$ is a finite base for $\mathbf{K}$, (2) γ is a selection function for $\boldsymbol{B}$, (3) the partial meet contraction $\sim_{\gamma}$ on $\boldsymbol{B}$ generates the operation ÷ on $\mathbf{K}$, (4) γ is a completed selection function, and (5) γ is transitively, maximizingly relational by $\underline{\ll}$. The proof of part 1 coincides with that of part 1 of Theorem 14.1.

Part 2: In order to prove that γ is a selection function for $\boldsymbol{B}$, we need to show that if $\boldsymbol{B} \perp \alpha \neq \varnothing$, then $\gamma(\boldsymbol{B} \perp \alpha)$ is non-empty.

It follows by *success* and *inclusion* that there is some $\boldsymbol{B}'$ such that $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \subseteq \boldsymbol{B}' \in \boldsymbol{B} \perp \alpha$. By *lemma* 14.3, there is some δ such that $\{\boldsymbol{B}'\} = \boldsymbol{B} \perp \delta$ and $\vdash \alpha \rightarrow \delta$. By *lemma* 14.8, $\boldsymbol{B}' = \boldsymbol{B} \cap (\mathbf{K} \div \delta)$.

Let $\boldsymbol{B}''$ be any element of $\boldsymbol{B} \perp \alpha$. By *lemma* 14.3, there is some ε such that $\{\boldsymbol{B}''\} = \boldsymbol{B} \perp \varepsilon$ and $\vdash \alpha \rightarrow \varepsilon$. By *lemma* 14.2, $\{\boldsymbol{B}', \boldsymbol{B}''\} = \boldsymbol{B} \perp (\delta \& \varepsilon)$.

By *lemma* 14.7, it follows from $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \subseteq \boldsymbol{B} \cap (\mathbf{K} \div \delta) \in \boldsymbol{B} \perp \alpha$ that $\mathbf{K} \div \delta$ is a maximally preservative α-removal. By *success* (G÷4), it is also a δ&ε-removal. Since $\vdash \alpha \rightarrow (\delta \& \varepsilon)$ it follows by *regularity 2* that $\mathbf{K} \div \delta$ is a maximally preservative δ&ε-removal. Thus $\mathbf{K} \div (\delta \& \varepsilon) \subseteq \mathbf{K} \div \delta$. We therefore have $\{\boldsymbol{B}', \boldsymbol{B}''\} = \boldsymbol{B} \perp (\delta \& \varepsilon)$ and (by *lemma* 14.7) $\mathbf{K} \div (\delta \& \varepsilon) \subseteq Cn\,(\boldsymbol{B} \cap (\mathbf{K} \div \delta)) = Cn\left(\boldsymbol{B}'\right)$. By the definition of $\underline{\ll}$, $\boldsymbol{B}'' \underline{\ll} \boldsymbol{B}'$.

Since this holds for all $\boldsymbol{B}'' \in \boldsymbol{B} \perp \alpha$, we may conclude by the construction of γ that $\boldsymbol{B}' \in \gamma\,(\boldsymbol{B} \perp \alpha)$, so that $\gamma(\boldsymbol{B} \perp \alpha)$ is non-empty.

Part 3: By *lemma* 14.7, $Cn\,(\boldsymbol{B} \cap (\mathbf{K} \div \alpha)) = \mathbf{K} \div \alpha$. It is therefore sufficient to prove that $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) = \bigcap \gamma\,(\boldsymbol{B} \perp \alpha)$.

If α is a logical theorem, then let $\beta \notin \mathbf{K}$. It follows by *vacuity* (G÷3) that $\mathbf{K} \div \beta = \mathbf{K}$. By *conservativity*, if $\mathbf{K} \div \beta \nsubseteq \mathbf{K} \div \alpha$, then there is some δ such that $\mathbf{K} \div \delta \nvdash \alpha$. By *closure* (G÷1), this is impossible. Thus $\mathbf{K} \div \beta \subseteq \mathbf{K} \div \alpha$, i.e. $\mathbf{K} \subseteq \mathbf{K} \div \alpha$. With *inclusion* (G÷2), this yields $\mathbf{K} = \mathbf{K} \div \alpha$. By the definition of partial meet contraction, $\bigcap\gamma(\boldsymbol{B} \perp \alpha) = \boldsymbol{B}$. Using the result of part 1 of the present proof, we obtain $Cn(\bigcap\gamma(\boldsymbol{B} \perp \alpha)) = Cn(\boldsymbol{B}) = \mathbf{K} = \mathbf{K} \div \alpha$.

It remains to prove the principal case, in which α is not a logical theorem.

Part 3a: We are going to show that $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \subseteq \bigcap\gamma\,(\boldsymbol{B} \perp \alpha)$, i.e., that if $X \in \gamma\,(\boldsymbol{B} \perp \alpha)$, then $\boldsymbol{B} \cap (\mathbf{K} \div a) \subseteq X$.

Let $X \in \gamma\,(\boldsymbol{B} \perp \alpha)$. By *lemma* 14.3, $\{X\} = B \perp \delta$ for some δ such that $\vdash \alpha \rightarrow \delta$. By *lemma* 14.8, $X = \boldsymbol{B} \cap (\mathbf{K} \div \delta)$ and by *lemma* 14.7, $Cn(X) = \mathbf{K}\div\delta$.

Let $Y_1, \dots Y_n$ be the elements of $\boldsymbol{B}\perp\alpha$ apart from X. By *lemma* 14.3, for each Y_k there is some ε_k such that $\alpha \rightarrow \varepsilon_k$ and $\{Y_k\} = \boldsymbol{B}\perp\varepsilon_k$. By *lemma* 14.8, $Y_k = \boldsymbol{B} \cap (\mathbf{K} \div \varepsilon_k)$. By *lemma* 14.7, $Cn(Y_k) = \mathbf{K}\div\varepsilon_k$.

By repeated uses of *lemma* 14.2, $\boldsymbol{B} \perp \alpha = \boldsymbol{B} \perp (\delta\&\varepsilon_1\& \dots \&\varepsilon_n)$. Thus, for all ϕ, $\boldsymbol{B} \cap (\mathbf{K} \div \phi) \vdash \alpha$ iff $\boldsymbol{B} \cap (\mathbf{K} \div \phi) \vdash \delta\&\varepsilon_1\& \dots \&\varepsilon_n$. By *lemma* 14.7, $\mathbf{K}\div\phi \vdash \alpha$ iff $\mathbf{K}\div\phi \vdash \delta\&\varepsilon_1\& \dots \&\varepsilon_n$. By *symmetry*, $\mathbf{K}\div\alpha = \mathbf{K}\div(\delta\&\varepsilon_1\& \dots \&\varepsilon_n)$.

If $\mathbf{K}\div(\delta\&\varepsilon_1\& \dots \&\varepsilon_n) \nvdash \delta$, then it follows from $\{X\} = \boldsymbol{B}\perp\delta$ that $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) = \boldsymbol{B} \cap (\mathbf{K} \div (\delta\&\varepsilon_1\& \dots \&\varepsilon_n)) \subseteq X$.

In the remaining case, when $\mathbf{K}\div(\delta\&\varepsilon_1\& \dots \&\varepsilon_n) \vdash \delta$, there is by *success* (G÷4) some ε_k such that $\mathbf{K}\div(\delta\&\varepsilon_1\& \dots \&\varepsilon_n) \nvdash \varepsilon_k$. Since $\{Yk\} = \boldsymbol{B} \perp \varepsilon_k$,

$$\boldsymbol{B} \cap (\mathbf{K} \div (\delta\&\varepsilon_1\& \dots \&\varepsilon_n)) \subseteq \boldsymbol{B} \cap (\mathbf{K} \div \varepsilon_k)\,,$$

thus by *lemma* 14.7, $\mathbf{K} \div (\delta\&\varepsilon_1\& \dots \&\varepsilon_n) \subseteq \mathbf{K} \div \varepsilon_k$.

We are now going to show that $\mathbf{K}\div\varepsilon_k$ is a maximally preservative $(\delta\&\varepsilon_1\& \dots \&\varepsilon_n)$-removal. Suppose that it is not. Then there is some ϕ such that $\mathbf{K} \div \varepsilon_k \subset \mathbf{K} \div \phi \nvdash (\delta\&\varepsilon_1\& \dots \&\varepsilon_n)$. Clearly, $\boldsymbol{B} \cap (\mathbf{K} \div \varepsilon_k) \subseteq \boldsymbol{B} \cap (\mathbf{K} \div \phi)$. Since $\boldsymbol{B} \cap (\mathbf{K} \div \varepsilon_k) = Y_k \in \boldsymbol{B} \perp (\delta\&\varepsilon_1\& \dots \&\varepsilon_n)$, it follows from

$$\mathbf{K} \div \phi \nvdash (\delta\&\varepsilon_1\& \dots \&\varepsilon_n)$$

that $\boldsymbol{B} \cap (\mathbf{K} \div \varepsilon_k) = \boldsymbol{B} \cap (\mathbf{K} \div \phi)$. On the other hand, it follows from the construction of $\boldsymbol{B}$ that $\&\,(\mathbf{K} \div \phi) \in \boldsymbol{B} \cap (\mathbf{K} \div \phi)$, whereas by $\mathbf{K} \div \varepsilon_k \subset \mathbf{K} \div \phi$ we have $\&\,(\mathbf{K} \div \phi) \notin \boldsymbol{B} \cap (\mathbf{K} \div \varepsilon_k)$, so that $\boldsymbol{B} \cap (\mathbf{K} \div \varepsilon_k) \neq \boldsymbol{B} \cap (\mathbf{K} \div \phi)$. By this contradiction, we can conclude that $\mathbf{K}\div\varepsilon_k$ is a maximally preservative $(\delta\&\varepsilon_1\& \dots \&\varepsilon_n)$-removal.

Since $\mathbf{K}\div\varepsilon_k$ is also a $(\delta\&\varepsilon_k)$-removal, it follows by *regularity 1* that all maximally preservative $\delta\&\varepsilon_k$-removals are maximally preservative $\delta\&\varepsilon_1\& \dots \&\varepsilon_n$-removals.

We are next going to show that $\mathbf{K} \div (\delta\&\varepsilon_1\& \dots \&\varepsilon_n) \subseteq \mathbf{K} \div (\delta\&\varepsilon_k)$. Suppose that this is not the case. Then by *conservativity* there is some ψ such that

$$\mathbf{K} \div (\delta\&\varepsilon_k) \subseteq \mathbf{K} \div \psi \nvdash \delta\&\varepsilon_k \text{ and } \mathbf{K} \div (\delta\&\varepsilon_1\& \dots \&\varepsilon_n) \cup \mathbf{K} \div \psi \vdash \delta\&\varepsilon_k.$$

Since $\mathbf{K}\div\psi$ is a preservative $\delta\&\varepsilon_k$-removal, there must, by *finitude*, be some maximally preservative $\delta\&\varepsilon_k$-removal $\mathbf{K}\div\psi'$ such that $\mathbf{K} \div \psi \subseteq \mathbf{K} \div \psi'$. Then clearly $\mathbf{K}\div(\delta\&\varepsilon_1\& \dots \&\varepsilon_n) \cup \mathbf{K}\div\psi' \vdash \delta\&\varepsilon_k$. However, we have just shown that all maximally preservative $\delta\&\varepsilon_k$-removals are maximally preservative $\delta\&\varepsilon_1\& \dots \&\varepsilon_n$-removals. Thus $\mathbf{K} \div (\delta\&\varepsilon_1\& \dots \&\varepsilon_n) \subseteq \mathbf{K} \div \psi'$, contradicting

$$\mathbf{K} \div (\delta\&\varepsilon_1\&\ldots\&\varepsilon_n) \cup \mathbf{K} \div \psi' \vdash \delta\&\varepsilon_k \text{ and } \mathbf{K} \div \psi' \nvdash \delta\&\varepsilon_k.$$

We can conclude that $\mathbf{K} \div (\delta\&\varepsilon_1\&\ldots\&\varepsilon_n) \subseteq \mathbf{K} \div (\delta\&\varepsilon_k)$.

From the definition of γ, it follows from $Y_k \in \boldsymbol{B} \perp \alpha$ and $X \in \gamma\,(\boldsymbol{B} \perp \alpha)$ that $Y_k \underline{\ll} X$ and $X \underline{\ll} X$. By the definition of $\underline{\ll}$ and *lemma* 14.9 there is some β such that $\{X, Y_k\} = \boldsymbol{B} \perp \beta$ and $\mathbf{K} \div \beta \subseteq Cn(X) = \mathbf{K} \div \delta$.

By *lemma* 14.2, $\{X, Y_k\} = \boldsymbol{B} \perp (\delta\&\varepsilon_k)$. It follows from $\boldsymbol{B} \perp \beta = \boldsymbol{B} \perp (\delta\&\varepsilon_k)$ that for all ζ, $\boldsymbol{B} \cap (\mathbf{K} \div \zeta) \vdash \beta$ iff $\boldsymbol{B} \cap (\mathbf{K} \div \zeta) \vdash \delta\&\varepsilon_k$. By *lemma* 14.7, $\mathbf{K}\div\zeta \vdash \beta$ iff $\mathbf{K}\div\zeta \vdash \delta\&\varepsilon_k$. By *symmetry*, $\mathbf{K}\div\beta = \mathbf{K}\div(\delta\&\varepsilon_k)$. Thus, $\mathbf{K} \div (\delta\&\varepsilon_k) \subseteq \mathbf{K} \div \delta$. We therefore have $\mathbf{K} \div \alpha = \mathbf{K} \div (\delta\&\varepsilon_1\&\ldots\&\varepsilon_n) \subseteq \mathbf{K} \div (\delta\&\varepsilon_k) \subseteq \mathbf{K} \div \delta = Cn(X)$, i.e., $\mathbf{K} \div \alpha \subseteq Cn(X)$ so that $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \subseteq \boldsymbol{B} \cap (Cn(X)) = X$. This holds for all $X \in \gamma\,(\boldsymbol{B} \perp \alpha)$, finishing this part of the proof.

Part 3b: In order to show that $\bigcap \gamma\,(\boldsymbol{B} \perp \alpha) \subseteq \boldsymbol{B} \cap (\mathbf{K} \div \alpha)$, we will assume that $\zeta \notin \boldsymbol{B} \cap (\mathbf{K} \div \alpha)$ and prove that there is some X such that $\zeta \notin X \in \gamma\,(\boldsymbol{B} \perp \alpha)$.

By the constructions of $\boldsymbol{B}$ and γ, this is trivially true unless $\zeta = \&\,(\mathbf{K} \div \beta)$ for some β. In that case, it follows by *closure* (G÷1) from $\&\,(\mathbf{K} \div \beta) \notin \boldsymbol{B} \cap (\mathbf{K} \div \alpha)$ that $\mathbf{K} \div \beta \nsubseteq \mathbf{K} \div \alpha$. By *conservativity*, there is some δ such that $\mathbf{K} \div \alpha \subseteq \mathbf{K} \div \delta \nvdash \alpha$ and $(\mathbf{K} \div \beta) \cup (\mathbf{K} \div \delta) \vdash \alpha$. It follows by *finitude* that there is some maximally preservative α-removal $\mathbf{K}\div\delta'$ such that $(\mathbf{K} \div \beta) \cup (\mathbf{K} \div \delta') \vdash \alpha$. It follows from $\mathbf{K}\div\delta' \nvdash \alpha$ and $(\mathbf{K} \div \beta) \cup (\mathbf{K} \div \delta') \vdash \alpha$ that $\mathbf{K} \div \beta \nsubseteq \mathbf{K} \div \delta'$.

Suppose that $\boldsymbol{B} \cap (\mathbf{K} \div \delta') \notin \boldsymbol{B} \perp \alpha$. There is then some W such that $\boldsymbol{B} \cap (\mathbf{K} \div \delta') \subset W \in \boldsymbol{B} \perp \alpha$. By *lemma* 14.3 there is some λ such that $\{W\} = \boldsymbol{B} \perp \lambda$. By *lemma* 14.8, $W = \boldsymbol{B} \cap (\mathbf{K} \div \lambda)$. By *closure* and *lemma* 14.7, $\mathbf{K}\div\delta' \subset \mathbf{K}\div\lambda \vdash \alpha$, contrary to the condition that $\mathbf{K}\div\delta'$ is a maximally preservative α-removal. We may conclude that $\boldsymbol{B} \cap (\mathbf{K} \div \delta') \in \boldsymbol{B} \perp \alpha$.

By *lemma* 14.3, $\{\boldsymbol{B} \cap (\mathbf{K} \div \delta')\} = \boldsymbol{B} \perp \varepsilon$ for some ε such that $\vdash \alpha \rightarrow \varepsilon$, and by *lemma* 14.8 $\boldsymbol{B} \cap (\mathbf{K} \div \delta') = \boldsymbol{B} \cap (\mathbf{K} \div \varepsilon)$ so that by *lemma* 14.7 $\mathbf{K}\div\delta' = \mathbf{K}\div\varepsilon$.

Next, let $Y \in \boldsymbol{B}\perp\alpha$. By *lemmas* 14.3 and 14.8 there is some ϕ such that $\vdash \alpha \rightarrow \phi$, $Y = \boldsymbol{B} \cap (\mathbf{K} \div \phi)$, and $\{Y\} = \boldsymbol{B} \perp \phi$. By *lemma* 14.7, $Cn(Y) = \mathbf{K}\div\phi$. By *lemma* 14.2, $\{\boldsymbol{B} \cap (\mathbf{K} \div \varepsilon), \boldsymbol{B} \cap (\mathbf{K} \div \phi)\} = \boldsymbol{B} \perp (\varepsilon\&\phi)$. We have $\vdash \alpha \rightarrow (\varepsilon\&\phi)$, and $\mathbf{K}\div\varepsilon$ is both a maximally preservative α-removal and, by *success* (G÷4), an $\varepsilon\&\phi$-removal. It follows by *regularity 2* that $\mathbf{K}\div\varepsilon$ is a maximally preservative $\varepsilon\&\phi$-removal, so that $\mathbf{K} \div (\varepsilon\&\phi) \subseteq \mathbf{K} \div \varepsilon$. By the definition of $\underline{\ll}$, $(\boldsymbol{B} \cap (\mathbf{K} \div \phi)) \underline{\ll} (\boldsymbol{B} \cap (\mathbf{K} \div \varepsilon))$. Since this holds for all elements $Y = \boldsymbol{B} \cap (\mathbf{K} \div \phi) \in \boldsymbol{B} \perp \alpha$, it follows by the definition of γ that $(\boldsymbol{B} \cap (\mathbf{K} \div \varepsilon)) \in \gamma(\boldsymbol{B} \perp \alpha)$. We have already shown that $\mathbf{K}\div\beta \nsubseteq \mathbf{K}\div\delta' = \mathbf{K}\div\varepsilon$, so that by *closure* (G÷1), $\&\,(\mathbf{K} \div \beta) \notin (\boldsymbol{B} \cap (\mathbf{K} \div \varepsilon))$. This concludes this part of the proof.

Part 4: To prove that γ has the completion property, let $\boldsymbol{B}\sim_\gamma\alpha \subseteq X \in \boldsymbol{B} \perp \alpha$. Let $Y \in \boldsymbol{B} \perp \alpha$. Then $\{X, Y\} \subseteq \boldsymbol{B} \perp \alpha$, and by part 3 of the present proof, $\mathbf{K}\div\alpha = Cn(\boldsymbol{B} \sim_\gamma \alpha)$ so that $\mathbf{K} \div \alpha \subseteq Cn(X)$. It follows by the definition of $\underline{\ll}$ that $Y \underline{\ll} X$. Since this holds for all $Y \in \boldsymbol{B} \perp \alpha$, it follows by the definition of γ that $X \in \gamma(\boldsymbol{B} \perp \alpha)$.

Part 5: It follows directly by the construction that γ is relational by $\underline{\ll}$ and that $\underline{\ll}$ has the maximizing property. It remains to be shown that $\underline{\ll}$ is transitive.

We will use the symbol $\leq$ as follows:

$Y \leq X$ iff there is some β such that $\{X, Y\} = \boldsymbol{B} \perp \beta$ and $\mathbf{K} \div \beta \subseteq Cn(X)$.

By *lemma* 14.9, $X \underline{\ll} Y$ iff either $X \subset Y$ or $X \leq Y$. The proof of transitivity can therefore be divided into the following four cases:

(1) If $X \subset Y$ and $Y \subset Z$, then either $X \subset Z$ or $X \leq Z$.
(2) If $X \leq Y$ and $Y \leq Z$, then either $X \subset Z$ or $X \leq Z$.
(3) If $X \subset Y$ and $Y \leq Z$, then either $X \subset Z$ or $X \leq Z$.
(4) If $X \leq Y$ and $Y \subset Z$, then either $X \subset Z$ or $X \leq Z$.

If $X = \boldsymbol{B}$, then $X \not\subset Y$, and $X \leq Y$ implies $X = Y$, so that $Y \underline{\ll} Z$ implies $X \underline{\ll} Z$. If $Y = \boldsymbol{B}$, then $Y \not\subset Z$, and $Y \leq Z$ implies $Y = Z$, so that $X \underline{\ll} Y$ implies $X \underline{\ll} Z$. If $Z = \boldsymbol{B}$, then either $X \subset Z$ or $X = Z$, in both cases yielding $X \underline{\ll} Z$. Thus, in the proofs of the four cases we can assume that $X \neq \boldsymbol{B}$, $Y \neq \boldsymbol{B}$, and $Z \neq \boldsymbol{B}$.

Case 1: Trivial.

Case 2: Suppose that $X \leq Y$ and $Y \leq Z$. By *lemma* 14.4 there are sentences a, b, and c such that $\{X\} = \boldsymbol{B} \perp a$, $\{Y\} = \boldsymbol{B} \perp b$, and $\{Z\} = \boldsymbol{B} \perp c$. By *lemmas* 14.7 and 14.8, $\mathbf{K} \div a = Cn(X)$, $\mathbf{K} \div b = Cn(Y)$, and $\mathbf{K} \div c = Cn(Z)$. By *lemma* 14.2, $\{X, Y\} = \boldsymbol{B} \perp (a\&b)$, and $\{Y, Z\} = \boldsymbol{B} \perp (b\&c)$.

By $Y \leq Z$, there is some β such that $\{Y, Z\} = \boldsymbol{B} \perp \beta$ and $\mathbf{K} \div \beta \subseteq Cn(Z)$. By $\boldsymbol{B} \perp \beta = \boldsymbol{B} \perp (b\&c)$ it follows that for all ε, $\boldsymbol{B} \cap (\mathbf{K} \div \varepsilon) \vdash \beta$ iff $\boldsymbol{B} \cap (\mathbf{K} \div \varepsilon) \vdash (b\&c)$. By *lemma* 14.7, $\mathbf{K} \div \varepsilon \vdash \beta$ iff $\mathbf{K} \div \varepsilon \vdash (b\&c)$. By *symmetry*, $\mathbf{K} \div \beta = \mathbf{K} \div (b\&c)$. Thus we have $\mathbf{K} \div (b\&c) \subseteq Cn(Z)$. By a similar proof it follows that $\mathbf{K} \div (a\&b) \subseteq Cn(Y)$. From $\mathbf{K} \div (a\&b) \subseteq \mathbf{K} \div b$ and $\boldsymbol{B} \cap (\mathbf{K} \div b) = Y \in \boldsymbol{B} \perp (a\&b)$ it follows by *lemma* 14.6 that $\mathbf{K} \div b$ is a maximally preservative $a\&b$-removal.

We are now going to show that $Z \not\subset X$. Suppose to the contrary that $Z \subset X$. Then $\mathbf{K} \div c \subset \mathbf{K} \div a$. It follows from the above construction of a and c by *lemma* 14.4 that $\vdash a \rightarrow c$, so that $\vdash a\&b \rightarrow b\&c$. It follows from $\mathbf{K} \div (a\&b) \subseteq Cn(Y)$ and $Y \in \boldsymbol{B} \perp b$ that $\mathbf{K} \div (a\&b) \nvdash b\&c$. Furthermore, it follows by *lemma* 14.6 from $\mathbf{K} \div (b\&c) \subseteq \mathbf{K} \div c$ and $Z = \boldsymbol{B} \cap (\mathbf{K} \div c) \in \boldsymbol{B} \perp b\&c$ that $\mathbf{K} \div c$ is a maximally preservative $b\&c$-removal. We may then conclude from *regularity 1* (since $\mathbf{K} \div b$ is both a $b\&c$-removal and a maximally preservative $a\&b$-removal) that $\mathbf{K} \div c$ is a maximally preservative $a\&b$-removal, contrary to our assumption that $\mathbf{K} \div c \subset \mathbf{K} \div a \nvdash a\&b$. By this contradiction, we may conclude that $Z \not\subset X$.

Having excluded $Z \subset X$, we have three remaining subcases under case 2: $X \subset Z$, $X = Z$ and $X \not\subseteq Z \not\subseteq X$. If $X \subset Z$, then we are done. If $X = Z$, then we can use $\{X\} = \boldsymbol{B} \perp a$ and $\mathbf{K} \div a \subseteq Cn(X)$ to obtain $X \leq Z$ directly from the definition of $\leq$. The remaining subcase is $X \not\subseteq Z \not\subseteq X$.

When $X \not\subseteq Z \not\subseteq X$ it follows by *lemma* 14.2 that $\{X, Z\} = \boldsymbol{B} \perp (a\&c)$ and $\{X, Y, Z\} = \boldsymbol{B} \perp (a\&b\&c)$. We are first going to prove that $\mathbf{K} \div (a\&b\&c) \subseteq \mathbf{K} \div c$. By *success* (G÷4), $\mathbf{K} \div (a\&b\&c)$ does not imply all three of a, b, and c. If it does not imply a, then it follows from $\{X\} = \boldsymbol{B} \perp a$ that $\&(\mathbf{K} \div (a\&b\&c)) \in X$, so that $\mathbf{K} \div (a\&b\&c) \subseteq Cn(X) = \mathbf{K} \div a$. By similar reasoning for b and c it follows that $\mathbf{K} \div (a\&b\&c)$ is a subset of at least one of $\mathbf{K} \div a$, $\mathbf{K} \div b$, and $\mathbf{K} \div c$.

If $\mathbf{K} \div (a\&b\&c) \subseteq \mathbf{K} \div a$, then by *success* (G÷4), $\mathbf{K} \div (a\&b\&c) \nvdash a\&b$. We have already shown that $\mathbf{K} \div b$ is a maximally preservative $a\&b$-removal, and in the same way we can show that $\mathbf{K}\div a$ is a maximally preservative $a\&b\&c$-removal. Since it is also an $a\&b$-removal, it follows by *regularity 1* that all maximally preservative $a\&b$-removals are also maximally preservative $a\&b\&c$-removals. Thus, $\mathbf{K}\div b$ is a maximally preservative $a\&b\&c$-removal, so that

$$\mathbf{K} \div (a\&b\&c) \subseteq \mathbf{K} \div b.$$

Thus, if $\mathbf{K} \div (a\&b\&c) \subseteq \mathbf{K} \div a$ then $\mathbf{K} \div (a\&b\&c) \subseteq \mathbf{K} \div b$.

If $\mathbf{K} \div (a\&b\&c) \subseteq \mathbf{K} \div b$, then, by *success* (G÷4), $\mathbf{K}\div(a\&b\&c) \nvdash b\&c$. From $\mathbf{K} \div (b\&c) \subseteq \mathbf{K} \div c$ and $\boldsymbol{B} \cap (\mathbf{K} \div c) = Z \in \boldsymbol{B} \perp (b\&c)$ it follows by *lemma* 14.6 that $\mathbf{K}\div c$ is a maximally preservative $b\&c$-removal. In the same way, it follows that $\mathbf{K}\div b$ is a maximally preservative $a\&b\&c$-removal. Since it is also a $b\&c$-removal, it follows by *regularity 1* that all maximally preservative $b\&c$-removals are maximally preservative $a\&b\&c$-removals. Thus, $\mathbf{K}\div c$ is a maximally preservative $a\&b\&c$-removal, so that $\mathbf{K} \div (a\&b\&c) \subseteq \mathbf{K} \div c$.

Thus, $\mathbf{K}\div(a\&b\&c)$ is a subset of at least one of $\mathbf{K}\div a$, $\mathbf{K}\div b$, and $\mathbf{K}\div c$, and if it is a subset of $\mathbf{K}\div a$ then it is a subset of $\mathbf{K}\div b$ and if it is a subset of $\mathbf{K}\div b$ then it is a subset of $\mathbf{K}\div c$. We may conclude that $\mathbf{K} \div (a\&b\&c) \subseteq \mathbf{K} \div c$.

Since $\boldsymbol{B} \cap (\mathbf{K} \div c) \in \boldsymbol{B} \perp (a\&b\&c)$ it follows by *lemma* 14.6 that $\mathbf{K}\div c$ is a maximally preservative $a\&b\&c$-removal. It is also, by *success* (G÷4), an $a\&c$-removal. It follows by *regularity 2* that $\mathbf{K}\div c$ is a maximally preservative $a\&c$-removal, so that $\mathbf{K} \div (a\&c) \subseteq \mathbf{K} \div c$.

Since $\mathbf{K}\div c = Cn(Z)$ we therefore have $\{X, Z\} = \boldsymbol{B} \perp (a\&c)$ and $\mathbf{K} \div (a\&c) \subseteq Cn(Z)$, so that $X \leq Z$, concluding the proof of case 2.

Case 3: Suppose that $X \subset Y$ and $Y \leq Z$. By *lemma* 14.4 there are a, b, and c such that $\{X\} = \boldsymbol{B} \perp a$, $\{Y\} = \boldsymbol{B} \perp b$, $\{Z\} = \boldsymbol{B} \perp c$, and $\vdash b \rightarrow a$. By *lemmas* 14.7 and 14.8, $\mathbf{K}\div a = Cn(X)$, $\mathbf{K}\div b = Cn(Y)$, and $\mathbf{K}\div c = Cn(Z)$.

We are first going to show that $Z \not\subset X$. If $Z \subset X$, then $Z \subset Y$, contrary to $Y \leq Z$. Thus, $Z \not\subset X$. The subcases $X \subset Z$ and $X = Z$ are treated just as in case 2. It remains to treat the subcase when $X \not\subseteq Z \not\subseteq X$. In that case it follows by *lemma* 14.2 that $\{X, Z\} = \boldsymbol{B} \perp (a\&c)$ and $\{Y, Z\} = \boldsymbol{B} \perp (b\&c)$. In the same way as in case 2, it follows from $Y \leq Z$ that $\mathbf{K} \div (b\&c) \subseteq \mathbf{K} \div c$ and that $\mathbf{K}\div c$ is a maximally preservative $b\&c$-removal.

From $\vdash b \rightarrow a$ it follows that $\vdash b\&c \rightarrow a\&c$. By *success*, $\mathbf{K}\div c$ is an $a\&c$-removal. We can conclude by *regularity 2* that $\mathbf{K}\div c$ is a maximally preservative $a\&c$-removal, so that $\mathbf{K} \div (a\&c) \subseteq \mathbf{K} \div c$. Since $\mathbf{K}\div c = Cn(Z)$ we then have $\{X, Z\} = \boldsymbol{B} \perp (a\&c)$ and $\mathbf{K} \div (a\&c) \subseteq Cn(Z)$, so that $X \leq Z$. This concludes the proof of case 3.

Case 4: Suppose that $X \leq Y \subset Z$. By *lemma* 14.4 there are a, b, and c such that $\{X\} = \boldsymbol{B} \perp a$, $\{Y\} = \boldsymbol{B} \perp b$, $\{Z\} = \boldsymbol{B} \perp c$, and $\vdash c \rightarrow b$. By *lemmas* 14.7 and 14.8 we have $\mathbf{K}\div a = Cn(X)$, $\mathbf{K}\div b = Cn(Y)$, and $\mathbf{K}\div c = Cn(Z)$.

If $Z \subset X$, then $Y \subset X$, contrary to $X \leq Y$. Thus $Z \not\subset X$. The subcases $X = Z$ and $X \subset Z$ are treated as in case 2. In the remaining subcase, $X \not\subseteq Z \not\subseteq X$, it follows by *lemma* 14.2 that $\{X, Y\} = \boldsymbol{B} \perp (a\&b)$ and $\{X, Z\} = \boldsymbol{B} \perp (a\&c)$. By $X \leq Y$, we can show, just as in case 2, that $\mathbf{K} \div (a\&b) \subseteq \mathbf{K} \div b$ and that $\mathbf{K}\div b$ is a maximally preservative $a\&b$-removal.

Suppose that $\mathbf{K} \div (a\&c) \subseteq \mathbf{K} \div a$. Then, by *lemma* 14.6, $\mathbf{K}{\div}a$ is a maximally preservative $a\&c$-removal. By *success* (G÷4), $\mathbf{K}{\div}(a\&c) \nvdash a\&b$. From $\vdash c \to b$ it follows that $\vdash a\&c \to a\&b$. Since $\mathbf{K}{\div}a$ is both an $a\&b$-removal and a maximaly preservative $a\&c$-removal, it follows by *regularity 1* that all maximally preservative $a\&b$-removals are maximally preservative $a\&c$-removals. Thus, $\mathbf{K}{\div}b$ is a maximally preservative $a\&c$-removal. However, it follows from $Y \subset Z \nvdash c$ that $\mathbf{K} \div b \subset \mathbf{K} \div c \nvdash a\&c$, so that $\mathbf{K}{\div}b$ cannot be a maximally preservative $a\&c$-removal. From this contradiction we may conclude that $\mathbf{K}{\div}(a\&c) \nsubseteq \mathbf{K}{\div}a$. By *lemma* 14.7, $\boldsymbol{B} \cap (\mathbf{K} \div (a\&c)) \nsubseteq \boldsymbol{B} \cap (\mathbf{K} \div \alpha)$.

Since $\{\boldsymbol{B} \cap (\mathbf{K} \div a)\} = \boldsymbol{B} \perp a$, it follows from $\boldsymbol{B} \cap (\mathbf{K} \div (a\&c)) \nsubseteq \boldsymbol{B} \cap (\mathbf{K} \div \alpha)$ that $\boldsymbol{B} \cap (\mathbf{K} \div (a\&c)) \vdash a$, so that $\mathbf{K}{\div}(a\&c) \vdash a$. Therefore, by *success* (G÷4), $\mathbf{K}{\div}(a\&c) \nvdash c$. Since $\boldsymbol{B} \perp c = \{\boldsymbol{B} \cap (\mathbf{K} \div c)\}$, it follows that $\boldsymbol{B} \cap (\mathbf{K} \div (a\&c)) \subseteq \boldsymbol{B} \cap (\mathbf{K} \div c)$, and by *lemma* 14.7 that $\mathbf{K} \div (a\&c) \subseteq \mathbf{K} \div c$. Since $\mathbf{K}{\div}c = Cn(Z)$ we then have $\{X, Z\} = \boldsymbol{B} \perp (a\&c)$ and $\mathbf{K} \div (a\&c) \subseteq Cn(Z)$, so that $X \leq Z$. This concludes the proof.

Proof of Theorem 14.5, Left-to-Right Due to the corresponding part of the proof of Theorem 14.2, we only have to show that *hyperregularity* holds.

Suppose that $\vdash \alpha \to \beta$ and $\mathbf{K}{\div}\alpha \nvdash \beta$, i.e. $\boldsymbol{B} \sim_\gamma \alpha \nvdash \beta$. By the maxichoice property, $\boldsymbol{B} {\sim_\gamma} \alpha \in \boldsymbol{B} \perp a$. It follows from $\boldsymbol{B} \sim_\gamma \alpha \nvdash \beta$ that $\boldsymbol{B}{\sim_\gamma}\alpha \in \boldsymbol{B} \perp \beta$. Let $X \in \boldsymbol{B} \perp \beta$. It follows from $\vdash \alpha \to \beta$ that there is some X' such that $X \subseteq X' \in \boldsymbol{B} \perp \alpha$. It follows that $X \subseteq X' \underline{\ll} \boldsymbol{B}{\sim_\gamma}\alpha$. By the maximizing and transitive properties of $\underline{\ll}$, it follows that $X \underline{\ll} \boldsymbol{B} \sim_\gamma \alpha$. Since this holds for all $X \in \boldsymbol{B} \perp \beta$, and $\boldsymbol{B}{\sim_\gamma}\alpha \in \boldsymbol{B} \perp \beta$, $\boldsymbol{B}{\sim_\gamma}\alpha \in \gamma(\boldsymbol{B} \perp \beta)$. We may conclude by the maxichoice property that that $\boldsymbol{B} \sim_\gamma \beta = \boldsymbol{B} \sim_\gamma \alpha$, i.e. $\mathbf{K}{\div}\beta = \mathbf{K}{\div}\alpha$.

Proof of Theorem 14.5, Right-to-Left Let $\boldsymbol{B} = \{\&X \mid (\exists\alpha)(X = \mathbf{K} \div \alpha)\}$. Let $\underline{\ll}$ be the relation such that $Y \underline{\ll} X$ iff either $Y \subset X$ or there is some β such that $\{X, Y\} = \boldsymbol{B} \perp \beta$ and $\mathbf{K}{\div}\beta = Cn(X)$. Let γ be the function such that:

(1) If $\boldsymbol{B} \perp \alpha \neq \varnothing$, then $\gamma(\boldsymbol{B} \perp \alpha) = \{\boldsymbol{B}' \in \boldsymbol{B} \perp \alpha \mid \boldsymbol{B}'' \underline{\ll} \boldsymbol{B}' \text{ for all } \boldsymbol{B}'' \in \boldsymbol{B} \perp \alpha\}$
(2) Otherwise, $\gamma(\boldsymbol{B} \perp \alpha) = \{\boldsymbol{B}\}$

We need to show (1) that $\boldsymbol{B}$ is a finite base for $\mathbf{K}$, (2) that γ is a selection function for $\boldsymbol{B}$, (3) that γ is maxichoice, (4) that the partial meet contraction $\sim_\gamma$ on $\boldsymbol{B}$ generates the operation $\div$ on $\mathbf{K}$, and (5) that γ is transitively, maximizingly relational by $\underline{\ll}$. The proof of part 1 coincides with that of part 1 of Theorem 14.1.

Part 2: To prove that γ is a selection function, it is sufficient to show that if $\boldsymbol{B} \perp \alpha \neq \varnothing$, then $\gamma(\boldsymbol{B} \perp \alpha)$ is non-empty.

We are first going to show that $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \in \boldsymbol{B} \perp \alpha$. It follows by *inclusion* (G÷2) and *success* (G÷4) that that $\mathbf{K}{\div}\alpha$ is a subset of $\mathbf{K}$ that does not imply α. Therefore, $\boldsymbol{B} \cap (\mathbf{K} \div \alpha)$ is a subset of $\boldsymbol{B}$ that does not imply α. Suppose that it is not an element of $\boldsymbol{B} \perp \alpha$. Then there must be some $\phi \in \boldsymbol{B}$ such that $\phi \notin \boldsymbol{B} \cap (\mathbf{K} \div \alpha)$ and $(\boldsymbol{B} \cap (\mathbf{K} \div \alpha)) \cup \{\phi\} \nvdash \alpha$. By the construction of $\boldsymbol{B}$, $\phi = \&(\mathbf{K}{\div}\delta)$ for some δ. From $\&(\mathbf{K} \div \delta) \notin \boldsymbol{B} \cap (\mathbf{K} \div \alpha)$ it follows by *closure* (G÷1) that $\mathbf{K}{\div}\delta \nsubseteq \mathbf{K}{\div}\alpha$, and from $(\boldsymbol{B} \cap (\mathbf{K} \div \alpha)) \cup \{\&(\mathbf{K} \div \delta)\} \nvdash \alpha$ it follows that $(\mathbf{K} \div \alpha) \cup (\mathbf{K} \div \delta)\} \nvdash \alpha$.

This contradicts *strong conservativity*, and we may conclude that (for all α) $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \in \boldsymbol{B} \perp \alpha$.

By *lemma* 14.3 there is some ϕ such that $\vdash \alpha \rightarrow \phi$ and $\{\boldsymbol{B} \cap (\mathbf{K} \div \alpha)\} = \boldsymbol{B} \perp \phi$. By *lemma* 14.7, $\mathbf{K} \div \alpha \nvdash \phi$. Let $Y \in \boldsymbol{B} \perp \alpha$. By *lemma* 14.3 there is some ψ such that $\vdash \alpha \rightarrow \psi$ and $\{Y\} = \boldsymbol{B} \perp \psi$. By *lemma* 14.2, $\{\boldsymbol{B} \cap (\mathbf{K} \div \alpha), Y\} = \boldsymbol{B} \perp (\phi \& \psi)$. Since $\vdash \alpha \rightarrow (\phi \& \psi)$ and $\mathbf{K} \div \alpha \nvdash (\phi \& \psi)$ it follows by *hyperregularity* that $\mathbf{K} \div \alpha = \mathbf{K} \div (\phi \& \psi)$. By *lemma* 14.7 and our definition of $\underline{\ll}$, it follows that $Y \underline{\ll} \boldsymbol{B} \cap (\mathbf{K} \div \alpha)$. Since this holds for all $Y \in \boldsymbol{B} \perp \alpha$, it follows by the definition of γ that $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \in \gamma(\boldsymbol{B} \perp \alpha)$. Thus $\gamma(\boldsymbol{B} \perp \alpha)$ is non-empty whenever $\boldsymbol{B} \perp \alpha$ is non-empty.

Part 3: In order to prove that γ is maxichoice, suppose that it is not. Then there is some α such that there are distinct X and Y with $X, Y \in \gamma(\boldsymbol{B} \perp \alpha)$. It follows, by the definition of γ, that there are ϕ and ψ such that $\{X, Y\} = \boldsymbol{B} \perp \phi = \boldsymbol{B} \perp \psi$, $Cn(X) = \mathbf{K} \div \phi$ and $Cn(Y) = \mathbf{K} \div \psi$.

It follows from $\{X, Y\} = \boldsymbol{B} \perp \phi = \boldsymbol{B} \perp \psi$ that $\{X, Y\} = \boldsymbol{B} \perp (\phi \& \psi)$. By what was shown in part 2 of the present proof, $\boldsymbol{B} \cap (\mathbf{K} \div (\phi \& \psi)) \in \boldsymbol{B} \perp (\phi \& \psi)$. It follows from this and $\boldsymbol{B} \perp \phi = \boldsymbol{B} \perp (\phi \& \psi)$ that $\boldsymbol{B} \cap (\mathbf{K} \div (\phi \& \psi)) \nvdash \phi$. By *lemma* 14.7, $\mathbf{K} \div (\phi \& \psi) \nvdash \phi$. From this it follows by *hyperregularity* that $\mathbf{K} \div (\phi \& \psi) = \mathbf{K} \div \phi$. Similarly, $\mathbf{K} \div (\phi \& \psi) = \mathbf{K} \div \psi$, so that $\mathbf{K} \div \phi = \mathbf{K} \div \psi$. Thus $X = Y$, contrary to our conditions. We may conclude that γ is maxichoice.

Part 4: It was shown in part 2 of the present proof that for all α, $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \in \gamma(\boldsymbol{B} \perp \alpha)$. By part 3, $\{\boldsymbol{B} \cap (\mathbf{K} \div \alpha)\} = \gamma(\boldsymbol{B} \perp \alpha)$. It follows from this, by *lemma* 14.7, that $Cn(\boldsymbol{B} \sim_{\gamma} \alpha) = \mathbf{K} \div \alpha$.

Part 5: It follows directly by the construction that γ is relational by $\underline{\ll}$ and that the maximizing property holds. In the proof of transitivity, we will use the symbol $\leq$ as follows:

$Y \leq X$ iff there is some β such that $\{X, Y\} = \boldsymbol{B} \perp \beta$ and $\mathbf{K} \div \beta = Cn(X)$.

Thus, $X \underline{\ll} Y$ iff either $X \subset Y$ or $X \leq Y$. The proof of transitivity can therefore be divided into the following four cases:

(1) If $X \subset Y$ and $Y \subset Z$, then either $X \subset Z$ or $X \leq Z$.
(2) If $X \leq Y$ and $Y \leq Z$, then either $X \subset Z$ or $X \leq Z$.
(3) If $X \subset Y$ and $Y \leq Z$, then either $X \subset Z$ or $X \leq Z$.
(4) If $X \leq Y$ and $Y \subset Z$, then either $X \subset Z$ or $X \leq Z$.

If $X = \boldsymbol{B}$, then $X \not\subset Y$, and $X \leq Y$ implies $X = Y$, so that $Y \underline{\ll} Z$ implies $X \underline{\ll} Z$. If $Y = \boldsymbol{B}$, then $Y \not\subset Z$, and $Y \leq Z$ implies $Y = Z$, so that $X \underline{\ll} Y$ implies $X \underline{\ll} Z$. If $Z = \boldsymbol{B}$, then either $X \subset Z$ or $X = Z$, in both cases yielding $X \underline{\ll} Z$. Thus, in the proofs of the four cases we can assume that $X \neq \boldsymbol{B}$, $Y \neq \boldsymbol{B}$, and $Z \neq \boldsymbol{B}$.

Case 1: Trivial.

Case 2: Suppose that $X \leq Y$ and $Y \leq Z$. By *lemma* 14.4, there are sentences a, b, and c such that $\{X\} = \boldsymbol{B} \perp a$, $\{Y\} = \boldsymbol{B} \perp b$, and $\{Z\} = \boldsymbol{B} \perp c$. By *lemma* 14.2 we have $\{X, Y\} = \boldsymbol{B} \perp (a \& b)$ and $\{Y, Z\} = \boldsymbol{B} \perp (b \& c)$

By $X \leq Y$ there is some ϕ such that $\{X, Y\} = \boldsymbol{B} \perp \phi$ and $\mathbf{K}\div\phi = Cn(Y)$. It follows from $\boldsymbol{B} \perp \phi = \boldsymbol{B} \perp (a\&b)$ that for all ε, $\boldsymbol{B} \cap (\mathbf{K} \div \varepsilon) \vdash \phi$ iff $\boldsymbol{B} \cap (\mathbf{K} \div \varepsilon) \vdash (a\&b)$. By *lemma* 14.7, $\mathbf{K}\div\varepsilon \vdash \phi$ iff $\mathbf{K}\div\varepsilon \vdash (a\&b)$. By *symmetry*, $\mathbf{K}\div(a\&b) = \mathbf{K}\div\phi = Cn(Y)$. By a similar proof, $\mathbf{K}\div(b\&c) = Cn(Z)$.

Suppose that $Z \subset X$. By the construction from *lemma* 14.4 that was used above to obtain a, b, and c, $\vdash a \to c$. It follows that $\vdash a\&b \to b\&c$. Since $\mathbf{K}\div(a\&b) = Cn(Y)$ and $Y \nvdash b$, we have $\mathbf{K}\div(a\&b) \nvdash b\&c$. By *hyperregularity* we then have $\mathbf{K}\div(a\&b) = \mathbf{K}\div(b\&c)$, i.e., $Cn(Y) = Cn(Z)$, so that $Y = Z$. Then $X \leq Z$ follows directly from $X \leq Y$, and we are done.

If $X \subset Z$, then we are also done. If $X = Z$, then we can use $\{X\} = \boldsymbol{B} \perp \alpha$ and $\boldsymbol{B} \cap (\mathbf{K} \div \alpha) \in \boldsymbol{B} \perp \alpha$, that was obtained in part 2 of the present proof, in the definition of $\leq$, and obtain $X \leq X$.

Finally, we have the case when $Z \not\subseteq X \not\subseteq Z$. Then *lemma* 14.2 yields $\{X, Z\} = \boldsymbol{B} \perp (a\&c)$ and $\{X, Y, Z\} = \boldsymbol{B} \perp (a\&b\&c)$. By *hyperregularity* and $\mathbf{K}\div(a\&b) \nvdash b$ it follows that $\mathbf{K}\div(a\&b) = \mathbf{K}\div b$. Similarly, $\mathbf{K}\div(b\&c) = \mathbf{K}\div c$. By *success* (G÷4), $\mathbf{K}\div(a\&b\&c) \nvdash a$ or $\mathbf{K}\div(a\&b\&c) \nvdash b$ or $\mathbf{K}\div(a\&b\&c) \nvdash c$. By *hyperregularity*, $\mathbf{K}\div(a\&b\&c)$ is identical to one of $\mathbf{K}\div a$, $\mathbf{K}\div b$, and $\mathbf{K}\div c$. Similarly, $\mathbf{K}\div(a\&c)$ is identical to one of $\mathbf{K}\div a$ and $\mathbf{K}\div c$.

Suppose that $\mathbf{K}\div(a\&c) = \mathbf{K}\div a$. It then follows (since, by *hyperregularity*, for all κ and λ, either $\mathbf{K}\div(\kappa\&\lambda) = \mathbf{K}\div\kappa$ or $\mathbf{K}\div(\kappa\&\lambda) = \mathbf{K}\div\lambda$):

(A) Either $\mathbf{K}\div(a\&b\&c) = \mathbf{K}\div(a\&b) = \mathbf{K}\div b$ or $\mathbf{K}\div(a\&b\&c) = \mathbf{K}\div c$.
(B) Either $\mathbf{K}\div(a\&b\&c) = \mathbf{K}\div(a\&c) = \mathbf{K}\div a$ or $\mathbf{K}\div(a\&b\&c) = \mathbf{K}\div b$.
(C) Either $\mathbf{K}\div(a\&b\&c) = \mathbf{K}\div(b\&c) = \mathbf{K}\div c$ or $\mathbf{K}\div(a\&b\&c) = \mathbf{K}\div a$.

Since the three conditions are incompatible, we may conclude that $\mathbf{K}\div(a\&c) \neq \mathbf{K}\div a$. By parts 3 and 4 of the present proof, since $\boldsymbol{B} \perp (a\&c) = \{X, Z\}$, $\mathbf{K}\div(a\&c)$ is either $Cn(X)$ or $Cn(Z)$. In the same way it follows from $\boldsymbol{B} \perp a = \{X\}$ that $\mathbf{K}\div a = Cn(X)$ and from $\boldsymbol{B} \perp c = \{Z\}$ that $\mathbf{K}\div c = Cn(Z)$. Thus, $\mathbf{K}\div(a\&c) = Cn(Z)$. It follows that $X \leq Z$.

Case 3: Suppose that $X \subset Y \leq Z$. By *lemma* 14.4, there are sentences a, b, and c such that $\{X\} = \boldsymbol{B} \perp a$, $\{Y\} = \boldsymbol{B} \perp b$, $\{Z\} = \boldsymbol{B} \perp c$, and $\vdash b \to a$. $Z \subseteq X$ is impossible, because then $Z \subset Y$, which contradicts $Y \leq Z$. If $X \subset Z$, then we are done. In the remaining case, when $Z \not\subseteq X \not\subseteq Z$, *lemma* 14.2 yields $\{X, Z\} = \boldsymbol{B} \perp (a\&c)$ and $\{Y, Z\} = \boldsymbol{B} \perp (b\&c)$. In the same way as in case 2 we obtain $\mathbf{K}\div(b\&c) = Cn(Z)$. From $\vdash b \to a$ it follows that $\vdash b\&c \to a\&c$. Since $Z \nvdash c$ we have $\mathbf{K}\div(b\&c) \nvdash a\&c$. It follows by *hyperregularity* that $\mathbf{K}\div(b\&c) = \mathbf{K}\div(a\&c)$, so that $\mathbf{K}\div(a\&c) = Cn(Z)$. It follows that $X \leq Z$.

Case 4: Suppose that $X \leq Y \subset Z$. By *lemma* 14.4, there are sentences a, b, and c such that $\{X\} = \boldsymbol{B} \perp a, \{Y\} = \boldsymbol{B} \perp b, \{Z\} = \boldsymbol{B} \perp c$, and $\vdash c \to b$. $Z \subseteq X$ is impossible, because then $Y \subset X$, which contradicts $X \leq Y$. If $X \subset Z$, then we are done. In the remaining case, when $Z \not\subseteq X \not\subseteq Z$, *lemma* 14.2 yields $\{X, Y\} = \boldsymbol{B} \perp (a\&b)$ and $\{X, Z\} = \boldsymbol{B} \perp (a\&c)$. In the same way as in case 2, we obtain $\mathbf{K}\div(a\&b) = Cn(Y)$. We can also prove, in the same way as in case 2, that $\mathbf{K}\div(a\&c)$ is identical to either $\mathbf{K}\div a$ or $\mathbf{K}\div c$.

First, let $\mathbf{K}\div(a\&c) = \mathbf{K}\div a$. From $\vdash c \to b$ it follows that $\vdash a\&c \to a\&b$. Since $\mathbf{K}\div a \nvdash a$ we have $\mathbf{K}\div(a\&c) \nvdash a\&b$. It then follows by *hyperregularity* that

$\mathbf{K}\div(a\&c) = \mathbf{K}\div(a\&b)$, so that $\mathbf{K}\div(a\&b) = \mathbf{K}\div a$. By parts 3 and 4 of the present proof, $\mathbf{K}\div a = Cn(X)$. We therefore have $Cn(X) = Cn(Y)$, so that $X = Y$ and $X \subset Z$.

In the other case, when $\mathbf{K}\div(a\&c) = \mathbf{K}\div c$, we use parts 3 and 4 of the present theorem to obtain $\mathbf{K}\div c = Cn(Z)$. It follows from $\{X,Z\} = \boldsymbol{B}\perp(a\&c)$ and $\mathbf{K}\div(a\&c) = Cn(Z)$ that $X \underline{\ll} Z$.

Proof of Theorem 14.6 Let $\boldsymbol{B}$ be a base for $\mathbf{K}$ and γ a selection function for $\boldsymbol{B}$ that is transitively maximizingly relational by $\underline{\ll}$.

PART 1: We are first going to show that $\left(\boldsymbol{B}\sim_{\gamma}\alpha\right) \cap \left(\boldsymbol{B}\sim_{\gamma}\beta\right) \subseteq \boldsymbol{B}\sim_{\gamma}(\alpha\&\beta)$. This is trivial if $\vdash \alpha$ or $\vdash \beta$. For the principal case, in which $\nvdash \alpha$ and $\nvdash \beta$, suppose to the contrary that there is some ζ such that $\zeta \in \boldsymbol{B}\sim_{\gamma}\alpha$, $\zeta \in \boldsymbol{B}\sim_{\gamma}\beta$ and $\zeta \notin \boldsymbol{B}\sim_{\gamma}(\alpha\&\beta)$. It follows from $\zeta \notin \boldsymbol{B}\sim_{\gamma}(\alpha\&\beta)$ that there is some X such that $\zeta \notin X \in \gamma(\boldsymbol{B}\perp(\alpha\&\beta))$. By *lemma* 14.1, either $X \in \boldsymbol{B}\perp\alpha$ or $X \in \boldsymbol{B}\perp\beta$. Without loss of generality we may assume that $X \in \boldsymbol{B}\perp\alpha$.

Since $\zeta \in \boldsymbol{B}\sim_{\gamma}\alpha$, $X \notin \gamma(\boldsymbol{B}\perp\alpha)$. Let $Y \in \gamma(\boldsymbol{B}\perp\alpha)$. From $Y \in \boldsymbol{B}\perp\alpha$ it follows that there is some Y' such that $Y \subseteq Y' \in \boldsymbol{B}\perp(\alpha\&\beta)$. If $Y = Y'$ we have $X \ll Y'$ directly, and if $Y \subset Y'$ the maximizing property yields $Y \ll Y'$, which with $X \ll Y$ and transitivity yields $X \ll Y'$. Thus, in both cases $X \ll Y'$, $X \in \gamma(\boldsymbol{B}\perp(\alpha\&\beta))$ and $Y' \in \boldsymbol{B}\perp(\alpha\&\beta)$, contrary to our definitions. This contradiction concludes the proof that $\left(\boldsymbol{B}\sim_{\gamma}\alpha\right) \cap \left(\boldsymbol{B}\sim_{\gamma}\beta\right) \subseteq \boldsymbol{B}\sim_{\gamma}(\alpha\&\beta)$.

Now let $\mathbf{K}\div\delta \subseteq (\mathbf{K}\div\alpha) \cap (\mathbf{K}\div\beta)$. Then $\boldsymbol{B} \cap (\mathbf{K}\div\delta) \subseteq \mathbf{K}\div\alpha$, from which it follows that $\boldsymbol{B} \cap (\mathbf{K}\div\delta) \subseteq \boldsymbol{B}\sim_{\gamma}\alpha$. Similarly, $\boldsymbol{B} \cap (\mathbf{K}\div\delta) \subseteq \boldsymbol{B}\sim_{\gamma}\beta$. Since $\left(\boldsymbol{B}\sim_{\gamma}\alpha\right) \cap \left(\boldsymbol{B}\sim_{\gamma}\beta\right) \subseteq \boldsymbol{B}\sim_{\gamma}(\alpha\&\beta)$, we have $\boldsymbol{B} \cap (\mathbf{K}\div\delta) \subseteq \boldsymbol{B}\sim_{\gamma}(\alpha\&\beta)$, thus $Cn\left(\boldsymbol{B} \cap (\mathbf{K}\div\delta)\right) \subseteq Cn\left(\boldsymbol{B}\sim_{\gamma}(\alpha\&\beta)\right)$, i.e. $\mathbf{K}\div\delta \subseteq \mathbf{K}\div(\alpha\&\beta)$.

PART 2: This is trivial if $\vdash \alpha$, $\vdash \beta$, $\alpha \notin \mathbf{K}$ or $\beta \notin \mathbf{K}$. In the remaining case, suppose that $\alpha \notin \mathbf{K}\div(\alpha\&\beta)$. Then $\alpha \notin \bigcap\gamma(\boldsymbol{B}\perp(\alpha\&\beta))$, and there is some X such that $\alpha \notin X \in \gamma(\boldsymbol{B}\perp(\alpha\&\beta))$. Clearly, $X \in \boldsymbol{B}\perp\alpha$. We are going to prove that $\gamma(\boldsymbol{B}\perp\alpha) \subseteq \gamma(\boldsymbol{B}\perp(\alpha\&\beta))$. Let $Y \in \gamma(\boldsymbol{B}\perp\alpha)$.

We are first going to show that $Y \in \boldsymbol{B}\perp(\alpha\&\beta)$. Suppose to the contrary that $Y \notin \boldsymbol{B}\perp(\alpha\&\beta)$. Then there is some Y' such that $Y \subset Y' \in \boldsymbol{B}\perp(\alpha\&\beta)$. By the maximizing property of $\underline{\ll}$, $Y \ll Y'$. By $X \in \boldsymbol{B}\perp\alpha$ and $Y \in \gamma(\boldsymbol{B}\perp\alpha)$ it follows that $X \underline{\ll} Y$. By transitivity, $X \ll Y'$, which contradicts $X \in \gamma(\boldsymbol{B}\perp(\alpha\&\beta))$ and $Y' \in \boldsymbol{B}\perp(\alpha\&\beta)$. It follows that $Y \in \boldsymbol{B}\perp(\alpha\&\beta)$.

It follows by transitivity from $Y \in \boldsymbol{B}\perp(\alpha\&\beta)$, $X \in \gamma(\boldsymbol{B}\perp(\alpha\&\beta))$ and $X \underline{\ll} Y$ that $Y \in \gamma(\boldsymbol{B}\perp(\alpha\&\beta))$. Thus, $\gamma(\boldsymbol{B}\perp\alpha) \subseteq \gamma(\boldsymbol{B}\perp(\alpha\&\beta))$. This yields $\bigcap(\gamma(\boldsymbol{B}\perp(\alpha\&\beta)) \subseteq \bigcap(\gamma(\boldsymbol{B}\perp\alpha))$ and $\mathbf{K}\div(\alpha\&\beta) \subseteq \mathbf{K}\div\alpha$.

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