# 8 Classical Orthogonal Polynomials

Example 7.2.5 discussed only one of the many types of the so-called classical orthogonal polynomials. Historically, these polynomials were discovered as solutions to differential equations arising in various physical problems. Such polynomials can be produced by starting with $1, x, x^2, \dots$ and employing the Gram-Schmidt process. However, there is a more elegant, albeit less general, approach that simultaneously studies most polynomials of interest to physicists. We will employ this approach.[1]

## 8.1 General Properties

Most relevant properties of the polynomials of interest are contained in

**Theorem 8.1.1** *Consider the functions*

$$F_n(x) = \frac{1}{w(x)} \frac{d^n}{dx^n}\left(ws^n\right) \quad \text{for } n = 0, 1, 2, \dots, \tag{8.1}$$

*where*

1. $F_1(x)$ *is a first-degree polynomial in* $x$,
2. $s(x)$ *is a polynomial in* $x$ *of degree less than or equal to* 2 *with only real roots*,
3. $w(x)$ *is a strictly positive function*, *integrable in the interval* $(a, b)$, *that satisfies the boundary conditions* $w(a)s(a) = 0 = w(b)s(b)$.

*Then* $F_n(x)$ *is a polynomial of degree* $n$ *in* $x$ *and is orthogonal—on the interval* $(a, b)$, *with weight* $w(x)$—*to any polynomial* $p_k(x)$ *of degree* $k < n$, *i.e.*,

$$\int_a^b p_k(x) F_n(x) w(x)\, dx = 0 \quad \text{for } k < n.$$

*These polynomials are collectively called* ***classical orthogonal polynomials***.

[1] This approach is due to F.G. Tricomi [Tric 55]. See also [Denn 67].

S. Hassani, *Mathematical Physics*, DOI 10.1007/978-3-319-01195-0_8,

Before proving the theorem, we need two lemmas:[2]

**Lemma 8.1.2** *The following identity holds*:

$$\frac{d^m}{dx^m}\left(ws^n p_{\le k}\right) = ws^{n-m} p_{\le k+m}, \quad m \le n.$$

*Proof* See Problem 8.1. □

**Lemma 8.1.3** *All the derivatives $d^m/dx^m(ws^n)$ vanish at $x = a$ and $x = b$, for all values of $m < n$.*

*Proof* Set $k = 0$ in the identity of the previous lemma and let $p_{\le 0} = 1$. Then we have $\frac{d^m}{dx^m}(ws^n) = ws^{n-m} p_{\le m}$. The RHS vanishes at $x = a$ and $x = b$ due to the third condition stated in the theorem. □

*Proof of the theorem* We prove the orthogonality first. The proof involves multiple use of integration by parts:

$$\begin{aligned}
\int_a^b p_k(x) F_n(x) w(x)\, dx &= \int_a^b p_k(x) \frac{1}{w}\left[\frac{d^n}{dx^n}\left(ws^n\right)\right] w\, dx \\
&= \int_a^b p_k(x) \frac{d}{dx}\left[\frac{d^{n-1}}{dx^{n-1}}\left(ws^n\right)\right] dx \\
&= p_k(x) \underbrace{\frac{d^{n-1}}{dx^{n-1}}\left(ws^n\right)\Big|_a^b}_{=0 \text{ by Lemma 8.1.3}} \\
&\quad - \int_a^b \frac{dp_k}{dx} \frac{d^{n-1}}{dx^{n-1}}\left(ws^n\right) dx.
\end{aligned}$$

This shows that each integration by parts transfers one differentiation from $ws^n$ to $p_k$ and introduces a minus sign. Thus, after $k$ integrations by parts, we get

$$\begin{aligned}
\int_a^b p_k(x) F_n(x) w(x)\, dx &= (-1)^k \int_a^b \frac{d^k p_k}{dx^k} \frac{d^{n-k}}{dx^{n-k}}\left(ws^n\right) dx \\
&= C \int_a^b \frac{d}{dx}\left[\frac{d^{n-k-1}}{dx^{n-k-1}}\left(ws^n\right)\right] dx \\
&= C \frac{d^{n-k-1}}{dx^{n-k-1}}\left(ws^n\right)\Big|_a^b = 0,
\end{aligned}$$

where we have used the fact that the $k$th derivative of a polynomial of degree $k$ is a constant. Note that $n - k - 1 \ge 0$ because $k < n$, so that the last line of the equation is well-defined. The last equality follows from Lemma 8.1.3.

[2] Recall that $p_{\le k}$ is a generic polynomial with degree less than or equal to $k$.

To prove the first part of the theorem, we use Lemma 8.1.2 with $k = 0$, $p_{\leq 0} \equiv p_0 = 1$, and $m = n$ to get

$$\frac{d^n}{dx^n}\big(ws^n\big) = wp_{\leq n}, \quad \text{or} \quad F_n(x) = \frac{1}{w}\frac{d^n}{dx^n}\big(ws^n\big) = p_{\leq n}.$$

To prove that $F_n(x)$ is a polynomial of degree precisely equal to $n$, we write $F_n(x) = p_{\leq n-1}(x) + k_n^{(n)} x^n$, multiply both sides by $w(x)F_n(x)$, and integrate over $(a, b)$:

$$\int_a^b \big[F_n(x)\big]^2 w(x)\,dx$$
$$= \int_a^b p_{\leq n-1} F_n(x) w(x)\,dx + k_n^{(n)} \int_a^b x^n F_n(x) w(x)\,dx.$$

The LHS is a positive quantity because both $w(x)$ and $[F_n(x)]^2$ are positive, and the first integral on the RHS vanishes by the first part of the proof. Therefore, the second term on the RHS cannot be zero. In particular, $k_n^{(n)} \neq 0$, and $F_n(x)$ is of degree $n$. □

It is customary to introduce a normalization constant in the definition of $F_n(x)$, and write

$$F_n(x) = \frac{1}{K_n w}\frac{d^n}{dx^n}\big(ws^n\big). \tag{8.2}$$

This equation is called the **generalized Rodriguez formula**. For historical reasons, different polynomial functions are normalized differently, which is why $K_n$ is introduced here.

generalized Rodriguez formula

From Theorem 8.1.1 it is clear that the sequence $\{F_n(x)\}_{n=0}^{\infty}$ forms an orthogonal set of polynomials on $[a, b]$ with weight function $w(x)$.

All the varieties of classical orthogonal polynomials were discovered as solutions of differential equations. Here, we give a single generic differential equation satisfied by all the $F_n$'s. The proof is outlined in Problem 8.4.

differential equation for classical orthogonal polynomials

**Proposition 8.1.4** *Let $k_1^{(1)}$ be the coefficient of $x$ in $F_1(x)$ and $\sigma_2$ the coefficient of $x^2$ in $s(x)$. Then the orthogonal polynomials $F_n$ satisfy the differential equation*

$$\frac{d}{dx}\left(ws\frac{dF_n}{dx}\right) = w\lambda_n F_n(x) \quad \textit{where} \quad \lambda_n = K_1 k_1^{(1)} n + \sigma_2 n(n-1).$$

We shall study the differential equation above in the context of the Sturm-Liouville problem (see Chap. 19), which is an eigenvalue problem involving differential operators.

## 8.2 Classification

Let us now investigate the consequences of various choices of $s(x)$. We start with $F_1(x)$, and note that it satisfies Eq. (8.2) with $n = 1$:

$$F_1(x) = \frac{1}{K_1 w}\frac{d}{dx}(ws), \quad \text{or} \quad \frac{1}{ws}\frac{d}{dx}(ws) = \frac{K_1 F_1(x)}{s},$$

which can be integrated to yield $ws = A\exp(\int K_1 F_1(x)\,dx/s)$ where $A$ is a constant. On the other hand, being a polynomial of degree 1, $F_1(x)$ can be written as $F_1(x) = k_1^{(1)}x + k_1^{(0)}$. It follows that

$$\begin{aligned} w(x)s(x) &= A\exp\left(\int \frac{K_1(k_1^{(1)}x + k_1^{(0)})}{s}\,dx\right), \\ w(a)s(a) &= 0 = w(b)s(b). \end{aligned} \tag{8.3}$$

Next we look at the three choices for $s(x)$: a constant, a polynomial of degree 1, and a polynomial of degree 2. For a constant $s(x)$, Eq. (8.3) can be easily integrated:

$$\begin{aligned} w(x)s(x) &= A\exp\left(\int \frac{K_1(k_1^{(1)}x + k_1^{(0)})}{s}\,dx\right) \equiv A\exp\left(\int (2\alpha x + \beta)\,dx\right) \\ &= Ae^{\alpha x^2 + \beta x + C} = Be^{\alpha x^2 + \beta x}, \\ 2\alpha &\equiv K_1 k_1^{(1)}/s, \quad \beta \equiv K_1 k_1^{(0)}/s, \quad B \equiv Ae^C. \end{aligned}$$

The interval $(a, b)$ is determined by $w(a)s(a) = 0 = w(b)s(b)$, which yields

$$Be^{\alpha a^2 + \beta a} = 0 = Be^{\alpha b^2 + \beta b}.$$

For nonzero $B$, the only way that this equality can hold is for $\alpha$ to be negative and for $a$ and $b$ to be infinite. Since $a < b$, we must take $a = -\infty$ and $b = +\infty$. With $y = \sqrt{|\alpha|}(x + \beta/(2\alpha))$ and choosing $B = s\exp(\beta^2/(4\alpha))$, we obtain $w(y) = \exp(-y^2)$. We also take the constant $s$ to be 1. This is always possible by a proper choice of constants such as $B$.

If the degree of $s$ is 1, then $s(x) = \sigma_1 x + \sigma_0$ and

$$\begin{aligned} w(x)(\sigma_1 x + \sigma_0) &= A\exp\left(\int \frac{K_1(k_1^{(1)}x + k_1^{(0)})}{\sigma_1 x + \sigma_0}\,dx\right) \\ &= A\exp\left[\int \left(\frac{K_1 k_1^{(1)}}{\sigma_1} + \frac{K_1 k_1^{(0)} - K_1 k_1^{(1)}\sigma_0/\sigma_1}{\sigma_1 x + \sigma_0}\right) dx\right] \\ &\equiv B(\sigma_1 x + \sigma_0)^\rho e^{\gamma x}, \end{aligned}$$

where $\gamma = K_1 k_1^{(1)}/\sigma_1$, $\rho = K_1 k_1^{(0)}/\sigma_1 - K_1 k_1^{(1)}\sigma_0/\sigma_1^2$, and $B$ is $A$ modified by the constant of integration. The last equation above must satisfy the boundary conditions at $a$ and $b$:

$$B(\sigma_1 a + \sigma_0)^\rho e^{\gamma a} = 0 = B(\sigma_1 b + \sigma_0)^\rho e^{\gamma b},$$

**Table 8.1** Special cases of Jacobi polynomials

| $\mu$ | $\nu$ | $w(x)$ | Polynomial |
|---|---|---|---|
| 0 | 0 | 1 | Legendre, $P_n(x)$ |
| $\lambda - \frac{1}{2}$ | $\lambda - \frac{1}{2}$ | $(1-x^2)^{\lambda-1/2}$ | Gegenbauer, $C_n^\lambda(x)$, $\lambda > -\frac{1}{2}$ |
| $-\frac{1}{2}$ | $-\frac{1}{2}$ | $(1-x^2)^{-1/2}$ | Chebyshev of the first kind, $T_n(x)$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $(1-x^2)^{1/2}$ | Chebyshev of the second kind, $U_n(x)$ |

which give $a = -\sigma_0/\sigma_1$, $\rho > 0$, $\gamma < 0$, and $b = +\infty$. With appropriate redefinition of variables and parameters, we can write

$$w(y) = y^\nu e^{-y}, \quad \nu > -1, \qquad s(x) = x, \quad a = 0, \quad b = +\infty.$$

Similarly, we can obtain the weight function and the interval of integration for the case when $s(x)$ is of degree 2. This result, as well as the results obtained above, are collected in the following proposition.

Hermite, Laguerre, and Jacobi polynomials

**Proposition 8.2.1** *If the conditions of Theorem 8.1.1 prevail, then*

(a) *For $s(x)$ of degree zero we get $w(x) = e^{-x^2}$ with $s(x) = 1$, $a = -\infty$, and $b = +\infty$. The resulting polynomials are called* ***Hermite polynomials*** *and are denoted by $H_n(x)$.*
(b) *For $s(x)$ of degree 1, we obtain $w(x) = x^\nu e^{-x}$ with $\nu > -1$, $s(x) = x$, $a = 0$, and $b = +\infty$. The resulting polynomials are called* ***Laguerre polynomials*** *and are denoted by $L_n^\nu(x)$.*
(c) *For $s(x)$ of degree 2, we get $w(x) = (1+x)^\mu(1-x)^\nu$ with $\mu, \nu > -1$, $s(x) = 1 - x^2$, $a = -1$, and $b = +1$. The resulting polynomials are called* ***Jacobi polynomials*** *and are denoted by $P_n^{\mu,\nu}(x)$.*

Jacobi polynomials are themselves divided into other subcategories depending on the values of $\mu$ and $\nu$. The most common and widely used of these are collected in Table 8.1. Note that the definition of each of the preceding polynomials involves a "standardization," which boils down to a particular choice of $K_n$ in the generalized Rodriguez formula.

## 8.3 Recurrence Relations

Besides the recurrence relations obtained in Sect. 7.2, we can use the differential equation of Proposition 8.1.4 to construct new recurrence relations involving derivatives. These relations apply only to *classical* orthogonal polynomials, and not to general ones. We start with Eq. (7.12)

$$F_{n+1}(x) = (\alpha_n x + \beta_n)F_n(x) + \gamma_n F_{n-1}(x), \tag{8.4}$$

differentiate both sides twice, and substitute for the second derivative from the differential equation of Proposition 8.1.4. This will yield

$$2ws\alpha_n F_n' + \left[\alpha_n \frac{d}{dx}(ws) + w\lambda_n(\alpha_n x + \beta_n)\right] F_n - w\lambda_{n+1}F_{n+1} + w\gamma_n\lambda_{n-1}F_{n-1} = 0. \tag{8.5}$$

#### Historical Notes

Karl Gustav Jacob Jacobi
1804–1851

**Karl Gustav Jacob Jacobi** (1804–1851) was the second son born to a well-to-do Jewish banking family in Potsdam. An obviously bright young man, Jacobi was soon moved to the highest class in spite of his youth and remained at the gymnasium for four years only because he could not enter the university until he was sixteen. He excelled at the University of Berlin in all the classical subjects as well as mathematical studies, the topic he soon chose as his career. He passed the examination to become a secondary school teacher, then later the examination that allowed university teaching, and joined the faculty at Berlin at the age of twenty. Since promotion there appeared unlikely, he moved in 1826 to the University of Königsberg in search of a more permanent position. He was known as a lively and creative lecturer who often injected his latest research topics into the lectures. He began what is now a common practice at most universities—the research seminar—for the most advanced students and his faculty collaborators. The Jacobi "school", together with the influence of Bessel and Neumann (also at Königsberg), sparked a renewal of mathematical excellence in Germany.

In 1843 Jacobi fell gravely ill with diabetes. After seeing his condition, Dirichlet, with the help of von Humboldt, secured a donation to enable Jacobi to spend several months in Italy, a therapy recommended by his doctor. The friendly atmosphere and healthful climate there soon improved his condition. Jacobi was later given royal permission to move from Königsberg to Berlin so that his health would not be affected by the harsh winters in the former location. A salary bonus given to Jacobi to offset the higher cost of living in the capital was revoked after he made some politically sensitive remarks in an impromptu speech. A permanent position at Berlin was also refused, and the reduced salary and lack of security caused considerable hardship for Jacobi and his family. Only after he accepted a position in Vienna did the Prussian government recognize the desirability of keeping the distinguished mathematician within its borders, offering him special concessions that together with his love for his homeland convinced Jacobi to stay. In 1851 Jacobi died after contracting both influenza and smallpox.

Jacobi's mathematical reputation began largely with his heated competition with Abel in the study of elliptic functions. Legendre, formerly the star of such studies, wrote Jacobi of his happiness at having "lived long enough to witness these magnanimous contests between two young athletes equally strong". Although Jacobi and Abel could reasonably be considered contemporary researchers who arrived at many of the same results independently, Jacobi suggested the names "Abelian functions" and "Abelian theorem" in a review he wrote for Crelle's Journal. Jacobi also extended his discoveries in elliptic functions to number theory and the theory of integration. He also worked in other areas of number theory, such as the theory of quadratic forms and the representation of integers as sums of squares and cubes. He presented the well-known *Jacobian*, or functional determinant, in 1841. To physicists, Jacobi is probably best known for his work in dynamics with the form introduced by Hamilton. Although elegant and quite general, Hamiltonian dynamics did not lend itself to easy solution of many practical problems in mechanics. In the spirit of Lagrange, Poisson, and others, Jacobi investigated transformations of Hamilton's equations that preserved their canonical nature (loosely speaking, that preserved the Poisson brackets in each representation). After much work and a little simplification, the resulting equations of motion, now known as *Hamilton-Jacobi equations*, allowed Jacobi to solve several important problems in ordinary and celestial mechanics. Clebsch and later Helmholtz amplified their use in other areas of physics.

We can get another recurrence relation involving derivatives by substituting (8.4) in (8.5) and simplifying:

$$2ws\alpha_n F_n' + \left[\alpha_n \frac{d}{dx}(ws) + w(\lambda_n - \lambda_{n+1})(\alpha_n x + \beta_n)\right] F_n$$
$$+ w\gamma_n(\lambda_{n-1} - \lambda_{n+1})F_{n-1} = 0. \tag{8.6}$$

Two other recurrence relations can be obtained by differentiating equations (8.6) and (8.5), respectively, and using the differential equation for $F_n$. Now solve the first equation so obtained for $\gamma_n(d/dx)(wF_{n-1})$ and substitute the result in the second equation. After simplification, the result will be

$$2w\alpha_n\lambda_n F_n + \frac{d}{dx}\left\{\left[\alpha_n \frac{d}{dx}(ws) + w(\lambda_n - \lambda_{n-1})(\alpha_n x + \beta_n)\right] F_n\right\}$$
$$+ (\lambda_{n-1} - \lambda_{n+1})\frac{d}{dx}(wF_{n+1}) = 0. \tag{8.7}$$

Finally, we record one more useful recurrence relation:

$$A_n(x)F_n - \lambda_{n+1}(\alpha_n x + \beta_n)\frac{dw}{dx}F_{n+1} + \gamma_n\lambda_{n-1}(\alpha_n x + \beta_n)\frac{dw}{dx}F_{n-1}$$
$$+ B_n(x)F_{n+1}' + \gamma_n D_n(x)F_{n-1}' = 0, \tag{8.8}$$

where

$$A_n(x) = (\alpha_n x + \beta_n)\left[2w\alpha_n\lambda_n + \alpha_n \frac{d^2}{dx^2}(ws) + \lambda_n(\alpha_n x + \beta_n)\frac{dw}{dx}\right]$$
$$- \alpha_n^2 \frac{d}{dx}(ws),$$
$$B_n(x) = \alpha_n \frac{d}{dx}(ws) - w(\alpha_n x + \beta_n)(\lambda_{n+1} - \lambda_n),$$
$$D_n(x) = w(\alpha_n x + \beta_n)(\lambda_{n-1} - \lambda_n) - \alpha_n \frac{d}{dx}(ws).$$

Details of the derivation of this relation are left for the reader. All these recurrence relations seem to be very complicated. However, complexity is the price we pay for generality. When we work with specific orthogonal polynomials, the equations simplify considerably. For instance, for Hermite and Legendre polynomials Eq. (8.6) yields, respectively,

useful recurrence relations for Hermite and Legendre polynomials

$$H_n' = 2nH_{n-1}, \quad \text{and} \quad \left(1 - x^2\right)P_n' + nxP_n - nP_{n-1} = 0. \tag{8.9}$$

Also, applying Eq. (8.7) to Legendre polynomials gives

$$P_{n+1}' - xP_n' - (n+1)P_n = 0, \tag{8.10}$$

and Eq. (8.8) yields

$$P_{n+1}' - P_{n-1}' - (2n+1)P_n = 0. \tag{8.11}$$

It is possible to find many more recurrence relations by manipulating the existing recurrence relations.

Before studying specific orthogonal polynomials, let us pause for a moment to appreciate the generality and elegance of the preceding discussion. With a few assumptions and a single defining equation we have severely restricted the choice of the weight function and with it the choice of the interval $(a, b)$. We have nevertheless exhausted the list of the so-called classical orthogonal polynomials.

## 8.4 Details of Specific Examples

We now construct the specific polynomials used frequently in physics. We have seen that the four parameters $K_n$, $k_n^{(n)}$, $k_n^{(n-1)}$, and $h_n$ determine all the properties of the polynomials. Once $K_n$ is fixed by some standardization, we can determine all the other parameters: $k_n^{(n)}$ and $k_n^{(n-1)}$ will be given by the generalized Rodriguez formula, and $h_n$ can be calculated as follows:

$$\begin{aligned} h_n &= \int_a^b F_n^2(x)w(x)\,dx = \int_a^b \bigl(k_n^{(n)}x^n + \cdots\bigr)F_n(x)w(x)\,dx \\ &= k_n^{(n)} \int_a^b wx^n \frac{1}{K_n w}\frac{d^n}{dx^n}\bigl(ws^n\bigr)\,dx = \frac{k_n^{(n)}}{K_n}\int_a^b x^n \frac{d}{dx}\left[\frac{d^{n-1}}{dx^{n-1}}\bigl(ws^n\bigr)\right]dx \\ &= \frac{k_n^{(n)}}{K_n}x^n \frac{d^{n-1}}{dx^{n-1}}\bigl(ws^n\bigr)\bigg|_a^b - \frac{k_n^{(n)}}{K_n}\int_a^b \frac{d}{dx}\bigl(x^n\bigr)\frac{d^{n-1}}{dx^{n-1}}\bigl(ws^n\bigr)\,dx. \end{aligned}$$

The first term of the last line is zero by Lemma 8.1.3. It is clear that each integration by parts introduces a minus sign and shifts one differentiation from $ws^n$ to $x^n$. Thus, after $n$ integrations by parts and noting that $d^0/dx^0(ws^n) = ws^n$ and $d^n/dx^n(x^n) = n!$, we obtain

$$h_n = \frac{(-1)^n k_n^{(n)}\, n!}{K_n}\int_a^b ws^n\,dx. \tag{8.12}$$

### 8.4.1 Hermite Polynomials

summary of properties of Hermite polynomials

The Hermite polynomials are standardized such that $K_n = (-1)^n$. Thus, the generalized Rodriguez formula (8.2) and Proposition 8.2.1 give

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\bigl(e^{-x^2}\bigr). \tag{8.13}$$

It is clear that each time $e^{-x^2}$ is differentiated, a factor of $-2x$ is introduced. The highest power of $x$ is obtained when we differentiate $e^{-x^2}$ $n$ times. This yields $(-1)^n e^{x^2}(-2x)^n e^{-x^2} = 2^n x^n \quad\Rightarrow\quad k_n^{(n)} = 2^n$.

To obtain $k_n^{(n-1)}$, we find it helpful to see whether the polynomial is even or odd. We substitute $-x$ for $x$ in Eq. (8.13) and get $H_n(-x) = (-1)^n H_n(x)$, which shows that if $n$ is even (odd), $H_n$ is an even (odd) polynomial, i.e., it can have only even (odd) powers of $x$. In either case, the next-highest power of $x$ in $H_n(x)$ is not $n-1$ but $n-2$. Thus, the coefficient of $x^{n-1}$ is zero for $H_n(x)$, and we have $k_n^{(n-1)}=0$. For $h_n$, we use (8.12) to obtain $h_n=\sqrt{\pi}\,2^n n!$.

Next we calculate the recurrence relation of Eq. (7.12). We can readily calculate the constants needed: $\alpha_n=2$, $\beta_n=0$, $\gamma_n=-2n$. Then substitute these in Eq. (7.12) to obtain

$$H_{n+1}(x)=2xH_n(x)-2nH_{n-1}(x). \tag{8.14}$$

Other recurrence relations can be obtained similarly.

Finally, the differential equation of $H_n(x)$ is obtained by first noting that $K_1=-1$, $\sigma_2=0$, $F_1(x)=2x \Rightarrow k_1^{(1)}=2$. All of this gives $\lambda_n=-2n$, which can be used in the equation of Proposition 8.1.4 to get

$$\frac{d^2H_n}{dx^2}-2x\frac{dH_n}{dx}+2nH_n=0. \tag{8.15}$$

### 8.4.2 Laguerre Polynomials

summary of properties of Laguerre polynomials

For Laguerre polynomials, the standardization is $K_n=n!$. Thus, the generalized Rodriguez formula (8.2) and Proposition 8.2.1 give

$$L_n^\nu(x)=\frac{1}{n!x^\nu e^{-x}}\frac{d^n}{dx^n}\left(x^\nu e^{-x}x^n\right)=\frac{1}{n!}x^{-\nu}e^x\frac{d^n}{dx^n}\left(x^{n+\nu}e^{-x}\right). \tag{8.16}$$

To find $k_n^{(n)}$ we note that differentiating $e^{-x}$ does not introduce any new powers of $x$ but only a factor of $-1$. Thus, the highest power of $x$ is obtained by leaving $x^{n+\nu}$ alone and differentiating $e^{-x}$ $n$ times. This gives

$$\frac{1}{n!}x^{-\nu}e^x x^{n+\nu}(-1)^n e^{-x}=\frac{(-1)^n}{n!}x^n \quad\Rightarrow\quad k_n^{(n)}=\frac{(-1)^n}{n!}.$$

We may try to check the evenness or oddness of $L_n^\nu(x)$; however, this will not be helpful because changing $x$ to $-x$ distorts the RHS of Eq. (8.16). In fact, $k_n^{(n-1)}\neq 0$ in this case, and it can be calculated by noticing that the next-highest power of $x$ is obtained by adding the first derivative of $x^{n+\nu}$ $n$ times and multiplying the result by $(-1)^{n-1}$, which comes from differentiating $e^{-x}$. We obtain

$$\frac{1}{n!}x^{-\nu}e^x\left[(-1)^{n-1}n(n+\nu)x^{n+\nu-1}e^{-x}\right]=\frac{(-1)^{n-1}(n+\nu)}{(n-1)!}x^{n-1},$$

and therefore $k_n^{(n-1)}=(-1)^{n-1}(n+\nu)/(n-1)!$.

Finally, for $h_n$ we get

$$h_n=\frac{(-1)^n[(-1)^n/n!]n!}{n!}\int_0^\infty x^\nu e^{-x}x^n\,dx=\frac{1}{n!}\int_0^\infty x^{n+\nu}e^{-x}\,dx.$$

If $\nu$ is not an integer (and it need not be), the integral on the RHS cannot be evaluated by elementary methods. In fact, this integral occurs so frequently in mathematical applications that it is given a special name, the **gamma** the gamma function **function**. A detailed discussion of this function can be found in Chap. 12. At this point, we simply note that

$$\Gamma(z+1) \equiv \int_0^\infty x^z e^{-x}\,dx, \qquad \Gamma(n+1) = n! \quad \text{for } n \in \mathbb{N}, \tag{8.17}$$

and write $h_n$ as

$$h_n = \frac{\Gamma(n+\nu+1)}{n!} = \frac{\Gamma(n+\nu+1)}{\Gamma(n+1)}.$$

The relevant parameters for the recurrence relation can be easily calculated:

$$\alpha_n = -\frac{1}{n+1}, \qquad \beta_n = \frac{2n+\nu+1}{n+1}, \qquad \gamma_n = -\frac{n+\nu}{n+1}.$$

Substituting these in Eq. (7.12) and simplifying yields

$$(n+1)L_{n+1}^\nu = (2n+\nu+1-x)L_n^\nu - (n+\nu)L_{n-1}^\nu.$$

With $k_1^{(1)} = -1$ and $\sigma_2 = 0$, we get $\lambda_n = -n$, and the differential equation of Proposition 8.1.4 becomes

$$x\frac{d^2L_n^\nu}{dx^2} + (\nu+1-x)\frac{dL_n^\nu}{dx} + nL_n^\nu = 0. \tag{8.18}$$

### 8.4.3 Legendre Polynomials

summary of properties of Legendre polynomials

Instead of discussing the Jacobi polynomials as a whole, we will discuss a special case of them, the Legendre polynomials $P_n(x)$, which are more widely used in physics.

With $\mu = 0 = \nu$, corresponding to the Legendre polynomials, the weight function for the Jacobi polynomials reduces to $w(x) = 1$. The standardization is $K_n = (-1)^n 2^n n!$. Thus, the generalized Rodriguez formula reads

$$P_n(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\left[\left(1-x^2\right)^n\right]. \tag{8.19}$$

To find $k_n^{(n)}$, we expand the expression in square brackets using the binomial theorem and take the $n$th derivative of the highest power of $x$. This yields

$$\begin{aligned} k_n^{(n)}x^n &= \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\left[\left(-x^2\right)^n\right] = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^{2n}\right) \\ &= \frac{1}{2^n n!}2n(2n-1)(2n-2)\cdots(n+1)x^n. \end{aligned}$$

After some algebra (see Problem 8.15), we get $k_n^{(n)} = \frac{2^n\Gamma(n+\frac{1}{2})}{n!\Gamma(\frac{1}{2})}$.

**Historical Notes**

Adrien-Marie Legendre
1752–1833

**Adrien-Marie Legendre** (1752–1833) came from a well-to-do Parisian family and received an excellent education in science and mathematics. His university work was advanced enough that his mentor used many of Legendre's essays in a treatise on mechanics. A man of modest fortune until the revolution, Legendre was able to devote himself to study and research without recourse to an academic position. In 1782 he won the prize of the Berlin Academy for calculating the trajectories of cannonballs taking air resistance into account. This essay brought him to the attention of Lagrange and helped pave the way to acceptance in French scientific circles, notably the Academy of Sciences, to which Legendre submitted numerous papers. In July 1784 he submitted a paper on planetary orbits that contained the now-famous *Legendre polynomials*, mentioning that Lagrange had been able to "present a more complete theory" in a recent paper by using Legendre's results. In the years that followed, Legendre concentrated his efforts in number theory, celestial mechanics, and the theory of elliptic functions. In addition, he was a prolific calculator, producing large tables of the values of special functions, and he also authored an elementary textbook that remained in use for many decades. In 1824 Legendre refused to vote for the government's candidate for *Institut National*. Because of this, his pension was stopped and he died in poverty and in pain at the age of 80 after several years of failing health.

Legendre produced a large number of useful ideas but did not always develop them in the most rigorous manner, claiming to hold the priority for an idea if he had presented merely a reasonable argument for it. Gauss, with whom he had several quarrels over priority, considered rigorous proof the standard of ownership. To Legendre's credit, however, he was an enthusiastic supporter of his young rivals Abel and Jacobi and gave their work considerable attention in his writings. Especially in the theory of elliptic functions, the area of competition with Abel and Jacobi, Legendre is considered more of a trailblazer than a great builder. Hermite wrote that Legendre "is considered the founder of the theory of elliptic functions" and "greatly smoothed the way for his successors", but notes that the recognition of the double periodicity of the inverse function, which allowed the great progress of others, was missing from Legendre's work.

Legendre also contributed to practical efforts in science and mathematics. He and two of his contemporaries were assigned in 1787 to a panel conducting geodetic work in cooperation with the observatories at Paris and Greenwich. Four years later the same panel members were appointed as the Academy's commissioners to undertake the measurements and calculations necessary to determine the length of the standard meter. Legendre's seemingly tireless skill at calculating produced large tables of the values of trigonometric and elliptic functions, logarithms, and solutions to various special equations.

In his famous textbook *Eléments de géométrie* (1794) he gave a simple proof that $\pi$ is irrational and conjectured that it is not the root of any algebraic equation of finite degree with rational coefficients. The textbook was somewhat dogmatic in its presentation of ordinary Euclidean thought and includes none of the non-Euclidean ideas beginning to be formed around that time. It was Legendre who first gave a rigorous proof of the theorem (assuming all of Euclid's postulates, of course) that the sum of the angles of a triangle is "equal to two right angles". Very little of his research in this area was of memorable quality. The same could possibly be argued for the balance of his writing, but one must acknowledge the very fruitful ideas he left behind in number theory and elliptic functions and, of course, the introduction of Legendre polynomials and the important *Legendre transformation* used both in thermodynamics and Hamiltonian mechanics.

To find $k_n^{(n-1)}$, we look at the evenness or oddness of the polynomials. By an investigation of the Rodriguez formula—as in our study of Hermite polynomials—we note that $P_n(-x) = (-1)^n P_n(x)$, which tells us that $P_n(x)$ is either even or odd. In either case, $x$ will not have an $(n-1)$st power. Therefore, $k_n^{(n-1)} = 0$.

We now calculate $h_n$ as given by (8.12):

$$h_n = \frac{(-1)^n k_n^{(n)} n!}{K_n} \int_{-1}^{1} \left(1 - x^2\right)^n dx = \frac{2^n \Gamma(n + \frac{1}{2}) / \Gamma(\frac{1}{2})}{2^n n!} \int_{-1}^{1} \left(1 - x^2\right)^n dx.$$

The integral can be evaluated by repeated integration by parts (see Problem 8.16). Substituting the result in the expression above yields $h_n = 2/(2n+1)$.

We need $\alpha_n$, $\beta_n$ and $\gamma_n$ for the recurrence relation:

$$\alpha_n = \frac{k_{n+1}^{(n+1)}}{k_n^{(n)}} = \frac{2^{n+1}\Gamma(n+1+\frac{1}{2})}{(n+1)!\Gamma(\frac{1}{2})} \frac{n!\Gamma(\frac{1}{2})}{2^n\Gamma(n+\frac{1}{2})} = \frac{2n+1}{n+1},$$

where we used the relation $\Gamma(z+1) = z\Gamma(z)$, an important property of the gamma function. We also have $\beta_n = 0$ (because $k_n^{(n-1)} = 0 = k_{n+1}^{(n)}$) and $\gamma_n = -n/(n+1)$. Therefore, the recurrence relation is

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x). \tag{8.20}$$

Now we use $K_1 = -2$, $P_1(x) = x \Rightarrow k_1^{(1)} = 1$, and $\sigma_2 = -1$ to obtain $\lambda_n = -n(n+1)$, which yields the following differential equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n}{dx}\right] = -n(n+1)P_n. \tag{8.21}$$

This can also be expressed as

$$(1-x^2)\frac{d^2P_n}{dx^2} - 2x\frac{dP_n}{dx} + n(n+1)P_n = 0. \tag{8.22}$$

### 8.4.4 Other Classical Orthogonal Polynomials

The rest of the classical orthogonal polynomials can be constructed similarly. For the sake of completeness, we merely quote the results.

**Jacobi Polynomials, $P_n^{\mu,\nu}(x)$**
**Standardization:**

$$K_n = (-2)^n n!$$

**Constants:**

$$k_n^{(n)} = 2^{-n}\frac{\Gamma(2n+\mu+\nu+1)}{n!\Gamma(n+\mu+\nu+1)}, \qquad k_n^{(n-1)} = \frac{n(\nu-\mu)}{2n+\mu+\nu}k_n,$$

$$h_n = \frac{2^{\mu+\nu+1}\Gamma(n+\mu+1)\Gamma(n+\nu+1)}{n!(2n+\mu+\nu+1)\Gamma(n+\mu+\nu+1)}$$

**Rodriguez formula:**

$$P_n^{\mu,\nu}(x) = \frac{(-1)^n}{2^n n!}(1+x)^{-\mu}(1-x)^{-\nu}\frac{d^n}{dx^n}\left[(1+x)^{\mu+n}(1-x)^{\nu+n}\right]$$

**Differential Equation:**

$$(1-x^2)\frac{d^2P_n^{\mu,\nu}}{dx^2} + \left[\mu-\nu-(\mu+\nu+2)x\right]\frac{dP_n^{\mu,\nu}}{dx} + n(n+\mu+\nu+1)P_n^{\mu,\nu} = 0$$

**A Recurrence Relation:**

$$\begin{aligned}
&2(n+1)(n+\mu+\nu+1)(2n+\mu+\nu)P_{n+1}^{\mu,\nu} \\
&\quad = (2n+\mu+\nu+1)\big[(2n+\mu+\nu)(2n+\mu+\nu+2)x+\nu^2-\mu^2\big]P_n^{\mu,\nu} \\
&\qquad - 2(n+\mu)(n+\nu)(2n+\mu+\nu+2)P_{n-1}^{\mu,\nu}
\end{aligned}$$

## Gegenbauer Polynomials, $C_n^\lambda(x)$

**Standardization:**

$$K_n = (-2)^n n! \frac{\Gamma(n+\lambda+\frac{1}{2})\Gamma(2\lambda)}{\Gamma(n+2\lambda)\Gamma(\lambda+\frac{1}{2})}$$

**Constants:**

$$k_n^{(n)} = \frac{2^n}{n!}\frac{\Gamma(n+\lambda)}{\Gamma(\lambda)}, \qquad k_n^{(n-1)} = 0, \qquad h_n = \frac{\sqrt{\pi}\,\Gamma(n+2\lambda)\Gamma(\lambda+\frac{1}{2})}{n!(n+\lambda)\Gamma(2\lambda)\Gamma(\lambda)}$$

**Rodriguez Formula:**

$$C_n^\lambda(x) = \frac{(-1)^n\Gamma(n+2\lambda)\Gamma(\lambda+\frac{1}{2})}{2^n n!\Gamma(n+\lambda+\frac{1}{2})\Gamma(2\lambda)}\big(1-x^2\big)^{-\lambda+1/2}\frac{d^n}{dx^n}\Big[\big(1-x^2\big)^{n+\lambda-1/2}\Big]$$

**Differential Equation:**

$$\big(1-x^2\big)\frac{d^2C_n^\lambda}{dx^2} - (2\lambda+1)x\frac{dC_n^\lambda}{dx} + n(n+2\lambda)C_n^\lambda = 0$$

**A Recurrence Relation:**

$$(n+1)C_{n+1}^\lambda = 2(n+\lambda)xC_n^\lambda - (n+2\lambda-1)C_{n-1}^\lambda$$

## Chebyshev Polynomials of the First Kind, $T_n(x)$

**Standardization:**

$$K_n = (-1)^n\frac{(2n)!}{2^n n!}$$

**Constants:**

$$k_n^{(n)} = 2^{n-1}, \qquad k_n^{(n-1)} = 0, \qquad h_n = \frac{\pi}{2}$$

**Rodriguez Formula:**

$$T_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}\big(1-x^2\big)^{1/2}\frac{d^n}{dx^n}\Big[\big(1-x^2\big)^{n-1/2}\Big]$$

**Differential Equation:**

$$\big(1-x^2\big)\frac{d^2T_n}{dx^2} - x\frac{dT_n}{dx} + n^2T_n = 0$$

**A Recurrence Relation:**

$$T_{n+1} = 2xT_n - T_{n-1}$$

**Chebyshev Polynomials of the Second Kind, $U_n(x)$**

**Standardization:**

$$K_n = (-1)^n \frac{(2n+1)!}{2^n (n+1)!}$$

**Constants:**

$$k_n^{(n)} = 2^n, \qquad k_n^{(n-1)} = 0, \qquad h_n = \frac{\pi}{2}$$

**Rodriguez Formula:**

$$U_n(x) = \frac{(-1)^n 2^n (n+1)!}{(2n+1)!} \left(1-x^2\right)^{-1/2} \frac{d^n}{dx^n}\left[\left(1-x^2\right)^{n+1/2}\right]$$

**Differential Equation:**

$$\left(1-x^2\right)\frac{d^2U_n}{dx^2} - 3x\frac{dU_n}{dx} + n(n+2)U_n = 0$$

**A Recurrence Relation:**

$$U_{n+1} = 2xU_n - U_{n-1}$$

## 8.5 Expansion in Terms of Orthogonal Polynomials

Having studied the different classical orthogonal polynomials, we can now use them to write an arbitrary function $f \in \mathcal{L}_w^2(a,b)$ as a series of these polynomials. If we denote a complete set of orthogonal (not necessarily classical) polynomials by $|C_k\rangle$ and the given function by $|f\rangle$, we may write

$$|f\rangle = \sum_{k=0}^{\infty} a_k |C_k\rangle, \tag{8.23}$$

where $a_k$ is found by multiplying both sides of the equation by $\langle C_i|$ and using the orthogonality of the $|C_k\rangle$'s:

$$\langle C_i|f\rangle = \sum_{k=0}^{\infty} a_k \langle C_i|C_k\rangle = a_i \langle C_i|C_i\rangle \quad \Rightarrow \quad a_i = \frac{\langle C_i|f\rangle}{\langle C_i|C_i\rangle}. \tag{8.24}$$

This is written in function form as

$$a_i = \frac{\int_a^b C_i^*(x) f(x) w(x)\, dx}{\int_a^b |C_i(x)|^2 w(x)\, dx}. \tag{8.25}$$

We can also "derive" the functional form of Eq. (8.23) by multiplying both of its sides by $\langle x|$ and using the fact that $\langle x|f\rangle = f(x)$ and $\langle x|C_k\rangle = C_k(x)$. The result will be

$$f(x) = \sum_{k=0}^{\infty} a_k C_k(x). \tag{8.26}$$

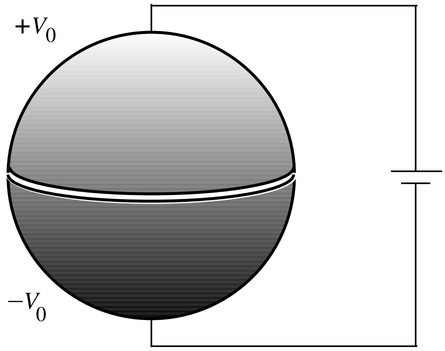


**Fig. 8.1** The voltage is $+V_0$ for the upper hemisphere, where $0 \le \theta < \pi/2$, or where $0 < \cos\theta \le 1$. It is $-V_0$ for the lower hemisphere, where $\pi/2 < \theta \le \pi$, or where $-1 \le \cos\theta < 0$

**Example 8.5.1** The solution of Laplace's equation in spherically symmetric electrostatic problems that are independent of the azimuthal angle is given by

$$\Phi(r,\theta) = \sum_{k=0}^{\infty} \left( \frac{b_k}{r^{k+1}} + c_k r^k \right) P_k(\cos\theta). \tag{8.27}$$

Consider two conducting hemispheres of radius $a$ separated by a small insulating gap at the equator. The upper hemisphere is held at potential $V_0$ and the lower one at $-V_0$, as shown in Fig. 8.1. We want to find the potential at points outside the resulting sphere. Since the potential must vanish at infinity, we expect the second term in Eq. (8.27) to be absent, i.e., $c_k = 0\ \forall k$. To find $b_k$, substitute $a$ for $r$ in (8.27) and let $\cos\theta \equiv x$. Then,

$$\Phi(a,x) = \sum_{k=0}^{\infty} \frac{b_k}{a^{k+1}} P_k(x),$$

where

$$\Phi(a,x) = \begin{cases} -V_0 & \text{if } -1 < x < 0, \\ +V_0 & \text{if } 0 < x < 1. \end{cases}$$

From Eq. (8.25), we have

$$\frac{b_k}{a^{k+1}} = \frac{\int_{-1}^{1} P_k(x)\Phi(a,x)\,dx}{\underbrace{\int_{-1}^{1} |P_k(x)|^2\,dx}_{=h_k}} = \frac{2k+1}{2} \int_{-1}^{1} P_k(x)\Phi(a,x)\,dx$$

$$= \frac{2k+1}{2} V_0 \left[ -\int_{-1}^{0} P_k(x)\,dx + \int_{0}^{1} P_k(x)\,dx \right].$$

To proceed, we rewrite the first integral:

$$\int_{-1}^{0} P_k(x)\,dx = -\int_{+1}^{0} P_k(-y)\,dy = \int_{0}^{1} P_k(-y)\,dy = (-1)^k \int_{0}^{1} P_k(x)\,dx,$$

where we made use of the parity property of $P_k(x)$. Therefore,

$$\frac{b_k}{a^{k+1}} = \frac{2k+1}{2} V_0\big[1-(-1)^k\big] \int_0^1 P_k(x)\,dx.$$

It is now clear that only odd polynomials contribute to the expansion. Using the result of Problem 8.27, we get

$$\frac{b_k}{a^{k+1}} = (-1)^{(k-1)/2} \frac{(2k+1)(k-1)!}{2^k(\frac{k+1}{2})!(\frac{k-1}{2})!} V_0, \quad k \text{ odd},$$

or

$$b_{2m+1} = (4m+3)a^{2m+2} V_0 (-1)^m \frac{(2m)!}{2^{2m+1} m!(m+1)!}.$$

Note that $\Phi(a,x)$ is an odd function; that is, $\Phi(a,-x) = -\Phi(a,x)$ as is evident from its definition. Thus, only odd polynomials appear in the expansion of $\Phi(a,x)$ to preserve this property. Having found the coefficients, we can write the potential:

$$\Phi(r,\theta) = V_0 \sum_{m=0}^{\infty} (-1)^m \frac{(4m+3)(2m)!}{2^{2m+1} m!(m+1)!} \left(\frac{a}{r}\right)^{2m+2} P_{2m+1}(\cos\theta).$$

The place where Legendre polynomials appear most naturally is, as mentioned above, in the solution of Laplace's equation in spherical coordinates. After the partial differential equation is transformed into three ordinary differential equations using the method of the separation of variables, the differential equation corresponding to the polar angle $\theta$ gives rise to solutions of which Legendre polynomials are special cases. This differential equation simplifies to Legendre differential equation if the substitution $x = \cos\theta$ is made; in that case, the solutions will be Legendre polynomials in $x$, or in $\cos\theta$. That is why the argument of $P_k(x)$ is restricted to the interval $[-1,+1]$.

expanding Dirac delta function in terms of Legendre polynomials

**Example 8.5.2** We can expand the Dirac delta function in terms of Legendre polynomial. We write

$$\delta(x) = \sum_{n=0}^{\infty} a_n P_n(x), \tag{8.28}$$

where

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} P_n(x)\delta(x)\,dx = \frac{2n+1}{2} P_n(0). \tag{8.29}$$

For odd $n$ this will give zero, because $P_n(x)$ is an odd polynomial. This is to be expected because $\delta(x)$ is an even function of $x$ [$\delta(x) = \delta(-x) = 0$ for $x \neq 0$]. To evaluate $P_n(0)$ for even $n$, we use the recurrence relation (8.20) for $x = 0$:

$$(n+1)P_{n+1}(0) = -nP_{n-1}(0),$$

or $nP_n(0) = -(n-1)P_{n-2}(0)$, or $P_n(0) = -\frac{n-1}{n}P_{n-2}(0)$. Iterating this $m$ times, we obtain

$$P_n(0) = (-1)^m \frac{(n-1)(n-3)\cdots(n-2m+1)}{n(n-2)(n-4)\cdots(n-2m+2)} P_{n-2m}(0).$$

For $n = 2m$, this yields

$$P_{2m}(0) = (-1)^m \frac{(2m-1)(2m-3)\cdots 3\cdot 1}{2m(2m-2)\cdots 4\cdot 2} P_0(0).$$

Now we "fill the gaps" in the numerator by multiplying it—and the denominator, of course—by the denominator. This yields

$$\begin{aligned} P_{2m}(0) &= (-1)^m \frac{2m(2m-1)(2m-2)\cdots 3\cdot 2\cdot 1}{[2m(2m-2)\cdots 4\cdot 2]^2} \\ &= (-1)^m \frac{(2m)!}{[2^m m!]^2} = (-1)^m \frac{(2m)!}{2^{2m}(m!)^2}, \end{aligned}$$

because $P_0(x) = 1$. Thus, we can write

$$\delta(x) = \sum_{m=0}^{\infty} \frac{4m+1}{2} (-1)^m \frac{(2m)!}{2^{2m}(m!)^2} P_{2m}(x).$$

We can also derive this expansion as follows. For any complete set of ortho*normal* vectors $\{|f_k\rangle\}_{k=1}^{\infty}$, we have

$$\begin{aligned} \delta(x-x') &= w(x)\langle x|x'\rangle = w(x)\langle x|\mathbf{1}|x'\rangle \\ &= w(x)\langle x|\Big(\sum_k |f_k\rangle\langle f_k|\Big)|x'\rangle = w(x)\sum_k f_k^*(x') f_k(x). \end{aligned}$$

Legendre polynomials are not orthonormal; but we can make them so by dividing $P_k(x)$ by $h_k^{1/2} = \sqrt{2/(2k+1)}$. Then, noting that $w(x) = 1$, we obtain

$$\delta(x-x') = \sum_{k=0}^{\infty} \frac{P_k(x')}{\sqrt{2/(2k+1)}} \frac{P_k(x)}{\sqrt{2/(2k+1)}} = \sum_{k=0}^{\infty} \frac{2k+1}{2} P_k(x') P_k(x).$$

For $x' = 0$ we get

$$\delta(x) = \sum_{k=0}^{\infty} \frac{2k+1}{2} P_k(0) P_k(x),$$

which agrees with Eqs. (8.28) and (8.29).

## 8.6 Generating Functions

It is possible to generate all orthogonal polynomials of a certain kind from a single function of two variables $g(x,t)$ by repeated differentiation of that function. Such a function is called a **generating function**. This generating

generating function

**Table 8.2** Generating functions for Hermite, Laguerre, Legendre, and both Chebyshev polynomials

| Polynomial | Generating function | $a_n$ |
|---|---|---|
| $H_n(x)$ | $\exp(-t^2 + 2xt)$ | $\frac{1}{n!}$ |
| $L_n^\nu(x)$ | $\frac{\exp[-xt/(1-t)]}{(1-t)^{\nu+1}}$ | 1 |
| $P_n(x)$ | $(t^2 - 2xt + 1)^{-1/2}$ | 1 |
| $T_n(x)$ | $(1 - t^2)(t^2 - 2xt + 1)^{-1}$ | 2 if $n \neq 0$, $a_0 = 1$ |
| $U_n(x)$ | $(t^2 - 2xt + 1)^{-1}$ | 1 |

function is assumed to be expandable in the form

$$g(x,t) = \sum_{n=0}^{\infty} a_n t^n F_n(x), \tag{8.30}$$

so that the $n$th derivative of $g(x,t)$ with respect to $t$ evaluated at $t = 0$ gives $F_n(x)$ to within a multiplicative constant. The constant $a_n$ is introduced for convenience. Clearly, for $g(x,t)$ to be useful, it must be in closed form. The derivation of such a function for general $F_n(x)$ is nontrivial, and we shall not attempt to derive such a general generating function. Instead, we simply quote these functions in Table 8.2, and leave the derivation of the generating functions of Hermite and Legendre polynomials as Problems 8.12 and 8.21. For Laguerre polynomials see [Hass 08, pp. 679–680].

## 8.7 Problems

**8.1** Let $n = 1$ in Eq. (8.1) and solve for $s\frac{dw}{dx}$. Now substitute this in the derivative of $ws^n p_{\le k}$ and show that the derivative is equal to $ws^{n-1} p_{\le k+1}$. Repeat this process $m$ times to prove Lemma 8.1.2.

**8.2** Find $w(x)$, $a$, and $b$ for the case of the classical orthogonal polynomials in which $s(x)$ is of second degree.

**8.3** Integrate by parts twice and use Lemma 8.1.2 to show that

$$\int_a^b F_m \big(wsF_n'\big)' \, dx = 0 \quad \text{for } m < n.$$

**8.4** Using Lemma 8.1.2 conclude that

(a) $(wsF_n')'/w$ is a polynomial of degree less than or equal to $n$.
(b) Write $(wsF_n')'/w$ as a linear combination of $F_i(x)$, and use their orthogonality and Problem 8.3 to show that the linear combination collapses to a single term.
(c) Multiply both sides of the differential equation so obtained by $F_n$ and integrate. The RHS becomes $h_n\lambda_n$. For the LHS, carry out the differentiation and note that $(ws)'/w = K_1 F_1$.

Now show that $K_1 F_1 F_n' + s F_n''$ is a polynomial of degree $n$, and that the LHS of the differential equation yields $\{K_1 k_1^{(1)} n + \sigma_2 n(n-1)\} h_n$. Now find $\lambda_n$.

**8.5** Derive the recurrence relation of Eq. (8.8). Hint: Differentiate Eq. (8.5) and substitute for $F_n''$ from the differential equation. Now multiply the resulting equation by $\alpha_n x + \beta_n$ and substitute for $(\alpha_n x + \beta_n) F_n'$ from one of the earlier recurrence relations.

**8.6** Using only the orthogonality of Hermite polynomials

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = \sqrt{\pi}\, 2^n n!\, \delta_{mn}$$

generate the first three of them.

**8.7** Use the generalized Rodriguez formula for Hermite polynomials and integration by parts to expand $x^{2k}$ and $x^{2k+1}$ in terms of Hermite polynomials.

**8.8** Use the recurrence relation for Hermite polynomials to show that

$$\int_{-\infty}^{\infty} x e^{-x^2} H_m(x) H_n(x)\, dx = \sqrt{\pi}\, 2^{n-1} n! \left[\delta_{m,n-1} + 2(n+1)\delta_{m,n+1}\right].$$

What happens when $m = n$?

**8.9** Apply the general formalism of the recurrence relations given in the book to Hermite polynomials to find the following:

$$H_n + H_{n-1}' - 2x H_{n-1} = 0.$$

**8.10** Show that

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} H_n^2(x)\, dx = \sqrt{\pi}\, 2^n \left(n + \frac{1}{2}\right) n!$$

**8.11** Use a recurrence relations for Hermite polynomials to show that

$$H_n(0) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^m \frac{(2m)!}{m!} & \text{if } n = 2m. \end{cases}$$

**8.12** Differentiate the expansion of $g(x,t)$ for Hermite polynomials with respect to $x$ (treating $t$ as a constant) and choose $a_n$ such that $n a_n = a_{n-1}$ to obtain a differential equation for $g$. Solve this differential equation. To determine the "constant" of integration use the result of Problem 8.11 to show that $g(0,t) = e^{-t^2}$.

**8.13** Use the expansion of the generating function for Hermite polynomials to obtain

$$\sum_{m,n=0}^{\infty} e^{-x^2} H_m(x) H_n(x) \frac{s^m t^n}{m!n!} = e^{-x^2+2x(s+t)-(s^2+t^2)}.$$

Then integrate both sides over $x$ and use the orthogonality of the Hermite polynomials to get

$$\sum_{n=0}^{\infty} \frac{(st)^n}{(n!)^2} \int_{-\infty}^{\infty} e^{-x^2} H_n^2(x)\, dx = \sqrt{\pi}\, e^{2st}.$$

Deduce from this the normalization constant $h_n$ of $H_n(x)$.

**8.14** Using the recurrence relation of Eq. (8.14) repeatedly, show that

$$\int_{-\infty}^{\infty} x^k e^{-x^2} H_m(x) H_{m+n}(x)\, dx = \begin{cases} 0 & \text{if } n > k, \\ \sqrt{\pi}\, 2^m (m+k)! & \text{if } n = k. \end{cases}$$

**8.15** Show that for Legendre polynomials, $k_n^{(n)} = 2^n \Gamma(n+\frac{1}{2})/[n!\Gamma(\frac{1}{2})]$. Hint: Multiply and divide the expression given in the book by $n!$; take a factor of 2 out of all terms in the numerator; the even terms yield a factor of $n!$, and the odd terms give a gamma function.

**8.16** Using integration by parts several times, show that

$$\int_{-1}^{1} (1-x^2)^n dx = \frac{2^m n(n-1)\cdots(n-m+1)}{3\cdot 5\cdot 7\cdots(2m-1)} \int_{-1}^{1} x^{2m} (1-x^2)^{n-m} dx.$$

Now show that

$$\int_{-1}^{1} (1-x^2)^n dx = \frac{2\Gamma(\frac{1}{2})n!}{(2n+1)\Gamma(n+\frac{1}{2})}.$$

**8.17** Given that $P_0(x) = 1$ and $P_1(x) = x$, find $P_2(x)$, $P_3(x)$, and $P_4(x)$ using an appropriate recurrence relation.

**8.18** Use the generalized Rodriguez formula to show that $P_0(1) = 1$ and $P_1(1) = 1$. Now use a recurrence relation to show that $P_n(1) = 1$ for all $n$. To be rigorous, you need to use mathematical induction.

**8.19** Apply the general formalism of the recurrence relations given in the book to find the following two relations for Legendre polynomials:

(a) $nP_n - xP_n' + P_{n-1}' = 0.$

(b) $(1-x^2)P_n' - nP_{n-1} + nxP_n = 0.$

**8.20** Show that

$$\int_{-1}^{1} x^n P_n(x)\, dx = \frac{2^{n+1}(n!)^2}{(2n+1)!}.$$

Hint: Use the definition of $h_n$ and $k_n^{(n)}$ and the fact that $P_n$ is orthogonal to any polynomial of degree lower than $n$.

**8.21** Differentiate the expansion of $g(x,t)$ for Legendre polynomials, and choose $a_n = 1$. For $P_n'$, you will substitute two different expressions to get two equations. First use Eq. (8.11) with $n+1$ replaced by $n$, to obtain

$$\left(1-t^2\right)\frac{dg}{dx} + tg = 2\sum_{n=2}^{\infty} nt^n P_{n-1} + 2t.$$

As an alternative, use Eq. (8.10) to substitute for $P_n'$ and get

$$(1-xt)\frac{dg}{dx} = \sum_{n=2}^{\infty} nt^n P_{n-1} + t.$$

Combine the last two equations to get $(t^2 - 2xt + 1)g' = tg$. Solve this differential equation and determine the "constant" of integration by using $P_n(1) = 1$ to show that $g(1,t) = 1/(1-t)$.

**8.22** Use the generating function for Legendre polynomials to show that $P_n(1) = 1$, $P_n(-1) = (-1)^n$, $P_n(0) = 0$ for odd $n$, and $P_n'(1) = n(n+1)/2$.

**8.23** Both electrostatic and gravitational potential energies depend on the quantity $1/|\mathbf{r}-\mathbf{r}'|$, where $\mathbf{r}'$ is the position vector of a point inside a charge or mass distribution and $\mathbf{r}$ is the position vector of the observation point.

(a) Let $\mathbf{r}$ lie along the $z$-axis, and use spherical coordinates and the definition of generating functions to show that

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r_>}\sum_{n=0}^{\infty}\left(\frac{r_<}{r_>}\right)^n P_n(\cos\theta),$$

where $r_<(r_>)$ is the smaller (larger) of $r$ and $r'$, and $\theta$ is the polar angle.

(b) The electrostatic or gravitational potential energy $\Phi(\mathbf{r})$ is given by

$$\Phi(\mathbf{r}) = k\iiint \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3x',$$

where $k$ is a constant and $\rho(\mathbf{r}')$ is the (charge or mass) density function. Use the result of part (a) to show that if the density depends only on $r'$, and not on any angle (i.e., $\rho$ is spherically symmetric), then $\Phi(\mathbf{r})$ reduces to the potential energy of a point charge at the origin for $r > r'$.

(c) What is $\Phi(\mathbf{r})$ for a spherically symmetric density which extends from the origin to $a$, with $a \gg r$ for any $r$ of interest?

(d) Show that the electric field $\mathbf{E}$ or gravitational field $\mathbf{g}$ (i.e., the negative gradient of $\Phi$) at any radial distance $r$ from the origin is given by $\frac{kQ(r)}{r^2}\hat{\mathbf{e}}_r$, where $Q(r)$ is the charge or mass enclosed in a sphere of radius $r$.

**8.24** Use the generating function for Legendre polynomials and their orthogonality to derive the relation

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} P_n^2(x)\,dx.$$

Integrate the LHS, expand the result in powers of $t$, and compare these powers on both sides to obtain the normalization constant $h_n$.

**8.25** Evaluate the following integrals using the expansion of the generating function for Legendre polynomials.

$$\text{(a)} \quad \int_0^{\pi} \frac{(a\cos\theta + b)\sin\theta\,d\theta}{\sqrt{a^2+2ab\cos\theta+b^2}}.$$

$$\text{(b)} \quad \int_0^{\pi} \frac{(a\cos^2\theta + b\sin^2\theta)\sin\theta\,d\theta}{\sqrt{a^2+2ab\cos\theta+b^2}}.$$

**8.26** Differentiate the expansion of the Legendre polynomial generating function with respect to $x$ and manipulate the resulting expression to obtain

$$\left(1-2xt+t^2\right)\sum_{n=0}^{\infty} t^n P_n'(x) = t\sum_{n=0}^{\infty} t^n P_n(x).$$

Equate equal powers of $t$ on both sides to derive the recurrence relation

$$P_{n+1}' + P_{n-1}' - 2xP_n' - P_n = 0.$$

**8.27** Show that

$$\int_0^1 P_k(x)\,dx = \begin{cases} \delta_{k0} & \text{if } k \text{ is even,} \\ \frac{(-1)^{(k+1)/2}(k-1)!}{2^k(\frac{k-1}{2})!(\frac{k+1}{2})!} & \text{if } k \text{ is odd.} \end{cases}$$

Hint: For even $k$, extend the region of integration to $(-1, 1)$ and use the orthogonality property. For odd $k$, note that

$$\frac{d^{k-1}}{dx^{k-1}}\left(1-x^2\right)^k\Big|_0^1$$

gives zero for the upper limit (by Lemma 8.1.3). For the lower limit, expand the expression using the binomial theorem, and carry out the differentiation, keeping in mind that only one term of the expansion contributes.

**8.28** Show that $g(x,t) = g(-x,-t)$ for both Hermite and Legendre polynomials. Now expand $g(x,t)$ and $g(-x,-t)$ and compare the coefficients of $t^n$ to obtain the **parity relations** for these polynomials:

parity relations

$$H_n(-x) = (-1)^n H_n(x) \quad \text{and} \quad P_n(-x) = (-1)^n P_n(x).$$

**8.29** Derive the orthogonality of Legendre polynomials directly from the differential equation they satisfy.

**8.30** Expand $|x|$ in the interval $(-1, +1)$ in terms of Legendre polynomials. Hint: Use the result of Problem 8.27.

**8.31** Apply the general formalism of the recurrence relations given in the book to find the following two relations for Laguerre polynomials:

(a) $nL_n^\nu - (n+\nu)L_{n-1}^\nu - x\frac{dL_n^\nu}{dx} = 0.$
(b) $(n+1)L_{n+1}^\nu - (2n+\nu+1-x)L_n^\nu + (n+\nu)L_{n-1}^\nu = 0.$

**8.32** From the generating function for Laguerre polynomials given in Table 8.2 deduce that $L_n^\nu(0) = \Gamma(n+\nu+1)/[n!\Gamma(\nu+1)]$.

**8.33** Let $L_n \equiv L_n^0$. Now differentiate both sides of

$$g(x,t) = \frac{e^{-xt/(1-t)}}{1-t} = \sum_0^\infty t^n L_n(x)$$

with respect to $x$ and compare powers of $t$ to obtain $L_n'(0) = -n$ and $L_n''(0) = \frac{1}{2}n(n-1)$. Hint: Differentiate $1/(1-t) = \sum_{n=0}^\infty t^n$ to get an expression for $(1-t)^{-2}$.

**8.34** Expand $e^{-kx}$ as a series of Laguerre polynomials $L_n^\nu(x)$. Find the coefficients by using (a) the orthogonality of $L_n^\nu(x)$ and (b) the generating function.

**8.35** Derive the recurrence relations given in the book for Jacobi, Gegenbauer, and Chebyshev polynomials.

**8.36** Show that $T_n(-x) = (-1)^n T_n(x)$ and $U_n(-x) = (-1)^n U_n(x)$. Hint: Use $g(x,t) = g(-x,-t)$.

**8.37** Show that $T_n(1) = 1$, $U_n(1) = n+1$, $T_n(-1) = (-1)^n$, $U_n(-1) = (-1)^n(n+1)$, $T_{2m}(0) = (-1)^m = U_{2m}(0)$, and $T_{2m+1}(0) = 0 = U_{2m+1}(0)$.