Chapter 7
Standing Waves

The two previous chapters have described wave motion with particular reference to *traveling* waves. Traveling waves carry energy and information from one point in space to another. This chapter begins a study of *standing* waves. Standing waves turn out to be the modes of vibration of important classes of musical instruments. Accordingly, the chapter quickly moves on to the vibration of guitar strings.

7.1 Standing Waves in General

A traveling wave moves; a standing wave does not. But that does not mean that a standing wave does not change with time. It just changes differently from a traveling wave. The difference between a traveling wave and a standing wave can be seen in a series of snapshots.

Figure 7.1 is a snapshot of a 500-Hz wave. The period of the wave is 2 ms (1,000/500). The wavelength is $344/500 = 0.69$ m, and you can see that wavelength in Fig. 7.1 because the horizontal axis is a spatial axis, x.

If this wave is a *traveling* wave, then it moves rigidly in time. If the wave is traveling from left to right, and we look at it 0.25 ms later (one-eighth of a period), then we find that the wave has moved to the position shown by the dashed line in Fig. 7.2.

If this wave is a *standing* wave and we look at it 0.25 ms later (one-eighth of a period), the wave looks like the dashed curve in Fig. 7.3. It has not moved. It has just changed its size.

We continue to track the standing wave. If we look at the standing wave after another 0.25 ms (total time of $t_3 = 0.5$ ms, or one quarter of a period), we find that the standing wave is momentarily totally flat, as shown in Fig. 7.4.

After another 0.25 ms (total time of $t_4 = 0.75$ ms or three-eighths of a period) we find that the 500-Hz standing wave has come out the other side. It has changed from the original solid curve to the dashed curve in Fig. 7.5.

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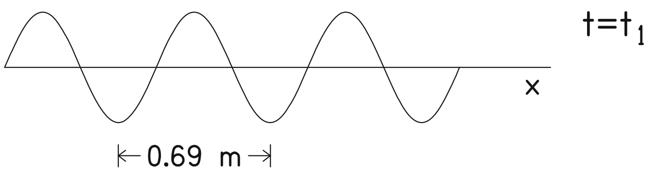

Fig. 7.1 A snapshot of a wave, taken at time t_1. This snapshot serves as the reference for snapshots to follow

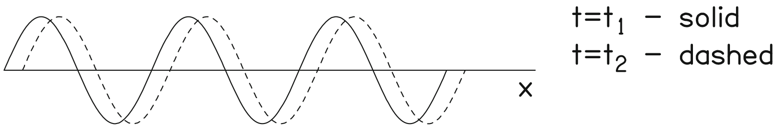

Fig. 7.2 As time goes on, the 500-Hz traveling wave moves from left to right as time increases from t_1 to t_2

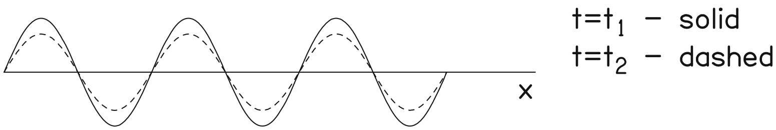

Fig. 7.3 In contrast to the traveling wave, the standing wave does not move

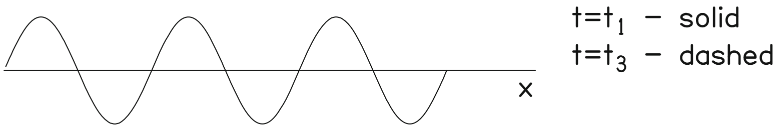

Fig. 7.4 The snapshot of the standing wave taken at time t_3 is shown by a *dashed line*, but that line is hidden by the x axis

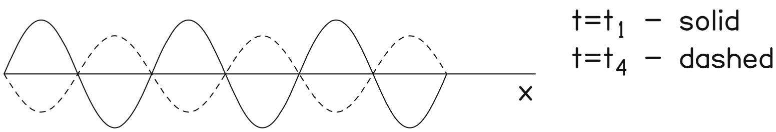

Fig. 7.5 The snapshot of the standing wave taken at time t_4 is shown by a *dashed line*

If we look at the 500-Hz standing wave after another 0.25 ms (total time of $t_5 = 1$ ms, or one-half of a period), we find that it has changed from the original solid curve to the dashed curve in Fig. 7.6.

Figure 7.6 shows how standing waves are normally represented, by two snapshots, one showing a maximum amplitude and the other, taken half a period later, showing a maximum in the other direction.

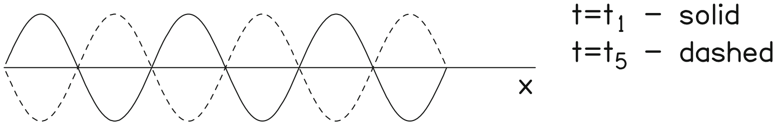

Fig. 7.6 The snapshot of the standing wave taken at time t_5 is shown by a *dashed line*. The difference in times, $t_5 - t_1$ equals half a period, $T/2$

As time goes on, the standing wave continues to evolve. The graphs of the continued motion look just like the graphs 7.2 through 7.6 played in reverse. For instance, after 1.25 ms the 500-Hz wave looks like it did at time t_4, and at 1.5 ms the wave looks like it did at time t_3. After a full period, the wave looks like it did at the beginning at time t_1.

A clear difference between a traveling wave and a standing wave is that with a standing wave there are places in space (values of location x) where there is never any displacement. No matter how long we wait at one of these places, there is never any action. Such places are called "nodes." By contrast, a traveling wave ultimately causes every point in space to experience some displacement. If there is no action now at your spot, be patient—the crest of a traveling wave will come by in a millisecond or so.

7.2 Standing Waves on a String

Standing waves don't just happen for no good reason. If there is a disturbance in a medium, such as a disturbance in the air or on a string, then the disturbance will travel as a traveling wave. A standing wave is caused by reflections in the system. For instance, if the string is stretched between two boundaries, then there will be reflections of the displacement wave at the boundaries. The key to understanding the standing wave are the conditions at the boundaries. Mathematicians call these "boundary conditions."

A guitar string is stretched between the bridge and the nut. A wave on the string is reflected from both these ends. The boundaries at the bridge and nut are places where the string is not allowed to vibrate. That is the key to the standing waves on the guitar string (or any other stretched string). The boundary conditions insist that there are no displacements at the ends of the string. The standing waves that are allowed to exist must be waves that satisfy those boundary conditions. That means that the wavelengths of the allowed vibrations must somehow "fit" onto the string.

The wave below has a wavelength that works. The wavelength is twice the length of the string so that half a wave fits perfectly onto the string. If the string is $L = 65\,\text{cm}$ long, then the wavelength is $\lambda = 130\,\text{cm}$ or $\lambda = 1.3\,\text{m}$. This is the longest wavelength that will fit on the string. This pattern of vibration of the string is the *first* mode. Because this is the mode with the longest wavelength, this mode has the lowest frequency. The frequency depends on the speed of sound on the string (Fig. 7.7).

Suppose that the speed of sound on the string is $v_s = 107$ m/s. Then the frequency of this mode of vibration is given by

$$f = v_s/\lambda = 107/1.3 = 82.3 \text{ Hz}. \tag{7.1}$$

This is the fundamental frequency of the lowest string on a guitar. The note is called "E."

Because the wavelength is twice the length of the string, we can write the formula as

$$f = f_1 = v_s/(2L) \tag{7.2}$$

The second mode of vibration of the string has a complete wavelength fitting into the length of the string. A better way to say it is that two half-wavelengths will fit (Fig. 7.8).

The frequency of the second mode of vibration of the string is then

$$f_2 = v_s/(L) = v_s/(2L) \times 2 = 107/1.3 \times 2 = 165\,\text{Hz}. \tag{7.3}$$

The third mode of vibration has three half-wavelengths fitting into the length of the string (Fig. 7.9).

The frequency of the third mode is

$$f_3 = v_s/\left(\frac{2}{3}L\right) = v_s/(2L) \times 3 = 107/(1.3) \times 3 = 247\,\text{Hz}. \tag{7.4}$$

Mode 1 has no nodes. Mode 2 has one node. Mode 3 has two nodes. If we were to continue, mode 4 would have three nodes, etc. (This counting of nodes does not

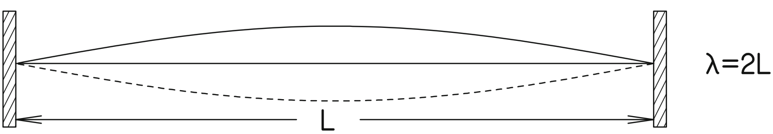

Fig. 7.7 The first mode of a stretched string gets half a wavelength into the length of the string

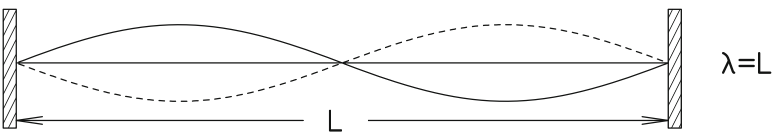

Fig. 7.8 The second mode of a stretched string gets a full wavelength into the length of the string

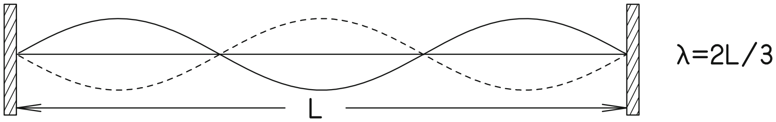

Fig. 7.9 The third mode of a stretched string gets one and a half wavelengths (three halves) into the length of the string

count the two fixed ends of the string where there can never be any displacement.) It should be clear that there is no limit to the number of possible modes of vibration.

The different modes of vibration of the guitar string radiate by means of the guitar body. These different modes create the different *harmonics* of the guitar tone. The first mode creates the fundamental tone or first harmonic. The second mode of vibration creates the second harmonic, and so on. Each of the harmonic frequencies is an integer multiple of the fundamental frequency.

Harmonics The paragraph above used the word "harmonic." The word has been used before in this book, as in "simple harmonic motion." However, the paragraph above gives a second meaning to the word. Two or more components of a tone are said to be harmonics if their frequencies are small integer multiples of some base frequency. For instance, 150, 300, and 450 Hz are all integer multiples of 150 Hz. The integers are 1, 2, and 3, and the frequencies correspond to the first, second, and third harmonics.

For the stretched string, one can express the frequency of the nth mode (equivalently the nth harmonic) as

$$f_n = v_s/(2L) \times n \tag{7.5}$$

This is a very simple relationship. As we study different musical instruments in the chapters that follow, the instruments will have physical modes of vibration that correspond to harmonics in the tone of the instrument, but the correspondence between modes and harmonics will not always be so simple.

7.3 The Guitar Player's Equation

To play melody and harmony, the guitar player must play notes with correct fundamental frequencies. The fundamental frequency of a note, or tone, is determined by the speed of sound and the length of the string. The speed of sound, in turn, depends on the tension in the string (F) and the linear mass density (μ) of the string according to the equation

$$v_s = \sqrt{\frac{F}{\mu}}. \tag{7.6}$$

Here, F is the tension measured in Newtons. Recall that Newtons is a metric unit of force, equivalent to about 1/4 pound in the English system of units. Linear mass density μ is the mass per unit length of the string. It is measured in kilograms per meter (kg/m).

Putting the information from Eqs. (7.2) and (7.6) together gives the guitar player's equation for the fundamental frequency of a tone,

$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}}. \tag{7.7}$$

Thus, three different string properties go into making the pitch of the guitar tone: length (L), tension (F), and linear density (μ).

7.4 The Stretched String: Some Observations

Modes: Really? The discussion of standing waves on a string made frequent use of the word "mode." This is not the first time you have seen this word. This word was introduced in Chap. 3 to describe a natural pattern of vibration of a physical system. Is it correct to use this word again to describe the standing wave? Absolutely—it is correct. The modes of vibration of a stretched string *are* standing waves. Recall that the requirements for a mode are that there be (1) a specific frequency, (2) a specific shape, and (3) an amplitude that can be determined by the user. You should be able to see how our definition of standing waves on the stretched string fulfills all those requirements.

The Musical Significance of the Stretched String: Harmonics Think about all the things there are in the world that you could bang on or blow on or otherwise cause to vibrate. Now think about the physical systems that are actually used in making music. Only small fraction of the possible things in the world make useful musical instruments, and the stretched string is one of them. The stretched string dominates our orchestras because of the violin family (includes violin, viola, cello, and bass viol) and it dominates our rock groups because of guitars. It is also the acoustical basis of the piano. This is really rather strange. The stretched string seems like an unlikely candidate for a musical instrument. It makes a very poor radiator. It has so little vibrating surface area that one always needs a sound board or amplification of some kind to make it at all useful musically.

There must be a reason why this one physical system, out of all the possibilities, is so dominant. The reason is that the modes of vibration have frequencies that are all integer multiples of a fundamental frequency. The second mode has a frequency that is twice the frequency of the first mode. The third mode has a frequency that is three times the frequency of the first mode, and so on. Because of this useful fact about the frequencies of the modes, the stretched string can create tones consisting of vibrating components that are harmonic. These components of a tone are sine waves and they are called "partials" because the entire tone is made up of many sine waves. Each sine component is only "part" of the entire wave. The partials of a tone made by a stretched string are harmonic because of the integer relationship between the frequencies of the partials. We can rewrite Eq. (7.5) to emphasize that fact,

$$f_n = nf_1. \tag{7.8}$$

This is an equation that is just about the frequencies of the modes, and ultimately about the frequencies of the partials. Most other physical systems have modes with frequencies that do not obey this special harmonic law.

Idealization and Reality When you pluck a guitar string your thumb pulls the string to the side and releases it. There is a restoring force that then causes the string to move toward equilibrium. The restoring force is responsible for the vibration of the string. Most of the restoring force is caused by the tension in the string. However, some of the restoring force comes from the *stiffness* of the string. To understand stiffness, imagine that the guitar string is not on the guitar but is lying loose on the table. There is no tension and yet if you try to kink the string tightly you encounter some resistance. The resistance is stiffness. This stiffness also contributes to the resistance to displacement when the string is stretched on the guitar in the usual way.

Our treatment of the stretched string, and the harmonic law concerning the mode frequencies, is based on the assumption that *all* of the force on the string comes from tension and none of it comes from stiffness. In other words, it is an idealization of the real string. Look at Eq. (7.7). There you see the importance of tension (F) but there is nothing about stiffness. When we discuss about string instruments later in this book, we will deal with real strings. As we will see at the time, what looks like a potential problem turns into a benefit.

Exercises

Exercise 1, High mode number

Draw the standing wave corresponding to the sixth mode of string vibration. How many nodes are there? (Don't count the two fixed ends of the string.)

Exercise 2, The A string

If the speed of sound on a string is 154 m/s, and the string length is 70 cm, what is the frequency of the first (or fundamental) mode of vibration? What is the frequency of the third harmonic?

Exercise 3, The guitar player's equation

Explain how the three parameters, μ, F, and L, are used in building, tuning, and playing a guitar.

Exercise 4, Standing and traveling

Explain how standing waves and traveling waves change with time. How are they different?

Exercise 5, How to play the guitar

To play different notes, a guitar player presses fingers down on the string to make it shorter. What is called an "open string" occurs when the player does not press down with a finger but uses the full string length (as the instrument was manufactured). The open E string has fundamental frequency of 82 Hz. Suppose it is 70 cm long. How short must the string be to play the note called "A" with a frequency of 110 Hz?

Exercise 6, A tale of two strings

The lowest fundamental frequency on a guitar is 82 Hz—the open E string. The top string on a guitar is also an E string. The open top string sounds the E two octaves higher. How much faster is the speed of sound on the top E string compared to the lowest string? Recall that going up by one octave corresponds to a doubling of the frequency.

Exercise 7, Bending notes

Bending an note means to change its frequency slightly in a slow and continuous way for musical effect. To play the blues you have to bend notes. How does a rock guitar player bend notes? What is the physical principle involved?

Exercise 8, The effect of tension

A guitar string has a fundamental frequency of 100 Hz. What does the frequency become if the tension in the string is doubled?

Exercise 9, More unit analysis

Chapter 5 introduced unit analysis with some simple examples. You might like to try unit analysis on a not-simple example such as Eq. (7.6) for the speed of sound on a string, v_s, given in meters per second.

$$v_s = \sqrt{\frac{F}{\mu}}.$$

The denominator is the linear mass density, and its units are kilograms per meter or (kg/m). The numerator is a force with units of Newtons, but that is not very helpful. The key to the units of force is Newton's second law, which says that force is equal to mass multiplied by an acceleration. Mass has units of kg, and acceleration has units of m/[(s)(s)]. Therefore, a Newton of force corresponds to kg m/[(s)(s)]. Show that the units on the right-hand side of this equation are m/s as expected. Don't forget to take the square root of the units. Note that a unit such as m/[(s)(s)] would normally be written as m/s^2.

Exercise 10, Role of the guitar body

From the *guitar player's equation*, Eq. (7.7), it looks as though the frequency of a guitar note, and hence its pitch is entirely determined by the vibrating string and is not related in any way to the body of the guitar. Can that possibly be right?

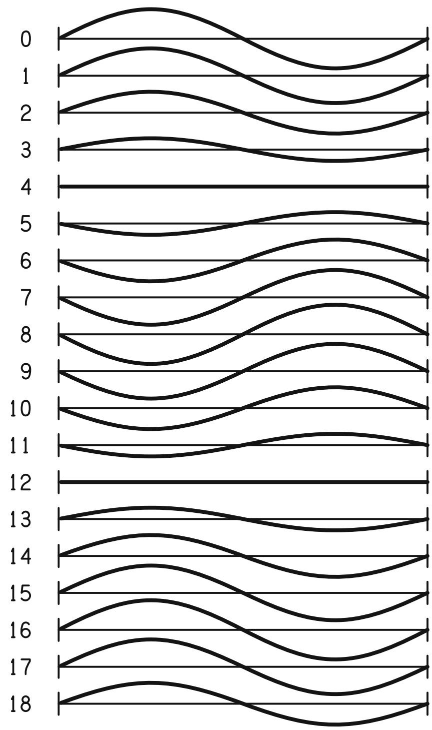

Fig. 7.10 Second mode of vibration of a stretched string at 19 different instants of time separated by 1 ms

Exercise 11, Other musical instruments

Not all musical instruments use strings for vibration, however they still use standing waves to make sound. Two of the most common non-string instrument categories are wind instruments and drums. What vibrates in these other instruments?

Exercise 12, One more guitar equation

Combine Eqs. (7.5) and (7.6) to obtain an equation for the frequency of the nth harmonic of a guitar tone.

Exercise 13, Second mode through time

Figure 7.10 shows 19 consecutive snapshots of the second mode of vibration taken at 1 ms intervals. Show that the frequency of this mode is 62.5 Hz. Show that the frequency of the fundamental mode is 31.25 Hz.

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