Chapter 3
Vibrations 2

Chapter 2 introduced the concept of simple harmonic motion. This concept allowed us to define the terms used to discuss vibrating systems—terms like amplitude and frequency. Chapter 3 extends this discussion to additional properties of real vibrating systems. It ends with the topics of spectrum and resonance, essential concepts in any form of physics, especially acoustics.

3.1 Damping

Ideal simple harmonic motion goes on forever. It is a kind of perpetual motion machine with an amplitude that never changes. It just keeps on going and going and going. Real, passive mechanical systems, like a mass suspended from a spring, are not like this. The vibrations (or oscillations) of a real, free mechanical system are damped by frictional forces, including air resistance. The amplitude of such systems is not constant but decreases gradually with time. Figure 3.1 shows motion that would be simple harmonic (sine wave) motion except that it is damped.

Because of damping in real systems, our theoretical picture of simple harmonic motion is an idealization. Idealizations are common in science. They help us deal with certain truths about the world without the encumbrances of a lot of messy real-world details. The ideal simple harmonic motion is a useful idealization because many systems vibrate for many cycles before decaying appreciably. Also, it is often possible to compensate for the damping (or decay) of vibrations by adding energy to the system from outside. Then the system is *active*, not passive or freely vibrating.

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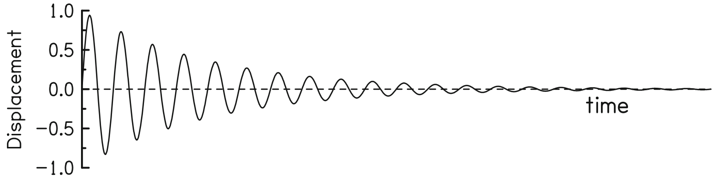


Fig. 3.1 Rapidly damped simple harmonic motion—like clinking glasses

3.2 Natural Modes of Vibration

The concept of “modes of vibration” is important in mechanics and acoustics. You already know something about it. The spring and mass system, with the mass confined to move in one dimension, is a system that has one mode of vibration. A mode is described by its properties:

1. *Frequency*: Most important, a mode has a specific frequency. The spring and mass system has a frequency given by the following formula:

$$f = \frac{1}{2\pi}\sqrt{\frac{s}{m}} \tag{3.1}$$

Here s is the stiffness of the spring, measured in units of Newtons per meter. A Newton is the unit of force in the metric system of units. It is equivalent to about 1/4 pound in the English system of units. Quantity m is the mass measured in kilograms. The prefactor $\frac{1}{2\pi}$ can be calculated from the fact that π is approximately equal to 3.14159. Thus, the prefactor is the number 0.159. Insight into Eq. (3.1) appears in Appendix F.

2. *Shape*: A mode of vibration has a shape. In the case of a mass hanging from a spring, the mode shape is simply the up and down motion of the spring. That is pretty obvious. In the future we will encounter musical systems with modal shapes that are not so obvious.
3. *Amplitude*: If we put a lot of energy into a mode of vibration, its amplitude will be large. But the amplitude is not a “property” of a mode. It only says how much action there is in the mode.

A mode is rather like your bank account. You have an account, and it exists. It has your name on it, and it has a number, just like the frequency of a mode. You can put a lot of money into your bank account or you can take money out of it. Whether the “amplitude” of money in your account is large or small, it is still your account. That is its property.

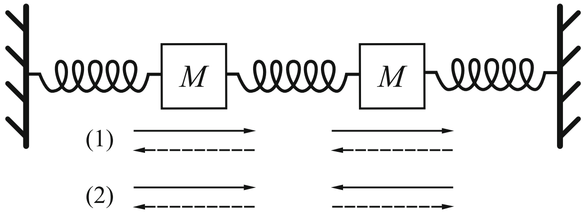


Fig. 3.2 A spring and mass system with two modes of vibration. Motion of the masses in (1) low- and (2) high-frequency modes are shown by *arrows*

3.3 Multimode Systems

The one-dimensional spring and mass system has one mode of vibration. Most physical systems, including musical systems, have more than one mode. Using springs and masses we can construct a one-dimensional system with two modes of vibration. Figure 3.2 shows such a system. By saying that the system is one-dimensional, we mean that the masses are required to move along a straight line path.

A system of two masses and two springs has two modes of vibration, which means that there are two separate and distinct natural frequencies for this system, one for each mode. The shapes of the two modes are quite different. In the mode with the lower frequency (1), the mode shape has the two masses moving together in lock step. That means that the spring in the middle is not stretched or compressed. In the mode with the higher frequency (2), the two masses move contrary to one another. When the first mass moves to the left, the second mass moves to the right and vice versa. Those mode shapes are shown by the arrows in Fig. 3.2.

Possibly you are now doing a bit of *inductive reasoning* and thinking that if a spring and mass system with one mass has one mode, and a system with two masses has two modes, then a system with three masses would have three modes. If this is what you are thinking, you would be right. And so on it goes with four and five masses, etc.

Systems of springs and masses are discrete systems. Most familiar vibrating objects, like drumheads, guitar strings, and window panes, are continuous systems. Such systems have many modes of vibration but sometimes only a few of them are important. It is often possible to identify the *effective springs* and the *effective masses* of such continuous system. The tuning fork provides an example.

3.4 The Tuning Fork

The tuning fork is a vibrating system with several modes of vibration, but one is much more important than the others.

1. *Main mode*: The main mode of a tuning fork has a frequency that is stamped on the fork so that the fork becomes a reference for that frequency. It makes

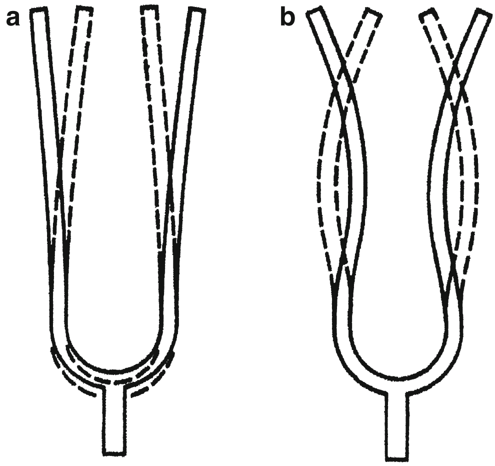


Fig. 3.3 (**a**) The main mode of a tuning fork. (**b**) The first clang mode

a good reference because it is extremely stable. We actually know how Mozart had his piano tuned because we have his tuning fork. We are rather sure that the frequency of the main mode of this tuning fork has not changed since Mozart's time (1756–1791). A few hundred years should not make any difference to a tuning fork.

2. *Other modes*: There are other modes of the tuning fork. Their frequencies are higher than the main mode. When a tuning fork is struck, these frequencies can also be heard. One of them, the "first clang mode," is particularly evident.
3. *Mode addition*: When a physical system vibrates in several modes at once, the resulting vibration is just the sum of the individual modes. Physicists sometimes call this addition property "superposition." The result of the adding is that the main mode and the clang modes of the tuning fork coexist, and a listener hears all the modes.

Example:

Suppose we have a tuning fork with a main mode having a frequency of 256 Hz and with a first clang mode having a frequency of 1,997 Hz. Suppose further that we observe the fork when the amplitude of the main mode is 1 mm and the amplitude of the first clang mode is one-third of that or 0.33 mm. Algebraically, the vibration is given by the sum of two sine waves with the correct frequencies and amplitudes. It is

$$x(t) = 1.0 \sin(360 \cdot 256\, t) + 0.33 \sin(360 \cdot 1997\, t) \tag{3.2}$$

Figure 3.4 shows how this addition works. Part (a) shows vibration in the main mode with a frequency of 256 Hz and an amplitude of 1 mm. Part (b) shows vibration in the first clang mode with a frequency of 1,997 Hz and an amplitude of 0.33 mm. The sum of Parts (a) and (b) is given in Part (c). If we put a microphone a few centimeters away from the tuning fork we could capture that sum.

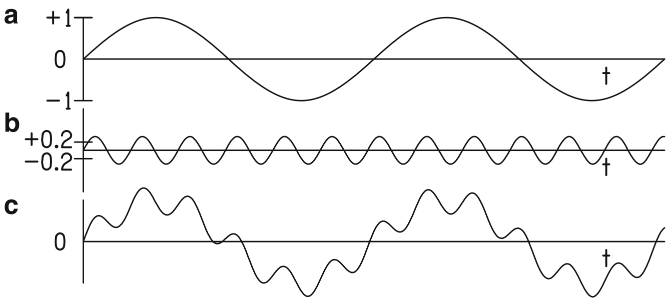


Fig. 3.4 Waveforms (**a**) and (**b**) are added together to make (**c**). The waveforms are functions of time, t

4. *Damping*: The clang modes do not interfere much with the use of the tuning fork because these modes are rapidly damped. Their amplitudes become small soon after the fork is struck. By contrast, the main mode lasts a long time—many minutes! One minute after the fork is struck, only the main mode can be heard. The fork effectively becomes a single mode system, just like a single spring and mass.
5. *Mode shapes*: The shapes of the main mode and the first clang mode appear in Fig. 3.3. They are drawn the way we always draw modes of vibration, with a solid line indicating the maximum displacement in one direction and a dashed line indicating the displacement half a period later in time, when the vibration has a maximum in the opposite direction.

3.4.1 Wave Addition Example

Adding two waves, as in Fig. 3.4, means that at every point in time the values of the two waves along the vertical axes are added together to get the sum. An example is given in Fig. 3.5.
Helpful rules for this kind of point-by-point addition are:

1. When one of the two waves is zero, the sum must be equal to the value of the other wave. This rule is illustrated by filled circles in the figure.
2. When both waves are positive the sum is even more positive, and when both waves are negative the sum is even more negative. These rules are illustrated by the square and triangle symbols.

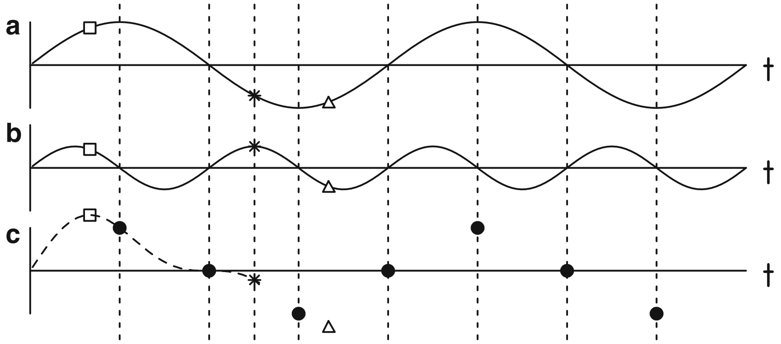


Fig. 3.5 Waves (**a**) and (**b**) are added together to get (**c**). Most of (**c**) is missing—a creative opportunity for the reader

3. When the waves are of opposite sign, the sum tends to be small. This rule is illustrated by the star symbols.
4. To eyeball an accurate plot one can draw a series of vertical lines (fixed points in time) and plot a series of dots that show the sum of the waves. Then by connecting the dots one gets the summed wave, as shown by the dashed line in part (c) of the figure.

3.5 The Spectrum

The waveform representation of a vibration (shown in Fig. 3.4c) is a function of time. An alternative representation is the spectral representation. It describes the vibration as a function of frequency. The amplitude spectrum shows the amplitude of each mode plotted against the frequency of the mode.

Remember that the vibration shown in Fig. 3.4c was the sum of a 256 Hz sine vibration with an amplitude of 1 mm and a 1,997 Hz sine vibration with an amplitude of 0.33 mm. The amplitude spectrum then looks like Fig. 3.6.

Notice how the heights of the waveforms in parts (a) and (b) of Fig. 3.4 directly translate into the representations of amplitudes in Fig. 3.6.

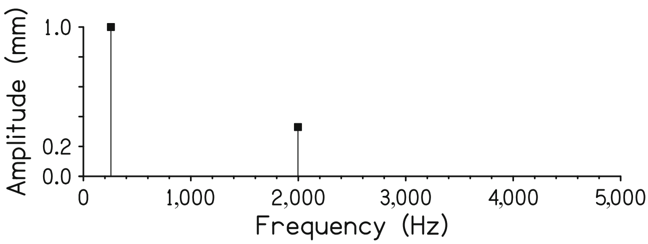


Fig. 3.6 The spectrum of the tuning fork has two lines, one for each mode

3.6 Resonance

The word “resonance” is often used (and misused!) in popular acoustical discussions. It is a word that has a technical meaning, and we shall deal with that technical definition right now.

The concept of resonance requires that there be two physical systems, a *driven* system and a *driving* system. As you can imagine, the driving system feeds energy into the driven system. For example, for the acoustic guitar, the driving system is a vibrating string, and the driven system is the guitar body. The body, with its top plate and sound hole, is responsible for radiating the sound. The guitar body vibrates because energy is fed into it from the string.

The concept of resonance requires that the driven system have some modes of vibration. Therefore, there are special frequencies where this driven system naturally vibrates if once started and left alone. Resonance occurs when the driving system attempts to drive the driven system at one of these special frequencies. When the driven system is driven at a frequency that it likes, the driven system responds enthusiastically and the amplitude of the resulting vibration can become huge.

3.6.1 A Wet Example

Imagine a bathtub that is half full of water. You are kneeling next to the tub and your goal is to slop a lot of water out of the tub and onto the bathroom floor. (The rules of this game are that you can only put one hand into the tub.) You could splash around randomly in the tub and get some water out, but if you really want to slop a *lot of water* you would use the concept of resonance.

A little experimenting would quickly show you that there is a mode of vibration of water in the tub, sloshing from one end to the other. This mode has a particular natural frequency. Your best slopping strategy is to use the palm of your hand to drive this mode of vibration—moving your hand back and forth (sine-wave motion) in the tub at this frequency. It would not take you long to learn what the natural frequency is and to move your hand at the correct frequency to match the natural frequency. You would see a resonant behavior develop as the amplitude of the water motion became larger and larger. Moving your hand at any other frequency would be less effective. Very soon you and the floor would be all wet.

3.6.2 Breaking Glassware

A popular motif in comedy films has a soprano singer, or other source of intense sound, breaking all the glassware in the room. Even bottles of gin miraculously

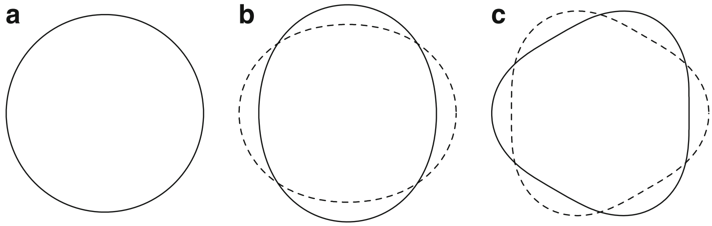


Fig. 3.7 (**a**) A quiet goblet, top view, has no vibration. (**b**) The main mode of vibration has four nodes. (**c**) The second mode has six nodes. The amplitudes are greatly exaggerated in the figure

burst when exposed to this intense sound. The fact is, it isn't that easy to break things with sound. However, it can be done. We can break a crystal goblet.

A crystal goblet is a continuous system with many modes of vibration, but it has a main mode of vibration that is particularly important. This is the mode that is excited if you ding the goblet with your fingernail. It is the mode that is continuously excited if you run a wet finger around the rim. The shape of the mode looks like Fig. 3.7b.

As you can tell from dinging the goblet, this mode of vibration takes a long time to decay away. A long decay time indicates that the resonance is "sharp," meaning that this mode of vibration can be excited by a sine-wave driver only if the frequency of the driver is very close to the natural frequency of the mode. In order to break the goblet, we cause it to have a very large amplitude in this mode of vibration by driving it with a loudspeaker at precisely the natural frequency of this mode. In other words, we use the phenomenon of resonance. We say that we are driving the goblet at its resonance frequency. The goblet is so happy to vibrate at this frequency that it responds with a large amplitude and finally vibrates itself to death.

3.6.3 Sympathetic Strings

Several Asian musical instruments employ the principle of resonance by adding *sympathetic strings* to the instrument, in addition to the main strings that are actually played. The sympathetic strings are tuned to frequencies that will occur in the piece of music to be played. The term "sympathetic" is apt because these strings might be said to vibrate "in sympathy" with the tones played on the main strings. The best known of these instruments is the Indian **sitar** (Fig. 3.8) a fretted, plucked string instrument which has about 20 strings, only six or seven of which are played. The **sarod** is another Indian instrument without frets, and having four or five melody strings, two drone strings, and about ten sympathetic, resonating strings. The **rubab** is a similar instrument from Afghanistan with three melody strings, three drones, and about ten sympathetic strings.

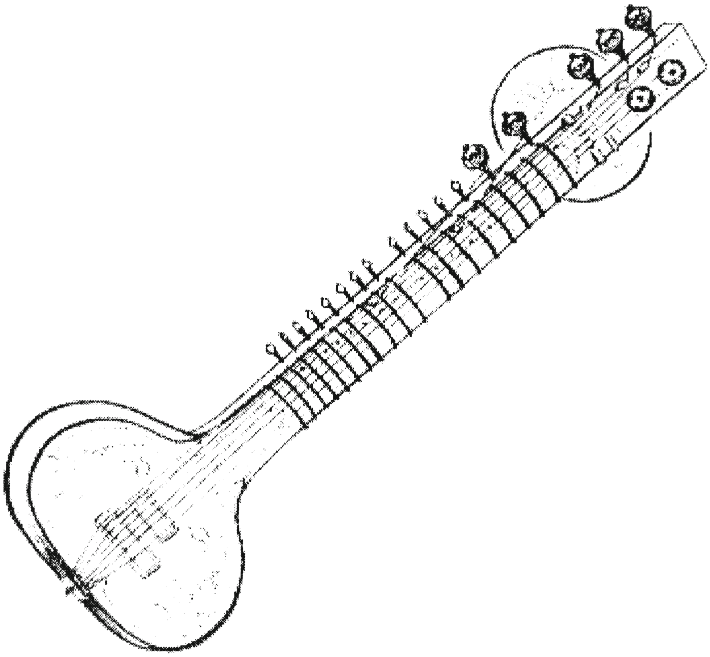


Fig. 3.8 The sitar. This instrument appears to have 18 strings

An alternative description of sympathetic strings would be to call them "resonators." A *resonator* is always a driven system, specifically designed to enhance the vibration or sound of the driving system.

Exercises

Exercise 1, Natural vibes

Find the natural frequency of a spring and mass system where the spring has a stiffness of 100 N/m and the mass is 250 g (0.25 kg).

Exercise 2, Filing the fork

Think about the main mode of the tuning fork. What part of the tuning fork acts as the mass? What part acts as the spring? What happens to the frequency of the tuning fork if you file away metal from the ends of the tines? What happens to the frequency if you file away metal near the junction of the two tines?

The fact that we can think about a continuous system like a tuning fork in terms of a single mass and spring shows the conceptual power of modal analysis. But which part of the system plays the role of the mass and which part plays the role of the spring depend on the shape of the mode we are considering.

Exercise 3, Lifetime

Look at the damped vibrations in Fig. 3.1. If the frequency of vibration is 200 Hz, how long does it take for the wave to die away? This exercise requires you to find your own definition for "die away." How much decrease is needed in order to say that the vibration has disappeared?

Exercise 4, A simple spectrum

Draw the spectrum for a 1,000-Hz sine tone.

Exercise 5, More complex spectra

Draw the spectra for the following waves:

(a) $x(t) = 1.3\sin(360 \cdot 575\,t) + 0.8\sin(360 \cdot 1200\,t)$
(b) $x(t) = 0.5\sin(360 \cdot 100\,t) + 2.2\sin(360 \cdot 780\,t) + 0.8\sin(360 \cdot 1650\,t)$

Exercise 6, Resonance—an anthropomorphism

If your friend makes a statement and you agree with the statement, it might be said that the two of you are in resonance. Explain how this metaphor is an extension of the concept of resonance for physical systems as explained in this chapter.

Exercise 7, Playground Resonance

(a) Explain how the idea of resonance is used to great effect when pushing a child in a swing.
(b) Is the swing example an exact example of resonance?

Exercise 8, Square root dependence

According to Eq. (3.1), the frequency of a spring and mass system depends on the ratio of spring stiffness to mass. However, it does not depend on this ratio in a linear way. It is a square-root dependence.

(a) What happens to the frequency if you change the system by doubling the stiffness and doubling the mass, both at once?

(b) What happens to the frequency if you double the mass?

Exercise 9, Adding waveforms

Figure 3.9 shows two waveforms, (a) and (b). They show displacement as a function of time for two simultaneous modes of vibration. (A) Add these waveforms—point by point—to find the resulting vibration pattern. (B) If the frequency of the wave in part (a) is 100 Hz, what is the frequency of the wave in part (b)?

Exercise 10, Resonance?

Which of the following are examples of resonance?

(a) You strike a Tibetan singing bowl and it vibrates in several of its modes.
(b) You run a wet finger around the rim of a crystal goblet and it vibrates in its main mode.
(c) You strike a tuning fork and it ultimately vibrates only in its main mode.
(d) You bow a violin string and it vibrates at its natural frequency.

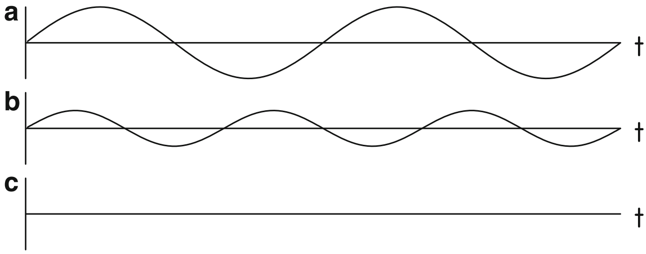
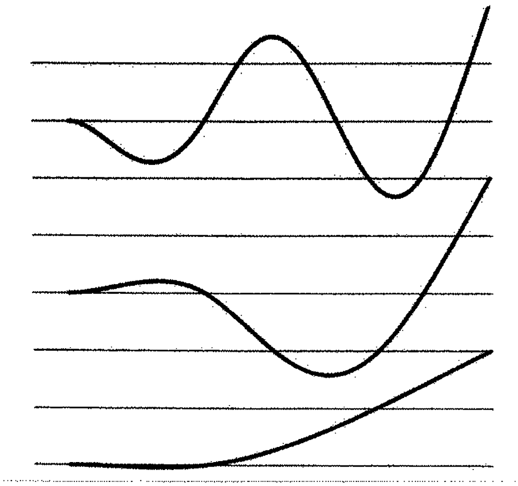


Fig. 3.9 Waveforms to be added for Exercise 9

Fig. 3.10 Modes of a kalimba tine for Exercise 11. The tine is fixed in the bridge at the left side



Exercise 11, Kalimba tine

The kalimba is a westernized version of the African mbira or "thumb piano." It consists of metal tines mounted on a wooden box. The performer uses his thumbs to pluck the tines and the box acts as a resonator. Figure 3.10 shows the first three modes of vibration of a kalimba tine, as calculated by David Chapman and published in 2012 in the Journal of the Acoustical Society of America. Which mode do you think has the highest frequency? Which has the lowest?

Exercise 12, Kalimba box

By itself, the kalimba tine does not radiate sound effectively. The kalimba box solves that problem. The box is hollow and has a sound hole like a guitar. The kalimba box vibrates, as forced by the tines, and its large surface area and sound hole become effective radiators of the sound. The box also has resonances because of the modes of vibration of air inside the box. Based on your experience with hollow objects, do you think that the resonance frequencies become higher or lower if the box is made bigger and its volume increases?

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