

# Chapter 8

## Standing Waves in Pipes

### 8.1 Pipe with Both Ends Open

A cylindrical pipe (a flute, a marimba resonator, an automobile exhaust pipe) filled with ordinary air is a system that shows standing waves. If both ends of the pipe are open, the boundary conditions insist that the pressure at the open end is zero. Zero means that it is not overpressure (positive); it is not underpressure (negative); it is ordinary atmospheric pressure (zero).

A snapshot of the first mode of vibration looks like Fig. 8.1.

Here the solid curve is a snapshot taken at a time when air has rushed in from the two ends and piled up in the middle of the pipe creating a pressure maximum there. The dashed curve is a snapshot taken half a cycle later when the air has rushed out of the open ends and created a partial vacuum in the middle of the pipe.

A snapshot of the second mode of vibration looks like Fig. 8.2.

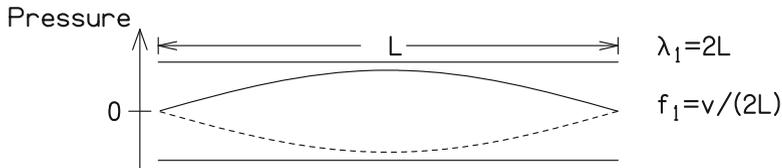
It should be clear that in this mode, the air is sloshing from one side of the pipe to the other, like the water in a bathtub. The solid line shows the pressure sloshed to the left side with a pressure valley on the right. Half a cycle later (dashed line) the pressure has sloshed to the right side.

The graphs of the modes of air in a cylindrical pipe are just the same as the graphs of the modes of a vibrating string. The physical systems are different, but the mathematical analysis is the same. If you know about the string there is almost nothing more to learn.

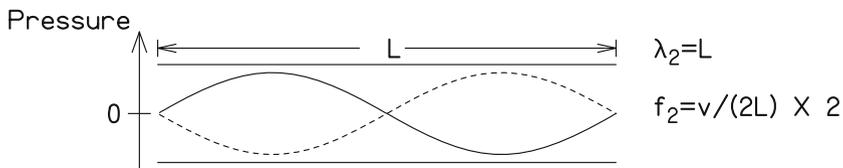
Just as in the case of the stretched string, the modes have wavelengths given by the equation  $\lambda_n = 2L/n$  where  $n$  is an integer indicating the mode number. Therefore the frequencies are given by the formula

$$f_n = [v/(2L)]n \quad (8.1)$$

The only new thing to learn is that when dealing with the vibrations of air in a pipe, you must use the speed of sound in air for  $v$ . For the vibrations of a string you must use the speed of a transverse vibration wave on the string.



**Fig. 8.1** Open–open pipe. The first mode has no nodes as we count them



**Fig. 8.2** Open–open pipe. The second mode has one node

**Example:**

If a cylindrical pipe is 90 cm long and open at both ends, then what is the frequency of the third mode of vibration?

**Answer:**

$$f_3 = v / (2L) \times 3 = 344 / (2 \times 0.9) \times 3 = 573 \text{ Hz.} \quad (8.2)$$

## 8.2 Pipe with One End Open and One End Closed

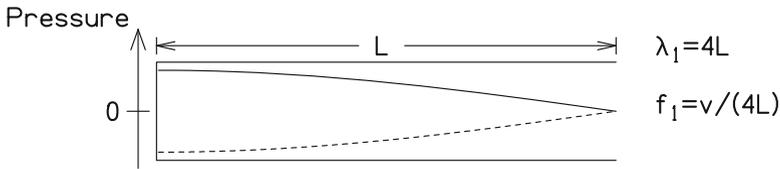
Closing off one end of a pipe changes the physical system dramatically. The first mode in an open–closed pipe looks like Fig. 8.3. There is a pressure maximum at the closed end of a pipe as the air rushes in and jams against this closed end. Half a cycle later there is a pressure minimum at the closed end. Hence, the boundary condition for a standing wave at the closed end is an *antinode*. At the open end the boundary condition is zero pressure as usual.

The second mode looks like Fig. 8.4.

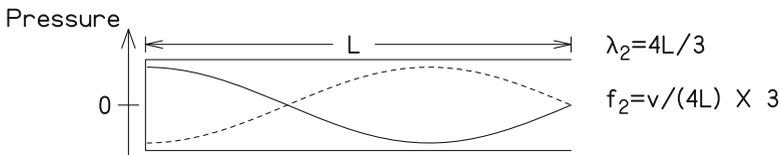
The third mode of vibration looks like Fig. 8.5.

There is an evident pattern. In the first mode one-quarter wavelength fits into the length of the pipe. In the second mode three-quarters of a wavelength fit. In the third mode five-quarters fit, and so on.

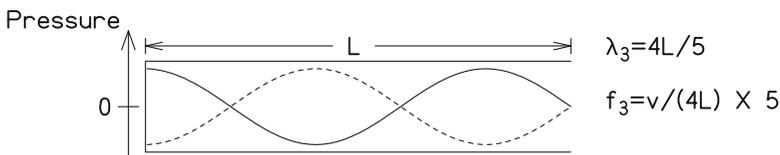
In connection with the frequencies  $f_1$ ,  $f_2$ , and  $f_3$ , ..., of modes 1, 2 and 3, ..., symbol  $v$  again represents the speed of sound in air. The frequency  $f_1$  for the open–closed pipe is  $v / (4L)$  and this can be compared with the frequency for the open–open pipe, namely,  $f_1 = v / (2L)$ .



**Fig. 8.3** Open-closed pipe. The first mode has only 1/4 wavelength in the pipe



**Fig. 8.4** Open-closed pipe. The second mode has a single node, 3/4 wavelength



**Fig. 8.5** Open-closed pipe. The third mode has 5/4 wavelengths in the pipe

**Harmonics Again** The modes of vibration of air in pipes have frequencies that are in a harmonic relationship. Therefore, used as musical instruments, pipes create tones with harmonic components. The pipe that is open at one end and closed at the other illustrates the warning given in Chap. 7 about more complicated relationships between modes and harmonics. Here, the first mode creates the first harmonic, the second mode creates the third harmonic, the third mode creates the fifth harmonic, and so on, all-odd numbered harmonics.

### 8.3 Playing a Pipe

If you blow across the open end of a pipe you can produce a tone. (Players of pan pipes are expert at this.) The tone occurs because blowing across the pipe excites the modes of vibration of the air in the pipe. More or less, all of the modes are excited at once, though it is usually the mode with the lowest frequency that is excited the most. Therefore, more or less all the modal frequencies are present in the tone at the same time. The *pitch* of the tone (what musicians call the “note”) is normally given by the frequency of the lowest mode. This rule about pitch holds good both for pipes that are open at both ends and for pipes that are open at one end and closed at the other.

Shock exciting the modes of a pipe, by banging the open end of a pipe into your palm, also excites all the modes. Again, all the different mode frequencies are present simultaneously, though the lowest frequency modes dominate.

## 8.4 Thinking Critically

There is something rather strange about what this chapter has said about standing waves in pipes. It is actually worse than strange, and you ought to be questioning it.

This chapter related the frequencies of standing waves to the length of pipes, but nothing was ever said about the diameter of the pipes. Imagine a pipe that is 90 cm long and 1 km in diameter. How would this pipe be any different from open air in the great outdoors? One can't imagine that such a system would have standing waves like those shown in the figures above. And why doesn't the diameter appear in the equations for the frequencies of the standing waves? How does this make any sense?

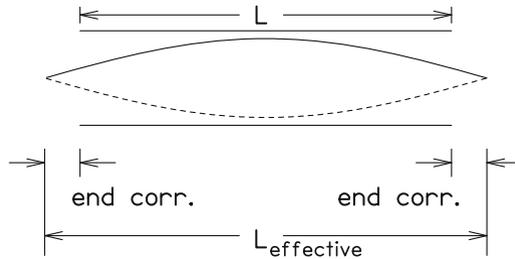
These are good questions. In fact, the treatment of the modes in a pipe makes the assumption that the pipe diameter is small compared to the length. Such a pipe is called "thin." The fact that the diameter is small means that the pipe appears to be a one-dimensional system and the only dimension of interest is the length. That is why the length is the only dimension that appears in the equations.

If the diameter of the pipe is not much smaller than the length, then the picture described in this chapter does not apply. If the diameter is much smaller than the length, but not *very much* smaller, then the picture described in this chapter is a decent approximation to reality, and it can be improved by including open-end corrections, as described below.

## 8.5 Open-End Corrections

Our treatment of an open end of a pipe has not been exactly correct. We have assumed that the pressure goes to zero (equivalent to the outside atmospheric pressure) right at the open end of the pipe. In fact, the pressure wave extends a little bit beyond the end of the pipe. The point of zero pressure actually occurs slightly outside the pipe. The amount by which the pressure wave extends beyond the open end of the pipe depends on the pipe diameter. Therefore, for the purpose of calculating the resonant frequencies of a pipe with one or two open ends, the effective pipe length is longer than the measured pipe length. The measured pipe length should be increased by the end correction to find the effective pipe length. In equation form, if  $L$  is the measured length of the pipe and  $L_{end}$  is the end correction, then the effective length of the pipe is given by

$$L_{effective} = L + L_{end} \quad (8.3)$$



**Fig. 8.6** Because the standing wave in a pipe extends beyond the ends of the pipe, the effective pipe length  $L_{\text{effective}}$  is longer than the measured length  $L$ . The difference is the end correction. For a pipe like this one with two open ends, the end correction must be added twice to obtain the effective length

The end correction is equal to 0.61 times the radius of the pipe (0.305 times the diameter) for one open end. If both ends are open the end correction is twice as large (Fig. 8.6).

**Example:**

A pipe is 1 m long and open at both ends. Its diameter is 5 cm (0.05 m). What is the lowest resonant frequency of the pipe?

**Answer:**

The lowest resonant frequency is given by  $v/(2L_{\text{effective}})$ .

The effective length of the pipe is  $L_{\text{effective}} = 1 + 0.05 \times 0.305 \times 2$  or 1.0305 m. Therefore, the lowest resonant frequency is

$$f = v/(2L_{\text{effective}}) = 344/(2 \cdot 1.0305) = 167 \text{ Hz.} \quad (8.4)$$

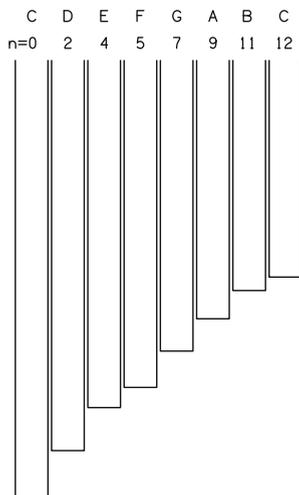
Note that without the end correction we would have calculated a frequency of 172 Hz. [ $v/(2L) = 344/2 = 172$ ]. The difference is significant.

**Making a pan flute**

The goal of this example is to construct a pan flute that plays the tones of a major scale starting on middle C and ending an octave higher. Such a scale has eight notes, called C, D, E, F, G, A, B and C, and so the pan flute will be made from eight tubes. They will be closed at one end.

From Appendix E we learn that middle C (called  $C_4$ ) has a fundamental frequency of 261.6 Hz. To make the other notes, we choose an equal-tempered scale. Therefore, all the fundamental frequencies are given by the equation

$$f_m = 261.6 \cdot 2^{m/12}$$



**Fig. 8.7** A pan flute (pan pipe) that plays an equal-tempered major scale using pipes that are closed at one end. Except for the small end correction, the tube lengths decrease exponentially for tones of increasing frequency

where  $m$  is the number of semitones,  $m = 0, 2, 4, 5, 7, 9, 11, 12$ . Notice that  $m = 0$  produces middle C and  $m = 12$  produces a C that is an octave higher. This equation, together with the values of  $m$  leads to the eight fundamental frequencies of the scale. Each of the notes of the scale will have a set of odd harmonics that will occur naturally without any effort on our part.

To make the flute, we will use lengths of tube with an inside diameter of 1 cm (10 mm). The diameter determines the end correction. To find out the lengths of the tubes, we use the formula for a pipe open at only one end, including the end correction,

$$f_m = v/[4(L_m + L_{end})],$$

where  $L_m$  is the length of the tube that we have to cut.

To find that length, we can combine the two equations and solve for  $L_m$ ,

$$L_m = v/(4 \times 261.6 \times 2^{m/12}) - L_{end}$$

or

$$L_m = 344000/(1046 \times 2^{m/12}) - 3,$$

where  $L_m$  is the tube length in mm, 344000 is the speed of sound in mm/s, and 3 mm is the end correction for a tube with an inside diameter of 10 mm.

The last equation leads to tube lengths of 326, 290, 258, 243, 216, 192, 171, and 161 mm, as shown in Figure 8.7.

## Exercises

### *Exercise 1, 1-ft pipe*

- (a) You have a pipe, 1 ft long and open at both ends. What frequency do you hear if you blow across this pipe? (Note: Use the fact that the speed of sound is  $v = 1130$  ft/s. You may ignore the end correction for this part and the other parts of this exercise.)
- (b) You close off one end of the pipe and blow across the open end again. Show that the fundamental frequency is an octave lower than in part (a) where both ends were open. (Recall the definition of an “octave:” Frequency  $f_a$  is an octave higher than frequency  $f_b$  if  $f_a$  is twice  $f_b$ , i.e.,  $f_a = 2f_b$ . Then  $f_b$  is an octave lower than  $f_a$ .)
- (c) Explain why the open–closed pipe makes a tone with only odd harmonics (1, 3, 5, ...) whereas an open–open pipe makes a tone with all the harmonics (1, 2, 3, ...)

### *Exercise 2, Nodes and modes*

Find the mathematical relationship between the number of nodes for the standing waves in an open–closed pipe and the mode number. Compare with the relationship for an open–open pipe. (Don’t count the points of zero pressure that occur at open ends.)

### *Exercise 3, Open–closed pipe*

Draw the standing wave for the fourth mode of an open–closed pipe. How many nodes does it have? (Don’t count the point of zero pressure at the open end.) What is a mathematical expression for the frequency?

### *Exercise 4, Open–open pipe*

Draw the standing wave for the fourth mode of an open–open pipe. How many nodes does it have? What is a mathematical expression for the frequency?

### *Exercise 5, The end correction*

In what way does the end correction depend on the ratio of the pipe diameter to the pipe length? Does the end correction tend to zero if the pipe diameter becomes smaller and smaller? Does the end correction tend to zero if the pipe length becomes longer and longer?

### *Exercise 6, Pipes of Pan*

Pan pipes are cylindrical tubes in an array. Blowing across a tube produces a tone with frequencies corresponding to the expected modes. The tubes are stacked in pairs, with one tube open–open and the other open–closed. Blowing across the pair of tubes creates two tones, with fundamental frequencies that are an octave apart. The two tubes of a pair are almost the same length, but not quite. Which of the two should be slightly longer? [Hint: Remember that the end correction is applied to open ends only.]

### *Exercise 7, The organ pipe is too long!*

According to Appendix C, a massive pipe organ has a pipe that sounds a frequency of 16 Hz. That is the fundamental frequency of the lowest note. (a) Show that the formula  $f_1 = v/(2L)$  implies that this pipe needs to be 10.75 m long. (b) Show that this length is more than 35 ft. (c) Unfortunately, you are confronted with an installation where the vertical space available for the pipe organ is only 30 ft. What can you do to solve this problem? (d) What do you sacrifice with this solution?

*Exercise 8, Boston Symphony Hall*

The front of the famous Boston Symphony Hall features 48 exposed organ pipes, and 33 of them appear to have the same length! How can the designer make organ pipes that look that way and still have the pipes play different notes?

*Exercise 9, How to make the pipe more complicated*

The mode frequencies for a stretched string depend on the string length, the mass density, and the tension. By contrast, the mode frequencies for air in a pipe depend only on the pipe length—much less complicated. What could be done to make the acoustics of a pipe seem more analogous to the acoustics of a string?

*Exercise 10, Popping a pipe*

Imagine that you have a piece of thin plumbing pipe, about 25 cm (10 in.) long and open at both ends. You can make a tonal sound by striking an open end on your palm (first remove burrs from the end of the pipe). Alternatively you could put a finger into the pipe and pop it out like a cork. Both methods shock excite the modes of the pipe. Do you expect that both methods lead to the same pitch?

*Exercise 11, The clarinet*

The clarinet is a woodwind instrument with a cylindrical bore about 60 cm long. When all the tone holes are closed, it resembles an open–closed pipe. Calculate the lowest frequency for the clarinet. Compare with the lowest frequency given in Appendix C.

*Exercise 12, Percussive bass*

You want to make a bass musical instrument from PVC pipe. The pipe should produce the musical note C2 with a frequency of 65.4 Hz when you slap an end with a wet sponge. You have PVC pipe with an inside diameter of 1.5 in. How long should you cut the pipe? Remember the end correction for the (only one) open end.

*Exercise 13, Slide whistle*

The slide whistle is made from a cylindrical tube that is closed at one end by a stopper. The stopper can be retracted to make the tube longer. When the stopper is fully retracted, the inside length is 287 mm. The rod that retracts the stopper has a total travel distance of 232 mm. (a) Show that the lower and upper frequencies are about 300 and 1,560 Hz. (b) The manufacturer claims that the range of the whistle is more than two octaves. Do you agree?

