11 Calculus of Residues

One of the most powerful tools made available by complex analysis is the theory of residues, which makes possible the routine evaluation of certain definite integrals that are impossible to calculate otherwise. The derivation, application, and analysis of this tool constitute the main focus of this chapter. In the preceding chapter we saw examples in which integrals were related to expansion coefficients of Laurent series. Here we will develop a systematic way of evaluating both real and complex integrals.

11.1 Residues

Recall that a singular point z_0 of $f : \mathbb{C} \to \mathbb{C}$ is a point at which f fails to be analytic. If in addition, there is some neighborhood of z_0 in which f is analytic at every point (except of course at z_0 itself), then z_0 is called an **isolated singularity** of f. Almost all the singularities we have encountered so far have been isolated singularities. However, we will see later—when discussing multivalued functions—that singularities that are not isolated do exist.

isolated singularity

Let z_0 be an isolated singularity of f. Then there exists an $r > 0$ such that within the "annular" region $0 < |z - z_0| < r$, the function f has the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \equiv \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}} \quad \text{and} \quad b_n = \frac{1}{2\pi i} \oint_C f(\xi)(\xi - z_0)^{n-1}\, d\xi.$$

In particular,

$$b_1 = \frac{1}{2\pi i} \oint_C f(\xi)\, d\xi, \tag{11.1}$$

where C is any simple closed contour around z_0, traversed in the positive sense, on and interior to which f is analytic except at the point z_0 itself. The complex number b_1, which is essentially the integral of $f(z)$ along the

S. Hassani, *Mathematical Physics*, DOI 10.1007/978-3-319-01195-0_11,

residue defined

contour, is called the **residue** of f at the isolated singular point z_0. It is important to note that the residue is independent of the contour C as long as z_0 is the only isolated singular point within C.

Historical Notes

Pierre Alphonse Laurent (1813–1854) graduated from the Ecole Polytechnique near the top of his class and became a second lieutenant in the engineering corps. On his return from the war in Algeria, he took part in the effort to improve the port at Le Havre, spending six years there directing various parts of the project. Laurent's superior officers admired the breadth of his practical experience and the good judgment it afforded the young engineer. During this period he wrote his first scientific paper, on the calculus of variations, and submitted it to the French Academy of Sciences for the grand prix in mathematics. Unfortunately the competition had already closed (although the judges had not yet declared a winner), and Laurent's submission was not successful. However, the paper so impressed Cauchy that he recommended its publication, also without success.

The paper for which Laurent is most well known suffered a similar fate. In it he described a more general form of a theorem earlier proven by Cauchy for the power series expansion of a function. Laurent realized that one could generalize this result to hold in any annular region between two singular or discontinuous points by using both positive and negative powers in the series, thus allowing treatment of regions beyond the first singular or discontinuous point. Again, Cauchy argued for the paper's publication without success. The passage of time provided a more just reward, however, and the use of Laurent series became a fundamental tool in complex analysis.

Laurent later worked in the theory of light waves and contended with Cauchy over the interpretation of the differential equations the latter had formulated to explain the behavior of light. Little came of his work in this area, however, and Laurent died at the age of forty-two, a captain serving on the committee on fortifications in Paris. His widow pressed to have two more of his papers read to the Academy, only one of which was published.

We use the notation $\operatorname{Res}[f(z_0)]$ to denote the residue of f at the isolated singular point z_0. Equation (11.1) can then be written as

$$\oint_C f(z)\,dz = 2\pi i \operatorname{Res}\bigl[f(z_0)\bigr].$$

What if there are several isolated singular points within the simple closed contour C? The following theorem provides the answer.

residue theorem

Theorem 11.1.1 (The residue theorem) *Let C be a positively oriented simple closed contour within and on which a function f is analytic except at a finite number of isolated singular points $z_1, z_2, \ldots, z_m$ interior to C. Then*

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{m} \operatorname{Res}\bigl[f(z_k)\bigr]. \tag{11.2}$$

Proof Let C_k be the positively traversed circle around z_k. Then Fig. 11.1 and the Cauchy-Goursat theorem yield

$$0 = \oint_{C'} f(z)\,dz = -\oint_{\text{circles}} f(z)\,dz + \oint_{\text{lines}} f(z)\,dz + \oint_C f(z)\,dz,$$

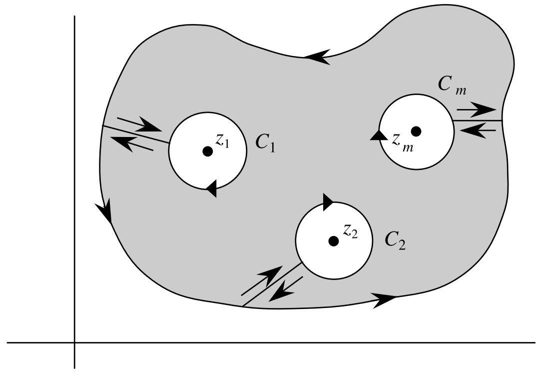

Fig. 11.1 Singularities are avoided by going around them

where C' is the union of all the contours, and the minus sign on the first integral is due to the fact that the interiors of all circles lie to our right as we traverse their boundaries. The two equal an opposite contributions of each line cancel out, and we obtain

$$\oint_C f(z)\,dz = \sum_{k=1}^{m} \oint_{C_k} f(z)\,dz = \sum_{k=1}^{m} 2\pi i \operatorname{Res}\bigl[f(z_k)\bigr],$$

where in the last step the definition of residue at z_k has been used. □

Example 11.1.2 Let us evaluate the integral $\oint_C (2z-3)\,dz/[z(z-1)]$ where C is the circle $|z|=2$. There are two isolated singularities in C, $z_1 = 0$ and $z_2 = 1$. To find $\operatorname{Res}[f(z_1)]$, we expand around the origin:

$$\frac{2z-3}{z(z-1)} = \frac{3}{z} - \frac{1}{z-1} = \frac{3}{z} + \frac{1}{1-z} = \frac{3}{z} + 1 + z + \cdots \quad \text{for } |z| < 1.$$

This gives $\operatorname{Res}[f(z_1)] = 3$. Similarly, expanding around $z = 1$ gives

$$\frac{2z-3}{z(z-1)} = \frac{3}{z-1+1} - \frac{1}{z-1} = -\frac{1}{z-1} + 3\sum_{k=0}^{\infty} (-1)^n (z-1)^n,$$

which yields $\operatorname{Res}[f(z_2)] = -1$. Thus,

$$\oint_C \frac{2z-3}{z(z-1)}\,dz = 2\pi i \bigl\{\operatorname{Res}\bigl[f(z_1)\bigr] + \operatorname{Res}\bigl[f(z_2)\bigr]\bigr\} = 2\pi i(3-1) = 4\pi i.$$

11.2 Classification of Isolated Singularities

Let $f : \mathbb{C} \to \mathbb{C}$ have an isolated singularity at z_0. Then there exist a real number $r > 0$ and an annular region $0 < |z - z_0| < r$ such that f can be represented by the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}. \tag{11.3}$$

principal part of a function

The second sum in Eq. (11.3), involving negative powers of $(z - z_0)$, is called the **principal part** of f at z_0. We can use the principal part to distinguish three types of isolated singularities. The behavior of the function near the isolated singularity is fundamentally different in each case.

removable singular point

1. If $b_n = 0$ for all $n \geq 1$, z_0 is called a **removable singular point** of f. In this case, the Laurent series contains only nonnegative powers of $(z - z_0)$, and setting $f(z_0) = a_0$ makes the function analytic at z_0. For example, the function $f(z) = (e^z - 1 - z)/z^2$, which is indeterminate at $z = 0$, becomes entire if we set $f(0) = \frac{1}{2}$, because its Laurent series $f(z) = \frac{1}{2} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots$ has no negative power.

poles defined

2. If $b_n = 0$ for all $n > m$ and $b_m \neq 0$, z_0 is called a **pole of order** m. In this case, the expansion takes the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m}$$

simple pole

for $0 < |z - z_0| < r$. In particular, if $m = 1$, z_0 is called a **simple pole**.

essential singularity

3. If the principal part of f at z_0 has an infinite number of nonzero terms, the point z_0 is called an **essential singularity**. A prototype of functions that have essential singularities is

$$\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z^n}\right),$$

strange behavior of functions with essential singularity

which has an essential singularity at $z = 0$ and a residue of 1 there. To see how strange such functions are, we let a be *any* real number, and consider $z = 1/(\ln a + 2n\pi i)$ for $n = 0, \pm 1, \pm 2, \ldots$. For such a z we have $e^{1/z} = e^{\ln a + 2n\pi i} = a e^{2n\pi i} = a$. In particular, as $n \to \infty$, z gets arbitrarily close to the origin. Thus, in an arbitrarily small neighborhood of the origin, there are infinitely many points at which the function $\exp(1/z)$ takes on an arbitrary value a. In other words, as $z \to 0$, the function gets arbitrarily close to *any* real number! This result holds for all functions with essential singularities.

Example 11.2.1 (Order of poles)

(a) The function $(z^2 - 3z + 5)/(z - 1)$ has a Laurent series around $z = 1$ containing only three terms:

$$\frac{z^2 - 3z + 5}{z - 1} = -1 + (z - 1) + \frac{3}{z - 1}.$$

Thus, it has a simple pole at $z = 1$, with a residue of 3.

(b) The function $\sin z/z^6$ has a Laurent series

$$\frac{\sin z}{z^6} = \frac{1}{z^6}\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!} = \frac{1}{z^5} - \frac{1}{6z^3} + \frac{1}{(5!)z} - \frac{z}{7!} + \cdots$$

about $z = 0$. The principal part has three terms. The pole, at $z = 0$, is of order 5, and the function has a residue of $1/120$ at $z = 0$.

(c) The function $(z^2 - 5z + 6)/(z - 2)$ has a removable singularity at $z = 2$, because

$$\frac{z^2 - 5z + 6}{z - 2} = \frac{(z-2)(z-3)}{z-2} = z - 3 = -1 + (z - 2)$$

and $b_n = 0$ for all n.

Many functions can be written as the ratio of two polynomials. A function of this form is called a **rational function**. If the degree of the numerator is larger than the denominator, the ratio can be written as a polynomial plus a rational function the degree of whose numerator is not larger than that of the denominator. When we talk about rational functions, we exclude the polynomials. So, we assume that the degree of the numerator is less than or equal to the degree of the denominator. Such rational functions f have the property that as z goes to infinity, f does not go to infinity. Stated equivalently, $f(1/z)$ does not go to infinity at the origin.

rational function

Let f be a function whose only singularities in the entire complex plane are *finite poles*, i.e., the point at infinity is not a pole of the function. This means that $f(1/z)$ does not have a pole at the origin. Let $\{z_j\}_{j=1}^n$ be the poles of f such that z_j is of order m_j. Expand the function about z_1 in a Laurent series

$$f(z) = \frac{b_1}{z - z_1} + \cdots + \frac{b_{m_1}}{(z - z_1)^{m_1}} + \sum_{k=0}^{\infty} a_k (z - z_1)^k \equiv \frac{P_1(z)}{(z - z_1)^{m_1}} + g_1(z),$$

where

$$P_1(z) \equiv b_1(z - z_1)^{m_1 - 1} + b_2(z - z_1)^{m_1 - 2} + \cdots + b_{m_1 - 1}(z - z_1) + b_{m_1}$$

is a polynomial of degree $m_1 - 1$ in z and g_1 is analytic at z_1. It should be clear that the remaining poles of f are in g_1. So, expand g_1 about z_2 in a Laurent series. A similar argument as above yields $g_1(z) = P_2(z)/(z - z_2)^{m_2} + g_2(z)$ where $P_2(z)$ is a polynomial of degree $m_2 - 1$ in z and g_2 is analytic at z_1 and z_2. Continuing in this manner, we get

$$f(z) = \frac{P_1(z)}{(z - z_1)^{m_1}} + \frac{P_2(z)}{(z - z_2)^{m_2}} + \cdots + \frac{P_n(z)}{(z - z_n)^{m_n}} + g(z),$$

where g has no poles. Since all poles of f have been isolated in the sum, g must be analytic everywhere in $\mathbb{C}$, i.e., an entire function. Now substitute $1/t$ for z, take the limit $t \to 0$, and note that, since the degree of P_i is

$m_i - 1$, all the terms in the preceding equation go to zero except possibly $g(1/t)$. Moreover,

$$\lim_{t\to 0} g(1/t) \neq \infty,$$

because, by assumption, the point at infinity is not a pole of f. Thus, g is a bounded entire function. By Proposition 10.5.5, g must be a constant. Taking a common denominator for all the terms yields a ratio of two polynomials. We have proved the following:

Proposition 11.2.2 *A function whose only singularities are poles in a finite region of the complex plane is a rational function.*

The isolated singularity that is most important in applications is a pole. For a function that has a pole of order m at z_0, the calculation of residues is routine. Such a calculation, in turn, enables us to evaluate many integrals effortlessly. How do we calculate the residue of a function f having a pole of order m at z_0?

It is clear that if f has a pole of order m, then $g : \mathbb{C} \to \mathbb{C}$ defined by $g(z) \equiv (z - z_0)^m f(z)$ is analytic at z_0. Thus, for any simple closed contour C that contains z_0 but no other singular point of f, we have

$$\operatorname{Res}\big[f(z_0)\big] = \frac{1}{2\pi i}\oint_C f(z)dz = \frac{1}{2\pi i}\oint_C \frac{g(z)dz}{(z-z_0)^m} = \frac{g^{(m-1)}(z_0)}{(m-1)!}.$$

In terms of f this yields[1]

Theorem 11.2.3 *If $f(z)$ has a pole of order m at z_0, then*

$$\operatorname{Res}\big[f(z_0)\big] = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\big[(z-z_0)^m f(z)\big]. \tag{11.4}$$

For the special, but important, case of a simple pole, we obtain

$$\operatorname{Res}\big[f(z_0)\big] = \lim_{z\to z_0}\big[(z-z_0) f(z)\big]. \tag{11.5}$$

11.3 Evaluation of Definite Integrals

The most widespread application of residues occurs in the evaluation of real definite integrals. It is possible to "complexify" certain real definite integrals and relate them to contour integrations in the complex plane. We will discuss this method shortly; however, we first need a lemma.

[1]The limit is taken because in many cases the mere substitution of z_0 may result in an indeterminate form.

Lemma 11.3.1 (Jordan's lemma) *Let C_R be a semicircle of radius R in the upper half of the complex plane* (*UHP*) *and centered at the origin. Let f be a function that tends uniformly to zero faster than $1/|z|$ for $\arg(z) \in [0, \pi]$ as $|z| \to \infty$. Let α be a nonnegative real number. Then*

Jordan's lemma

$$\lim_{R\to\infty} I_R \equiv \lim_{R\to\infty} \int_{C_R} e^{i\alpha z} f(z)\, dz = 0.$$

Proof For $z \in C_R$ we write $z = Re^{i\theta}$, $dz = iRe^{i\theta} d\theta$, and

$$i\alpha z = i\alpha(R\cos\theta + iR\sin\theta) = i\alpha R\cos\theta - \alpha R\sin\theta$$

and substitute in the absolute value of the integral to show that

$$|I_R| \le \int_0^\pi e^{-\alpha R\sin\theta} R \left| f\left(Re^{i\theta}\right)\right| d\theta.$$

By assumption, $R|f(Re^{i\theta})| < \epsilon(R)$ independent of θ, where $\epsilon(R)$ is an arbitrary positive number that tends to zero as $R \to \infty$. By breaking up the interval of integration into two equal pieces and changing θ to $\pi - \theta$ in the second integral, one can show that

$$|I_R| < 2\epsilon(R) \int_0^{\pi/2} e^{-\alpha R\sin\theta} d\theta.$$

Furthermore, by plotting $\sin\theta$ and $2\theta/\pi$ on the same graph, one can easily see that $\sin\theta \ge 2\theta/\pi$ for $0 \le \theta \le \pi/2$. Thus,

$$|I_R| < 2\epsilon(R) \int_0^{\pi/2} e^{-(2\alpha R/\pi)\theta} d\theta = \frac{\pi\epsilon(R)}{\alpha R}\left(1 - e^{-\alpha R}\right),$$

which goes to zero as R gets larger and larger. □

Note that Jordan's lemma applies for $\alpha = 0$ as well, because $(1 - e^{-\alpha R}) \to \alpha R$ as $\alpha \to 0$. If $\alpha < 0$, the lemma is still valid if the semicircle C_R is taken in the lower half of the complex plane (LHP) and $f(z)$ goes to zero uniformly for $\pi \le \arg(z) \le 2\pi$.

We are now in a position to apply the residue theorem to the evaluation of definite integrals. The three types of integrals most commonly encountered are discussed separately below. In all cases we assume that Jordan's lemma holds.

11.3.1 Integrals of Rational Functions

The first type of integral we can evaluate using the residue theorem is of the form

$$I_1 = \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx,$$

where $p(x)$ and $q(x)$ are real polynomials, and $q(x) \neq 0$ for any real x. We can then write

$$I_1 = \lim_{R\to\infty} \int_{-R}^{R} \frac{p(x)}{q(x)}\, dx = \lim_{R\to\infty} \int_{C_x} \frac{p(z)}{q(z)}\, dz,$$

where C_x is the (open) contour lying on the real axis from $-R$ to $+R$. Since Jordan's lemma holds by assumption, we can close that contour by adding to it the semicircle of radius R [see Fig. 11.2(a)]. This will not affect the value of the integral, because in the limit $R \to \infty$, the contribution of the integral of the semicircle tends to zero. We close the contour in the UHP if $q(z)$ has at least one zero there. We then get

$$I_1 = \lim_{R\to\infty} \oint_C \frac{p(z)}{q(z)}\, dz = 2\pi i \sum_{j=1}^{k} \mathrm{Res}\left[\frac{p(z_j)}{q(z_j)}\right],$$

where C is the closed contour composed of the interval $(-R, R)$ and the semicircle C_R, and $\{z_j\}_{j=1}^k$ are the zeros of $q(z)$ in the UHP. We may instead close the contour in the LHP,[2] in which case

$$I_1 = -2\pi i \sum_{j=1}^{k} \mathrm{Res}\left[\frac{p(z_j)}{q(z_j)}\right],$$

where $\{z_j\}_{j=1}^k$ are the zeros of $q(z)$ in the LHP. The minus sign indicates that in the LHP we (are forced to) integrate clockwise.

Example 11.3.2 Let us evaluate the integral $I = \int_0^\infty x^2\, dx/[(x^2+1)(x^2+9)]$. Since the integrand is even, we can extend the interval of integration to all real numbers (and divide the result by 2). It is shown below that Jordan's lemma holds. Therefore, we write the contour integral corresponding to I:

$$I = \frac{1}{2} \oint_C \frac{z^2\, dz}{(z^2+1)(z^2+9)},$$

where C is as shown in Fig. 11.2(a). Note that the contour is traversed in the positive sense. This is always true for the UHP. The singularities of the function in the UHP are the *simple* poles i and $3i$ corresponding to the simple zeros of the denominator. The residues at these poles are

$$\mathrm{Res}\big[f(i)\big] = \lim_{z\to i}(z-i)\frac{z^2}{(z-i)(z+i)(z^2+9)} = -\frac{1}{16i},$$

$$\mathrm{Res}\big[f(3i)\big] = \lim_{z\to 3i}(z-3i)\frac{z^2}{(z^2+1)(z-3i)(z+3i)} = \frac{3}{16i}.$$

[2]Provided that Jordan's lemma holds there.

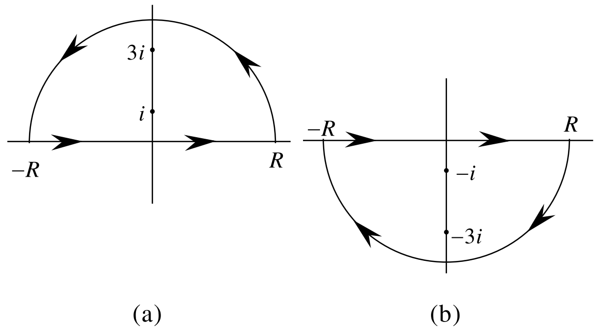

Fig. 11.2 (**a**) The large semicircle is chosen in the UHP. (**b**) Note how the direction of contour integration is forced to be clockwise when the semicircle is chosen in the LHP

Thus, we obtain

$$I=\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+9)}=\frac{1}{2}\oint_C \frac{z^2\,dz}{(z^2+1)(z^2+9)}$$
$$=\pi i\left(-\frac{1}{16i}+\frac{3}{16i}\right)=\frac{\pi}{8}.$$

It is instructive to obtain the same results using the LHP. In this case, the contour is as shown in Fig. 11.2(b) and is taken clockwise, so we have to introduce a minus sign. The singular points are at $z=-i$ and $z=-3i$. These are simple poles at which the residues of the function are

$$\operatorname{Res}\bigl[f(-i)\bigr]=\lim_{z\to -i}(z+i)\frac{z^2}{(z-i)(z+i)(z^2+9)}=\frac{1}{16i},$$
$$\operatorname{Res}\bigl[f(-3i)\bigr]=\lim_{z\to -3i}(z+3i)\frac{z^2}{(z^2+1)(z-3i)(z+3i)}=-\frac{3}{16i}.$$

Therefore,

$$I=\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+9)}=\frac{1}{2}\oint_C \frac{z^2\,dz}{(z^2+1)(z^2+9)}$$
$$=-\pi i\left(\frac{1}{16i}-\frac{3}{16i}\right)=\frac{\pi}{8}.$$

To show that Jordan's lemma applies to this integral, we have only to establish that $\lim_{R\to\infty} R|f(Re^{i\theta})|=0$. In the case at hand, $\alpha=0$ because there is no exponential function in the integrand. Thus,

$$R\bigl|f\bigl(Re^{i\theta}\bigr)\bigr|=R\left|\frac{R^2e^{2i\theta}}{(R^2e^{2i\theta}+1)(R^2e^{2i\theta}+9)}\right|=\frac{R^3}{|R^2e^{2i\theta}+1||R^2e^{2i\theta}+9|},$$

which clearly goes to zero as $R\to\infty$.

Example 11.3.3 Let us now consider a slightly more complicated integral:

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)^2},$$

which turns into $\oint_C z^2\, dz/[(z^2+1)(z^2+4)^2]$ as a contour integral. The poles in the UHP are at $z = i$ and $z = 2i$. The former is a simple pole, and the latter is a pole of order 2. Thus, using Eqs. (11.5) and (11.4), we obtain

$$\begin{aligned}
\mathrm{Res}\big[f(i)\big] &= \lim_{z\to i}(z-i)\frac{z^2}{(z-i)(z+i)(z^2+4)^2} = -\frac{1}{18i},\\
\mathrm{Res}\big[f(2i)\big] &= \frac{1}{(2-1)!}\lim_{z\to 2i}\frac{d}{dz}\left[(z-2i)^2\frac{z^2}{(z^2+1)(z+2i)^2(z-2i)^2}\right]\\
&= \lim_{z\to 2i}\frac{d}{dz}\left[\frac{z^2}{(z^2+1)(z+2i)^2}\right] = \frac{5}{72i},
\end{aligned}$$

and

$$\int_{-\infty}^{\infty}\frac{x^2\, dx}{(x^2+1)(x^2+4)^2} = 2\pi i\left(-\frac{1}{18i}+\frac{5}{72i}\right) = \frac{\pi}{36}.$$

Closing the contour in the LHP would yield the same result.

11.3.2 Products of Rational and Trigonometric Functions

The second type of integral we can evaluate using the residue theorem is of the form

$$\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\cos ax\, dx \quad \text{or} \quad \int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\sin ax\, dx,$$

where a is a real number, $p(x)$ and $q(x)$ are real polynomials in x, and $q(x)$ has no real zeros. These integrals are the real and imaginary parts of

$$I_2 = \int_{-\infty}^{\infty}\frac{p(x)}{q(x)}e^{iax}\, dx.$$

The presence of e^{iax} dictates the choice of the half-plane: If $a \ge 0$, we choose the UHP; otherwise, we choose the LHP. We must, of course, have enough powers of x in the denominator to render $R|p(Re^{i\theta})/q(Re^{i\theta})|$ uniformly convergent to zero.

Example 11.3.4 Let us evaluate $\int_{-\infty}^{\infty}[\cos ax/(x^2+1)^2]\, dx$ where $a \neq 0$. This integral is the real part of the integral $I_2 = \int_{-\infty}^{\infty} e^{iax}\, dx/(x^2+1)^2$. When $a > 0$, we close in the UHP as advised by Jordan's lemma. Then we proceed as for integrals of rational functions. Thus, we have

$$I_2 = \oint_C \frac{e^{iaz}}{(z^2+1)^2}\, dz = 2\pi i\,\mathrm{Res}\big[f(i)\big] \quad \text{for } a > 0$$

because there is only one pole (of order 2) in the UHP at $z = i$. We next calculate the residue:

$$\mathrm{Res}\big[f(i)\big] = \lim_{z\to i}\frac{d}{dz}\left[(z-i)^2\frac{e^{iaz}}{(z-i)^2(z+i)^2}\right]$$

$$= \lim_{z \to i} \frac{d}{dz}\left[\frac{e^{iaz}}{(z+i)^2}\right] = \lim_{z \to i}\left[\frac{(z+i)iae^{iaz} - 2e^{iaz}}{(z+i)^3}\right]$$
$$= \frac{e^{-a}}{4i}(1+a).$$

Substituting this in the expression for I_2, we obtain $I_2 = \frac{\pi}{2}e^{-a}(1+a)$ for $a > 0$.

When $a < 0$, we have to close the contour in the LHP, where the pole of order 2 is at $z = -i$ and the contour is taken clockwise. Thus, we get

$$I_2 = \oint_C \frac{e^{iaz}}{(z^2+1)^2}\,dz = -2\pi i \operatorname{Res}\big[f(-i)\big] \quad \text{for } a < 0.$$

For the residue we obtain

$$\operatorname{Res}\big[f(-i)\big] = \lim_{z \to -i} \frac{d}{dz}\left[(z+i)^2 \frac{e^{iaz}}{(z-i)^2(z+i)^2}\right] = -\frac{e^a}{4i}(1-a),$$

and the expression for I_2 becomes $I_2 = \frac{\pi}{2}e^a(1-a)$ for $a < 0$. We can combine the two results and write

$$\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+1)^2}\,dx = \operatorname{Re}(I_2) = I_2 = \frac{\pi}{2}\big(1+|a|\big)e^{-|a|}.$$

Example 11.3.5 As another example, let us evaluate

$$\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4+4}\,dx \quad \text{where } a \neq 0.$$

This is the imaginary part of the integral $I_2 = \int_{-\infty}^{\infty} xe^{iax}\,dx/(x^4+4)$, which, in terms of z and for the closed contour in the UHP (when $a > 0$), becomes

$$I_2 = \oint_C \frac{ze^{iaz}}{z^4+4}\,dz = 2\pi i \sum_{j=1}^{m} \operatorname{Res}\big[f(z_j)\big] \quad \text{for } a > 0. \tag{11.6}$$

The singularities are determined by the zeros of the denominator: $z^4 + 4 = 0$, or $z = 1 \pm i, -1 \pm i$. Of these four simple poles only two, $1+i$ and $-1+i$, are in the UHP. We now calculate the residues:

$$\operatorname{Res}\big[f(1+i)\big]$$
$$= \lim_{z \to 1+i} (z-1-i)\frac{ze^{iaz}}{(z-1-i)(z-1+i)(z+1-i)(z+1+i)}$$
$$= \frac{(1+i)e^{ia(1+i)}}{(2i)(2)(2+2i)} = \frac{e^{ia}e^{-a}}{8i},$$

$$\begin{aligned}
&\mathrm{Res}\big[f(-1+i)\big]\\
&\quad=\lim_{z\to -1+i}(z+1-i)\frac{ze^{iaz}}{(z+1-i)(z+1+i)(z-1-i)(z-1+i)}\\
&\quad=\frac{(-1+i)e^{ia(-1+i)}}{(2i)(-2)(-2+2i)}=-\frac{e^{-ia}e^{-a}}{8i}.
\end{aligned}$$

Substituting in Eq. (11.6), we obtain

$$I_2=2\pi i\frac{e^{-a}}{8i}\big(e^{ia}-e^{-ia}\big)=i\frac{\pi}{2}e^{-a}\sin a.$$

Thus,

$$\int_{-\infty}^{\infty}\frac{x\sin ax}{x^4+4}\,dx=\mathrm{Im}(I_2)=\frac{\pi}{2}e^{-a}\sin a\quad\text{for } a>0. \tag{11.7}$$

For $a<0$, we could close the contour in the LHP. But there is an easier way of getting to the answer. We note that $-a>0$, and Eq. (11.7) yields

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{x\sin ax}{x^4+4}\,dx&=-\int_{-\infty}^{\infty}\frac{x\sin[(-a)x]}{x^4+4}\,dx\\
&=-\frac{\pi}{2}e^{-(-a)}\sin(-a)=\frac{\pi}{2}e^{a}\sin a.
\end{aligned}$$

We can collect the two cases in

$$\int_{-\infty}^{\infty}\frac{x\sin ax}{x^4+4}\,dx=\frac{\pi}{2}e^{-|a|}\sin a.$$

11.3.3 Functions of Trigonometric Functions

The third type of integral we can evaluate using the residue theorem involves only trigonometric functions and is of the form

$$\int_0^{2\pi}F(\sin\theta,\cos\theta)\,d\theta,$$

where F is some (typically rational) function of its arguments. Since θ varies from 0 to 2π, we can consider it an argument of a point z on the unit circle centered at the origin. Then $z=e^{i\theta}$ and $e^{-i\theta}=1/z$, and we can substitute $\cos\theta=(z+1/z)/2$, $\sin\theta=(z-1/z)/(2i)$, and $d\theta=dz/(iz)$ in the original integral, to obtain

$$\oint_C F\left(\frac{z-1/z}{2i},\frac{z+1/z}{2}\right)\frac{dz}{iz}.$$

This integral can often be evaluated using the method of residues.

Example 11.3.6 Let us evaluate the integral $\int_0^{2\pi} d\theta/(1 + a\cos\theta)$ where $|a| < 1$. Substituting for $\cos\theta$ and $d\theta$ in terms of z, we obtain

$$\oint_C \frac{dz/iz}{1 + a[(z^2+1)/(2z)]} = \frac{2}{i}\oint_C \frac{dz}{2z + az^2 + a},$$

where C is the unit circle centered at the origin. The singularities of the integrand are the zeros of its denominator:

$$z_1 = \frac{-1 + \sqrt{1-a^2}}{a} \quad \text{and} \quad z_2 = \frac{-1 - \sqrt{1-a^2}}{a}.$$

For $|a| < 1$ it is clear that z_2 will lie outside the unit circle C; therefore, it does not contribute to the integral. But z_1 lies inside, and we obtain

$$\oint_C \frac{dz}{2z + az^2 + a} = 2\pi i \operatorname{Res}\big[f(z_1)\big].$$

The residue of the simple pole at z_1 can be calculated:

$$\begin{aligned}\operatorname{Res}\big[f(z_1)\big] &= \lim_{z\to z_1}(z - z_1)\frac{1}{a(z-z_1)(z-z_2)} = \frac{1}{a}\left(\frac{1}{z_1 - z_2}\right)\\ &= \frac{1}{a}\left(\frac{a}{2\sqrt{1-a^2}}\right) = \frac{1}{2\sqrt{1-a^2}}.\end{aligned}$$

It follows that

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2}{i}\oint_C \frac{dz}{2z + az^2 + a} = \frac{2}{i}2\pi i\left(\frac{1}{2\sqrt{1-a^2}}\right) = \frac{2\pi}{\sqrt{1-a^2}}.$$

Example 11.3.7 As another example, let us consider the integral

$$I = \int_0^{\pi} \frac{d\theta}{(a + \cos\theta)^2} \quad \text{where } a > 1.$$

Since $\cos\theta$ is an even function of θ, we may write

$$I = \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{(a + \cos\theta)^2} \quad \text{where } a > 1.$$

This integration is over a complete cycle around the origin, and we can make the usual substitution:

$$I = \frac{1}{2}\oint_C \frac{dz/iz}{[a + (z^2+1)/2z]^2} = \frac{2}{i}\oint_C \frac{z\,dz}{(z^2 + 2az + 1)^2}.$$

The denominator has the roots $z_1 = -a + \sqrt{a^2 - 1}$ and $z_2 = -a - \sqrt{a^2 - 1}$, which are both of order 2. The second root is outside the unit circle because $a > 1$. Also, it is easily verified that for all $a > 1$, z_1 is inside the unit circle.

Since z_1 is a pole of order 2, we have

$$\begin{aligned}\mathrm{Res}\big[f(z_1)\big] &= \lim_{z\to z_1}\frac{d}{dz}\left[(z-z_1)^2\frac{z}{(z-z_1)^2(z-z_2)^2}\right]\\ &= \lim_{z\to z_1}\frac{d}{dz}\left[\frac{z}{(z-z_2)^2}\right] = \frac{1}{(z_1-z_2)^2}-\frac{2z_1}{(z_1-z_2)^3}\\ &= \frac{a}{4(a^2-1)^{3/2}}.\end{aligned}$$

We thus obtain $I = \frac{2}{i}2\pi i\,\mathrm{Res}[f(z_1)] = \frac{\pi a}{(a^2-1)^{3/2}}$.

11.3.4 Some Other Integrals

The three types of definite integrals discussed above do not exhaust all possible applications of the residue theorem. There are other integrals that do not fit into any of the foregoing three categories but are still manageable. As the next two examples demonstrate, a clever choice of contours allows evaluation of other types of integrals.

Example 11.3.8 Let us evaluate the Gaussian integral

$$I = \int_{-\infty}^{\infty} e^{iax-bx^2}dx \quad \text{where } a, b\in\mathbb{R},\ b>0.$$

Completing squares in the exponent, we have

$$I = \int_{-\infty}^{\infty} e^{-b[x-ia/(2b)]^2-a^2/4b}dx = e^{-a^2/4b}\lim_{R\to\infty}\int_{-R}^{R} e^{-b[x-ia/(2b)]^2}dx.$$

If we change the variable of integration to $z = x - ia/(2b)$, we obtain

$$I = e^{-a^2/(4b)}\lim_{R\to\infty}\int_{-R-ia/(2b)}^{R-ia/(2b)} e^{-bz^2}dz.$$

Let us now define I_R:

$$I_R \equiv \int_{-R-ia/(2b)}^{R-ia/(2b)} e^{-bz^2}dz.$$

This is an integral along a straight line C_1 that is parallel to the x-axis (see Fig. 11.3). We close the contour as shown and note that e^{-bz^2} is analytic throughout the interior of the closed contour (it is an entire function!). Thus, the contour integral must vanish by the Cauchy-Goursat theorem. So we obtain

$$I_R + \int_{C_3} e^{-bz^2}dz + \int_{R}^{-R} e^{-bx^2}dx + \int_{C_4} e^{-bz^2}dz = 0.$$

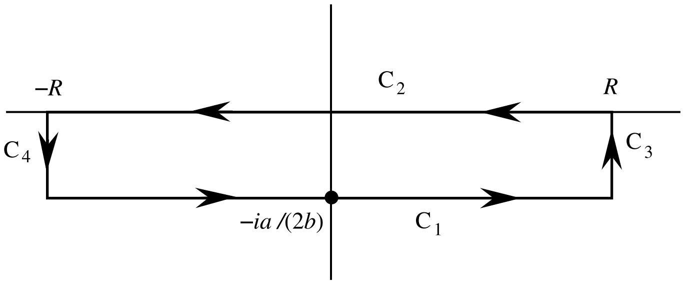

Fig. 11.3 The contour for the evaluation of the Gaussian integral

Along C_3, $z = R + iy$ and

$$\int_{C_3} e^{-bz^2}\,dz = \int_{-ia/(2b)}^{0} e^{-b(R+iy)^2}\,i\,dy = ie^{-bR^2}\int_{-ia/(2b)}^{0} e^{by^2-2ibRy}\,dy$$

which clearly tends to zero as $R \to \infty$. We get a similar result for the integral along C_4. Therefore, we have

$$I_R = \int_{-R}^{R} e^{-bx^2}\,dx \quad\Rightarrow\quad \lim_{R\to\infty} I_R = \int_{-\infty}^{\infty} e^{-bx^2}\,dx = \sqrt{\frac{\pi}{b}}.$$

Finally, we get

$$\int_{-\infty}^{\infty} e^{iax-bx^2}\,dx = \sqrt{\frac{\pi}{b}}\,e^{-a^2/(4b)}.$$

Example 11.3.9 Let us evaluate $I = \int_0^\infty dx/(x^3+1)$. If the integrand were even, we could extend the lower limit of integration to $-\infty$ and close the contour in the UHP. Since this is not the case, we need to use a different trick. To get a hint as to how to close the contour, we study the singularities of the integrand. These are simply the roots of the denominator: $z^3 = -1$ or $z_n = e^{i(2n+1)\pi/3}$ with $n = 0, 1, 2$. These, as well as a contour that has only z_0 as an interior point, are shown in Fig. 11.4. We thus have

$$I + \int_{C_R} \frac{dz}{z^3+1} + \int_{C_2} \frac{dz}{z^3+1} = 2\pi i \operatorname{Res}\bigl[f(z_0)\bigr]. \tag{11.8}$$

The C_R integral vanishes, as usual. Along C_2, $z = re^{i\alpha}$, with constant α, so that $dz = e^{i\alpha}dr$ and

$$\int_{C_2} \frac{dz}{z^3+1} = \int_{\infty}^{0} \frac{e^{i\alpha}dr}{(re^{i\alpha})^3+1} = -e^{i\alpha}\int_0^\infty \frac{dr}{r^3e^{3i\alpha}+1}.$$

In particular, if we choose $3\alpha = 2\pi$, we obtain

$$\int_{C_2} \frac{dz}{z^3+1} = -e^{i2\pi/3}\int_0^\infty \frac{dr}{r^3+1} = -e^{i2\pi/3} I.$$

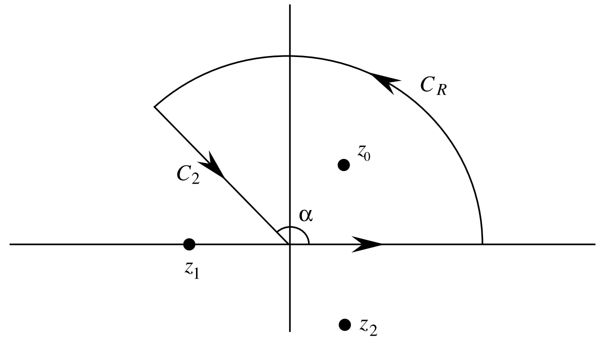

Fig. 11.4 The contour is chosen so that only one of the poles lies inside

Substituting this in Eq. (11.8) gives

$$\left(1 - e^{i2\pi/3}\right) I = 2\pi i \operatorname{Res}\left[f(z_0)\right] \quad \Rightarrow \quad I = \frac{2\pi i}{1 - e^{i2\pi/3}} \operatorname{Res}\left[f(z_0)\right].$$

On the other hand,

$$\operatorname{Res}\left[f(z_0)\right] = \lim_{z \to z_0} (z - z_0) \frac{1}{(z - z_0)(z - z_1)(z - z_2)}$$
$$= \frac{1}{(z_0 - z_1)(z_0 - z_2)} = \frac{1}{(e^{i\pi/3} - e^{i\pi})(e^{i\pi/3} - e^{i5\pi/3})}.$$

These last two equations yield

$$I = \frac{2\pi i}{1 - e^{i2\pi/3}} \frac{1}{(e^{i\pi/3} - e^{i\pi})(e^{i\pi/3} - e^{i5\pi/3})} = \frac{2\pi}{3\sqrt{3}}.$$

11.3.5 Principal Value of an Integral

So far we have discussed only integrals of functions that have no singularities on the contour. Let us now investigate the consequences of the presence of singular points on the contour. Consider the integral

$$\int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\, dx, \tag{11.9}$$

where x_0 is a real number and f is analytic at x_0. To avoid x_0—which causes the integrand to diverge—we bypass it by indenting the contour as shown in Fig. 11.5 and denoting the new contour by C_u. The contour C_0 is simply a semicircle of radius ϵ. For the contour C_u, we have

$$\int_{C_u} \frac{f(z)}{z - x_0}\, dz = \int_{-\infty}^{x_0 - \epsilon} \frac{f(x)}{x - x_0}\, dx + \int_{x_0 + \epsilon}^{\infty} \frac{f(x)}{x - x_0}\, dx + \int_{C_0} \frac{f(z)}{z - x_0}\, dz.$$

principal value of an integral

In the limit $\epsilon \to 0$, the sum of the first two terms on the RHS—when it exists—defines the **principal value** of the integral in Eq. (11.9):

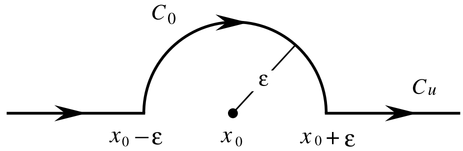

Fig. 11.5 The contour C_u avoids x_0

$$P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}\,dx=\lim_{\epsilon\to 0}\left[\int_{-\infty}^{x_0-\epsilon}\frac{f(x)}{x-x_0}\,dx+\int_{x_0+\epsilon}^{\infty}\frac{f(x)}{x-x_0}\,dx\right].$$

The integral over the semicircle is calculated by noting that $z-x_0=\epsilon e^{i\theta}$ and $dz=i\epsilon e^{i\theta}d\theta$: $\int_{C_0}f(z)\,dz/(z-x_0)=-i\pi f(x_0)$. Therefore,

$$\int_{C_u}\frac{f(z)}{z-x_0}\,dz=P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}\,dx-i\pi f(x_0). \tag{11.10}$$

On the other hand, if C_0 is taken below the singularity on a contour C_d, say, we obtain

$$\int_{C_d}\frac{f(z)}{z-x_0}\,dz=P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}\,dx+i\pi f(x_0).$$

We see that the contour integral depends on how the singular point x_0 is avoided. However, the principal value, if it exists, is unique. To calculate this principal value we close the contour by adding a large semicircle to it as before, assuming that the contribution from this semicircle goes to zero by Jordan's lemma. The contours C_u and C_d are replaced by a closed contour, and the value of the integral will be given by the residue theorem. We therefore have

$$P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}\,dx=\pm i\pi f(x_0)+2\pi i\sum_{j=1}^{m}\operatorname{Res}\left[\frac{f(z_j)}{z_j-x_0}\right], \tag{11.11}$$

where $\{z_j\}_{j=1}^m$ are the poles of $f(z)$, the plus sign corresponds to placing the infinitesimal semicircle in the UHP, as shown in Fig. 11.5, and the minus sign corresponds to the other choice.

Example 11.3.10 Let us use the principal-value method to evaluate the integral

$$I=\int_0^{\infty}\frac{\sin x}{x}\,dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx.$$

It appears that $x=0$ is a singular point of the integrand; in reality, however, it is only a removable singularity, as can be verified by the Taylor expansion of $\sin x/x$. To make use of the principal-value method, we write

$$I=\frac{1}{2}\operatorname{Im}\left(\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx\right)=\frac{1}{2}\operatorname{Im}\left(P\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx\right).$$

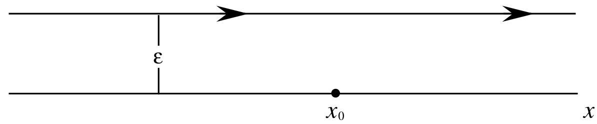

Fig. 11.6 The equivalent contour obtained by "stretching" C_u, the contour of Fig. 11.5

We now use Eq. (11.11) with the small circle in the UHP, noting that there are no singularities for e^{ix}/x there. This yields

$$P\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx = i\pi e^{(0)} = i\pi.$$

Therefore,

$$\int_0^{\infty}\frac{\sin x}{x}\,dx = \frac{1}{2}\,\mathrm{Im}(i\pi) = \frac{\pi}{2}.$$

The principal value of an integral can be written more compactly if we deform the contour C_u by stretching it into that shown in Fig. 11.6. For small enough ϵ, such a deformation will not change the number of singularities within the infinite closed contour. Thus, the LHS of Eq. (11.10) will have limits of integration $-\infty + i\epsilon$ and $+\infty + i\epsilon$. If we change the variable of integration to $\xi = z - i\epsilon$, this integral becomes

$$\int_{-\infty}^{\infty}\frac{f(\xi + i\epsilon)}{\xi + i\epsilon - x_0}\,d\xi = \int_{-\infty}^{\infty}\frac{f(\xi)\,d\xi}{\xi - x_0 + i\epsilon} = \int_{-\infty}^{\infty}\frac{f(z)\,dz}{z - x_0 + i\epsilon}, \tag{11.12}$$

where in the last step we changed the dummy integration variable back to z. Note that since f is assumed to be continuous at all points on the contour, $f(\xi + i\epsilon) \to f(\xi)$ for small ϵ. The last integral of Eq. (11.12) shows that there is no singularity on the new x-axis; we have pushed the singularity down to $x_0 - i\epsilon$. In other words, we have given the singularity on the x-axis a small negative imaginary part. We can thus rewrite Eq. (11.10) as

$$P\int_{-\infty}^{\infty}\frac{f(x)}{x - x_0}\,dx = i\pi f(x_0) + \int_{-\infty}^{\infty}\frac{f(x)\,dx}{x - x_0 + i\epsilon},$$

where x is used instead of z in the last integral because we are indeed integrating along the new x-axis—assuming that no other singularities are present in the UHP. A similar argument, this time for the LHP, introduces a minus sign for the first term on the RHS and for the ϵ term in the denominator. Therefore,

Proposition 11.3.11 *The principal value of an integral with one simple pole on the real axis is*

$$P\int_{-\infty}^{\infty}\frac{f(x)}{x - x_0}\,dx = \pm i\pi f(x_0) + \int_{-\infty}^{\infty}\frac{f(x)\,dx}{x - x_0 \pm i\epsilon}, \tag{11.13}$$

where the plus (minus) sign refers to the UHP (LHP).

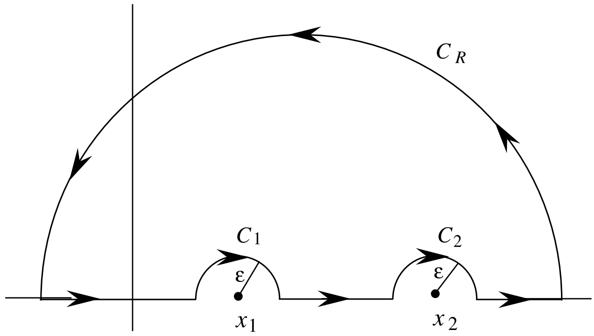

Fig. 11.7 One of the four choices of contours for evaluating the principal value of the integral when there are two poles on the real axis

This result is sometimes abbreviated as

$$\frac{1}{x - x_0 \pm i\epsilon} = P\frac{1}{x - x_0} \mp i\pi\delta(x - x_0). \tag{11.14}$$

Example 11.3.12 Let us use residues to evaluate the function

$$f(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{ikx}\,dx}{x - i\epsilon}, \quad \epsilon > 0.$$

The integral representation of the θ (step) function

We have to close the contour by adding a large semicircle. Whether we do this in the UHP or the LHP is dictated by the sign of k: If $k > 0$, we close in the UHP. Thus,

$$f(k) = \frac{1}{2\pi i}\int_C \frac{e^{ikz}\,dz}{z - i\epsilon} = \text{Res}\left[\frac{e^{ikz}}{z - i\epsilon}\right]_{z\to i\epsilon}$$
$$= \lim_{z\to i\epsilon}\left[(z - i\epsilon)\frac{e^{ikz}}{z - i\epsilon}\right] = e^{-k\epsilon} \xrightarrow[\epsilon\to 0]{} 1.$$

On the other hand, if $k < 0$, we must close in the LHP, in which the integrand is analytic. Thus, by the Cauchy-Goursat theorem, the integral vanishes. Therefore, we have

$$f(k) = \begin{cases} 1 & \text{if } k > 0, \\ 0 & \text{if } k < 0. \end{cases}$$

theta (or step) function

This is precisely the definition of the **theta function** (or step function). Thus, we have obtained an integral representation of that function:

$$\theta(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{ixt}}{t - i\epsilon}\,dt.$$

Now suppose that there are two singular points on the real axis, at x_1 and x_2. Let us avoid x_1 and x_2 by making little semicircles, as before, letting both semicircles be in the UHP (see Fig. 11.7). Without writing the integrands, we can represent the contour integral by

$$\int_{-\infty}^{x_1-\epsilon} + \int_{C_1} + \int_{x_1+\epsilon}^{x_2-\epsilon} + \int_{C_2} + \int_{x_2+\epsilon}^{\infty} + \int_{C_R} = 2\pi i \sum \text{Res}.$$

The principal value of the integral is naturally defined to be the sum of all integrals having ϵ in their limits. The contribution from the small semicircle C_1 can be calculated by substituting $z - x_1 = \epsilon e^{i\theta}$ in the integral:

$$\int_{C_1} \frac{f(z)\,dz}{(z-x_1)(z-x_2)} = \int_{\pi}^{0} \frac{f(x_1+\epsilon e^{i\theta}) i\epsilon e^{i\theta}\,d\theta}{\epsilon e^{i\theta}(x_1+\epsilon e^{i\theta}-x_2)} = -i\pi \frac{f(x_1)}{x_1-x_2},$$

with a similar result for C_2. Putting everything together, we get

$$P\int_{-\infty}^{\infty} \frac{f(x)}{(x-x_1)(x-x_2)}\,dx - i\pi \frac{f(x_2)-f(x_1)}{x_2-x_1} = 2\pi i \sum \text{Res}.$$

If we include the case where both C_1 and C_2 are in the LHP, we get

$$P\int_{-\infty}^{\infty} \frac{f(x)}{(x-x_1)(x-x_2)}\,dx = \pm i\pi \frac{f(x_2)-f(x_1)}{x_2-x_1} + 2\pi i \sum \text{Res}, \tag{11.15}$$

where the plus sign is for the case where C_1 and C_2 are in the UHP and the minus sign for the case where both are in the LHP. We can also obtain the result for the case where the two singularities coincide by taking the limit $x_1 \to x_2$. Then the RHS of the last equation becomes a derivative, and we obtain

$$P\int_{-\infty}^{\infty} \frac{f(x)}{(x-x_0)^2}\,dx = \pm i\pi f'(x_0) + 2\pi i \sum \text{Res}.$$

Example 11.3.13 An expression encountered in the study of Green's functions or propagators (which we shall discuss later in the book) is

$$\int_{-\infty}^{\infty} \frac{e^{itx}\,dx}{x^2-k^2},$$

where k and t are real constants. We want to calculate the principal value of this integral. We use Eq. (11.15) and note that for $t > 0$, we need to close the contour in the UHP, where there are no poles:

$$P\int_{-\infty}^{\infty} \frac{e^{itx}\,dx}{x^2-k^2} = P\int_{-\infty}^{\infty} \frac{e^{itx}\,dx}{(x-k)(x+k)} = i\pi \frac{e^{ikt}-e^{-ikt}}{2k} = -\pi \frac{\sin kt}{k}.$$

When $t < 0$, we have to close the contour in the LHP, where again there are no poles:

$$P\int_{-\infty}^{\infty} \frac{e^{itx}\,dx}{x^2-k^2} = P\int_{-\infty}^{\infty} \frac{e^{itx}\,dx}{(x-k)(x+k)} = -i\pi \frac{e^{ikt}-e^{-ikt}}{2k} = \pi \frac{\sin kt}{k}.$$

The two results above can be combined into a single relation:

$$P\int_{-\infty}^{\infty} \frac{e^{itx}\,dx}{x^2-k^2} = -\pi \frac{\sin k|t|}{k}.$$

11.4 Problems

11.1 Evaluate each of the following integrals, for all of which C is the circle $|z| = 3$.

(a) $\displaystyle\oint_C \frac{4z-3}{z(z-2)}\,dz.$ (b) $\displaystyle\oint_C \frac{e^z}{z(z-i\pi)}\,dz.$

(c) $\displaystyle\oint_C \frac{\cos z}{z(z-\pi)}\,dz.$ (d) $\displaystyle\oint_C \frac{z^2+1}{z(z-1)}\,dz.$

(e) $\displaystyle\oint_C \frac{\cosh z}{z^2+\pi^2}\,dz.$ (f) $\displaystyle\oint_C \frac{1-\cos z}{z^2}\,dz.$

(g) $\displaystyle\oint_C \frac{\sinh z}{z^4}\,dz.$ (h) $\displaystyle\oint_C z\cos\left(\frac{1}{z}\right)dz.$

(i) $\displaystyle\oint_C \frac{dz}{z^3(z+5)}.$ (j) $\displaystyle\oint_C \tan z\,dz.$

(k) $\displaystyle\oint_C \frac{dz}{\sinh 2z}.$ (l) $\displaystyle\oint_C \frac{e^z}{z^2}\,dz.$

(m) $\displaystyle\oint_C \frac{dz}{z^2\sin z}.$ (n) $\displaystyle\oint_C \frac{e^z\,dz}{(z-1)(z-2)}.$

11.2 Let $h(z)$ be analytic and have a simple zero at $z = z_0$, and let $g(z)$ be analytic there. Let $f(z) = g(z)/h(z)$, and show that

$$\operatorname{Res}\big[f(z_0)\big] = \frac{g(z_0)}{h'(z_0)}.$$

11.3 Find the residue of $f(z) = 1/\cos z$ at each of its poles.

11.4 Evaluate the integral $\int_0^\infty dx/[(x^2+1)(x^2+4)]$ by closing the contour (a) in the UHP and (b) in the LHP.

11.5 Evaluate the following integrals, in which a and b are nonzero real constants.

(a) $\displaystyle\int_0^\infty \frac{2x^2+1}{x^4+5x^2+6}\,dx.$ (b) $\displaystyle\int_0^\infty \frac{dx}{6x^4+5x^2+1}.$

(c) $\displaystyle\int_0^\infty \frac{dx}{x^4+1}.$ (d) $\displaystyle\int_0^\infty \frac{\cos x\,dx}{(x^2+a^2)^2(x^2+b^2)}.$

(e) $\displaystyle\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx.$ (f) $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2}.$

(g) $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2(x^2+2)}.$ (h) $\displaystyle\int_0^\infty \frac{2x^2-1}{x^6+1}\,dx.$

(i) $\displaystyle\int_0^\infty \frac{x^2 dx}{(x^2+a^2)^2}$. (j) $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{(x^2+4x+13)^2}$.

(k) $\displaystyle\int_0^\infty \frac{x^3 \sin ax}{x^6+1}\,dx$. (l) $\displaystyle\int_0^\infty \frac{x^2+1}{x^2+4}\,dx$.

(m) $\displaystyle\int_{-\infty}^\infty \frac{x\cos x\,dx}{x^2-2x+10}$. (n) $\displaystyle\int_{-\infty}^\infty \frac{x\sin x\,dx}{x^2-2x+10}$.

(o) $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$. (p) $\displaystyle\int_0^\infty \frac{x^2 dx}{(x^2+4)^2(x^2+25)}$.

(q) $\displaystyle\int_0^\infty \frac{\cos ax}{x^2+b^2}\,dx$. (r) $\displaystyle\int_0^\infty \frac{dx}{(x^2+4)^2}$.

11.6 Evaluate each of the following integrals by turning it into a contour integral around a unit circle.

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$.

(b) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$ where $a>1$.

(c) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta}$.

(d) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a+b\cos^2\theta)^2}$ where $a, b > 0$.

(e) $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\,d\theta$.

(f) $\displaystyle\int_0^{\pi} \frac{d\phi}{1-2a\cos\phi+a^2}$ where $a \neq \pm 1$.

(g) $\displaystyle\int_0^{\pi} \frac{\cos^2 3\phi\,d\phi}{1-2a\cos\phi+a^2}$ where $a \neq \pm 1$.

(h) $\displaystyle\int_0^{\pi} \frac{\cos 2\phi\,d\phi}{1-2a\cos\phi+a^2}$ where $a \neq \pm 1$.

(i) $\displaystyle\int_0^{\pi} \tan(x+ia)\,dx$ where $a \in \mathbb{R}$.

(j) $\displaystyle\int_0^{\pi} e^{\cos\phi}\cos(n\phi-\sin\phi)\,d\phi$ where $n \in \mathbb{Z}$.

11.7 Evaluate the integral $I = \int_{-\infty}^{\infty} e^{\alpha x}\,dx/(1+e^x)$ for $0<\alpha<1$. Hint: Choose a closed (long) rectangle that encloses only one of the zeros of the denominator. Show that the contributions of the short sides of the rectangle are zero.

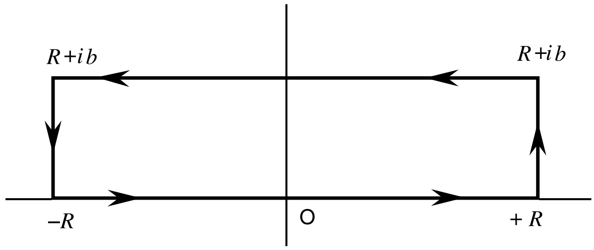

Fig. 11.8 The contour used in Problem 11.8

11.8 Derive the integration formula $\int_0^\infty e^{-x^2}\cos(2bx)\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$ where $b \neq 0$ by integrating the function e^{-z^2} around the rectangular path shown in Fig. 11.8.

11.9 Use the result of Example 11.3.12 to show that $\theta'(k) = \delta(k)$.

11.10 Find the principal values of the following integrals.

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x\,dx}{(x^2+4)(x-1)}$. (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^3}\,dx$ where $a \geq 0$.

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{x\cos x}{x^2-5x+6}\,dx$. (d) $\displaystyle\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx$.

11.11 Evaluate the following integrals.

(a) $\displaystyle\int_0^\infty \frac{x^2-b^2}{x^2+b^2}\left(\frac{\sin ax}{x}\right)dx$. (b) $\displaystyle\int_0^\infty \frac{\sin ax}{x(x^2+b^2)}\,dx$.

(c) $\displaystyle\int_0^\infty \frac{\sin ax}{x(x^2+b^2)^2}\,dx$. (d) $\displaystyle\int_0^\infty \frac{\cos 2ax - \cos 2bx}{x^2}\,dx$.

(e) $\displaystyle\int_0^\infty \frac{\sin^2 x\,dx}{x^2}$. (f) $\displaystyle\int_0^\infty \frac{\sin^3 x\,dx}{x^3}$.