

The subject of complex analysis is an extremely rich and powerful area of mathematics. We have already seen some of this richness and power in the previous chapter. This chapter concludes our discussion of complex analysis by introducing some other topics with varying degrees of importance.

12.1 Meromorphic Functions

Complex functions that have only simple poles as their singularities are numerous in applications and are called **meromorphic functions**. In this section, we derive an important result for such functions.

Suppose that $f(z)$ has simple poles at $\{z_j\}_{j=1}^N$, where N could be infinity. Then, assuming that $z \neq z_j$ for all j , and noting that the residue of $f(\xi)/(\xi - z)$ at $\xi = z$ is simply $f(z)$, the residue theorem yields

$$\frac{1}{2\pi i} \int_{C_n} \frac{f(\xi)}{\xi - z} d\xi = f(z) + \sum_{j=1}^n \text{Res} \left(\frac{f(\xi)}{\xi - z} \right)_{\xi=z_j},$$

where C_n is a circle containing the first n poles, and it is assumed that the poles are arranged in order of increasing absolute values. Since the poles of f are assumed to be simple, we have

$$\begin{aligned} \text{Res} \left(\frac{f(\xi)}{\xi - z} \right)_{\xi=z_j} &= \lim_{\xi \rightarrow z_j} (\xi - z_j) \frac{f(\xi)}{\xi - z} = \frac{1}{z_j - z} \lim_{\xi \rightarrow z_j} [(\xi - z_j) f(\xi)] \\ &= \frac{1}{z_j - z} \text{Res}[f(\xi)]_{\xi=z_j} \equiv \frac{r_j}{z_j - z}, \end{aligned}$$

where r_j is, by definition, the residue of $f(\xi)$ at $\xi = z_j$. Substituting in the preceding equation gives

$$f(z) = \frac{1}{2\pi i} \int_{C_n} \frac{f(\xi)}{\xi - z} d\xi - \sum_{j=1}^n \frac{r_j}{z_j - z}.$$

Taking the difference between this and the same equation evaluated at $z = 0$ (assumed to be none of the poles),¹ we can write

$$f(z) - f(0) = \frac{z}{2\pi i} \int_{C_n} \frac{f(\xi)}{\xi(\xi - z)} d\xi + \sum_{j=1}^n r_j \left(\frac{1}{z - z_j} + \frac{1}{z_j} \right).$$

If $|f(\xi)|$ approaches a finite value as $|\xi| \rightarrow \infty$, the integral vanishes for an infinite circle (which includes all poles now), and we obtain what is called the **Mittag-Leffler expansion** of the meromorphic function f :

$$f(z) = f(0) + \sum_{j=1}^N r_j \left(\frac{1}{z - z_j} + \frac{1}{z_j} \right). \quad (12.1)$$

Now we let g be an entire function with *simple zeros*. We claim that (a) $(dg/dz)/g(z)$ is a meromorphic function that is bounded for all values of z , and (b) its residues are all unity. To see this, note that g is of the form²

$$g(z) = (z - z_1)(z - z_2) \cdots (z - z_N) f(z),$$

where z_1, \dots, z_N are all the zeros of g , and f is an analytic function that does not vanish anywhere in the complex plane. It is now easy to see that

$$\frac{g'(z)}{g(z)} = \sum_{j=1}^N \frac{1}{z - z_j} + \frac{f'(z)}{f(z)}.$$

This expression has both properties (a) and (b) mentioned above. Furthermore, the last term is an entire function that is bounded for all \mathbb{C} . Therefore, it must be a constant by Proposition 10.5.5. This derivation also verifies Eq. (12.1), which in the case at hand can be written as

$$\frac{d}{dz} \ln g(z) = \frac{g'(z)}{g(z)} = \frac{d}{dz} \ln g(0) + \sum_{j=1}^N \left(\frac{1}{z - z_j} + \frac{1}{z_j} \right),$$

whose solution is readily found and is given in the following

Proposition 12.1.1 *If g is an entire function with simple zeros $\{z_j\}_{j=1}^N$, then*

$$g(z) = g(0)e^{cz} \prod_{j=1}^N \left(1 - \frac{z}{z_j} \right) e^{z/z_j} \quad \text{where} \quad c = \frac{(dg/dz)|_{z=0}}{g(0)} \quad (12.2)$$

and it is assumed that $z_j \neq 0$ for all j .

¹This is not a restrictive assumption because we can always move our coordinate system so that the origin avoids all poles.

²One can “prove” this by factoring the simple zeros one by one, writing $g(z) = (z - z_1)f_1(z)$ and noting that $g(z_2) = 0$, with $z_2 \neq z_1$, implies that $f_1(z) = (z - z_2)f_2(z)$, etc.

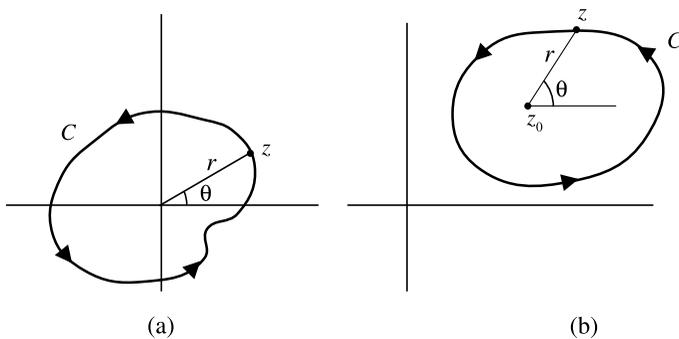


Fig. 12.1 (a) The origin is a branch point of the natural log function. (b) z_0 is a branch point of $f(z)$ if $f(z_0 + re^{i\theta}) \neq f(z_0 + re^{i(\theta+2\pi)})$

12.2 Multivalued Functions

The arbitrariness, up to a multiple of 2π , of the angle $\theta = \arg(z)$ in $z = re^{i\theta}$ leads to functions that can take different values at the same point. Consider, for example, the function $f(z) = \sqrt{z}$. Writing z in polar coordinates, we obtain $f(z) = f(r, \theta) = (re^{i\theta})^{1/2} = \sqrt{r}e^{i\theta/2}$. This shows that for the same $z = (r, \theta) = (r, \theta + 2\pi)$, we get two different values, $f(r, \theta)$ and $f(r, \theta + 2\pi) = -f(r, \theta)$.

This may be disturbing at first. After all, the definition of a function (mapping) ensures that for any point in the domain a *unique* image is obtained. Here two different images are obtained for the same z . Riemann found a cure for this complex “double vision” by introducing what is now called Riemann sheets. We will discuss these briefly below, but first let us take a closer look at a prototype of multivalued functions. Consider the natural log function, $\ln z$. For $z = re^{i\theta}$ this is defined as $\ln z = \ln r + i\theta = \ln |z| + i \arg(z)$ where $\arg(z)$ is defined only to within a multiple of 2π ; that is, $\arg(z) = \theta + 2n\pi$, for $n = 0, \pm 1, \pm 2, \dots$

We can see the peculiar nature of the logarithmic function by considering a closed curve around the origin, as shown in Fig. 12.1(a). Starting at an arbitrary point z on the curve, we move counterclockwise, noticing the constant increase in the angle θ , until we reach the initial point. Now, the angle is $\theta + 2\pi$. Thus, the process of moving around the origin has changed the value of the log function by $2\pi i$, i.e., $(\ln z)_{\text{final}} - (\ln z)_{\text{initial}} = 2\pi i$. Note that in this process z does not change, because

$$z_{\text{final}} = re^{i(\theta+2\pi)} = re^{i\theta} e^{2\pi i} = re^{i\theta} = z_{\text{initial}}.$$

Definition 12.2.1 A **branch point** of a function $f : \mathbb{C} \rightarrow \mathbb{C}$ is a complex number z_0 with the property that for any (small enough) closed curve C encircling z_0 and for any point $z \equiv z_0 + re^{i\theta}$ on the curve, $f(z_0 + re^{i\theta}) \neq f(z_0 + re^{i(\theta+2\pi)})$.

Historical Notes

Victor-Alexandre Puiseux (1820–1883) was the first to take up the subject of multivalued functions. In 1850 Puiseux published a celebrated paper on complex algebraic functions given by $f(u, z) = 0$, f a polynomial in u and z . He first made clear the distinction

between **poles** and **branch points** that Cauchy had barely perceived, and introduced the notion of an **essential singular point**, to which Weierstrass independently had called attention. Though Cauchy, in the 1846 paper, did consider the variation of simple multivalued functions along paths that enclosed branch points, Puiseux clarified this subject too.

Puiseux also showed that the development of a function of z about a branch point $z = a$ must involve *fractional powers* of $z - a$. He then improved on Cauchy’s theorem on the expansion of a function in a Maclaurin series. By his significant investigations of many-valued functions and their branch points in the complex plane, and by his initial work on integrals of such functions, Puiseux brought Cauchy’s pioneering work in function theory to the end of what might be called the first stage. The difficulties in the theory of multiple-valued functions and integrals of such functions were still to be overcome. Cauchy did write other papers on the integrals of multivalued functions in which he attempted to follow up on Puiseux’s work; and though he introduced the notion of branch cuts (*lignes d’arrêt*), he was still confused about the distinction between poles and branch points. This subject of algebraic functions and their integrals was to be pursued by Riemann. Puiseux was a keen mountaineer and was the first to scale the Alpine peak that is now named after him.

Thus, $z = 0$ is a branch point of the logarithmic function. Studying the behavior of $\ln(1/z) = -\ln z$ around $z = 0$ will reveal that the point “at infinity” is also a branch point of $\ln z$. If $z_0 \neq 0$ is any other point of the complex plane, then choosing C to be a small loop, we get

$$\begin{aligned} \ln(z_0 + re^{i\phi}) &= \ln\left[z_0\left(1 + \frac{re^{i\phi}}{z_0}\right)\right] = \ln z_0 + \ln\left(1 + \frac{re^{i\phi}}{z_0}\right) \\ &\approx \ln z_0 + \frac{re^{i\phi}}{z_0} \quad \text{for } r \ll |z_0|. \end{aligned}$$

It is now clear that $\ln(z_0 + re^{i\theta}) = \ln(z_0 + re^{i(\theta+2\pi)})$. We therefore conclude that any point of the complex plane other than the origin cannot be a branch point of the natural log function.

12.2.1 Riemann Surfaces

The idea of a Riemann surface begins with the removal of all points that lie on the line (or any other curve) joining two branch points. For $\ln z$ this means the removal of all points lying on a curve that starts at $z = 0$ and extends all the way to infinity. Such a curve is called a **branch cut**, or simply a *cut*.

branch cut or simply
“cut”

Let us concentrate on $\ln z$ and take the cut to be along the negative half of the real axis. Let us also define the functions

$$\begin{aligned} f_n(z) &= f_n(r, \theta) \\ &= \ln r + i(\theta + 2n\pi) \quad \text{for } -\pi < \theta < \pi; r > 0; n = 0, \pm 1, \dots, \end{aligned}$$

so $f_n(z)$ takes on the same values for $-\pi < \theta < \pi$ that $\ln z$ takes in the range $(2n - 1)\pi < \theta < (2n + 1)\pi$. We have replaced the multivalued logarithmic function by a series of different functions that are analytic in the cut z -plane.

This process of cutting the z -plane and then defining a sequence of functions eliminates the contradiction caused by the existence of branch points, since we are no longer allowed to completely encircle a branch point.

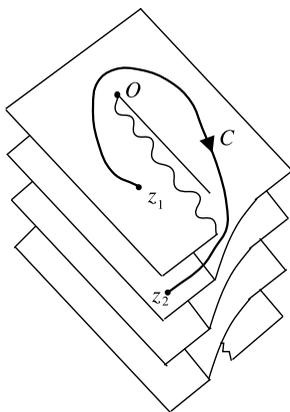


Fig. 12.2 A few sheets of the Riemann surface of the logarithmic function. The path C encircling the origin O ends up on the lower sheet

A complete circulation involves crossing the cut, which, in turn, violates the domain of definition of $f_n(z)$.

We have made good progress. We have replaced the (nonanalytic) multivalued function $\ln z$ with a series of analytic (in their domain of definition) functions $f_n(z)$. However, there is a problem left: $f_n(z)$ has a discontinuity at the cut. In fact, just above the cut $f_n(r, \pi - \epsilon) = \ln r + i(\pi - \epsilon + 2n\pi)$ with $\epsilon > 0$, and just below it $f_n(r, -\pi + \epsilon) = \ln r + i(-\pi + \epsilon + 2n\pi)$, so that

$$\lim_{\epsilon \rightarrow 0} [f_n(r, \pi - \epsilon) - f_n(r, -\pi + \epsilon)] = 2\pi i.$$

To cure this we make the observation that the value of $f_n(z)$ just above the cut is the same as the value of $f_{n+1}(z)$ just below the cut. This suggests the following geometrical construction, due to Riemann: Superpose an infinite series of cut complex planes one on top of the other, each plane corresponding to a different value of n . The adjacent planes are connected along the cut such that the upper lip of the cut in the $(n - 1)$ th plane is connected to the lower lip of the cut in the n th plane. All planes contain the two branch points. That is, the branch points appear as “hinges” at which all the planes are joined. With this geometrical construction, if we cross the cut, we end up on a different plane adjacent to the previous one (Fig. 12.2).

The geometric surface thus constructed is called a **Riemann surface**; each plane is called a **Riemann sheet** and is denoted by R_j , for $j = 0, \pm 1, \pm 2, \dots$. A single-valued function defined on a Riemann sheet is called a **branch** of the original multivalued function.

Riemann surfaces and sheets

We have achieved the following: From a multivalued function we have constructed a sequence of single-valued functions, each defined in a single complex plane; from this sequence of functions we have constructed a single complex function defined on a single Riemann surface. Thus, the logarithmic function is analytic throughout the Riemann surface except at the branch points, which are simply the function’s singular points.

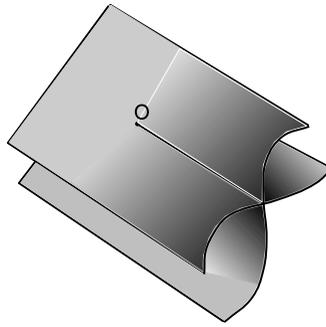


Fig. 12.3 The Riemann surface for $f(z) = z^{1/2}$

It is now easy to see the geometrical significance of branch points. A complete cycle around a branch point takes us to another Riemann sheet, where the function takes on a different form. On the other hand, a complete cycle around an ordinary point either never crosses the cut, or if it does, it will cross it back to the original sheet.

Let us now briefly consider two of the more common multivalued functions and their Riemann surfaces.

Example 12.2.2 (The function $f(z) = z^{1/n}$) The only branch points for the function $f(z) = z^{1/n}$ are $z = 0$ and the point at infinity. Defining $f_k(z) \equiv r^{1/n} e^{i(\theta + 2k\pi/n)}$ for $k = 0, 1, \dots, n-1$ and $0 < \theta < 2\pi$ and following the same procedure as for the logarithmic function, we see that there must be n Riemann sheets, labeled R_0, R_1, \dots, R_{n-1} , in the Riemann surface. The lower edge of R_{n-1} is pasted to the upper edge of R_0 along the cut, which is taken to be along the positive real axis. The Riemann surface for $n = 2$ is shown in Fig. 12.3.

It is clear that for any noninteger value of α the function $f(z) = z^\alpha$ has a branch point at $z = 0$ and another at the point at infinity. For irrational α the number of Riemann sheets is infinite.

Example 12.2.3 (The function $f(z) = (z^2 - 1)^{1/2}$) The branch points for the function $f(z) = (z^2 - 1)^{1/2}$ are at $z_1 = +1$ and $z_2 = -1$ (see Fig. 12.4). Writing $z - 1 = r_1 e^{i\theta_1}$ and $z + 1 = r_2 e^{i\theta_2}$, we have

$$f(z) = (r_1 e^{i\theta_1})^{1/2} (r_2 e^{i\theta_2})^{1/2} = \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2}.$$

The cut is along the real axis from $z = -1$ to $z = +1$. There are two Riemann sheets in the Riemann surface. Clearly, only cycles of 2π involving *one* branch point will cross the cut and therefore end up on a different sheet. Any closed curve that has both z_1 and z_2 as interior points will remain entirely on the original sheet.

evaluation of integrals
involving cuts

The notion of branch cuts can be used to evaluate certain integrals that do not fit into the three categories discussed in Chap. 11. The basic idea is to circumvent the cut by constructing a contour that is infinitesimally close to the cut and circles around branch points.

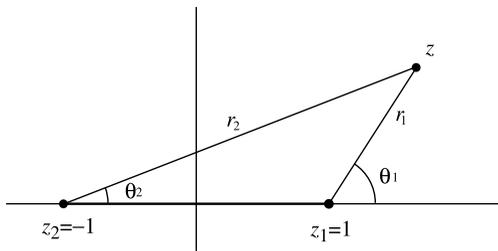


Fig. 12.4 The cut for the function $f(z) = (z^2 - 1)^{1/2}$ is from z_1 to z_2 . Paths that circle only one of the points cross the cut and end up on the other sheet

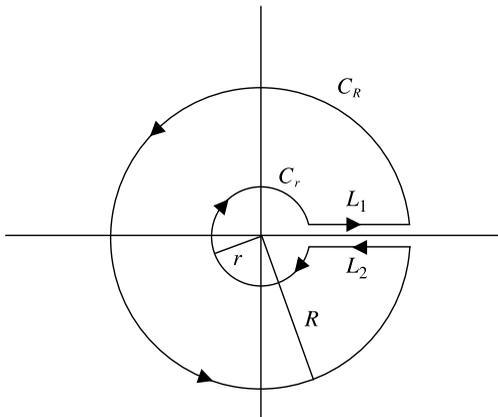


Fig. 12.5 The contour for the evaluation of the integrals of Examples 12.2.4 and 12.2.5

Example 12.2.4 To evaluate the integral $I = \int_0^\infty x^\alpha dx / (x^2 + 1)$ for $|\alpha| < 1$, consider the complex integral $I' = \oint_C z^\alpha dz / (z^2 + 1)$ where C is as shown in Fig. 12.5 and the cut is taken along the positive real axis. To evaluate the contribution from C_R and C_r , we let ρ stand for either r or R . Then we have

$$I_\rho = \int_{C_\rho} \frac{(\rho e^{i\theta})^\alpha}{(\rho e^{i\theta})^2 + 1} i\rho e^{i\theta} d\theta = i \int_0^{2\pi} \frac{\rho^{\alpha+1} e^{i(\alpha+1)\theta}}{\rho^2 e^{2i\theta} + 1} d\theta.$$

It is clear that since $|\alpha| < 1$, $I_\rho \rightarrow 0$ as $\rho \rightarrow 0$ or $\rho \rightarrow \infty$.

The contributions from L_1 and L_2 do not cancel one another because the value of the function changes above and below the cut. To evaluate these two integrals we have to choose a branch of the function. Let us choose that branch on which $z^\alpha = |z|^\alpha e^{i\alpha\theta}$ for $0 < \theta < 2\pi$. Along L_1 , $\theta \approx 0$ or $z^\alpha = x^\alpha$, and along L_2 , $\theta \approx 2\pi$ or $z^\alpha = (x e^{2\pi i})^\alpha$. Thus,

$$\begin{aligned} \oint_C \frac{z^\alpha}{z^2 + 1} dz &= \int_0^\infty \frac{x^\alpha}{x^2 + 1} dx + \int_\infty^0 \frac{x^\alpha e^{2\pi i\alpha}}{(x e^{2\pi i})^2 + 1} dx \\ &= (1 - e^{2\pi i\alpha}) \int_0^\infty \frac{x^\alpha}{x^2 + 1} dx. \end{aligned} \tag{12.3}$$

The LHS of this equation can be obtained using the residue theorem. There are two simple poles, at $z = +i$ and $z = -i$ with residues $\text{Res}[f(i)] = (e^{i\pi/2})^\alpha/2i$ and $\text{Res}[f(-i)] = -(e^{i3\pi/2})^\alpha/2i$. Thus,

$$\oint_C \frac{z^\alpha}{z^2+1} dz = 2\pi i \left(\frac{e^{i\alpha\pi/2}}{2i} - \frac{e^{i3\alpha\pi/2}}{2i} \right) = \pi (e^{i\alpha\pi/2} - e^{i3\alpha\pi/2}).$$

Combining this with Eq. (12.3), we obtain

$$\int_0^\infty \frac{x^\alpha}{x^2+1} dx = \frac{\pi(e^{i\alpha\pi/2} - e^{i3\alpha\pi/2})}{1 - e^{2i\pi\alpha}} = \frac{\pi}{2} \sec \frac{\alpha\pi}{2}.$$

If we had chosen a different branch of the function, both the LHS and the RHS of Eq. (12.3) would have been different, but the final result would still have been the same.

Example 12.2.5 Here is another integral involving a branch cut:

$$I = \int_0^\infty \frac{x^{-a}}{x+1} dx \quad \text{for } 0 < a < 1.$$

To evaluate this integral we use the zeroth branch of the function and the contour of the previous example (Fig. 12.5). Thus, writing $z = \rho e^{i\theta}$, we have

$$\begin{aligned} 2\pi i \text{Res}[f(-1)] &= \oint_C \frac{z^{-a}}{z+1} dz = \int_0^\infty \frac{\rho^{-a}}{\rho+1} d\rho + \oint_{C_R} \frac{z^{-a}}{z+1} dz \\ &\quad + \int_\infty^0 \frac{(\rho e^{2i\pi})^{-a}}{\rho e^{2i\pi} + 1} e^{2i\pi} d\rho + \oint_{C_r} \frac{z^{-a}}{z+1} dz. \end{aligned} \quad (12.4)$$

The contributions from both circles vanish by the same argument used in the previous example. On the other hand, $\text{Res}[f(-1)] = (-1)^{-a}$. For the branch we are using, $-1 = e^{i\pi}$. Thus, $\text{Res}[f(-1)] = e^{-ia\pi}$. The RHS of Eq. (12.4) yields

$$\int_0^\infty \frac{\rho^{-a}}{\rho+1} d\rho - e^{-2i\pi a} \int_0^\infty \frac{\rho^{-a}}{\rho+1} d\rho = (1 - e^{-2i\pi a})I.$$

It follows from (12.4) that $(1 - e^{-2i\pi a})I = 2\pi i e^{-i\pi a}$, or

$$\int_0^\infty \frac{x^{-a}}{x+1} dx = \frac{\pi}{\sin a\pi} \quad \text{for } 0 < a < 1.$$

Example 12.2.6 Let us evaluate $I = \int_0^\infty \ln x dx / (x^2 + a^2)$ with $a > 0$. We choose the zeroth branch of the logarithmic function, in which $-\pi < \theta < \pi$, and use the contour of Fig. 12.6.

For L_1 , $z = \rho e^{i\pi}$ (note that $\rho > 0$), and for L_2 , $z = \rho$. Thus, we have

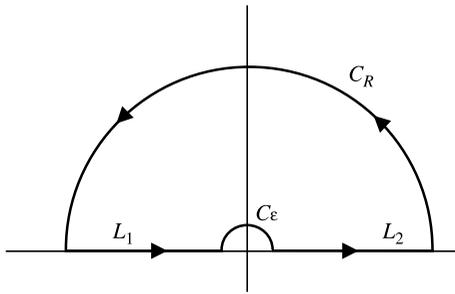


Fig. 12.6 The contour for the evaluation of the integral of Example 12.2.6

$$\begin{aligned}
 2\pi i \operatorname{Res}[f(ia)] &= \oint_C \frac{\ln z}{z^2 + a^2} dz \\
 &= \int_{-\infty}^{\epsilon} \frac{\ln(\rho e^{i\pi})}{(\rho e^{i\pi})^2 + a^2} e^{i\pi} d\rho + \int_{C_\epsilon} \frac{\ln z}{z^2 + a^2} dz \\
 &\quad + \int_{\epsilon}^{\infty} \frac{\ln \rho}{\rho^2 + a^2} d\rho + \int_{C_R} \frac{\ln z}{z^2 + a^2} dz, \quad (12.5)
 \end{aligned}$$

where $z = ia$ is the only singularity—a simple pole—in the UHP. Now we note that

$$\begin{aligned}
 \int_{-\infty}^{\epsilon} \frac{\ln(\rho e^{i\pi})}{(\rho e^{i\pi})^2 + a^2} e^{i\pi} d\rho &= \int_{\epsilon}^{\infty} \frac{\ln \rho + i\pi}{\rho^2 + a^2} d\rho \\
 &= \int_{\epsilon}^{\infty} \frac{\ln \rho}{\rho^2 + a^2} d\rho + i\pi \int_{\epsilon}^{\infty} \frac{d\rho}{\rho^2 + a^2}.
 \end{aligned}$$

The contributions from the circles tend to zero. On the other hand,

$$\operatorname{Res}[f(ia)] = \lim_{z \rightarrow ia} (z - ia) \frac{\ln z}{(z - ia)(z + ia)} = \frac{\ln(ia)}{2ia} = \frac{1}{2ia} \left(\ln a + i \frac{\pi}{2} \right).$$

Substituting the last two results in Eq. (12.5), we obtain

$$\frac{\pi}{a} \left(\ln a + i \frac{\pi}{2} \right) = 2 \int_{\epsilon}^{\infty} \frac{\ln \rho}{\rho^2 + a^2} d\rho + i\pi \int_{\epsilon}^{\infty} \frac{d\rho}{\rho^2 + a^2}.$$

It can also easily be shown that $\int_0^{\infty} d\rho/(\rho^2 + a^2) = \pi/(2a)$. Thus, in the limit $\epsilon \rightarrow 0$, we get $I = \frac{\pi}{2a} \ln a$. The sign of a is irrelevant because it appears as a square in the integral. Thus, we can write

$$\int_0^{\infty} \frac{\ln x}{x^2 + a^2} dx = \frac{\pi}{2|a|} \ln |a|, \quad a \neq 0.$$

12.3 Analytic Continuation

Analytic functions have certain unique properties, some of which we have already noted. For instance, the Cauchy integral formula gives the value of an analytic function inside a simple closed contour once its value on the contour is known. We have also seen that we can deform the contours of integration as long as we do not encounter any singularities of the function.

Combining these two properties and assuming that $f : \mathbb{C} \rightarrow \mathbb{C}$ is analytic within a region $S \subset \mathbb{C}$, we can ask the following question: Is it possible to extend f beyond S ? We shall see in this section that the answer is yes in many cases of interest.³ First consider the following:

equal on a piece, equal
all over

Theorem 12.3.1 *Let $f_1, f_2 : \mathbb{C} \rightarrow \mathbb{C}$ be analytic in a region S . If $f_1 = f_2$ in a neighborhood of a point $z \in S$, or for a segment of a curve in S , then $f_1 = f_2$ for all $z \in S$.*

Proof Let $g = f_1 - f_2$, and $U = \{z \in S \mid g(z) = 0\}$. Then U is a subset of S that includes the neighborhood of z (or the line segment) in which $f_1 = f_2$. If U is the entire region S , we are done. Otherwise, U has a boundary beyond which $g(z) \neq 0$. Since all points within the boundary satisfy $g(z) = 0$, and since g is continuous (more than that, it is analytic) on S , g must vanish also *on* the boundary. But the boundary points are not isolated: Any small circle around any one of them includes points of U as well as points outside U . Thus, g must vanish on a neighborhood of any boundary point, implying that g vanishes for some points outside U . This contradicts our assumption. Thus, U must include the entire region S . \square

A consequence of this theorem is the following corollary.

Corollary 12.3.2 *The behavior of a function that is analytic in a region $S \subset \mathbb{C}$ is completely determined by its behavior in a (small) neighborhood of an arbitrary point in that region.*

This process of determining the behavior of an analytic function outside the region in which it was originally defined is called **analytic continuation**. Although there are infinitely many ways of analytically continuing beyond regions of definition, the values of all functions obtained as a result of diverse continuations are the same at any given point. This follows from Theorem 12.3.1.

Let $f_1, f_2 : \mathbb{C} \rightarrow \mathbb{C}$ be analytic in regions S_1 and S_2 , respectively. Suppose that f_1 and f_2 have different functional forms in their respective regions of analyticity. If there is an overlap between S_1 and S_2 and if $f_1 = f_2$ within that overlap, then the (unique) analytic continuation of f_1 into S_2 must be f_2 , and vice versa. In fact, we may regard f_1 and f_2 as a single

³Provided that S is not discrete (countable). (See [Lang 85, p. 91].)

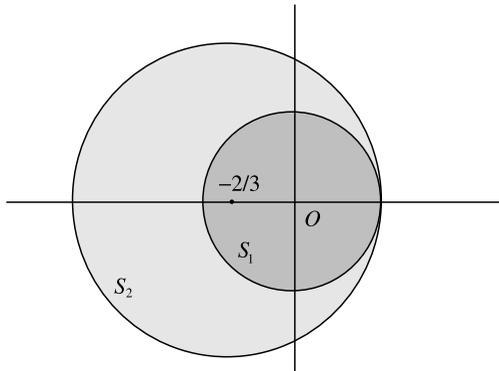


Fig. 12.7 The function defined in the smaller circle is continued analytically into the larger circle

function $f : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$f(z) = \begin{cases} f_1(z) & \text{when } z \in S_1, \\ f_2(z) & \text{when } z \in S_2. \end{cases}$$

Clearly, f is analytic for the combined region $S = S_1 \cup S_2$. We then say that f_1 and f_2 are analytic continuations of one another.

Example 12.3.3 Consider the function $f_1(z) = \sum_{n=0}^{\infty} z^n$, which is analytic for $|z| < 1$. We have seen that it converges to $1/(1-z)$ for $|z| < 1$. Thus, we have $f_1(z) = 1/(1-z)$ when $|z| < 1$, and f_1 is not defined for $|z| > 1$.

Now let us consider a second function,

$$f_2(z) = \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^{n+1} \left(z + \frac{2}{3}\right)^n,$$

which converges for $|z + \frac{2}{3}| < \frac{5}{3}$. To see what it converges to, we note that

$$f_2(z) = \frac{3}{5} \sum_{n=0}^{\infty} \left[\frac{3}{5} \left(z + \frac{2}{3}\right) \right]^n.$$

Thus,

$$f_2(z) = \frac{\frac{3}{5}}{1 - \frac{3}{5} \left(z + \frac{2}{3}\right)} = \frac{1}{1-z} \quad \text{when } \left|z + \frac{2}{3}\right| < \frac{5}{3}.$$

We observe that although $f_1(z)$ and $f_2(z)$ have different series representations in the two overlapping regions (see Fig. 12.7), they represent the same function, $f(z) = 1/(1-z)$. We can therefore write

$$f(z) = \begin{cases} f_1(z) & \text{when } |z| < 1, \\ f_2(z) & \text{when } |z + \frac{2}{3}| < \frac{5}{3}, \end{cases}$$

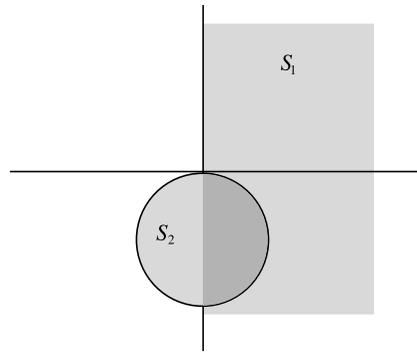


Fig. 12.8 The functions f_1 and f_2 are analytic continuations of each other: f_1 analytically continues f_2 into the *right half-plane*, and f_2 analytically continues f_1 into the semicircle in the *left half-plane*

and f_1 and f_2 are analytic continuations of one another. In fact, $f(z) = 1/(1-z)$ is the analytic continuation of both f_1 and f_2 for all of \mathbb{C} except $z = 1$. Figure 12.7 shows S_i , the region of definition of f_i , for $i = 1, 2$.

Example 12.3.4 The function $f_1(z) = \int_0^\infty e^{-zt} dt$ exists only if $\operatorname{Re}(z) > 0$, in which case $f_1(z) = 1/z$. Its region of definition S_1 is shown in Fig. 12.8 and is simply the right half-plane.

Now we define f_2 by a geometric series: $f_2(z) = i \sum_{n=0}^\infty [(z+i)/i]^n$ where $|z+i| < 1$. This series converges, within its circle of convergence S_2 , to

$$i \frac{1}{1 - (z+i)/i} = \frac{1}{z}.$$

Thus, we have

$$\frac{1}{z} = \begin{cases} f_1(z) & \text{when } z \in S_1, \\ f_2(z) & \text{when } z \in S_2. \end{cases}$$

The two functions are analytic continuations of one another, and $f(z) = 1/z$ is the analytic continuation of both f_1 and f_2 for all $z \in \mathbb{C}$ except $z = 0$.

12.3.1 The Schwarz Reflection Principle

A result that is useful in some physical applications is referred to as a dispersion relation. To derive such a relation we need to know the behavior of analytic functions on either side of the real axis. This is found using the Schwarz reflection principle, for which we need the following result.

Proposition 12.3.5 *Let f_i be analytic throughout S_i , where $i = 1, 2$. Let B be the boundary between S_1 and S_2 (Fig. 12.9) and assume that f_1 and f_2 are continuous on B and coincide there. Then the two functions are analytic*

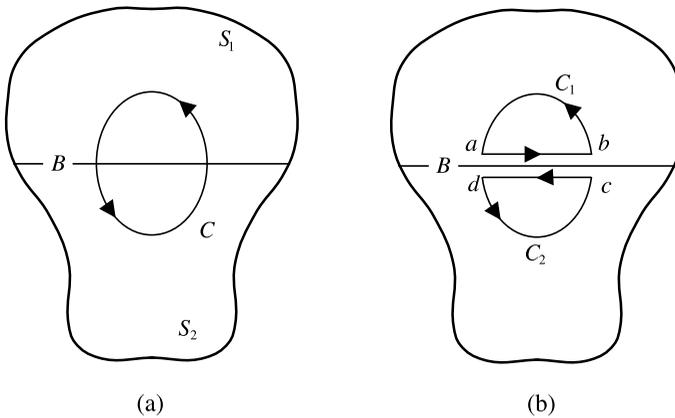


Fig. 12.9 (a) Regions S_1 and S_2 separated by the boundary B and the contour C . (b) The contour C splits up into C_1 and C_2

continuations of one another and together they define a (unique) function

$$f(z) = \begin{cases} f_1(z) & \text{when } z \in S_1 \cup B, \\ f_2(z) & \text{when } z \in S_2 \cup B, \end{cases}$$

which is analytic throughout the entire region $S_1 \cup S_2 \cup B$.

Proof The proof consists of showing that the function integrates to zero along any closed curve in $S_1 \cup S_2 \cup B$. Once this is done, one can use Morera’s theorem to conclude analyticity. The case when the closed curve is entirely in either S_1 or S_2 is trivial. When the curve is partially in S_1 and partially in S_2 the proof becomes only slightly more complicated, because one has to split up the contour C into C_1 and C_2 of Fig. 12.9(b). The details are left as an exercise. \square

Theorem 12.3.6 (Schwarz reflection principle) *Let f be a function that is analytic in a region S that has a segment of the real axis as part of its boundary B . If $f(z)$ is real whenever z is real, then the analytic continuation g of f into S^* (the mirror image of S with respect to the real axis) exists and is given by* Schwarz reflection principle

$$g(z) = (f(z^*))^* \equiv f^*(z^*), \quad \text{where } z \in S^*.$$

Proof First, we show that g is analytic in S^* . Let

$$f(z) \equiv u(x, y) + iv(x, y), \quad g(z) \equiv U(x, y) + iV(x, y).$$

Then $f(z^*) = f(x, -y) = u(x, -y) + iv(x, -y)$ and $g(z) = f^*(z^*)$ imply that $U(x, y) = u(x, -y)$ and $V(x, y) = -v(x, -y)$. Therefore,

$$\frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial(-y)} = \frac{\partial V}{\partial y},$$

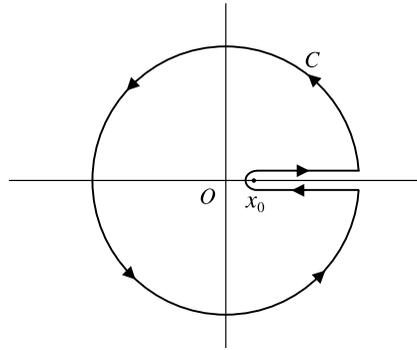


Fig. 12.10 The contour used for dispersion relations

$$\frac{\partial U}{\partial y} = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = -\frac{\partial V}{\partial x}.$$

These are the Cauchy-Riemann conditions for $g(z)$. Thus, g is analytic.

Next, we note that $f(x, 0) = g(x, 0)$, implying that f and g agree on the real axis. Proposition 12.3.5 then implies that f and g are analytic continuations of one another. \square

It follows from this theorem that there exists an analytic function h such that

$$h(z) = \begin{cases} f(z) & \text{when } z \in S, \\ g(z) & \text{when } z \in S^*. \end{cases}$$

We note that $h(z^*) = g(z^*) = f^*(z) = h^*(z)$.

12.3.2 Dispersion Relations

Let f be analytic throughout the complex plane except at a cut along the real axis extending from x_0 to infinity. For a point z not on the x -axis, the Cauchy integral formula gives $f(z) = (2\pi i)^{-1} \int_C f(\xi) d\xi / (\xi - z)$ where C is the contour shown in Fig. 12.10.

We assume that f drops to zero fast enough that the contribution from the large circle tends to zero. The reader may show that the contribution from the small half-circle around x_0 also vanishes. Then

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \left[\int_{x_0+i\epsilon}^{\infty+i\epsilon} \frac{f(\xi)}{\xi - z} d\xi - \int_{x_0-i\epsilon}^{\infty-i\epsilon} \frac{f(\xi)}{\xi - z} d\xi \right] \\ &= \frac{1}{2\pi i} \left[\int_{x_0}^{\infty} \frac{f(x+i\epsilon)}{x - z + i\epsilon} dx - \int_{x_0}^{\infty} \frac{f(x-i\epsilon)}{x - z - i\epsilon} dx \right]. \end{aligned}$$

Since z is not on the real axis, we can ignore the $i\epsilon$ terms in the denominators, so that $f(z) = (2\pi i)^{-1} \int_{x_0}^{\infty} [f(x+i\epsilon) - f(x-i\epsilon)] dx / (x - z)$. The Schwarz reflection principle in the form $f^*(z) = f(z^*)$ can now be used to

yield

$$f(x + i\epsilon) - f(x - i\epsilon) = f(x + i\epsilon) - f^*(x + i\epsilon) = 2i \operatorname{Im}[f(x + i\epsilon)].$$

The final result is

$$f(z) = \frac{1}{\pi} \int_{x_0}^{\infty} \frac{\operatorname{Im}[f(x + i\epsilon)]}{x - z} dx.$$

This is one form of a **dispersion relation**. It expresses the value of a function at any point of the cut complex plane in terms of an integral of the imaginary part of the function on the upper edge of the cut. dispersion relation

When there are no residues in the UHP, we can obtain other forms of dispersion relations by equating the real and imaginary parts of Eq. (11.11). The result is

$$\begin{aligned} \operatorname{Re}[f(x_0)] &= \pm \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Im}[f(x)]}{x - x_0} dx, \\ \operatorname{Im}[f(x_0)] &= \mp \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Re}[f(x)]}{x - x_0} dx, \end{aligned} \tag{12.6}$$

where the upper (lower) sign corresponds to placing the small semicircle around x_0 in the UHP (LHP). The real and imaginary parts of f , as related by Eq. (12.6), are sometimes said to be the **Hilbert transform** of one another. Hilbert transform

In some applications, the imaginary part of f is an odd function of its argument. Then the first equation in (12.6) can be written as

$$\operatorname{Re}[f(x_0)] = \pm \frac{2}{\pi} P \int_0^{\infty} \frac{x \operatorname{Im}[f(x)]}{x^2 - x_0^2} dx.$$

To arrive at dispersion relations, the following condition must hold:

$$\lim_{R \rightarrow \infty} R |f(Re^{i\theta})| = 0,$$

where R is the radius of the large semicircle in the UHP (or LHP). If f does not satisfy this prerequisite, it is still possible to obtain a dispersion relation called a **dispersion relation with one subtraction**. This can be done by introducing an extra factor of x in the denominator of the integrand. We start with Eq. (11.15), confining ourselves to the UHP and assuming that there are no poles there, so that the sum over residues is dropped: dispersion relation with one subtraction

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{1}{i\pi} P \int_{-\infty}^{\infty} \frac{f(x)}{(x - x_1)(x - x_2)} dx.$$

The reader may check that by equating the real and imaginary parts on both sides, letting $x_1 = 0$ and $x_2 = x_0$, and changing x to $-x$ in the first half of the interval of integration, we obtain

$$\begin{aligned} &\frac{\operatorname{Re}[f(x_0)]}{x_0} \\ &= \frac{\operatorname{Re}[f(0)]}{x_0} + \frac{1}{\pi} \left[P \int_0^{\infty} \frac{\operatorname{Im}[f(-x)]}{x(x + x_0)} dx + P \int_0^{\infty} \frac{\operatorname{Im}[f(x)]}{x(x - x_0)} dx \right]. \end{aligned}$$

For the case where $\text{Im}[f(-x)] = -\text{Im}[f(x)]$, this equation yields

$$\text{Re}[f(x_0)] = \text{Re}[f(0)] + \frac{2x_0^2}{\pi} P \int_0^\infty \frac{\text{Im}[f(x)]}{x(x^2 - x_0^2)} dx. \quad (12.7)$$

Example 12.3.7 In optics, it has been shown that the imaginary part of the forward-scattering light amplitude with frequency ω is related, by the so-called **optical theorem**, to the total cross section for the absorption of light of that frequency:

$$\text{Im}[f(\omega)] = \frac{\omega}{4\pi} \sigma_{\text{tot}}(\omega).$$

Substituting this in Eq. (12.7) yields

$$\text{Re}[f(\omega_0)] = \text{Re}[f(0)] + \frac{\omega_0^2}{2\pi^2} P \int_0^\infty \frac{\sigma_{\text{tot}}(\omega)}{\omega^2 - \omega_0^2} d\omega. \quad (12.8)$$

Thus, the real part of the (coherent) forward scattering of light, that is, the real part of the *index of refraction*, can be computed from Eq. (12.8) by either measuring or calculating $\sigma_{\text{tot}}(\omega)$, the simpler quantity describing the absorption of light in the medium. Equation (12.8) is the original **Kramers-Kronig relation**.

12.4 The Gamma and Beta Functions

We have already encountered the gamma function. In this section, we derive some useful relations involving the gamma function and the closely related beta function. The gamma function is a generalization of the factorial function—which is defined only for positive integers—to the system of complex numbers. By differentiating the integral

$$I(\alpha) \equiv \int_0^\infty e^{-\alpha t} dt = 1/\alpha$$

with respect to α repeatedly and setting $\alpha = 1$ at the end, we get $\int_0^\infty t^n e^{-t} dt = n!$. This fact motivates the generalization

$$\Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t} dt \quad \text{for } \text{Re}(z) > 0, \quad (12.9)$$

where Γ is called the **gamma (or factorial) function**. It is also called *Euler's integral of the second kind*. It is clear from its definition that

$$\Gamma(n+1) = n! \quad (12.10)$$

if n is a positive integer. The restriction $\text{Re}(z) > 0$ assures the convergence of the integral.

An immediate consequence of Eq. (12.9) is obtained by integrating it by parts:

$$\Gamma(z+1) = z\Gamma(z). \quad (12.11)$$

This also leads to Eq. (12.10) by iteration and the fact that $\Gamma(1) = 1$.

Another consequence is the analyticity of $\Gamma(z)$. Differentiating Eq. (12.11) with respect to z , we obtain

$$\frac{d\Gamma(z+1)}{dz} = \Gamma(z) + z \frac{d\Gamma(z)}{dz}.$$

Thus, $d\Gamma(z)/dz$ exists and is finite if and only if $d\Gamma(z+1)/dz$ is finite (recall that $z \neq 0$). The procedure of showing the latter is outlined in Problem 12.16. Therefore, $\Gamma(z)$ is analytic whenever $\Gamma(z+1)$ is. To see the singularities of $\Gamma(z)$, we note that

$$\Gamma(z+n) = z(z+1)(z+2)\cdots(z+n-1)\Gamma(z),$$

or

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)(z+2)\cdots(z+n-1)}. \quad (12.12)$$

The numerator is analytic as long as $\operatorname{Re}(z+n) > 0$, or $\operatorname{Re}(z) > -n$. Thus, for $\operatorname{Re}(z) > -n$, the singularities of $\Gamma(z)$ are the poles at $z = 0, -1, -2, \dots, -n+1$. Since n is arbitrary, we conclude that

Box 12.4.1 $\Gamma(z)$ is analytic at all $z \in \mathbb{C}$ except at $z = 0, -1, -2, \dots$, where $\Gamma(z)$ has simple poles.

A useful result is obtained by setting $z = \frac{1}{2}$ in Eq. (12.9):

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \quad (12.13)$$

This can be obtained by making the substitution $u = \sqrt{t}$ in the integral.

We can derive an expression for the logarithmic derivative of the gamma function that involves an infinite series. To do so, we use Eq. (12.2) noting that $1/\Gamma(z+1)$ is an entire function with simple zeros at $\{-k\}_{k=1}^{\infty}$. Equation (12.2) gives

$$\frac{1}{\Gamma(z+1)} = e^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k},$$

where γ is a constant to be determined. Using Eq. (12.11), we obtain

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}. \quad (12.14)$$

To determine γ , let $z = 1$ in Eq. (12.14) and evaluate the resulting product numerically. The result is $\gamma = 0.57721566\dots$, the so-called **Euler-Mascheroni constant** **Mascheroni constant**.

Differentiating the logarithm of both sides of Eq. (12.14), we obtain

$$\frac{d}{dz} \ln[\Gamma(z)] = -\frac{1}{z} - \gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{z+k} \right). \quad (12.15)$$

Other properties of the gamma function are derivable from the results presented here. Those derivations are left as problems.

The **beta function**, or **Euler's integral of the first kind**, is defined for complex numbers a and b as follows:

$$B(a, b) \equiv \int_0^1 t^{a-1} (1-t)^{b-1} dt \quad \text{where } \operatorname{Re}(a), \operatorname{Re}(b) > 0. \quad (12.16)$$

By changing t to $1/t$, we can also write

$$B(a, b) \equiv \int_1^{\infty} t^{-a-b} (t-1)^{b-1} dt. \quad (12.17)$$

Since $0 \leq t \leq 1$ in Eq. (12.16), we can define θ by $t = \sin^2 \theta$. This gives

$$B(a, b) = 2 \int_0^{\pi/2} \sin^{2a-1} \theta \cos^{2b-1} \theta d\theta. \quad (12.18)$$

This relation can be used to establish a connection between the gamma and beta functions. We note that

$$\Gamma(a) = \int_0^{\infty} t^{a-1} e^{-t} dt = 2 \int_0^{\infty} x^{2a-1} e^{-x^2} dx,$$

where in the last step we changed the variable to $x = \sqrt{t}$. Multiply $\Gamma(a)$ by $\Gamma(b)$ and express the resulting double integral in terms of polar coordinates to obtain $\Gamma(a)\Gamma(b) = \Gamma(a+b)B(a, b)$, or

gamma function and
beta function are related

$$B(a, b) = B(b, a) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (12.19)$$

Let us now establish the following useful relation:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \quad (12.20)$$

With $a = z$ and $b = 1 - z$, and using $u = \tan \theta$, Eqs. (12.18) and (12.19) give

$$\Gamma(z)\Gamma(1-z) = B(z, 1-z) = 2 \int_0^{\infty} \frac{u^{2z-1}}{u^2+1} du \quad \text{for } 0 < \operatorname{Re}(z) < 1.$$

Using the result obtained in Example 12.2.4, we immediately get Eq. (12.20), valid for $0 < \operatorname{Re}(z) < 1$. By analytic continuation we then generalize Eq. (12.20) to values of z for which both sides are analytic.

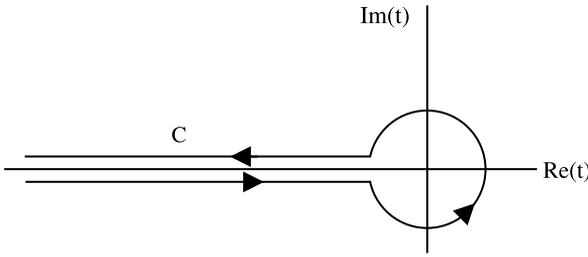


Fig. 12.11 The contour C used in evaluating the reciprocal gamma function

Example 12.4.2 As an illustration of the use of Eq. (12.20), let us show that $\Gamma(z)$ can also be written as

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C \frac{e^t}{t^z} dt, \quad (12.21)$$

where C is the contour shown in Fig. 12.11. From Eqs. (12.9) and (12.20) it follows that

$$\begin{aligned} \frac{1}{\Gamma(z)} &= \frac{\sin \pi z}{\pi} \Gamma(1-z) = \frac{\sin \pi z}{\pi} \int_0^\infty e^{-r} r^{-z} dr \\ &= \frac{e^{i\pi z} - e^{-i\pi z}}{2\pi i} \int_0^\infty \frac{e^{-r}}{r^z} dr. \end{aligned}$$

The contour integral of Eq. (12.21) can be evaluated by noting that above the real axis, $t = re^{i\pi} = -r$, below it $t = re^{-i\pi} = -r$, and, as the reader may check, that the contribution from the small circle at the origin is zero; so

$$\begin{aligned} \int_C \frac{e^t}{t^z} dt &= \int_0^\infty \frac{e^{-r}}{(re^{i\pi})^z} (-dr) + \int_\infty^0 \frac{e^{-r}}{(re^{-i\pi})^z} (-dr) \\ &= -e^{-i\pi z} \int_0^\infty \frac{e^{-r}}{r^z} dr + e^{i\pi z} \int_0^\infty \frac{e^{-r}}{r^z} dr. \end{aligned}$$

Comparison with the last equation above yields the desired result.

Another useful relation can be obtained by combining Eqs. (12.11) and (12.20):

$$\Gamma(z)\Gamma(1-z) = \Gamma(z)(-z)\Gamma(-z) = \frac{\pi}{\sin \pi z}.$$

Thus,

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z \sin \pi z}. \quad (12.22)$$

Once we know $\Gamma(x)$ for positive values of real x , we can use Eq. (12.22) to find $\Gamma(x)$ for $x < 0$. Thus, for instance, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ gives $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$. Equation (12.22) also shows that the gamma function has simple poles wherever z is a negative integer.

12.5 Method of Steepest Descent

It is shown in statistical mechanics ([Hill 87, pp. 150–152]) that the partition function, which generates all the thermodynamical quantities, can be written as a contour integral. Debye found a very elegant technique of approximating this contour integral, which we investigate in this section. Consider the integral

$$I(\alpha) \equiv \int_C e^{\alpha f(z)} g(z) dz \quad (12.23)$$

where $|\alpha|$ is large and f and g are analytic in some region of \mathbb{C} containing the contour C . Since this integral occurs frequently in physical applications, it would be helpful if we could find a general approximation for it that is applicable for all f and g . The fact that $|\alpha|$ is large will be of great help. By redefining $f(z)$, if necessary, we can assume that $\alpha = |\alpha|e^{i \arg(\alpha)}$ is real and positive [absorb $e^{i \arg(\alpha)}$ into the function $f(z)$ if need be].

The exponent of the integrand can be written as

$$\alpha f(z) = \alpha u(x, y) + i\alpha v(x, y).$$

Since α is large and positive, we expect the exponential to be the largest at the maximum of $u(x, y)$. Thus, if we deform the contour so that it passes through a point z_0 at which $u(x, y)$ is maximum, the contribution to the integral may come mostly from the neighborhood of z_0 . This opens up the possibility of expanding the exponent about z_0 and keeping the lowest terms in the expansion, which is what we are after. There is one catch, however. Because of the largeness of α , the imaginary part of αf in the exponent will oscillate violently as $v(x, y)$ changes even by a small amount. This oscillation can make the contribution of the real part of $f(z_0)$ negligibly small and render the whole procedure useless. Thus, we want to tame the variation of $\exp[i\alpha v(x, y)]$ by making $v(x, y)$ vary as slowly as possible. A necessary condition is for the derivative of v to vanish at z_0 . This and the fact that the real part is to have a maximum at z_0 lead to

$$\left. \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{df}{dz} \right|_{z_0} = 0. \quad (12.24)$$

However, we do not stop here but demand that the imaginary part of f be constant along the deformed contour: $\text{Im}[f(z)] = \text{Im}[f(z_0)]$ or $v(x, y) = v(x_0, y_0)$.

Equation (12.24) and the Cauchy-Riemann conditions imply that $\partial u/\partial x = 0 = \partial u/\partial y$ at z_0 . Thus, it might appear that z_0 is a maximum (or minimum) of the surface described by the function $u(x, y)$. This is not true: For the surface to have a maximum (minimum), both second derivatives, $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$, must be negative (positive). But that is impossible because $u(x, y)$ is harmonic—the sum of these two derivatives is zero. Recall that a point at which the derivatives vanish but that is neither a maximum nor a minimum is called a *saddle point*. That is why the procedure described below is sometimes called the **saddle point approximation**.

saddle point
approximation

We are interested in values of z close to z_0 . So let us expand $f(z)$ in a Taylor series about z_0 , use Eq. (12.24), and keep terms only up to the second, to obtain

$$f(z) = f(z_0) + \frac{1}{2}(z - z_0)^2 f''(z_0). \quad (12.25)$$

Let us assume that $f''(z_0) \neq 0$, and define

$$z - z_0 = r_1 e^{i\theta_1} \quad \text{and} \quad \frac{1}{2} f''(z_0) = r_2 e^{i\theta_2} \quad (12.26)$$

and substitute in the above expansion to obtain

$$f(z) - f(z_0) = r_1^2 r_2 e^{i(2\theta_1 + \theta_2)}, \quad (12.27)$$

or

$$\begin{aligned} \operatorname{Re}[f(z) - f(z_0)] &= r_1^2 r_2 \cos(2\theta_1 + \theta_2), \\ \operatorname{Im}[f(z) - f(z_0)] &= r_1^2 r_2 \sin(2\theta_1 + \theta_2). \end{aligned} \quad (12.28)$$

The constancy of $\operatorname{Im}[f(z)]$ implies that $\sin(2\theta_1 + \theta_2) = 0$, or $2\theta_1 + \theta_2 = n\pi$. Thus, for $\theta_1 = -\theta_2/2 + n\pi/2$ where $n = 0, 1, 2, 3$, the imaginary part of f is constant. The angle θ_2 is determined by the second equation in (12.26). Once we determine n , the path of saddle point integration will be specified.

To get insight into this specification, consider $z - z_0 = r_1 e^{i(-\theta_2/2 + n\pi/2)}$, and eliminate r_1 from its real and imaginary parts to obtain

$$y - y_0 = \left[\tan\left(\frac{n\pi}{2} - \frac{\theta_2}{2}\right) \right] (x - x_0).$$

This is the equation of a line passing through $z_0 = (x_0, y_0)$ and making an angle of $\theta_1 = (n\pi - \theta_2)/2$ with the real axis. For $n = 0, 2$ we get one line, and for $n = 1, 3$ we get another that is perpendicular to the first (see Fig. 12.12). It is to be emphasized that along both these lines the imaginary part of $f(z)$ remains constant. To choose the correct line, we need to look at the real part of the function. Also note that these “lines” are small segments of (or tangents to) the deformed contour at z_0 .

We are looking for directions along which $\operatorname{Re}(f)$ goes through a relative maximum at z_0 . In fact, we are after a path on which the function decreases maximally. This occurs when $\operatorname{Re}[f(z)] - \operatorname{Re}[f(z_0)]$ take the largest negative value. Equation (12.28) determines such a path: It is that path on which $\cos(2\theta_1 + \theta_2) = -1$, or when $n = 1, 3$. There is only one such path in the region of interest, and the procedure is uniquely determined.⁴ Because the descent from the maximum value at z_0 is maximum along such a path, this procedure is called the method of **steepest descent**.

method of steepest descent

⁴The angle θ_1 is still ambiguous by π , because n can be 1 or 3. However, by a suitable sign convention described below, we can remove this ambiguity.

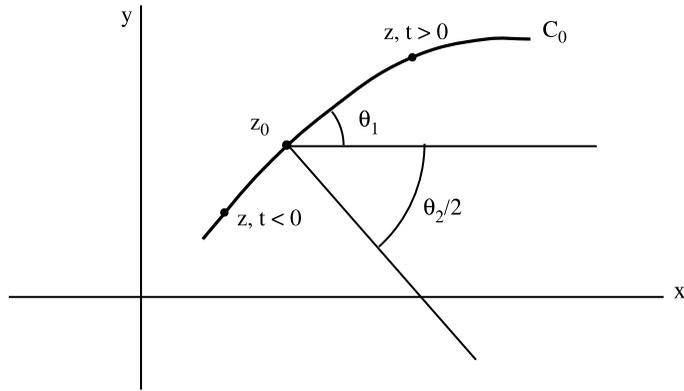


Fig. 12.12 A segment of the contour C_0 in the vicinity of z_0 . The lines mentioned in the text are small segments of the contour C_0 centered at z_0

Now that we have determined the contour, let us approximate the integral. Substituting $2\theta_1 + \theta_2 = \pi, 3\pi$ in Eq. (12.27), we get

$$f(z) - f(z_0) = -r_1^2 r_2 \equiv -t^2 = \frac{1}{2}(z - z_0)^2 f''(z_0). \quad (12.29)$$

Using this in Eq. (12.23) yields

$$I(\alpha) \approx \int_{C_0} e^{\alpha[f(z_0) - t^2]} g(z) dz = e^{\alpha f(z_0)} \int_{C_0} e^{-\alpha t^2} g(z) dz, \quad (12.30)$$

where C_0 is the deformed contour passing through z_0 .

To proceed, we need to solve for z in terms of t . From Eq. (12.29) we have

$$(z - z_0)^2 = -\frac{2}{f''(z_0)} t^2 = -\frac{t^2}{r_2} e^{-i\theta_2}.$$

Therefore, $|z - z_0| = |t|/\sqrt{r_2}$, or $z - z_0 = (|t|/\sqrt{r_2})e^{i\theta_1}$, by the first equation of (12.26). Let us agree that for $t > 0$, the point z on the contour will move in the direction that makes an angle of $0 \leq \theta_1 < \pi$, and that $t < 0$ corresponds to the opposite direction. This convention removes the remaining ambiguity of the angle θ_1 , and gives

$$z = z_0 + \frac{t}{\sqrt{r_2}} e^{i\theta_1}, \quad 0 \leq \theta_1 < \pi. \quad (12.31)$$

Using the Taylor expansion of $g(z)$ about z_0 , we can write

$$\begin{aligned} g(z) dz &= \left\{ \sum_{n=0}^{\infty} \frac{t^n}{r_2^{n/2} n!} e^{in\theta_1} g^{(n)}(z_0) \right\} \frac{e^{i\theta_1}}{\sqrt{r_2}} dt \\ &= \sum_{n=0}^{\infty} \frac{t^n}{r_2^{(n+1)/2} n!} e^{i(n+1)\theta_1} g^{(n)}(z_0) dt, \end{aligned}$$

and substituting this in Eq. (12.30) yields

$$\begin{aligned}
 I(\alpha) &\approx e^{\alpha f(z_0)} \int_{C_0} e^{-\alpha t^2} \left\{ \sum_{n=0}^{\infty} \frac{t^n}{r_2^{(n+1)/2} n!} e^{i(n+1)\theta_1} g^{(n)}(z_0) \right\} dt \\
 &= e^{\alpha f(z_0)} \sum_{n=0}^{\infty} \frac{e^{i(n+1)\theta_1}}{r_2^{(n+1)/2} n!} g^{(n)}(z_0) \int_{-\infty}^{\infty} e^{-\alpha t^2} t^n dt. \tag{12.32}
 \end{aligned}$$

The extension of the integral limits to infinity does not alter the result significantly because α is assumed large and positive. The integral in the sum is zero for odd n . When n is even, we make the substitution $u = \alpha t^2$ and show that

$$\int_{-\infty}^{\infty} e^{-\alpha t^2} t^n dt = \alpha^{-(n+1)/2} \Gamma[(n+1)/2].$$

With $n = 2k$, and using $r_2 = |f''(z_0)|/2$, the sum becomes

$$I(\alpha) \approx e^{\alpha f(z_0)} \sum_{k=0}^{\infty} \frac{2^{k+1/2} e^{i(2k+1)\theta_1}}{|f''(z_0)|^{k+1/2} (2k)!} g^{(2k)}(z_0) \Gamma\left(k + \frac{1}{2}\right) \alpha^{-k-1/2}. \tag{12.33}$$

This is called the **asymptotic expansion** of $I(\alpha)$. In most applications, only the first term of the above series is retained, giving asymptotic expansion of $I(\alpha)$

$$I(\alpha) \approx e^{\alpha f(z_0)} \sqrt{\frac{2\pi}{\alpha}} \frac{e^{i\theta_1} g(z_0)}{\sqrt{|f''(z_0)|}}. \tag{12.34}$$

Example 12.5.1 Let us approximate the integral

$$I(\alpha) \equiv \Gamma(\alpha + 1) = \int_0^{\infty} e^{-z} z^\alpha dz,$$

where α is a positive real number. First, we must rewrite the integral in the form of Eq. (12.23). We can do this by noting that $z^\alpha = e^{\alpha \ln z}$. Thus, we have

$$I(\alpha) = \int_0^{\infty} e^{\alpha \ln z - z} dz = \int_0^{\infty} e^{\alpha(\ln z - z/\alpha)} dz,$$

and we identify $f(z) = \ln z - z/\alpha$ and $g(z) = 1$. The saddle point is found from $f'(z) = 0$ or $z_0 = \alpha$. Furthermore, from

$$\frac{1}{2} f''(z_0) = \frac{1}{2} \left(-\frac{1}{\alpha^2} \right) = \frac{1}{2\alpha^2} e^{i\pi} \Rightarrow \theta_2 = \pi$$

and $2\theta_1 + \theta_2 = \pi, 3\pi$, as well as the condition $0 \leq \theta_1 < \pi$, we conclude that $\theta_1 = 0$.

Substitution in Eq. (12.34) yields

$$\begin{aligned}
 \Gamma(\alpha + 1) &\approx e^{\alpha f(z_0)} \sqrt{\frac{2\pi}{\alpha}} \frac{1}{\sqrt{|1/\alpha^2|}} \\
 &= \sqrt{2\pi} \alpha e^{\alpha(\ln \alpha - 1)} = \sqrt{2\pi} e^{-\alpha} \alpha^{\alpha+1/2}, \tag{12.35}
 \end{aligned}$$

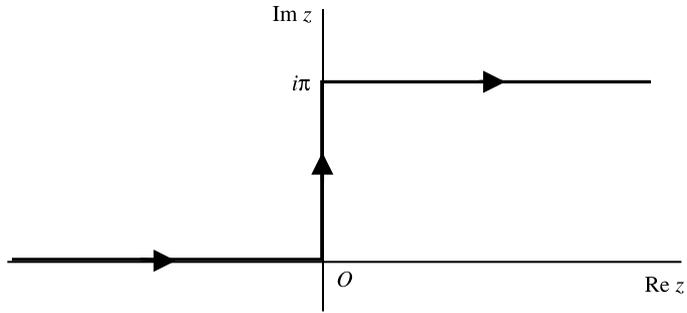


Fig. 12.13 The contour for the evaluation of the Hankel function of the first kind

Stirling approximation which is called the **Stirling approximation**. For $\alpha = n$, a positive integer, this yields the useful result

$$n! \approx \sqrt{2\pi} e^{-n} n^{n+1/2}$$

with the approximation getting better and better for larger and larger n .

Hankel function of the first kind **Example 12.5.2** The *Hankel function of the first kind* is defined as

$$H_v^{(1)}(\alpha) \equiv \frac{1}{i\pi} \int_C e^{(\alpha/2)(z-1/z)} \frac{dz}{z^{\nu+1}},$$

where C is the contour shown in Fig. 12.13.

We want to find the asymptotic expansion of this function, choosing the branch of the function in which $-\pi < \theta < \pi$.

We identify $f(z) = \frac{1}{2}(z - 1/z)$ and $g(z) = z^{-\nu-1}$. Next, the stationary points of f are calculated:

$$\frac{df}{dz} = \frac{1}{2} + \frac{1}{2z^2} = 0 \Rightarrow z_0 = \pm i.$$

The contour of integration suggests the saddle point $z_0 = +i$. The second derivative evaluated at the saddle point gives $f''(z_0) = -1/z_0^3 = -i = e^{-i\pi/2}$, or $\theta_2 = -\pi/2$. This, and the convention $0 \leq \theta_1 < \pi$, force us to choose $\theta_1 = 3\pi/4$. Substituting this in Eq. (12.34) and noting that $f(i) = i$ and $|f''(z_0)| = 1$, we obtain

$$H_v^{(1)}(\alpha) \equiv \frac{1}{i\pi} I(\alpha) \approx \frac{1}{i\pi} e^{\alpha i} \sqrt{\frac{2\pi}{\alpha}} e^{i3\pi/4} i^{-\nu-1} = \sqrt{\frac{2}{\alpha\pi}} e^{i(\alpha-\nu\pi/2-\pi/4)},$$

where we have used $i^{-\nu-1} = e^{-i(\nu+1)\pi/2}$.

Although Eq. (12.34) is adequate for most applications, we shall have occasions to demand a better approximation. One may try to keep higher-order terms of Eq. (12.33), but that infinite sum is in reality inconsistent. The reason is that in the product $g(z) dz$, we kept only the first power of t in the expansion of z . To restore consistency, let us expand $z(t)$ as well.

Suppose

$$z - z_0 = \sum_{m=1}^{\infty} b_m t^m \Rightarrow dz = \sum_{m=0}^{\infty} (m+1)b_{m+1} t^m dt,$$

so that

$$\begin{aligned} g(z) dz &= \sum_{n=0}^{\infty} \frac{t^n}{r_2^{n/2} n!} e^{in\theta_1} g^{(n)}(z_0) \sum_{m=0}^{\infty} (m+1)b_{m+1} t^m dt \\ &= \sum_{m,n=0}^{\infty} \frac{e^{in\theta_1}}{r_2^{n/2} n!} (m+1)b_{m+1} g^{(n)}(z_0) t^{m+n} dt. \end{aligned}$$

Now introduce $l = m + n$ and note that the summation over n goes up to l . This gives

$$g(z) dz = \sum_{l=0}^{\infty} \underbrace{\sum_{n=0}^l \frac{e^{in\theta_1}}{r_2^{n/2} n!} (l-n+1)b_{l-n+1} g^{(n)}(z_0)}_{\equiv a_l} t^l dt = \sum_{l=0}^{\infty} a_l t^l dt.$$

Substituting this in Eq. (12.30) and changing the contour integration into the integral from $-\infty$ to ∞ as before yields

$$\begin{aligned} I(\alpha) &\approx e^{\alpha f(z_0)} \sum_{k=0}^{\infty} a_{2k} \alpha^{-k-1/2} \Gamma\left(k + \frac{1}{2}\right), \\ a_{2k} &= \sum_{n=0}^{2k} \frac{e^{in\theta_1}}{r_2^{n/2} n!} (2k-n+1)b_{2k-n+1} g^{(n)}(z_0). \end{aligned} \tag{12.36}$$

The only thing left to do is to evaluate b_m . We shall not give a general formula for these coefficients. Instead, we shall calculate the first three of them. This should reveal to the reader the general method of approximating them to any order. We have already calculated b_1 in Eq. (12.31). To calculate b_2 , keep the next-highest term in the expansion of both z and t^2 . Thus write

$$z - z_0 = b_1 t + b_2 t^2, \quad t^2 = -\frac{1}{2} f''(z_0)(z - z_0)^2 - \frac{1}{6} f'''(z_0)(z - z_0)^3.$$

Now substitute the first equation in the second and equate the coefficients of equal powers of t on both sides. The second power of t gives nothing new: It merely reaffirms the value of b_1 . The coefficient of the third power of t is $-b_1 b_2 f''(z_0) - \frac{1}{6} b_1^3 f'''(z_0)$. Setting this equal to zero gives

$$b_2 = -\frac{b_1^2 f'''(z_0)}{6 f''(z_0)} = \frac{f'''(z_0)}{3 |f''(z_0)|^2} e^{4i\theta_1}, \tag{12.37}$$

where we substituted for b_1 from Eq. (12.31) and used $2\theta_1 + \theta_2 = \pi$.

To calculate b_3 , keep one more term in the expansion of both z and t^2 to obtain

$$z - z_0 = b_1 t + b_2 t^2 + b_3 t^3$$

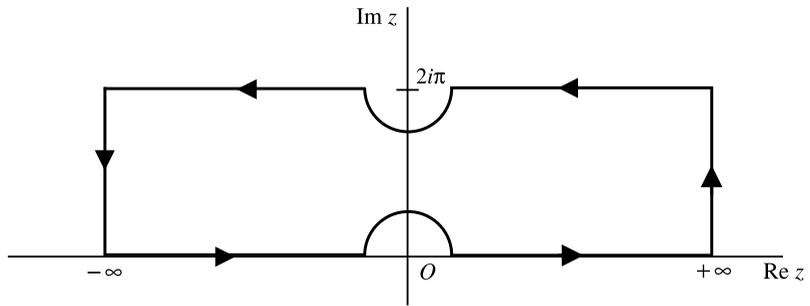


Fig. 12.14 Contour used for Problem 12.4

and

$$t^2 = -\frac{1}{2}f''(z_0)(z - z_0)^2 - \frac{1}{6}f'''(z_0)(z - z_0)^3 - \frac{1}{24}f^{(iv)}(z_0)(z - z_0)^4.$$

Once again substitute the first equation in the second and equate the coefficients of equal powers of t on both sides. The second and third powers of t give nothing new. Setting the coefficient of the fourth power of t equal to zero yields

$$\begin{aligned} b_3 &= b_1^3 \left\{ \frac{5[f'''(z_0)]^2}{72[f''(z_0)]^2} - \frac{f^{(iv)}}{24f''(z_0)} \right\} \\ &= \frac{\sqrt{2}e^{3i\theta_1}}{12|f''(z_0)|^{3/2}} \left\{ \frac{5[f'''(z_0)]^2}{3[f''(z_0)]^2} - \frac{f^{(iv)}}{f''(z_0)} \right\}. \end{aligned} \quad (12.38)$$

12.6 Problems

12.1 Derive Eq. (12.2) from its logarithmic derivative.

12.2 Show that the point at infinity is not a branch point for $f(z) = (z^2 - 1)^{1/2}$.

12.3 Find the following integrals, for which $0 \neq a \in \mathbb{R}$.

$$\begin{aligned} \text{(a)} \quad & \int_0^\infty \frac{\ln x}{(x^2 + a^2)^2} dx, & \text{(b)} \quad & \int_0^\infty \frac{\ln x}{(x^2 + a^2)^2 \sqrt{x}} dx, \\ \text{(c)} \quad & \int_0^\infty \frac{(\ln x)^2}{x^2 + a^2} dx. \end{aligned}$$

12.4 Use the contour in Fig. 12.14 to evaluate the following integrals.

$$\text{(a)} \quad \int_0^\infty \frac{\sin ax}{\sinh x} dx, \quad \text{(b)} \quad \int_0^\infty \frac{x \cos ax}{\sinh x} dx.$$

12.5 Show that $\int_0^\pi f(\sin \theta) d\theta = 2 \int_0^{\pi/2} f(\sin \theta) d\theta$ for an arbitrary function f defined in the interval $[-1, +1]$.

12.6 Find the principal value of the integral $\int_{-\infty}^{\infty} x \sin x dx / (x^2 - x_0^2)$ and evaluate

$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{(x - x_0 \pm i\epsilon)(x + x_0 \pm i\epsilon)} dx$$

for the four possible choices of signs.

12.7 Use analytic continuation, the analyticity of the exponential, hyperbolic, and trigonometric functions, and the analogous identities for real z to prove the following identities.

$$\begin{aligned} \text{(a)} \quad e^z &= \cosh z + \sinh z, & \text{(b)} \quad \cosh^2 z - \sinh^2 z &= 1, \\ \text{(c)} \quad \sin 2z &= 2 \sin z \cos z. \end{aligned}$$

12.8 Show that the function $1/z^2$ represents the analytic continuation into the domain $\mathbb{C} - \{0\}$ (all the complex plane minus the origin) of the function defined by $\sum_{n=0}^{\infty} (n+1)(z+1)^n$ where $|z+1| < 1$.

12.9 Find the analytic continuation into $\mathbb{C} - \{i, -i\}$ (all the complex plane except i and $-i$) of $f(z) = \int_0^{\infty} e^{-zt} \sin t dt$ where $\operatorname{Re}(z) > 0$.

12.10 Expand $f(z) = \sum_{n=0}^{\infty} z^n$ (defined in its circle of convergence) in a Taylor series about $z = a$. For what values of a does this expansion permit the function $f(z)$ to be continued analytically?

12.11 The two power series

$$f_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{and} \quad f_2(z) = i\pi + \sum_{n=1}^{\infty} (-1)^n \frac{(z-2)^n}{n}$$

have no common domain of convergence. Show that they are nevertheless analytic continuations of one another.

12.12 Prove that the functions defined by the two series

$$1 + az + a^2 z^2 + \dots \quad \text{and} \quad \frac{1}{1-z} - \frac{(1-a)z}{(1-z)^2} + \frac{(1-a)^2 z^2}{(1-z)^3} - \dots$$

are analytic continuations of one another.

12.13 Show that the function $f_1(z) = 1/(z^2 + 1)$, where $z \neq \pm i$, is the analytic continuation into $\mathbb{C} - \{i, -i\}$ of the function $f_2(z) = \sum_{n=0}^{\infty} (-1)^n z^{2n}$, where $|z| < 1$.

12.14 Find the analytic continuation into $\mathbb{C} - \{0\}$ of the function

$$f(z) = \int_0^{\infty} t e^{-zt} dt \quad \text{where } \operatorname{Re}(z) > 0.$$

12.15 Show that the integral in Eq. (12.9) converges. Hint: First show that $|\Gamma(z+1)| \leq \int_0^\infty t^x e^{-t} dt$ where $x = \operatorname{Re}(z)$. Now show that

$$\int_0^\infty t^x e^{-t} dt \leq \int_0^1 t^x e^{-t} dt + \int_0^\infty t^n e^{-t} dt \quad \text{for some integer } n > 0$$

and conclude that $\Gamma(z)$ is finite.

12.16 Show that $d\Gamma(z+1)/dz$ exists and is finite by establishing the following:

- $|\ln t| < t + 1/t$ for $t > 0$. Hint: For $t \geq 1$, show that $t - \ln t$ is a monotonically increasing function. For $t < 1$, make the substitution $t = 1/s$.
- Use the result from part (a) in the integral for $d\Gamma(z+1)/dz$ to show that $|d\Gamma(z+1)/dz|$ is finite. Hint: Differentiate inside the integral.

12.17 Derive Eq. (12.11) from Eq. (12.9).

12.18 Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, and that

$$(2k-1)!! \equiv (2k-1)(2k-3)\cdots 5 \cdot 3 \cdot 1 = \frac{2^k}{\sqrt{\pi}} \Gamma\left(\frac{2k+1}{2}\right).$$

12.19 Show that $\Gamma(z) = \int_0^1 [\ln(1/t)]^{z-1} dt$ with $\operatorname{Re}(z) > 0$.

12.20 Derive the identity $\int_0^\infty e^{-x^\alpha} dx = \Gamma[(\alpha+1)/\alpha]$.

12.21 Consider the function $f(z) = (1+z)^\alpha$.

- Show that $d^n f/dz^n|_{z=0} = \Gamma(\alpha+1)/\Gamma(\alpha-n+1)$, and use it to derive the relation

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, \quad \text{where}$$

$$\binom{\alpha}{n} \equiv \frac{\alpha!}{n!(\alpha-n)!} \equiv \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)}.$$

- Show that for general complex numbers a and b we can formally write

$$(a+b)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} a^n b^{\alpha-n}.$$

- Show that if α is a positive integer m , the series in part (b) truncates at $n = m$.

12.22 Prove that the residue of $\Gamma(z)$ at $z = -k$ is $r_k = (-1)^k/k!$. Hint: Use Eq. (12.12)

12.23 Derive the following relation for $z = x + iy$:

$$|\Gamma(z)| = \Gamma(x) \prod_{k=0}^{\infty} \left[1 + \frac{y^2}{(x+k)^2} \right]^{-1/2}.$$

12.24 Using the definition of $B(a, b)$, Eq. (12.16), show that $B(a, b) = B(b, a)$.

12.25 Integrate Eq. (12.21) by parts and derive Eq. (12.11).

12.26 For positive integers n , show that $\Gamma(\frac{1}{2} - n)\Gamma(\frac{1}{2} + n) = (-1)^n \pi$.

12.27 Show that

- (a) $B(a, b) = B(a+1, b) + B(a, b+1)$.
 (b) $B(a, b+1) = (\frac{b}{a+b})B(a, b)$.
 (c) $B(a, b)B(a+b, c) = B(b, c)B(a, b+c)$.

12.28 Verify that $\int_{-1}^1 (1+t)^a (1-t)^b dt = 2^{a+b+1} B(a+1, b+1)$.

12.29 Show that the volume of the solid formed by the surface $z = x^a y^b$, the xy -, yz -, and xz -planes, and the plane parallel to the z -axis and going through the points $(0, y_0)$ and $(x_0, 0)$ is

$$\frac{x_0^{a+1} y_0^{b+1}}{a+b+2} B(a+1, b+1).$$

12.30 Derive this relation:

$$\int_0^{\infty} \frac{\sinh^a x}{\cosh^b x} dx = \frac{1}{2} B\left(\frac{a+1}{2}, \frac{b-a}{2}\right) \quad \text{where } -1 < a < b.$$

Hint: Let $t = \tanh^2 x$ in Eq. (12.16).

12.31 The *Hankel function of the second kind* is defined as

Hankel function of the second kind

$$H_\nu^{(2)}(\alpha) \equiv \frac{1}{i\pi} \int_C e^{(\alpha/2)(z-1/z)} \frac{dz}{z^{\nu+1}},$$

where C is the contour shown in Fig. 12.15.

Find the asymptotic expansion of this function.

12.32 Find the asymptotic dependence of the *modified Bessel function of the first kind*, defined as

modified Bessel function of the first kind

$$I_\nu(\alpha) \equiv \frac{1}{2\pi i} \oint_C e^{(\alpha/2)(z+1/z)} \frac{dz}{z^{\nu+1}},$$

where C starts at $-\infty$, approaches the origin and circles it, and goes back to $-\infty$. Thus the negative real axis is excluded from the domain of analyticity.

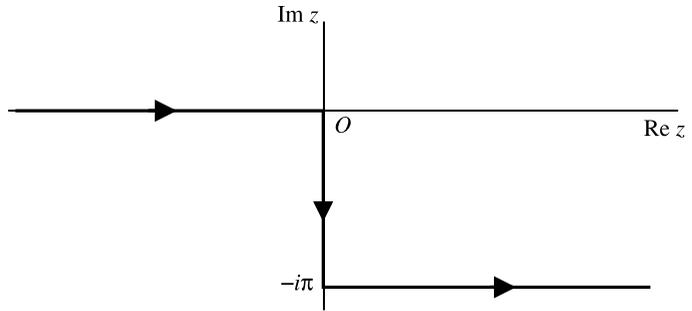


Fig. 12.15 The contour for the evaluation of the Hankel function of the second kind

modified Bessel function of the second kind

12.33 Find the asymptotic dependence of the *modified Bessel function of the second kind*:

$$K_\nu(\alpha) \equiv \frac{1}{2} \int_C e^{-(\alpha/2)(z+1/z)} \frac{dz}{z^{\nu+1}},$$

where C starts at ∞ , approaches the origin and circles it, and goes back to ∞ . Thus the positive real axis is excluded from the domain of analyticity.