# Chapter 9
# Fourier Analysis and Synthesis

The concepts of Fourier analysis and synthesis come from an amazingly powerful theorem by mathematician J.B.J. Fourier (1768–1830). Fourier's theorem states that any waveform is just a sum of sine waves. This statement has two implications, Fourier synthesis and Fourier analysis.

Fourier *synthesis* is something that you already know about from your study of vibrations and waves. You already know that a complicated vibration can be created by adding up simple harmonic motions of various frequencies. You already know that a complex wave can be synthesized by adding up sine waves. Fourier's theorem says that by adding up sine waves you can create any waveform your heart desires, *any waveform at all*.

The opposite of synthesis is *analysis*. In Fourier analysis one begins with a complex wave and discovers what the sine waves are that make it. Just as a chemist analyzes a material to discover how much of what elements are present, the acoustician analyzes a complex tone to discover how much of what sine frequencies are present. Fourier's theorem also says that any complex wave can be analyzed in *only one way*. There is only one set of sine wave frequencies, amplitudes, and phases that can come out of the analysis. For that reason, we can make a better statement of the theorem, "Any waveform is a *unique* sum of sine waves."

To summarize: In Fourier synthesis one begins with an amplitude spectrum and phase spectrum and uses those two spectra to generate a complex wave. The spectra are functions of frequency. The complex wave is a function of time.

In Fourier analysis one begins with a complex wave that is a function of time. The results of the analysis are amplitude and phase spectra, and these spectra are functions of frequency.

The following examples give (1) an equation for a wave (or signal), (2) a graph of the wave as a function of time, and (3) graphs showing the amplitude and phase spectra.

W.M. Hartmann, *Principles of Musical Acoustics*, Undergraduate Lecture Notes in Physics, DOI 10.1007/978-1-4614-6786-1_9,

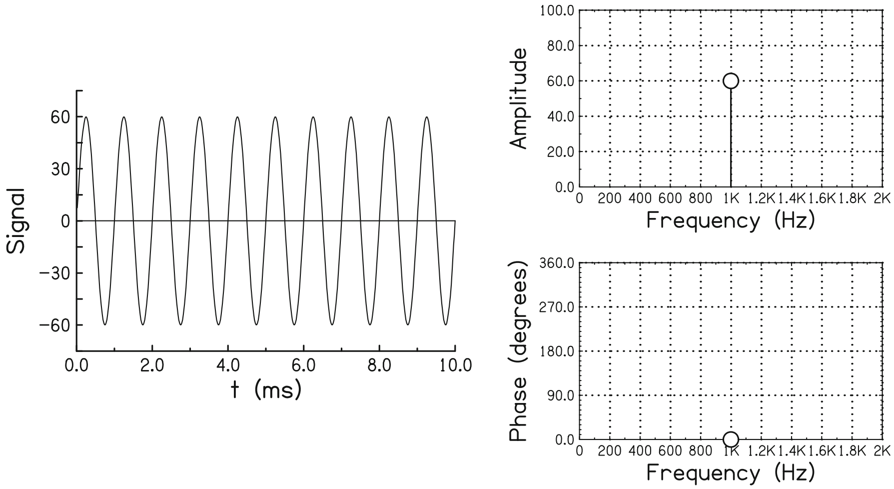


**Fig. 9.1** A sine wave has the simplest possible spectrum

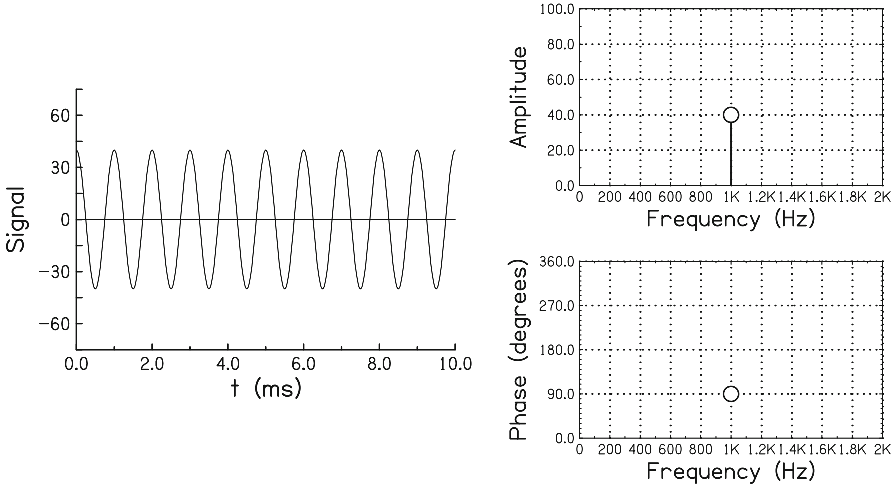


**Fig. 9.2** Another sine wave. This one starts at a peak because of the 90-degree phase

## 9.1 The Sine Wave

The simplest wave has a single component. That makes it a pure sine wave. A sine signal with an amplitude of 60 units, a frequency of 1,000 Hz, and and a phase of zero shown in Fig. 9.1 and is given by the equation

$$\text{Signal} = 60\,\sin(360 \cdot 1000\,t + 0). \tag{9.1}$$

A variation on the 1,000-Hz signal is a sine signal with an amplitude of 40 units, and a phase of 90°, as shown in Fig. 9.2. It is given by

$$\text{Signal} = 40\ \sin(360 \cdot 1000\,t + 90). \tag{9.2}$$

## 9.2 Complex Waves

We next consider a complex wave with two components. There is a fundamental component with a frequency of 200 Hz and an amplitude of 45 units. There is a third harmonic with a frequency of 600 Hz and an amplitude of 15 units. (You may wish to revisit the definition of harmonics in Chap. 7.) Both components have zero phase. Please notice the positions of all those numbers, 200, 600, 45, 15, 0, and 0, in Eq. (9.3) below. You can consider this equation to be a model mathematical description for all complex waves.

$$\text{Signal} = 45\ \sin(360 \cdot 200\,t + 0) + 15\ \sin(360 \cdot 600\,t + 0). \tag{9.3}$$

Phase plays a role in shaping the final waveform. Consider what happens if the phase of the third harmonic is changed to 90° as in Fig. 9.4:

$$\text{Signal} = 45\ \sin(360 \cdot 200\,t + 0) + 15\ \sin(360 \cdot 600\,t + 90). \tag{9.4}$$

Comparing the two final waveforms in Figs. 9.3 and 9.4 makes it clear that by changing the relative phase of the first and third harmonics we have changed the shape of the waveform. Curiously, this phase change does *not* change the sound of this wave. A law called "Ohm's law of phases" says that human listeners are insensitive to phase changes. For a signal with two low-numbered harmonics, like the first and third harmonics of 200 Hz, the law holds good. More about Ohm's law appears below in a discussion of the sound of periodic waves.

## 9.3 Periodicity

The periodicity of the complex wave with components at 200 and 600 Hz is 1/200 s. That's because after 1/200 s (5 ms) the 200-Hz component has gone through exactly one cycle, and the 600-Hz component has gone through exactly three. Both components are then ready to start over again. That's the basis of periodicity. The component at 200 Hz is the lowest frequency component. It also happens to be the fundamental. The component at 600 Hz is the third harmonic.

But sometimes the assignment of periodicity, fundamental, and harmonic numbers is not so simple. Think about a complex wave that has components at 400, 600, and 800 Hz, as shown in Fig. 9.5. After 1/400 s (2.5 ms) the 400-Hz component

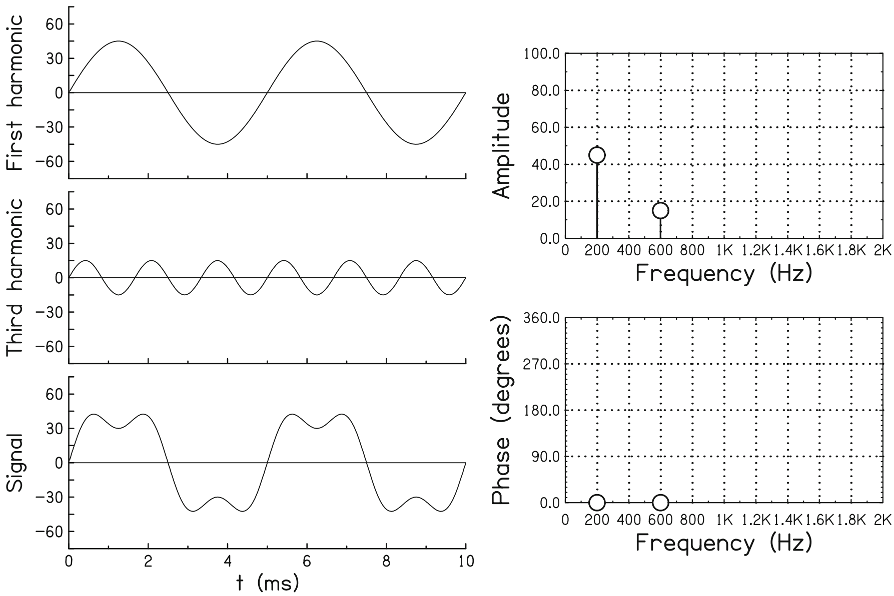


**Fig. 9.3** A complex wave with two components. The two components can be seen in the amplitude and phase plots on the *right*. The figure on the *left* shows the first harmonic or fundamental (200 Hz), the third harmonic (600 Hz), and the final signal waveform. The final signal is made by summing the fundamental component and the third-harmonic component

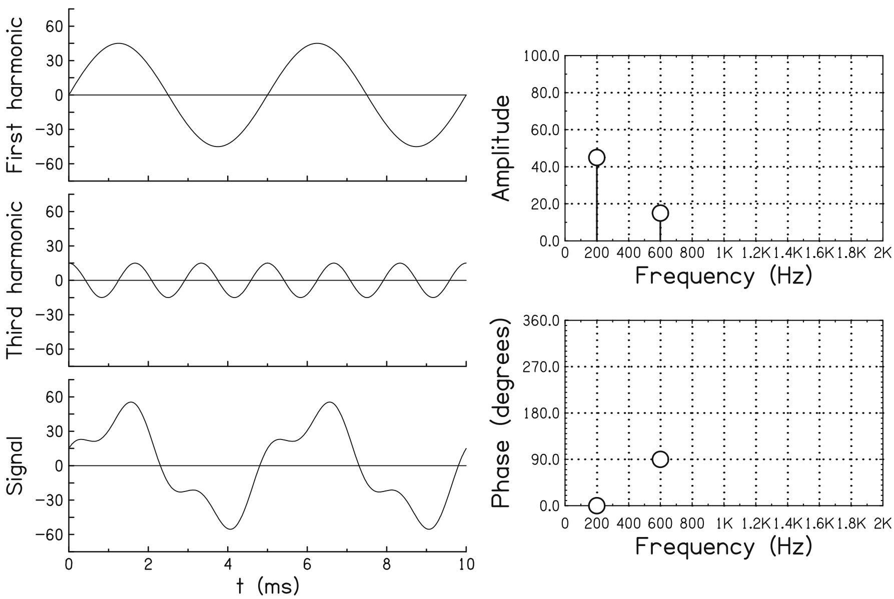


**Fig. 9.4** A complex wave that is the same as in Fig. 9.3, except for the phase of the third harmonic. This phase change causes a change in the waveform but causes no change in the sound of the wave

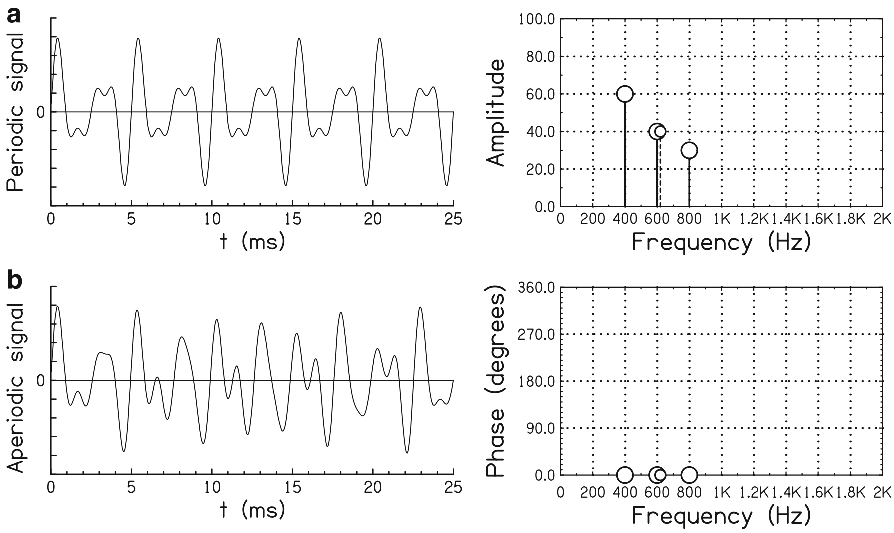


**Fig. 9.5** Part (**a**) shows a complex tone with three sine components, 400, 600, and 800 Hz. The component amplitudes decrease with increasing frequency, as shown in the spectrum on the *right*, and all components have zero phase. There is a missing fundamental, with frequency of 200 Hz. The period is 1/(200 Hz) or 5 ms. Part (**b**) shows an inharmonic tone with three sine components having frequencies of 400, 620, and 800 Hz. The amplitudes and phases are the same as in part (**a**), but the frequency of the third harmonic is increased from 600 to 620 Hz, as shown by the little *circles* and *dashed lines* in the spectra. Although part (**b**) starts out looking like part (**a**), no periodicity can be seen in part (**b**)

has gone through one cycle and the 800-Hz component has gone through two but the 600-Hz component has gone through 1.5 cycles. This is not an integer number of cycles and this component is not ready to start over again. The period of this complex wave is, in fact, 1/200 s, as can be seen in Fig. 9.5a. The fundamental frequency is 200 Hz. It just happens that there is no power at 200 Hz. The fundamental is missing. What is present are harmonics 2, 3, and 4.

To find the fundamental frequency $f_0$ (and the period $T = 1/f_0$) when you are given a set of components like 400, 600, and 800 Hz, you find the *largest common divisor*. The largest number that divides into these three frequencies an integer number of times is 200. That makes 200 Hz the fundamental. Of course, the number 100 will divide into those three frequencies too, but 100 is not the largest number.

## 9.4 The Sawtooth

You will remember the sawtooth wave from the discussion of the oscilloscope. It is a function of time, and on the basis of Fourier's theorem we expect that we can create it by adding up sine wave components. Furthermore, the sawtooth is a periodic waveform, and so we expect that the sine wave components will be harmonics. If the period is 0.0025 s (2.5 ms) we expect the components to be harmonics of a 400-Hz fundamental. Figure 9.6 shows an approximation to a sawtooth. It is an approximation because only the first 12 harmonics are included, 400, 800, 1,200, ..., 4,800 Hz. All the other harmonics have been omitted. It is simple to describe the amplitude spectrum of the sawtooth. The fundamental has an amplitude of 1, the second harmonic has amplitude 1/2, the third harmonic has amplitude of 1/3, and so on. The phase spectrum is even easier. All harmonics have a phase of 180°. Therefore, the waveform is described by an equation that begins,

$$\text{Signal} = 1.\ \sin(360{\cdot}400\,t+180)+\frac{1}{2}\ \sin(360{\cdot}800\,t+180)+\frac{1}{3}\ \sin(360{\cdot}1200\,t+180)+\ldots \tag{9.5}$$

An exercise at the end of the chapter asks you to extend this equation.

## 9.5 The Sounds

This chapter has described three kinds of periodic signals. How do they sound?

**The 1,000-Hz sine**: Figures 9.1 and 9.2 showed 1,000-Hz sine tones with different starting phases. The starting phase affects the sound for only an instant and cannot possibly matter in the long run. A frequency of 1,000 Hz is somewhat piercing and

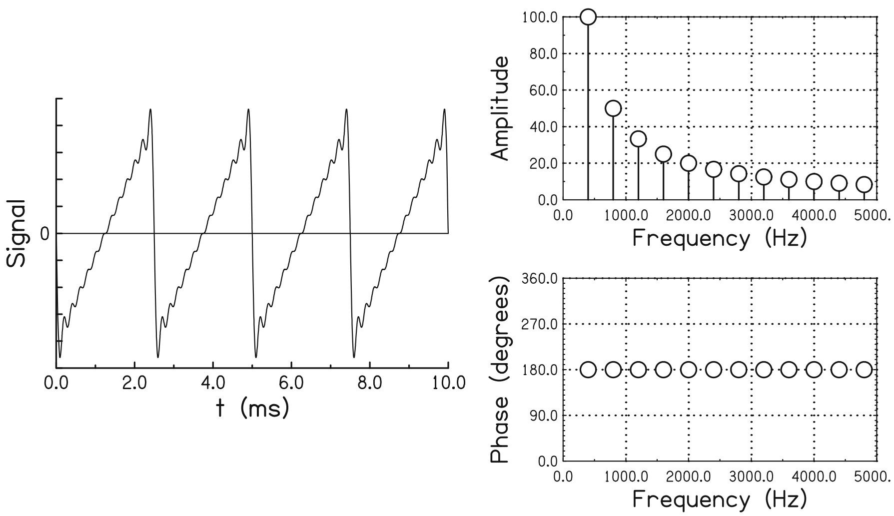


**Fig. 9.6** A sawtooth with a 2.5-ms period can be approximated by adding up the first 12 harmonics of a 400-Hz fundamental

unpleasant (3,000 Hz would be worse). A 100-Hz sine tone has a “dull” tone color. The fact that this simple sine waveshape, with only a single component, can have a tone color that spans the range from dull to piercing tells you that the color of a tone depends more on the frequencies that are present in that tone than on the shape of the waveform.

**The two-component complex, 200 plus 600 Hz**: Figures 9.3 and 9.4 showed complex tones consisting of a fundamental component (harmonic number 1) and a third harmonic having one-third the amplitude. If you listen carefully, you can hear both harmonics separately. That is not the usual situation. Usually one does not hear the individual harmonics of a complex periodic tone, but then a typical complex tone has many harmonics, not just two. If one starts with the complex tone having harmonics 1 and 3 and then adds harmonics 2 and 4, it is no longer possible to hear the first and third harmonics separately. Instead one hears a single tone with a complex tone color.

If one starts with harmonics 1 and 3, as before, and adds harmonics 5, 7, 9, 11, etc., then once again the third harmonic tends to disappear in the crowd. A listener normally hears only a single tone. Odd harmonics like this are the basis of the square wave and triangle wave, waveforms that come out of typical function generators (recall from the end of Chap. 4).

For a typical complex tone, the fundamental frequency (reciprocal of the period) determines the pitch, and the amplitudes of the components determine the tone color or steady-state timbre of the tone. High-frequency harmonics with large amplitudes lead to a bright tone color.

**Ohm’s Law** For periodic tones having fundamental frequencies of 100 Hz or greater and having only a dozen low-numbered harmonics, Ohm’s law of phase insensitivity holds well. Ohm’s law fails when there are systematic changes among high harmonics that are close together. For instance, the tone color of a waveform made from harmonics 20, 21, 22, 23 (and no others) can depend on whether the harmonics all have the same phase or whether they have random phases. These kinds of signals do not occur in nature. They can be constructed electronically for experiments in psychoacoustics. Despite failures with such specially constructed signals, Ohm’s “law” holds rather well for naturally occurring signals such as the tones of musical instruments. The human ear is insensitive to the relative phases among the harmonics. By contrast, changes in the amplitude spectrum make a big difference in the sound of a tone. As a result, the phase spectrum of complex tones is not treated with the same respect as the amplitude spectrum. Most of the time we ignore the phase spectrum.

**Phase-scrambled sawtooth** The 400-Hz sawtooth waveform in Fig. 9.6 has a pitch of 400 Hz and a bright tone color. It could serve as the starting point for the synthesis of a trumpet tone or violin tone. We can use the sawtooth waveform to illustrate Ohm’s law. Figure 9.7 shows a waveform having the same amplitude spectrum as the sawtooth in Fig. 9.6. However, the phases of the harmonics have been chosen randomly from the full range, 0–360°. Obviously the waveform looks

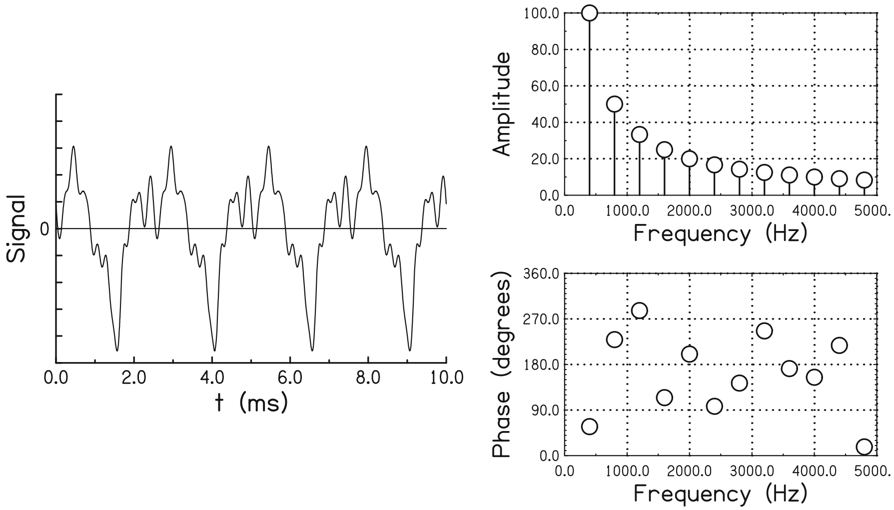


**Fig. 9.7** A 400-Hz signal with 12 harmonics having the same amplitudes as the sawtooth in Fig. 9.6. However, the phases are all different from sawtooth phases

entirely different, though the period is still the same. All that is no surprise. What is surprising is that this waveform sounds just like the sawtooth waveform. Ohm's law holds well in this case.

## 9.6 Harmonic and Inharmonic Spectra, Periodic and Aperiodic Tones

**Harmonic: Periodic** A tone made out of sine waves having frequencies of 200, 400, 600, and 800 Hz has harmonic components. These sine components (also called partials) are all harmonics of 200 Hz. Because the partials are harmonic, the tone is periodic. Its period is 1/200 s and the pitch is 200 Hz, or very nearly 200 Hz.

A tone made out of sine waves having frequencies of 400, 600, and 800 Hz (see Fig. 9.5a) also has harmonic components. These components are all harmonics of 200 Hz, but the fundamental (200 Hz) is not present in the spectrum. The tone has a period of 1/200 s and the pitch is very close to 200 Hz.

**Inharmonic: Aperiodic** If the partials of a tone are not simple integer multiples of a fundamental, then the partials are inharmonic and the tone is aperiodic. For instance, the tone with components 400, 620, and 800 Hz has inharmonic components and the tone is not periodic. Figure 9.5b shows the waveform evolving in a complicated way. No two "cycles" are the same. The waveform does not seem to repeat itself.

But hang on a minute! Maybe this tone is periodic after all. We could say that there is a fundamental frequency of 20 Hz and the components are the 20th, 31st, and 40th harmonics of 20 Hz. [400/20 = 20; 630/20 = 31; 800/20 = 40.] The period is 1/20 s, and we might expect a pitch of 20 Hz.

In strict mathematical terms there is nothing wrong with the argument in the paragraph above, and yet the prediction about the pitch is quite wrong. That problem points up the fact that what we choose to call harmonic and periodic is determined by our auditory perceptions. Harmonic numbers 2, 3, and 4 are numbers that are small enough and close enough together that the tone with components at 400, 600, and 800 can be considered periodic. Harmonics 20, 31, and 40 are so large and so far apart that we do not perceive the complex consisting of 400, 620, and 800 as a single tone, and we do not treat it as a single periodic entity. More about this important feature of the human auditory system will appear in Chap. 13 on pitch perception.

**The Tuning Fork Again** Recall the discussion of the tuning fork in Chap. 3. There we added up two sine components with frequencies of 256 and 1,997 Hz. The process was just the same as adding up the components of 200 Hz and 600 Hz in Figs. 9.3 and 9.4. But there is one big difference. The mode frequencies of the tuning fork were not harmonics. Therefore, the final wave for the tuning fork in Fig. 3.4c was not periodic. Because these components were not harmonics, there was no reason to be concerned about the phases of the harmonics, and the spectrum in Fig. 3.6 showed only the amplitudes. When two components are not harmonics, their relative phase varies continuously. By contrast, the harmonics of a periodic tone are locked together. Their relative phases determine the shape of the waveform, and there may be some reason to be interested in the phase spectrum.

## 9.7 Filters

Possibly you have been asking yourself, "What's the point of this Fourier analysis?" "What does it matter if a waveform like a sawtooth or whatever can be analyzed into components of different frequencies, or synthesized by adding up sine tones with those frequencies?" Ultimately, the answer to these questions is *filtering*. A filter passes some things and rejects other things. For instance, a filter in a coffee maker passes coffee and rejects coffee grounds. A filter in acoustics or electronics passes certain frequencies and tends to reject other frequencies. Filtering of some sort can be found everywhere in acoustics, including the human auditory system. Because filtering is selective of *frequency*, the frequency representation made possible by Fourier analysis is crucial. The fact that filters operate in the domain of frequency is the reason that the concepts of Fourier analysis and synthesis are so important.

From your study of vibrations you have learned about resonance. Systems with natural frequencies of vibration serve as filters in that they respond strongly to only certain frequencies. If a complicated waveform is impressed upon such a system, the system will select out mainly those frequencies that are its natural frequencies. Filters are particularly evident in audio or electronics. There are lowpass filters that pass only low frequencies and high-pass filters that pass only high frequencies. There are bandpass filters that act like resonant devices (resonators) because they pass only frequencies in a band near the resonant frequency.

**All the World's a Lowpass Filter** The consequences of Fourier's theorem are not limited to acoustics. The relationship between waveforms and frequency has enormous commercial and political consequences. The fact is, everything in the world is a lowpass filter in the sense that there is some upper limit to the frequency to which it will respond. In communication systems, the lowpass character limits the bandwidth, and this places a limitation on the rate at which information can be transmitted.

Figure 9.8a shows a data stream consisting of seven bits—ones or zeros. This data stream actually represents the capital letter "Y." It is shown here, transmitted at a rate of 10,000,000 bits per second, so that each bit lasts for 0.1 μs, which is 100 ns.

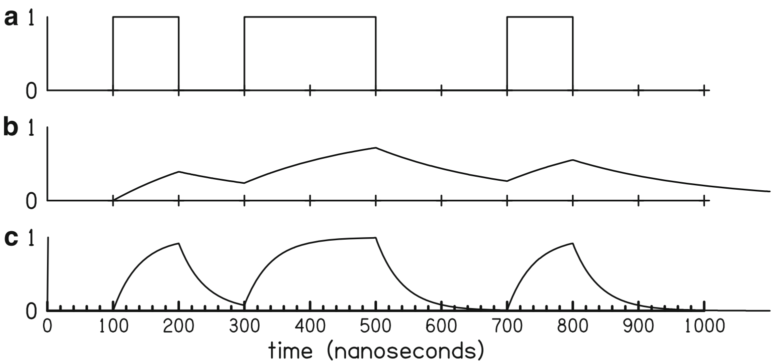


**Fig. 9.8** (**a**) A computer code (ASCII) for the letter "Y." (**b**) The computer signal from (**a**) after it is passed through a channel with a bandwidth of only 0.8 MHz. (**c**) The computer signal from (**a**) after it is passed through a channel with a bandwidth of 4.0 MHz

Suppose we try to transmit this signal through a communications channel with a bandwidth of only 800 KHz (0.8 MHz). This channel will not pass the high frequencies needed to make a clean signal, and the received signal will look like Fig. 9.8b. It is very likely to be unreadable at the receiving end, and data are lost. In order to transmit information reliably in such a channel we need to slow down the transmission rate so that the frequencies in the spectrum are not so high. Alternatively, if we can afford it, we can buy or rent a channel with a wider bandwidth. For instance if the bandwidth of the channel is 4 MHz, the received signal looks like Fig. 9.8c. That is very likely to be a useable signal. The receiving system can recover the data without error.

Thus, by using a wider bandwidth we have been able to transmit information at the high rate.

Every communication system in the world has some bandwidth limitation. The search for wider bandwidth has led telecommunications companies to new technologies, such as optical data transmission. The scarcity of bandwidth in broadcasting has led to governmental regulation worldwide. Essentially, everywhere you look—Fourier rules!

## 9.8 Continuous Spectra

The spectra that appear in this chapter have all been line spectra in that they consist of discrete frequencies, like 200, 400, 600 Hz. This section introduces continuous spectra where there are no gaps between frequencies. The concept is somewhat abstract, but it can be approached from what you already know.

(1) From Fig. 9.5 you know that a wave with components at 400, 600, and 800 Hz has a short period, only 1/200 s. Its harmonics are far apart—they are separated by 200 Hz.
(2) From Fig. 9.5 you also know that a wave with components at 400, 620, and 800 Hz is less periodic. It has a longer period, 1/20 s. Its harmonics are much closer together; they are separated by 20 Hz.
(3) The tuning fork with frequencies of 256 and 1,997 Hz is even less periodic. Its period is a full second, and that caused us to say that it was not really periodic. Its harmonics (to stretch the definition of the term) would be separated by only 1 Hz—in fact, they would be the 256th and the 1,997th harmonics of 1 Hz—kind of a silly idea, but it is mathematically correct.
(4) Finally, there is a signal like the letter "Y" in Fig. 9.8. That signal is not at all periodic. It never repeats itself. It's a one-shot event. Therefore, we have to say that the period is infinite, and, to follow the mathematical logic, the separation between the spectral components becomes zero. That's the concept of the spectral continuum. Just for illustration, the amplitude spectrum is shown in Fig. 9.9. It looks continuous, not like individual discrete lines. The phase spectrum is not shown.

## Exercises

*Exercise 1, A complex wave*

Write the equation for a wave having the first four harmonics of 250 Hz, all with 90° of phase and with amplitudes given by $1/n$ where $n$ is the harmonic number. [Hint: Follow the form in Eqs. (9.3) and (9.4).]

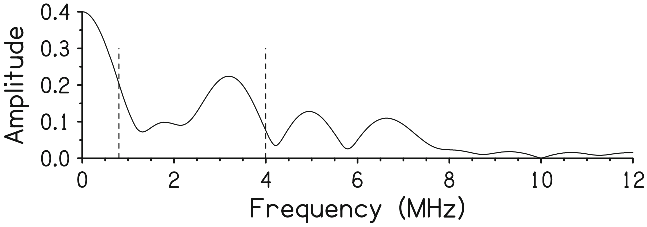


**Fig. 9.9** The amplitude spectrum of the signal for the letter "Y" from Fig. 9.8. The spectrum becomes zero at 10 MHz because each pulse has a duration that is a multiple of 0.1 μs. *Dashed vertical lines* appear at 0.8 and 4.0 MHz. You can see how much spectral strength is lost if the signal is filtered to reduce the amplitude of components above 0.8 MHz. That is why Fig. 9.8b is such a poor representation of Fig. 9.8a

*Exercise 2, Morphing a signal*

Figure 9.3 shows the signal described by the equation

$$\text{Signal} = 45\,\sin(360 \cdot 200\,t + 0) + 15\,\sin(360 \cdot 600\,t + 0). \tag{9.6}$$

Use that plot to imagine the signal described by

$$\text{Signal} = 45\,\sin(360 \cdot 200\,t + 0) + 45\,\sin(360 \cdot 600\,t + 0), \tag{9.7}$$

where the third harmonic is just as tall as the fundamental. Sketch that signal.

*Exercise 3, What's the period?*

(a) Use figures to show why a wave with components at 200 and 600 Hz has the period of $1/200$ s. Do the phases of the components change the periodicity?
(b) What is the period of a wave with components at 200 and 500 Hz?
(c) What is the period of a wave with components at 200 and 601 Hz?
(d) What is the period of a wave with components of 150, 450, and 750 Hz?
(e) What is the period of a wave with components of 220, 330, 550, and 660 Hz?

*Exercise 4, Physical vs psychological*

The discussion of the phase-scrambled sawtooth in the text says that the sawtooth waveform has a pitch of 400 Hz. Frequency is a physical quantity and it can properly be measured in Hz. Pitch, however, is a perceptual quantity, and it can only be measured by a human listener. So how can one measure a pitch in physical units of Hz?

*Exercise 5, Bozo and the waveform*

Bozo explains that according to Ohm's law the amplitude spectrum determines the shape of the waveform. Therefore, the picture on an oscilloscope screen does not depend much on the phases of the components. Set Bozo straight on this matter.

*Exercise 6, More components for the saw.*

Equation (9.5) gives the first three components of the sawtooth wave. What are the next three?

*Exercise 7, Drill.*

Consider the wave given by the equation:

$$\begin{aligned} \text{Tone} = {} & 10\ \sin(360 \cdot 200\,t + 30) + 15\ \sin(360 \cdot 400\,t + 60) \\ & + 5\ \sin(360 \cdot 600\,t - 90) + 3\ \sin(360 \cdot 800\,t + 240). \end{aligned} \tag{9.8}$$

(a) How many harmonics are there?
(b) How many vertical lines would a spectrum analyzer show?
(c) Which harmonic has the largest amplitude?
(d) What is the amplitude of the component with a frequency of 600 Hz?
(e) What is the phase of the component with a frequency of 400 Hz?
(f) Can a phase angle really be negative (e.g., $-90°$)?
(g) Could the phase of 240° just as well be $-120°$?
(h) What is the period of this tone.
(i) If the frequencies are in units of Hertz (e.g., 200, 400 ... Hz), is it essential that parameter $t$ be measured in seconds?

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