Clifford Algebras 27

The last chapter introduced the exterior product, which multiplied a p-vector and a q-vector to yield a $(p+q)$-vector. By directly summing the spaces of all such vectors, we obtained a vector space which was closed under multiplication. This led to a 2^n-dimensional algebra, which we called the exterior algebra (see Theorem 26.3.6).

In the meantime we revisited inner product and considered non-Euclidean inner products, which are of physical significance. In this chapter, we shall combine the exterior product with the inner product to create a new type of algebra, the Clifford algebra, which happens to have important applications in physics.

In our definition of exterior product in the previous chapter, we assumed that the number of vectors was equal to the number of linear functionals taken from the dual space [see Eq. (26.14)]. As a result of this complete pairing, we always ended up with a number. It is useful, however, to define an "incomplete" pairing in which the number of vectors and dual vectors are not the same. In particular, if we have a p-vector and a single 1-form, then we can pair the 1-form with one of the factors of the p-vector to get a $(p-1)$-vector. This process is important enough to warrant the following:

Definition 27.0.1 Let $\mathbf{A}$ be a p-vector and $\boldsymbol{\theta}$ a 1-form in a vector space $\mathcal{V}$. Then define $i_{\boldsymbol{\theta}} : \Lambda^p(\mathcal{V}^*) \to \Lambda^{p-1}(\mathcal{V}^*)$ by

interior product of a 1-form and a p-vector

$$i_{\boldsymbol{\theta}}\mathbf{A}(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{p-1}) = \mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{p-1}).$$

$i_{\boldsymbol{\theta}}\mathbf{A}$ is called the **interior product** or **contraction** of $\boldsymbol{\theta}$ and $\mathbf{A}$.

Note that if $\mathbf{A}$ is a 1-vector $\mathbf{v}$, then $i_{\boldsymbol{\theta}}\mathbf{v} = \langle \boldsymbol{\theta}, \mathbf{v} \rangle$, and if it is a real number α, then (by definition) $i_{\boldsymbol{\theta}}\alpha = 0$.

An immediate consequence of Definition 27.0.1 is the following:

Theorem 27.0.2 *Let $\mathbf{A}$ be a p-vector and $\mathbf{B}$ be a q-vector on a vector space $\mathcal{V}$. Then, $i_{\boldsymbol{\theta}}$ is an* ***antiderivation*** *with respect to the wedge product*:

antiderivation

$$i_{\boldsymbol{\theta}}(\mathbf{A} \wedge \mathbf{B}) = (i_{\boldsymbol{\theta}}\mathbf{A}) \wedge \mathbf{B} + (-1)^p \mathbf{A} \wedge (i_{\boldsymbol{\theta}}\mathbf{B}).$$

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If $(\mathcal{V}, \mathbf{g})$ is an inner product space, we can define the interior product of a 1-vector $\mathbf{v}$ and a p-vector $\mathbf{A}$. In fact, if $\mathbf{g}_* : \mathcal{V} \to \mathcal{V}^*$ is as defined in Eq. (26.30), then let

$$i_{\mathbf{v}}\mathbf{A} \equiv i_{\mathbf{g}_*(\mathbf{v})}\mathbf{A}. \tag{27.1}$$

In particular, if $\mathbf{A}$ is a vector $\mathbf{u}$, then $i_{\mathbf{v}}\mathbf{u} = \mathbf{g}(\mathbf{u}, \mathbf{v})$.

27.1 Construction of Clifford Algebras

Clifford product

Let $\mathcal{V}$ be a real vector space with inner product $\mathbf{g}$. Let $\mathbf{v} \in \mathcal{V}$ and $\mathbf{A} \in \Lambda^p(\mathcal{V}^*)$. Define the product $\vee : \mathcal{V} \times \Lambda^p(\mathcal{V}^*) \to \Lambda^{p+1}(\mathcal{V}^*) \oplus \Lambda^{p-1}(\mathcal{V}^*)$ by

$$\mathbf{v} \vee \mathbf{A} = \mathbf{v} \wedge \mathbf{A} + i_{\mathbf{v}}\mathbf{A} \tag{27.2}$$

where $i_{\mathbf{v}}\mathbf{A}$ is as defined in Eq. (27.1). This product is called the **Clifford product**.

The special case of $p = 1$ is of importance. For such a case, we obtain

$$\mathbf{v} \vee \mathbf{u} = \mathbf{v} \wedge \mathbf{u} + i_{\mathbf{v}}\mathbf{u} = \mathbf{v} \wedge \mathbf{u} + \mathbf{g}(\mathbf{u}, \mathbf{v}) \tag{27.3}$$

which can also be written as

$$\mathbf{v} \vee \mathbf{u} + \mathbf{u} \vee \mathbf{v} = 2\mathbf{g}(\mathbf{u}, \mathbf{v}). \tag{27.4}$$

This equation is sometimes taken as the definition of the Clifford product and the starting point of the Clifford algebra (to be discussed below).

We see that the Clifford product has been defined on the vector space which underlies the exterior *algebra*. However, the left factor in the Clifford product is just a vector, not a general member of the exterior algebra. Is it possible to define Clifford product of a q-vector and a p-vector? It is indeed possible if we assume that $\vee$ is associative and distributes over addition. To show this, pick a basis and write a q-vector in terms of that basis. Thus, let $\mathbf{A}$ be as before and let $\mathbf{B} \in \Lambda^q(\mathcal{V}^*)$, and write

$$\mathbf{B} = \frac{1}{q!} b^{j_1 \dots j_q} \mathbf{e}_{j_1} \wedge \dots \wedge \mathbf{e}_{j_q}.$$

Then

$$\begin{aligned} q!\mathbf{B} \vee \mathbf{A} &= b^{j_1 \dots j_q} \mathbf{e}_{j_1} \wedge \dots \wedge \mathbf{e}_{j_q} \vee \mathbf{A} \\ &= b^{j_1 \dots j_q} \mathbf{e}_{j_1} \wedge \dots \wedge \mathbf{e}_{j_{q-2}} \wedge (\mathbf{e}_{j_{q-1}} \vee \mathbf{e}_{j_q} + g_{j_q j_{q-1}}) \vee \mathbf{A} \\ &= b^{j_1 \dots j_q} \mathbf{e}_{j_1} \wedge \dots \wedge \mathbf{e}_{j_{q-2}} \wedge (\mathbf{e}_{j_{q-1}} \vee \mathbf{e}_{j_q} \vee \mathbf{A}) \end{aligned} \tag{27.5}$$

because $g_{j_q j_{q-1}}$ is symmetric under the exchange of j_q and j_{q-1} while $b^{j_1 \dots j_q}$ is antisymmetric. To continue, we use Eq. (27.2) and rewrite the term in the parentheses on the last line of Eq. (27.5):

$$\begin{aligned}\mathbf{e}_{j_{q-1}} \vee \mathbf{e}_{j_q} \vee \mathbf{A} &= \mathbf{e}_{j_{q-1}} \vee (\mathbf{e}_{j_q} \wedge \mathbf{A} + i_{\mathbf{e}_{j_q}} \mathbf{A}) \\ &= \mathbf{e}_{j_{q-1}} \vee (\mathbf{e}_{j_q} \wedge \mathbf{A}) + \mathbf{e}_{j_{q-1}} \vee (i_{\mathbf{e}_{j_q}} \mathbf{A}) \\ &= \mathbf{e}_{j_{q-1}} \wedge (\mathbf{e}_{j_q} \wedge \mathbf{A}) + i_{\mathbf{e}_{j_{q-1}}} (\mathbf{e}_{j_q} \wedge \mathbf{A}) \\ &\quad + \mathbf{e}_{j_{q-1}} \wedge (i_{\mathbf{e}_{j_q}} \mathbf{A}) + i_{\mathbf{e}_{j_{q-1}}} (i_{\mathbf{e}_{j_q}} \mathbf{A}) \\ &= \mathbf{e}_{j_{q-1}} \wedge \mathbf{e}_{j_q} \wedge \mathbf{A} + g_{j_q j_{q-1}} \mathbf{A} - \mathbf{e}_{j_q} \wedge (i_{\mathbf{e}_{j_{q-1}}} \mathbf{A}) \\ &\quad + \mathbf{e}_{j_{q-1}} \wedge (i_{\mathbf{e}_{j_q}} \mathbf{A}) + i_{\mathbf{e}_{j_{q-1}}} (i_{\mathbf{e}_{j_q}} \mathbf{A}),\end{aligned}$$

where in the last equality, we used the antiderivation property of the interior product (Theorem 27.0.2). Substituting the last equation in (27.5) yields

$$\begin{aligned}q!\mathbf{B} \vee \mathbf{A} = q!\mathbf{B} \wedge \mathbf{A} &+ b^{j_1 \dots j_q} \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_{q-2}} \wedge \mathbf{e}_{j_{q-1}} \wedge (i_{\mathbf{e}_{j_q}} \mathbf{A}) \\ &- b^{j_1 \dots j_q} \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_{q-2}} \wedge \mathbf{e}_{j_q} \wedge (i_{\mathbf{e}_{j_{q-1}}} \mathbf{A}) \\ &+ b^{j_1 \dots j_q} \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_{q-2}} \wedge \big[i_{\mathbf{e}_{j_{q-1}}} (i_{\mathbf{e}_{j_q}} \mathbf{A}) \big].\end{aligned} \tag{27.6}$$

The right-hand side is given entirely in terms of wedge products, which are known operations. Hence, the Clifford product of any p-vector and q-vector can be defined, and this product is in $\Lambda(\mathcal{V}^*)$ of Theorem 26.3.6. Thus, $\Lambda(\mathcal{V}^*)$ is an algebra not only under the wedge product but also under the Clifford product. With the latter as the multiplication rule, $\Lambda(\mathcal{V}^*)$ is called a **Clifford algebra** and denoted by $\mathcal{C}_V$.

Clifford algebra

Historical Notes

At the age of 15 **William Clifford** went to King's College, London where he excelled in mathematics, classics, English literature, and gymnastics. Three years later, he entered Trinity College, Cambridge, where he won not only prizes for mathematics but also one for a speech he delivered on Sir Walter Raleigh. In 1868, he was elected to a Fellowship at Trinity, and three years later, he was appointed to the chair of Mathematics and Mechanics at University College London. In 1874 he was elected a Fellow of the Royal Society. He was also an active member of the London Mathematical Society which held its meetings at University College.

William Clifford
1845–1879

Clifford read the work of Riemann and Lobachevsky on non-euclidean geometry, and became interested in the subject. Almost 50 years before the advent of Einstein's general theory of relativity, he wrote *On the space theory of matter* in which he argued that energy and matter are different aspects of the curvature of space.

Clifford generalised the quaternions (introduced by Hamilton two years before Clifford's birth) to what he called the biquaternions and he used them to study motion in non-euclidean spaces and on certain surfaces.

As a teacher, Clifford's reputation was outstanding and famous for his clarity of explanation of difficult mathematical problems. Not only was he a highly original teacher and researcher, he was also a philosopher of science. At the age of 23 he delivered a lecture to the Royal Institution entitled *Some of the conditions of mental development*, in which he tried to explain how scientific discovery comes about.

He was eccentric in appearance, habits and opinions. A fellow undergraduate describes him as follows: "His neatness and dexterity were unusually great, but the most remarkable thing was his great strength as compared with his weight. At one time he would pull up on the bar with either hand."

Like another British mathematician, Charles Dodgson, he took pleasure in entertaining children. Although he never achieved Dodgson's success in writing such books as *Alice's Adventures in Wonderland* (which the latter wrote under the pseudonym Lewis Carroll), Clifford wrote *The Little People*, a collection of fairy stories written to amuse children.

In 1876 Clifford suffered a physical collapse, which was made worse by overwork, and most likely, caused by it. He would spend the entire day teaching and doing administrative work, and the entire night doing research. Spending six months in Algeria and Spain allowed him to recover sufficiently to resume his work. But after 18 months he collapsed again, after which he spent some time in Mediterranean countries, but this was not enough to improve his health. After a couple of months in England in late 1878, he left for Madeira. The hoped-for recovery never materialized and he died a few months later.

We have shown that the Clifford product of a p-vector and a q-vector lies in $\Lambda(\mathcal{V}^*)$. This implies that the underlying vector space of the Clifford algebra is a subspace of $\Lambda(\mathcal{V}^*)$. However, it can be shown that the set of Clifford products exhaust the entire $\Lambda(\mathcal{V}^*)$; i.e., that the Clifford algebra is 2^N-dimensional. This follows from the fact that a p-vector A, which can be written as

$$\mathsf{A} = \frac{1}{p!} a^{i_1 i_2 \dots i_p} \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_p}, \tag{27.7}$$

can also be written as

$$\mathsf{A} \to \mathsf{a} = \frac{1}{p!} a^{i_1 i_2 \dots i_p} \mathbf{e}_{i_1} \vee \mathbf{e}_{i_2} \vee \cdots \vee \mathbf{e}_{i_p}, \tag{27.8}$$

where we have introduced a new notation to differentiate between members of the exterior algebra and the Clifford algebra. The details of the derivation of (27.8) from (27.7) are given as Problem 27.1.

27.1.1 The Dirac Equation

The interest in the Clifford algebra in the physics community came about after Dirac discovered the relativistic wave equation for an electron. As is usually the case, when a mathematical topic finds its way into physics, a healthy collaboration between physicists and mathematicians sets in and the topic becomes an active area of research in both fields. Dirac's discovery and its connection with the Clifford algebra has led to some fundamental results in many branches of mathematics. It is therefore worthwhile to see how Dirac discovered the equation that now bears his name.

The transition from classical to quantum mechanics is made by changing the energy E and momentum $\mathbf{p}$ to derivative operators:[1] $E \to i\partial/\partial t$ and $\mathbf{p} \to -i\nabla$ which act on the wave function ψ. Thus a non-relativistic free particle, whose energy and momentum are related by $E = p^2/2m$, is described by the Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = \frac{(-i\nabla)^2}{2m}\psi \quad \text{or} \quad i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi.$$

The relativistic energy-momentum relation, $E^2 - p^2 = m^2$, leads to Klein-Gordon equation whose time derivative is of second order. Although eventually accepted as a legitimate equation for relativistic particles, Klein-

[1] We are using the natural units for which the Planck constant (over 2π) and the speed of light are set to 1: $\hbar = 1 = c$.

Gordon equation was initially abandoned because, due to its second derivative in time, it gave rise to negative probabilities. Therefore, it was desirable to find a relativistic equation which was first order in time derivative, and Dirac found precisely such an equation.

Dirac's idea was to factor out $E^2 - p^2$ into $(E - p)(E + p)$ and to somehow incorporate the mass term in the factorization. We avoid writing E and $\mathbf{p}$ as derivatives, but consider them as commuting operators. Since it is not possible to include m in a straightforward factorization, Dirac came up with the ingenious idea of multiplying E and $\mathbf{p}$ operators by quantities to be determined by certain consistency conditions. More precisely, he considered an equation of the form

$$\left(\beta E + \sum_{i=1}^{3} \alpha_i p_i + m\right)\psi = 0,$$

and demanded that β and α_i be chosen in such way that

$$\left(\beta E + \sum_{j=1}^{3} \alpha_j p_j - m\right)\left(\beta E + \sum_{i=1}^{3} \alpha_i p_i + m\right)\psi = 0 \qquad (27.9)$$

reduce to

$$\left(E^2 - \sum_{i=1}^{3} p_i^2 - m^2\right)\psi = 0. \qquad (27.10)$$

Multiplying the two parentheses above, we obtain

$$\begin{aligned}
&\beta^2 E^2 + \sum_{i=1}^{3} \beta\alpha_i E p_i + \beta m E + \sum_{j=1}^{3} \alpha_j \beta E p_j + \sum_{i,j=1}^{3} \alpha_j \alpha_i p_j p_i \\
&\quad + \sum_{j=1}^{3} m\alpha_j p_j - \beta m E - \sum_{i=1}^{3} m\alpha_i p_i - m^2 \\
&= \beta^2 E^2 + \sum_{i=1}^{3} (\beta\alpha_i + \alpha_i\beta) E p_i + \frac{1}{2}\sum_{i,j=1}^{3} (\alpha_j\alpha_i + \alpha_i\alpha_j) p_i p_j - m^2.
\end{aligned}$$

For this to be equal to the expression in parentheses of Eq. (27.10), we need to have

$$\beta^2 = 1, \qquad \beta\alpha_i + \alpha_i\beta = 0, \qquad \frac{1}{2}(\alpha_j\alpha_i + \alpha_i\alpha_j) = -\delta_{ij}.$$

The last condition is the result of the fact that $p_j p_i$ is symmetric in ij, and therefore, its product with the antisymmetric part of $\alpha_i\alpha_j$ automatically vanishes. Letting $\beta \equiv \gamma^0$ and $\alpha_i = \gamma^i$, the above conditions can be condensed into the single condition

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}. \qquad (27.11)$$

This equation is identical to (27.4), hence the connection between the Dirac equation and Clifford algebra.

It is clear that Eq. (27.11) cannot hold if the γs are ordinary numbers. In fact, Dirac showed that they have to be 4×4 matrices, now called **Dirac γ matrices**. If the γs are 4×4 matrices, then ψ must be a column vector with 4 components. It turns out that two of these components correspond to the two components of the electron spin. It took a while before the other two components were identified as those of the antiparticle of the electron, namely positron.

Dirac γ matrices

27.2 General Properties of the Clifford Algebra

Equation (27.8) implies that a vector in $\Lambda(\mathcal{V}^*)$, being a direct sum of p-vectors for different p's, can be expressed as a linear combination of the basis vectors of $\Lambda^p(\mathcal{V}^*)$, where the basis vectors are given as Clifford product (rather than the wedge product) of the basis vectors of $\mathcal{V}$:

Theorem 27.2.1 *Let $\{\mathbf{e}_i\}_{i=1}^N$ be a basis of an inner product space $\mathcal{V}$. Then the 2^N vectors*

$$\mathbf{1}, \mathbf{e}_i, \mathbf{e}_i \vee \mathbf{e}_j (i < j), \mathbf{e}_i \vee \mathbf{e}_j \vee \mathbf{e}_k, (i < j < k), \dots, \mathbf{e}_1 \vee \mathbf{e}_2 \vee \cdots \vee \mathbf{e}_N$$

form a basis of $\mathcal{C}_V$.

Thus, if $\mathbf{u}$ is an arbitrary vector of $\Lambda(\mathcal{V}^*)$, then it can be expressed as follows:

$$\mathbf{u} = \alpha\mathbf{1} + u^i\mathbf{e}_i + u^{|i_1i_2|}\mathbf{e}_{i_1} \vee \mathbf{e}_{i_2} + \cdots + u^{|i_1\dots i_N|}\mathbf{e}_{i_1} \vee \cdots \vee \mathbf{e}_{i_N} \qquad (27.12)$$

where $|i_1 i_2 \dots i_p|$ means that the sum over repeated indices is over $i_1 < i_2 < \cdots < i_p$. Equation (27.12) is sometimes written as

$$\mathbf{u} = \alpha\mathbf{1} + u^i\mathbf{e}_i + \frac{1}{2!}u^{i_1i_2}\mathbf{e}_{i_1} \vee \mathbf{e}_{i_2} + \cdots + \frac{1}{N!}u^{i_1\dots i_N}\mathbf{e}_{i_1} \vee \cdots \vee \mathbf{e}_{i_N} \quad (27.13)$$

where the coefficients are assumed completely antisymmetric in all their indices, but the sum has no ordering.

Note the appearance of $\mathbf{1}$ in the sum multiplying the scalar α. This suggests changing (27.4) to

$$\mathbf{v} \vee \mathbf{u} + \mathbf{u} \vee \mathbf{v} = 2\mathbf{g}(\mathbf{u}, \mathbf{v})\mathbf{1}, \qquad (27.14)$$

which, when specialized to the basis vectors, becomes

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 2g_{ij}\mathbf{1}, \qquad \mathbf{e}_i^2 \equiv \mathbf{e}_i \vee \mathbf{e}_i = g_{ii}\mathbf{1}, \qquad (27.15)$$

where we have removed the multiplication sign $\vee$, a practice which we shall often adhere to from now on. Equation (27.15) completely frees the Clifford algebra from the exterior algebra, with which we started our discussion. This is easily seen in an orthonormal basis:

$$\mathbf{e}_i^2 = \pm\mathbf{1}, \qquad \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \quad \text{if } i \neq j. \qquad (27.16)$$

Multiplying elements **u** and **v**, each expressed as in (27.12) or (27.13), reduces to the multiplication of various Clifford products of the basis vectors. In such multiplications, one commutes the basis vectors which appear twice in the product using (27.16) until all repetitions disappear and one regains Clifford products of basis vectors appearing in (27.13). The following example should clarify this.

Example 27.2.2 Let $\mathcal{V}$ be a 2-dimensional Euclidean vector space (i.e., $g_{ij} = \delta_{ij}$) with the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. Consider two elements **u** and **v** of the Clifford algebra over $\mathcal{V}$. These can very generally be written as

$$\mathbf{u} = \alpha_u \mathbf{1} + \beta_u^1 \mathbf{e}_1 + \beta_u^2 \mathbf{e}_2 + \gamma_u \mathbf{e}_1\mathbf{e}_2$$
$$\mathbf{v} = \alpha_v \mathbf{1} + \beta_v^1 \mathbf{e}_1 + \beta_v^2 \mathbf{e}_2 + \gamma_v \mathbf{e}_1\mathbf{e}_2$$

and

$$\begin{aligned}
\mathbf{u} \vee \mathbf{v} &= \left(\alpha_u \mathbf{1} + \beta_u^1 \mathbf{e}_1 + \beta_u^2 \mathbf{e}_2 + \gamma_u \mathbf{e}_1\mathbf{e}_2\right) \vee \left(\alpha_v \mathbf{1} + \beta_v^1 \mathbf{e}_1 + \beta_v^2 \mathbf{e}_2 + \gamma_v \mathbf{e}_1\mathbf{e}_2\right) \\
&= \alpha_u\alpha_v \mathbf{1} + \alpha_u \beta_v^1 \mathbf{e}_1 + \alpha_u \beta_v^2 \mathbf{e}_2 + \alpha_u \gamma_v \mathbf{e}_1\mathbf{e}_2 \\
&\quad + \alpha_v \beta_u^1 \mathbf{e}_1 + \beta_u^1 \beta_v^1 \underbrace{\mathbf{e}_1\mathbf{e}_1}_{=1} + \beta_u^1 \beta_v^2 \mathbf{e}_1\mathbf{e}_2 + \beta_u^1 \gamma_v \underbrace{\mathbf{e}_1\mathbf{e}_1}_{=1} \mathbf{e}_2 \\
&\quad + \beta_u \alpha_v^2 \mathbf{e}_2 + \beta_u^2 \beta_v^1 \underbrace{\mathbf{e}_2\mathbf{e}_1}_{-\mathbf{e}_1\mathbf{e}_2} + \beta_u^2 \beta_v^2 \underbrace{\mathbf{e}_2\mathbf{e}_2}_{=1} + \beta_u^2 \gamma_v \overbrace{\underbrace{\mathbf{e}_2\mathbf{e}_1}_{-\mathbf{e}_1\mathbf{e}_2} \mathbf{e}_2}^{-\mathbf{e}_1} \\
&\quad + \gamma_u \alpha_v \mathbf{e}_1\mathbf{e}_2 + \gamma_u \beta_v^1 \overbrace{\mathbf{e}_1 \underbrace{\mathbf{e}_2\mathbf{e}_1}_{-\mathbf{e}_1\mathbf{e}_2}}^{-\mathbf{e}_2} + \gamma_u \beta_v^2 \mathbf{e}_1 \underbrace{\mathbf{e}_2\mathbf{e}_2}_{=1} + \gamma_u \gamma_v \overbrace{\mathbf{e}_1 \underbrace{\mathbf{e}_2\mathbf{e}_1}_{-\mathbf{e}_1\mathbf{e}_2} \mathbf{e}_2}^{=-1}.
\end{aligned}$$

We see that the right-hand side is a linear combination of $\mathbf{1}$, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_1\mathbf{e}_2$, as it should be since the Clifford algebra is closed under multiplication. Problem 27.2 asks you to find the coefficients of the linear combination.

Although we will primarily be dealing with real vector spaces, Eqs. (27.15) and (27.16) could be applied to complex vector spaces. Therefore, it is possible to have complex Clifford algebras, and we shall occasionally deal with such algebras as well. complex Clifford algebras

If **u** in (27.13) or (27.12) contains products of only even numbers of basis vectors, then it is called an **even element** of the algebra. The collection of all even elements of $\mathcal{C}_V$ is a subalgebra of $\mathcal{C}_V$ and is denoted by $\mathcal{C}_V^0$. The **odd elements** are denoted by $\mathcal{C}_V^1$, and although they form a *subspace* of $\mathcal{C}_V$, obviously, they do not form a subalgebra. As a vector space, $\mathcal{C}_V$ is the direct sum of the even and odd subspaces: even and odd elements

$$\mathcal{C}_V = \mathcal{C}_V^0 \oplus \mathcal{C}_V^1. \tag{27.17}$$

The discussion above can be made slightly more formal.

Definition 27.2.3 Let ω be the linear automorphism of $\mathcal{V}$ given by $\omega(\mathbf{a}) = -\mathbf{a}$ for all $\mathbf{a} \in \mathcal{V}$. The involution of the Clifford algebra $\mathcal{C}_V$ induced by ω is called the **degree involution** and is denoted by ω_V.

degree involution

Note that for any $\mathbf{u} \in \mathcal{C}_V$ given by (27.13), $\omega_V(\mathbf{u})$ is obtained by changing the sign of all the vectors in that equation. It is obvious that $\omega_V^2 = \iota$, where $\iota(\mathbf{u}) = \mathbf{u}$ for all $\mathbf{u} \in \mathcal{C}_V$. Thus, ω_V is indeed an involution of $\mathcal{C}_V$. Now, an involution has only two eigenvalues, ± 1, and the intersection of the eigenspaces of these eigenvalues is zero. Moreover, we can identify these eigenspaces as $\mathcal{C}_V^0$ and $\mathcal{C}_V^1$, where

$$\mathcal{C}_V^0 = \ker(\omega_V - \iota), \qquad \mathcal{C}_V^1 = \ker(\omega_V + \iota). \tag{27.18}$$

Consider two inner product spaces $\mathcal{V}$ and $\mathcal{U}$, and define the inner product on their direct sum $\mathcal{V} \oplus \mathcal{U}$ by

$$\langle \mathbf{v}_1 \oplus \mathbf{u}_1, \mathbf{v}_2 \oplus \mathbf{u}_2 \rangle \equiv \langle \mathbf{v}_1, \mathbf{v}_2 \rangle + \langle \mathbf{u}_1, \mathbf{u}_2 \rangle.$$

Then we have the following important theorem: (For a proof, see [Greu 78, p. 234])

Theorem 27.2.4 *Let $\mathcal{W} = \mathcal{V} \oplus \mathcal{U}$. Then the Clifford algebra $\mathcal{C}_W$ is isomorphic to the skew symmetric tensor product of $\mathcal{C}_V$ and $\mathcal{C}_U$:*

$$\mathcal{C}_W \equiv \mathcal{C}_{V\oplus U} \cong \mathcal{C}_V \hat{\otimes} \mathcal{C}_U.$$

The skew symmetric tensor product was defined for exterior algebras in Definition 26.3.7, but it can also be defined for Clifford algebras. One merely has to change $\wedge$ to $\vee$. Note that the caret over $\otimes$ signifies the *product* defined on the *space* $\mathcal{C}_V \otimes \mathcal{C}_U$. Thus, as vector spaces, $\mathcal{C}_{V\oplus U} \cong \mathcal{C}_V \otimes \mathcal{C}_U$. Since all Clifford algebras are direct sums of their even and odd subspaces, we have

$$\begin{aligned} \mathcal{C}_W^0 \oplus \mathcal{C}_W^1 &\cong (\mathcal{C}_V^0 \oplus \mathcal{C}_V^1) \otimes (\mathcal{C}_U^0 \oplus \mathcal{C}_U^1) \\ &\cong (\mathcal{C}_V^0 \otimes \mathcal{C}_U^0) \oplus (\mathcal{C}_V^0 \otimes \mathcal{C}_U^1) \oplus (\mathcal{C}_V^1 \otimes \mathcal{C}_U^0) \oplus (\mathcal{C}_V^1 \otimes \mathcal{C}_U^1). \end{aligned}$$

In particular,

$$\begin{aligned} \mathcal{C}_W^0 &\cong (\mathcal{C}_V^0 \otimes \mathcal{C}_U^0) \oplus (\mathcal{C}_V^1 \otimes \mathcal{C}_U^1) \\ \mathcal{C}_W^1 &\cong (\mathcal{C}_V^0 \otimes \mathcal{C}_U^1) \oplus (\mathcal{C}_V^1 \otimes \mathcal{C}_U^0). \end{aligned} \tag{27.19}$$

Furthermore, if we invoke the product of Definition 26.3.7 on the first equation above, we get

$$\mathcal{C}_W^0 \cong (\mathcal{C}_V^0 \hat{\otimes} \mathcal{C}_U^0) \oplus (\mathcal{C}_V^1 \hat{\otimes} \mathcal{C}_U^1). \tag{27.20}$$

The second equation in (27.19), when restricted to vector spaces themselves, yields

$$\mathcal{W} \cong (\mathbf{1}_V \otimes \mathcal{U}) \oplus (\mathcal{V} \otimes \mathbf{1}_U), \tag{27.21}$$

where $\mathbf{1}_V$ and $\mathbf{1}_U$ are the identities of $\mathcal{C}_V$ and $\mathcal{C}_U$, respectively.

Consider the linear map $\sigma_V : \mathcal{C}_V \to \mathcal{C}_V^{\mathrm{op}}$ given by

$$\sigma_V(\mathbf{a} \vee \mathbf{b}) = \sigma_V(\mathbf{b}) \vee \sigma_V(\mathbf{a}), \quad \sigma_V(\mathbf{v}) = \mathbf{v}, \quad \mathbf{a}, \mathbf{b} \in \mathcal{C}_V, \ \mathbf{v} \in \mathcal{V}. \quad (27.22)$$

It is straightforward to show that σ_V is an involution and that it commutes with the degree involution:

$$\sigma_V \circ \omega_V = \omega_V \circ \sigma_V \quad (27.23)$$

Definition 27.2.5 The **conjugation** involution is defined as $\sigma_V \circ \omega_V$. The **conjugate** of an element $\mathbf{a} \in \mathcal{C}_V$ is

conjugation involution

$$\bar{\mathbf{a}} = \sigma_V \circ \omega_V(\mathbf{a}).$$

In particular, $\bar{\mathbf{v}} = -\mathbf{v}$ if $\mathbf{v} \in \mathcal{V}$.

It is clear from the definition that

$$\overline{\mathbf{a} \vee \mathbf{b}} = \bar{\mathbf{b}} \vee \bar{\mathbf{a}}, \quad \mathbf{a}, \mathbf{b} \in \mathcal{C}_V.$$

We saw a special case of this relation in our discussion of the quaternions in Example 3.1.16.

27.2.1 Homomorphism with Other Algebras

Let $\mathcal{V}$ be an inner product space and $\mathcal{A}$ an algebra with identity. A *linear* map $\varphi : \mathcal{V} \to \mathcal{A}$ can always be extended to a unital homomorphism $\phi : \mathcal{C}_V \to \mathcal{A}$. Indeed, since $\mathcal{C}_V$ consists of sums of Clifford products of vectors in $\mathcal{V}$, one simply has to define the action of ϕ on a product of vectors in $\mathcal{V}$. The obvious choice is

$$\phi(\mathbf{v}_1 \vee \mathbf{v}_2 \vee \cdots \vee \mathbf{v}_m) = \varphi(\mathbf{v}_1)\varphi(\mathbf{v}_2)\ldots\varphi(\mathbf{v}_m)$$

where on the right-hand side the product in $\mathcal{A}$ is denoted by juxtaposition. For φ to be extendable to a unital homomorphism, it has to be compatible with Eq. (27.14); i.e., it has to satisfy

$$\phi(\mathbf{v} \vee \mathbf{u}) + \phi(\mathbf{u} \vee \mathbf{v}) = 2\mathbf{g}(\mathbf{u}, \mathbf{v})\phi(\mathbf{1})$$

or, denoting $\mathbf{g}(\mathbf{u}, \mathbf{v})$ by $\langle \mathbf{u}, \mathbf{v} \rangle$,

$$\varphi(\mathbf{v})\varphi(\mathbf{u}) + \varphi(\mathbf{u})\varphi(\mathbf{v}) = 2\langle \mathbf{u}, \mathbf{v} \rangle \mathbf{1}_A. \quad (27.24)$$

By setting $\mathbf{u} = \mathbf{v}$, we obtain an equivalent condition

$$\varphi(\mathbf{v})^2 = \langle \mathbf{v}, \mathbf{v} \rangle \mathbf{1}_A. \quad (27.25)$$

Example 27.2.6 Let $\varphi : \mathbb{R}^2 \to \mathcal{L}(\mathbb{R}^2)$ be a linear map. We want to extend it to a homomorphism $\phi : \mathcal{C}_{\mathbb{R}^2} \to \mathcal{L}(\mathbb{R}^2)$. It is convenient to identify $\mathcal{L}(\mathbb{R}^2)$ with the set of 2×2 matrices and write

$$\varphi(\mathbf{v}) = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}. \tag{27.26}$$

Let $\mathbf{v} = (\alpha, \beta)$. For the extension to be possible, according to Eq. (27.25), we must have

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \left(\alpha^2 + \beta^2\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

One convenient solution to this equation is $\alpha_{11} = \alpha = -\alpha_{22}$ and $\alpha_{12} = \alpha_{21} = \beta$. Hence, we write Eq. (27.26) as

$$\varphi(\alpha, \beta) = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}. \tag{27.27}$$

Now let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis of $\mathbb{R}^2$. Then $\{\mathbf{1}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 \vee \mathbf{e}_2\}$ is a basis of $\mathcal{C}_{\mathbb{R}^2}$. Furthermore, $\phi(\mathbf{1})$ is the 2×2 unit matrix, and by (27.27)

$$\varphi(\mathbf{e}_1) = \varphi(1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \varphi(\mathbf{e}_2) = \varphi(0, 1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\phi(\mathbf{e}_1 \vee \mathbf{e}_2) = \varphi(\mathbf{e}_1)\varphi(\mathbf{e}_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is easy to show (see Problem 27.7) that these 4 matrices form a basis of $\mathcal{L}(\mathbb{R}^2)$. Since ϕ maps a basis onto a basis, it is a linear isomorphism and therefore and algebra isomorphism. Thus, $\mathcal{C}_{\mathbb{R}^2} \cong \mathcal{L}(\mathbb{R}^2)$.

27.2.2 The Canonical Element

Strictly speaking, the identification of (27.8) with (27.7) should be considered as an isomorphism ϕ_V of $\Lambda(\mathcal{V}^*)$ and $\mathcal{C}_V$:

$$\phi_V(\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_p}) = \mathbf{e}_{i_1} \vee \mathbf{e}_{i_2} \vee \cdots \vee \mathbf{e}_{i_p}. \tag{27.28}$$

Invoking Proposition 2.6.7, we conclude that,

canonical element

Definition 27.2.7 Given a determinant function $\mathbf{\Delta}$ in $\mathcal{V}$, there is a unique element $\mathbf{e}_\Delta \in \mathcal{C}_V$ such that

$$\phi_V(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_N}) = \mathbf{\Delta}(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_N}) \cdot \mathbf{e}_\Delta. \tag{27.29}$$

$\mathbf{e}_\Delta$ is called the **canonical element** in $\mathcal{C}_V$ with respect to the determinant function $\mathbf{\Delta}$.

Now choose an orthogonal basis $\{\mathbf{e}_i\}_{i=1}^N$ in $\mathcal{V}$ for which $\boldsymbol{\Delta}(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_N}) = 1$. Then, (27.28) and (27.29) yield

$$\mathbf{e}_\Delta = \mathbf{e}_1 \vee \cdots \vee \mathbf{e}_N. \tag{27.30}$$

Next use the Lagrange identity (26.32) and write it in the form

$$\det\bigl(\langle \mathbf{x}_i, \mathbf{y}_j \rangle\bigr) = \lambda_\Delta \boldsymbol{\Delta}(\mathbf{x}_1, \dots, \mathbf{x}_N)\boldsymbol{\Delta}(\mathbf{y}_1, \dots, \mathbf{y}_N) \quad \mathbf{x}_i, \mathbf{y}_j \in \mathcal{V}. \tag{27.31}$$

Setting $\mathbf{x}_i = \mathbf{y}_i = \mathbf{e}_i$, and evaluating the determinant on the left-hand side of the equation, we obtain

$$\begin{aligned} \lambda_\Delta &= \langle \mathbf{e}_1, \mathbf{e}_1 \rangle \dots \langle \mathbf{e}_N, \mathbf{e}_N \rangle = g_{11} \dots g_{NN} \\ &= (\mathbf{e}_1 \vee \mathbf{e}_1) \dots (\mathbf{e}_N \vee \mathbf{e}_N) = \mathbf{e}_1^2 \dots \mathbf{e}_N^2, \end{aligned} \tag{27.32}$$

where we used Eq. (27.15). Using (27.32), together with (27.15) and (27.30), one can easily show that

$$\mathbf{e}_\Delta^2 \equiv \mathbf{e}_\Delta \vee \mathbf{e}_\Delta = (-1)^{N(N-1)/2} \lambda_\Delta \cdot \mathbf{1}. \tag{27.33}$$

Since $\lambda_\Delta \neq 0$, it follows that $\mathbf{e}_\Delta$ is invertible.

Equation (27.30) can be used to show that

$$\mathbf{e}_i \vee \mathbf{e}_\Delta = (-1)^{N-1} \mathbf{e}_\Delta \vee \mathbf{e}_i,$$

and since any vector in $\mathcal{V}$ is a linear combination of the basis $\{\mathbf{e}_i\}_{i=1}^N$, the equation holds for arbitrary vectors. We thus have the following:

Theorem 27.2.8 *The canonical element $\mathbf{e}_\Delta$ satisfies the relations*

$$\begin{aligned} \mathbf{e}_\Delta \vee \mathbf{v} &= (-1)^{N-1} \mathbf{v} \vee \mathbf{e}_\Delta, \quad \mathbf{v} \in \mathcal{V}, \\ \mathbf{e}_\Delta \vee \mathbf{u} &= \omega_V^{N-1}(\mathbf{u}) \vee \mathbf{e}_\Delta, \quad \mathbf{u} \in \mathcal{C}_V, \end{aligned}$$

where ω_V is the degree involution of Definition 27.2.3. In particular, $\mathbf{e}_\Delta \vee \mathbf{u} = \mathbf{u} \vee \mathbf{e}_\Delta$ if N is odd, and $\mathbf{e}_\Delta \vee \mathbf{u} = \omega_V(\mathbf{u}) \vee \mathbf{e}_\Delta$ if N is even.

27.2.3 Center and Anticenter

Definition 27.2.9 The **center** of the Clifford algebra $\mathcal{C}_V$, denoted by $\mathcal{Z}_V$, consists of elements $\mathbf{a} \in \mathcal{C}_V$ satisfying

$$\mathbf{a} \vee \mathbf{u} = \mathbf{u} \vee \mathbf{a} \quad \forall \mathbf{u} \in \mathcal{C}_V.$$

The **anticenter** of the Clifford algebra $\mathcal{C}_V$, denoted by $\overline{\mathcal{Z}}_V$, consists of elements $\mathbf{a} \in \mathcal{C}_V$ satisfying

$$\mathbf{a} \vee \mathbf{u} = \omega_V(\mathbf{u}) \vee \mathbf{a} \quad \forall \mathbf{u} \in \mathcal{C}_V.$$

Since $\mathcal{C}_V$ is generated by $\mathcal{V}$ (i.e., it is sums of products of elements in $\mathcal{V}$), it follows that

$$\mathbf{a} \in \mathcal{Z}_V \quad \text{if and only if } \mathbf{a} \vee \mathbf{x} = \mathbf{x} \vee \mathbf{a} \ \forall \mathbf{x} \in \mathcal{V},$$

and

$$\mathbf{a} \in \overline{\mathcal{Z}}_V \quad \text{if and only if } \mathbf{a} \vee \mathbf{x} = -\mathbf{x} \vee \mathbf{a} \ \forall \mathbf{x} \in \mathcal{V}.$$

It is easy to show that $\mathcal{Z}_V$ is a subalgebra of $\mathcal{C}_V$ and that both $\mathcal{Z}_V$ and $\overline{\mathcal{Z}}_V$ are invariant under the degree involution ω_V. Therefore, as in Eq. (27.17)

$$\begin{aligned} \mathcal{Z}_V &= \mathcal{Z}_V^0 \oplus_V \mathcal{Z}_V^1 \\ \overline{\mathcal{Z}}_V &= \overline{\mathcal{Z}}_V^0 \oplus_V \overline{\mathcal{Z}}_V^1 \end{aligned} \tag{27.34}$$

where $\oplus_V$ indicates a direct sum of vector spaces.

Proposition 27.2.10 $\overline{\mathcal{Z}}_V^1 = 0$. *That is, the anticenter of any Clifford algebra consists of even elements only.*

Proof If $\mathbf{a} \in \overline{\mathcal{Z}}_V^1$, then $\mathbf{a} \vee \mathbf{x} = -\mathbf{x} \vee \mathbf{a}$ for any $\mathbf{x} \in \mathcal{V}$. Equation (27.30) then gives

$$\mathbf{a} \vee \mathbf{e}_\Delta = (-1)^N \mathbf{e}_\Delta \vee \mathbf{a}.$$

On the other hand, Theorem 27.2.8 and $\omega_V(\mathbf{a}) = -\mathbf{a}$ for $\mathbf{a} \in \overline{\mathcal{Z}}_V^1 = \mathcal{C}_V^1 \cap \overline{\mathcal{Z}}_V$ yields

$$\mathbf{e}_\Delta \vee \mathbf{a} = \omega_V^{N-1}(\mathbf{a}) \vee \mathbf{e}_\Delta = (-1)^{N-1} \mathbf{a} \vee \mathbf{e}_\Delta.$$

The last two equations, therefore, give $\mathbf{a} \vee \mathbf{e}_\Delta = 0$, and since $\mathbf{e}_\Delta$ is invertible, we have $\mathbf{a} = 0$. □

Proposition 27.2.11 *If $\mathcal{V}$ is odd-dimensional, then $\mathbf{e}_\Delta \in \mathcal{Z}_V$, and if it is even-dimensional, then $\mathbf{e}_\Delta \in \overline{\mathcal{Z}}_V$.*

Proof The proof follows immediately from Theorem 27.2.8. □

Consider the linear map $\phi_V : \mathcal{C}_V \to \mathcal{C}_V$ given by

$$\phi_V(\mathbf{u}) = \mathbf{e}_\Delta \vee \mathbf{u}, \quad \mathbf{u} \in \mathcal{C}_V$$

and note that since $\mathbf{e}_\Delta$ is invertible, ϕ_V is a linear isomorphism. If N is odd, then Eq. (27.30) shows that $\phi_V : \mathcal{C}_V^0 \to \mathcal{C}_V^1$ and ϕ_V establishes an isomorphism of $\mathcal{C}_V^0$ and $\mathcal{C}_V^1$. Now restrict the map to $\mathcal{Z}_V$. Then, using Theorem 27.2.8, for $\mathbf{x} \in \mathcal{V}$, we obtain

$$\begin{aligned} \phi_V(\mathbf{u}) \vee \mathbf{x} &= \mathbf{e}_\Delta \vee \mathbf{u} \vee \mathbf{x} = \mathbf{e}_\Delta \vee \mathbf{x} \vee \mathbf{u} \\ &= (-1)^{N-1} \mathbf{x} \vee \mathbf{e}_\Delta \vee \mathbf{u} = (-1)^{N-1} \mathbf{x} \vee \phi_V(\mathbf{u}). \end{aligned}$$

Similarly,

$$\phi_V(\mathbf{v}) \vee \mathbf{x} = (-1)^N \mathbf{x} \vee \phi_V(\mathbf{v}) \quad \mathbf{v} \in \overline{\mathcal{Z}}_V, \ \mathbf{x} \in \mathcal{V}.$$

We have just proved

Proposition 27.2.12 *If N is odd, then ϕ_V restricts to linear automorphisms of $\mathcal{Z}_V$ and $\overline{\mathcal{Z}}_V$ and establishes an isomorphism between $\mathcal{Z}_V^0$ and $\mathcal{Z}_V^1$. If N is even, then ϕ_V interchanges $\mathcal{Z}_V$ and $\overline{\mathcal{Z}}_V$.*

Proposition 27.2.13 $\mathcal{Z}_V^0 = \mathrm{Span}\{\mathbf{1}\}$.

Proof We use induction on the dimension of $\mathcal{V}$. For $N = 1$, the proposition is trivial. Assume that it holds for $N - 1$, and choose $\mathbf{v} \in \mathcal{V}$ such that $\langle \mathbf{v}, \mathbf{v} \rangle \neq 0$. With $\mathcal{U}$ the orthogonal complement of $\mathbf{v}$, we can write

$$\mathcal{V} = \mathrm{Span}\{\mathbf{v}\} \oplus \mathcal{U}$$

and note that (27.20) and (27.21) become

$$\mathcal{C}_V^0 \cong \big(\mathbf{1} \hat{\otimes} \mathcal{C}_U^0\big) \oplus \big(\mathbf{v} \hat{\otimes} \mathcal{C}_U^1\big)$$
$$\mathcal{V} \cong (\mathbf{1} \otimes \mathcal{U}) \oplus (\mathbf{v} \otimes \mathbf{1}).$$

Identifying the left and right hand sides of these equations, we write

$$\mathbf{u} = \mathbf{1} \otimes \mathbf{b} + \mathbf{v} \otimes \mathbf{c}, \qquad \mathbf{u} \in \mathcal{C}_V^0, \ \mathbf{b} \in \mathcal{C}_U^0, \ \mathbf{c} \in \mathcal{C}_U^1$$
$$\mathbf{x} = \mathbf{1} \otimes \mathbf{y} + \mathbf{v} \otimes \mathbf{1}, \qquad \mathbf{x} \in \mathcal{V}, \ \mathbf{y} \in \mathcal{U}.$$

We now use the multiplication rule of Eq. (26.24), noting that $\mathbf{1}$ and $\mathbf{b}$ have even degrees while $\mathbf{v}$, $\mathbf{y}$, and $\mathbf{c}$ have odd degrees:

$$\begin{aligned}
\mathbf{u} \vee \mathbf{x} &= (\mathbf{1} \otimes \mathbf{b} + \mathbf{v} \otimes \mathbf{c}) \odot (\mathbf{1} \otimes \mathbf{y} + \mathbf{v} \otimes \mathbf{1}) \\
&= (\mathbf{1} \otimes \mathbf{b}) \odot (\mathbf{1} \otimes \mathbf{y}) + (\mathbf{1} \otimes \mathbf{b}) \odot (\mathbf{v} \otimes \mathbf{1}) \\
&\quad + (\mathbf{v} \otimes \mathbf{c}) \odot (\mathbf{1} \otimes \mathbf{y}) + (\mathbf{v} \otimes \mathbf{c}) \odot (\mathbf{v} \otimes \mathbf{1}) \\
&= \mathbf{1} \otimes (\mathbf{b} \vee \mathbf{y}) + \mathbf{v} \otimes \mathbf{b} + \mathbf{v} \otimes (\mathbf{c} \vee \mathbf{y}) - (\mathbf{v} \vee \mathbf{v}) \otimes \mathbf{c}.
\end{aligned}$$

Similarly,

$$\mathbf{x} \vee \mathbf{u} = \mathbf{1} \otimes (\mathbf{y} \vee \mathbf{b}) + \mathbf{v} \otimes \mathbf{b} - \mathbf{v} \otimes (\mathbf{y} \vee \mathbf{c}) + (\mathbf{v} \vee \mathbf{v}) \otimes \mathbf{c}.$$

Now assume that $\mathbf{u}$ is in the center of the Clifford algebra. Then the two equations above are equal, and noting that $\mathbf{v} \vee \mathbf{v} = \langle \mathbf{v}, \mathbf{v} \rangle \mathbf{1}$, we obtain

$$\mathbf{1} \otimes (\mathbf{b} \vee \mathbf{y} - \mathbf{y} \vee \mathbf{b} - 2\langle \mathbf{v}, \mathbf{v} \rangle \mathbf{c}) + \mathbf{v} \otimes (\mathbf{c} \vee \mathbf{y} + \mathbf{y} \vee \mathbf{c}) = 0,$$

or

$$\begin{aligned} \mathbf{b} \vee \mathbf{y} - \mathbf{y} \vee \mathbf{b} - 2\langle \mathbf{v}, \mathbf{v} \rangle \mathbf{c} &= 0 \\ \mathbf{c} \vee \mathbf{y} + \mathbf{y} \vee \mathbf{c} &= 0. \end{aligned} \tag{27.35}$$

The second equation implies that $\mathbf{c} \in \overline{\mathcal{Z}}_U$ and hence $\mathbf{c} \in \overline{\mathcal{Z}}^1_U$. Then by Proposition 27.2.10, $\mathbf{c} = 0$. The first relation in (27.35) now implies that $\mathbf{b} \in \mathcal{Z}_U$ and therefore $\mathbf{b} \in \mathcal{Z}^0_U$. By induction assumption, $\mathbf{b}$ is a multiple of the identity in $\mathcal{C}_U$. Thus,

$$\mathbf{u} = \mathbf{1} \otimes (\alpha \mathbf{1}_U) + \mathbf{v} \otimes \mathbf{0} = \alpha \mathbf{1},$$

i.e., $\mathbf{u} \in \operatorname{Span}\{\mathbf{1}\}$. □

Theorem 27.2.14 *Let $\mathcal{V}$ be an N-dimensional space. Then*

(a) *If N is odd, $\mathcal{Z}_V = \operatorname{Span}\{\mathbf{1}, \mathbf{e}_\Delta\}$, $\overline{\mathcal{Z}}_V = 0$.*
(b) *If N is even, $\mathcal{Z}_V = \operatorname{Span}\{\mathbf{1}\}$, $\overline{\mathcal{Z}}_V = \operatorname{Span}\{\mathbf{e}_\Delta\}$. Thus all Clifford algebras over an even-dimensional vector space are central.*

Proof Suppose that N is odd and $\mathbf{a} \in \overline{\mathcal{Z}}_V$. Then, $\mathbf{a} \vee \mathbf{x} = -\mathbf{x} \vee \mathbf{a}$, for any $\mathbf{x} \in \mathcal{V}$. Equation (27.30) then yields $\mathbf{a} \vee \mathbf{e}_\Delta = -\mathbf{e}_\Delta \vee \mathbf{a}$. On the other hand, the second equation of Theorem 27.2.8 implies that $\mathbf{a} \vee \mathbf{e}_\Delta = \mathbf{e}_\Delta \vee \mathbf{a}$. Hence, $\mathbf{a} \vee \mathbf{e}_\Delta = 0$, and since $\mathbf{e}_\Delta$ is invertible, we have $\mathbf{a} = 0$. This proves the second part of (a).

Next observe that by Proposition 27.2.13 $\mathcal{Z}^0_V = \operatorname{Span}\{\mathbf{1}\}$ and that, since N is assumed odd, by Proposition 27.2.12, ϕ_V is an automorphism of $\mathcal{Z}_V$ and an isomorphism between $\mathcal{Z}^0_V$ and $\mathcal{Z}^1_V$. Since $\mathcal{Z}^0_V = \operatorname{Span}\{\mathbf{1}\}$ and $\phi_V(\mathbf{1}) = \mathbf{e}_\Delta$, we must have $\mathcal{Z}^1_V = \operatorname{Span}\{\mathbf{e}_\Delta\}$.

Now consider the case when N is even. For $\mathbf{a} \in \mathcal{Z}_V$, we have $\mathbf{a} \vee \mathbf{e}_\Delta = \mathbf{e}_\Delta \vee \mathbf{a}$. On the other hand, by Theorem 27.2.8, $\mathbf{a} \vee \mathbf{e}_\Delta = \omega_V(\mathbf{a}) \vee \mathbf{e}_\Delta$. Since $\mathbf{e}_\Delta$ is invertible, we have $\omega_V(\mathbf{a}) = \mathbf{a}$ or $(\omega_V - \iota)\mathbf{a} = 0$. Therefore, by Eq. (27.18), $\mathbf{a} \in \mathcal{Z}^0_V$ and hence $\mathcal{Z}_V = \mathcal{Z}^0_V$. Proposition 27.2.13 now gives $\mathcal{Z}_V = \operatorname{Span}\{\mathbf{1}\}$.

Since ϕ_V interchanges $\mathcal{Z}_V$ and $\overline{\mathcal{Z}}_V$, we have

$$\overline{\mathcal{Z}}_V = \phi_V(\mathcal{Z}_V) = \phi_V\big(\operatorname{Span}\{\mathbf{1}\}\big) = \operatorname{Span}\big\{\phi_V(\mathbf{1})\big\} = \operatorname{Span}\{\mathbf{e}_\Delta\}.$$

This completes the proof. □

27.2.4 Isomorphisms

Let $(\mathcal{V}, \mathbf{g})$ be an inner product space. If we change the sign of the inner product, we get another inner product space $\widetilde{\mathcal{V}} \equiv (\mathcal{V}, -\mathbf{g})$. Next consider two vector spaces $\mathcal{V}$ and $\mathcal{U}$, and suppose that $\boldsymbol{\Delta}$ can be chosen in such a way that the canonical element of $\mathcal{C}_V$ satisfies $\mathbf{e}_\Delta^2 = \pm\mathbf{1}$. Then we have the following two theorems whose proof can be found in [Greu 78, pp. 244–245]:

Theorem 27.2.15 *Let* $\dim \mathcal{V} = 2m$, *and assume that* $\boldsymbol{\Delta}$ *can be chosen such that* $\lambda_\Delta = (-1)^m$. *Then the Clifford algebras* $\mathcal{C}_V$ *and* $\mathcal{C}_{\widetilde{V}}$ *are isomorphic.*

Theorem 27.2.16 *Let* $\mathcal{V}$ *be an even-dimensional inner product space and* $\mathcal{U}$ *any other inner product space. Then*

$$\mathcal{C}_{V\oplus U} \cong \mathcal{C}_V \otimes \mathcal{C}_U \quad \text{if } \mathbf{e}_\Delta^2 = \mathbf{1}$$

$$\mathcal{C}_{V\oplus \widetilde{U}} \cong \mathcal{C}_V \otimes \mathcal{C}_U \quad \text{if } \mathbf{e}_\Delta^2 = -\mathbf{1}.$$

Another theorem, which will be useful in the classification of Clifford algebras and whose proof is given in [Greu 78, p. 248], is the following.

Theorem 27.2.17 *Let* $\mathcal{V}$ *be an even-dimensional inner product space. Assume that* $\mathcal{V}$ *has an antisymmetric involution* ω *(so that* $\omega^t = -\omega$*). Then the Clifford algebra* $\mathcal{C}_V$ *is isomorphic to* $\mathcal{L}(\Lambda(\mathcal{V}_1))$, *the set of linear transformation of* $\Lambda(\mathcal{V}_1)$ *where* $\mathcal{V}_1 = \ker(\omega - \iota)$.

Recall from Theorem 26.3.6 that $\dim \Lambda(\mathcal{V}_1) = 2^{\dim \mathcal{V}_1}$, and that all real vector spaces of dimension N are isomorphic to $\mathbb{R}^N$. Therefore, we can identify $\mathcal{L}(\Lambda(\mathcal{V}_1))$ with $\mathcal{L}(\mathbb{R}^{2^{\dim \mathcal{V}_1}})$, and obtain the algebra isomorphism

$$\mathcal{C}_V \cong \mathcal{L}\big(\mathbb{R}^{2^{\dim \mathcal{V}_1}}\big) \tag{27.36}$$

for $\mathcal{V}$ of the theorem above.

27.3 General Classification of Clifford Algebras

In almost all our preceding discussion, we have assumed that our scalars come from $\mathbb{R}$. There is a good reason for that: the complex Clifford algebras are very limited and applications in physics almost exclusively focus on real Clifford algebras. In this subsection, we include complex numbers as our scalars and classify all complex Clifford algebras.

Definition 27.3.1 Let $\mathbb{F}$ denote either $\mathbb{C}$ or $\mathbb{R}$ and let $\mathcal{V}$ be a vector space over $\mathbb{F}$. Choose a basis $\{\mathbf{e}_i\}_{i=1}^n$ for $\mathcal{V}$ and let $\mathbf{v} = \sum_{i=1}^n \eta_i \mathbf{e}_i$ be a vector in $\mathcal{V}$. A **quadratic form** of index ν on $\mathcal{V}$ is a map $\mathbf{Q}_\nu : \mathcal{V} \to \mathbb{F}$ given by

quadratic form

$$\mathbf{Q}_\nu(\mathbf{v}) = -\sum_{i=1}^{\nu} \eta_i^2 + \sum_{i=\nu+1}^{n} \eta_i^2. \tag{27.37}$$

A quadratic form yields an inner product.[2] In fact, defining

$$2\mathbf{g}(\mathbf{u},\mathbf{v}) = \mathbf{Q}_\nu(\mathbf{u}+\mathbf{v},\mathbf{u}+\mathbf{v}) - \mathbf{Q}_\nu(\mathbf{u},\mathbf{u}) - \mathbf{Q}_\nu(\mathbf{v},\mathbf{v}),$$

it is easy to verify that $\mathbf{g}$ is indeed a symmetric bilinear map. Conversely, given an inner product $\mathbf{g}$ of index ν, one can construct a quadratic form: $\mathbf{Q}_\nu(\mathbf{v}) = \mathbf{g}(\mathbf{v},\mathbf{v})$. It turns out that the basis vectors chosen to define the quadratic from are automatically $\mathbf{g}$-orthonormal.

When $\mathbb{F} = \mathbb{C}$, there is only one kind of quadratic form: that with $\nu = 0$. This is because one can alway change η_k to $i\eta_k$ to turn all the negative terms in the sum to positive terms. However, when $\mathbb{F} = \mathbb{R}$, we obtain different quadratic forms depending on the index ν. Thus, the real quadratic form leads to the inner product of $\mathbb{R}^n_\nu$ introduced in Eq. (26.40) and the corresponding Clifford algebra will be discussed in detail in the next section.

Before we classify the easy case of complex Clifford algebras, let us observe some general properties of the Clifford algebra over $\mathbb{F}$, which we denote by $\mathcal{C}_V(\mathbb{F})$. First, we note that since $\mathbf{e}_i^2 \equiv \mathbf{e}_i \vee \mathbf{e}_i = \mathbf{g}(\mathbf{e}_i,\mathbf{e}_i)\mathbf{1}$, $\mathbf{e}_i^k \neq \mathbf{0}$ for any positive integer k. Therefore, $\mathcal{C}_V(\mathbb{F})$ cannot contain a radical for any $\mathcal{V}$ over $\mathbb{F}$. This means that $\mathcal{C}_V(\mathbb{F})$ is semi-simple. Moreover, Theorem 27.2.14 implies that $\mathcal{C}_V(\mathbb{F})$ is simple if $\mathcal{V}$ is even-dimensional.

Next, we look at the case of odd-dimensional vector spaces which is only slightly more complicated. In this case $\mathcal{Z}_V = \mathrm{Span}\{\mathbf{1},\mathbf{e}_\Delta\}$ by Theorem 27.2.14, and as we shall see presently, $\mathbf{e}_\Delta^2$ plays a significant role in the classification of $\mathcal{C}_V(\mathbb{F})$ when $\dim\mathcal{V}$ is odd. Equation (27.33) gives $\mathbf{e}_\Delta^2$ in terms of λ_Δ of Eq. (27.32). If $\mathbb{F} = \mathbb{C}$, then we can choose $\mathbf{g}(\mathbf{e}_i,\mathbf{e}_i) = 1$. In fact, for any non-null vector $\mathbf{v} \in \mathcal{V}$, we have

$$\mathbf{g}(\mathbf{e}_v,\mathbf{e}_v) \equiv \mathbf{g}\left(\frac{\mathbf{v}}{\sqrt{\mathbf{g}(\mathbf{v},\mathbf{v})}}, \frac{\mathbf{v}}{\sqrt{\mathbf{g}(\mathbf{v},\mathbf{v})}}\right) = 1.$$

Hence, we can always set $\lambda_\Delta = 1$ when $\mathbb{F} = \mathbb{C}$. We can't do this for the real case because $\sqrt{\mathbf{g}(\mathbf{v},\mathbf{v})}$ may be pure imaginary.

If $\mathbb{F} = \mathbb{R}$, then because of the Lagrange identities (26.42) and (27.31), $\lambda_\Delta = (-1)^\nu$ and the canonical element satisfies the relation

$$\mathbf{e}_\Delta^2 = (-1)^{N(N-1)/2+\nu}\mathbf{1}. \tag{27.38}$$

Thus $\mathbf{e}_\Delta^2 = \pm\mathbf{1}$ depending on the index ν and dimension N of $\mathcal{V}$.

We discuss the case of $\mathbf{e}_\Delta^2 = +\mathbf{1}$ first. The elements $\mathbf{P}_\pm = \frac{1}{2}(\mathbf{1} \pm \mathbf{e}_\Delta)$ are two orthogonal idempotents belonging to the center of $\mathcal{C}_V(\mathbb{F})$. Since $\mathbf{P}_+ + \mathbf{P}_- = \mathbf{1}$, we have the decomposition

$$\mathcal{C}_V(\mathbb{F}) = \mathcal{C}_V^+(\mathbb{F}) \oplus \mathcal{C}_V^-(\mathbb{F}) \equiv \mathbf{P}_+\mathcal{C}_V(\mathbb{F}) \oplus \mathbf{P}_-\mathcal{C}_V(\mathbb{F}), \tag{27.39}$$

where $\mathcal{C}_V^+(\mathbb{F})$ and $\mathcal{C}_V^-(\mathbb{F})$ are subalgebras (actually ideals) of $\mathcal{C}_V(\mathbb{F})$.

[2]Here we define an inner product simply as a symmetric bilinear map as in Definition 2.4.2.

Since $\mathbf{e}_\Delta$ is the product of an odd number of vectors, $\omega_V(\mathbf{e}_\Delta) = -\mathbf{e}_\Delta$. Hence, $\omega_V(\mathbf{P}_\pm) = \mathbf{P}_\mp$. Furthermore, ω_V, being an involution, has an inverse. Thus, it is an isomorphism of $\mathcal{C}_V^+(\mathbb{F})$ and $\mathcal{C}_V^-(\mathbb{F})$.

We now show that $\mathcal{C}_V^\pm(\mathbb{F})$ are central. We do this for $\mathcal{C}_V^+(\mathbb{F})$, with the other case following immediately from the proof of $\mathcal{C}_V^+(\mathbb{F})$. Let $\mathbf{a}_+ \in \mathcal{Z}_V^+(\mathbb{F})$, the center of $\mathcal{C}_V^+(\mathbb{F})$, and $\mathbf{b} \in \mathcal{C}_V(\mathbb{F})$. Then

$$\mathbf{a}_+\mathbf{b} = \mathbf{a}_+(\mathbf{b}_+ + \mathbf{b}_-) = \mathbf{a}_+\mathbf{b}_+ + \mathbf{a}_+\mathbf{b}_- = \mathbf{a}_+\mathbf{b}_+ = \mathbf{b}_+\mathbf{a}_+ = \mathbf{b}\mathbf{a}_+$$

because $\mathbf{0} = \mathbf{a}_+\mathbf{b}_- = \mathbf{b}_-\mathbf{a}_+$. It follows that $\mathbf{a}_+ \in \mathcal{Z}_V(\mathbb{F})$. Therefore,

$$\mathbf{a}_+ = \alpha\mathbf{1} + \beta\mathbf{e}_\Delta = \alpha(\mathbf{P}_+ + \mathbf{P}_-) + \beta(\mathbf{P}_+ - \mathbf{P}_-) = (\alpha+\beta)\mathbf{P}_+ + (\alpha-\beta)\mathbf{P}_-.$$

Since $\mathbf{a}_+$ has no component in $\mathbf{P}_-$, we must have $\alpha = \beta$ and $\mathbf{a}_+ = 2\alpha\mathbf{P}_+$. Hence, $\mathbf{a}_+ \in \text{Span}\{\mathbf{P}_+\}$. But $\mathbf{P}_+$ is the identity of $\mathcal{C}_V^+(\mathbb{F})$. It now follows that $\mathcal{C}_V^+(\mathbb{F})$ is central simple. Similarly, $\mathcal{C}_V^-(\mathbb{F})$ is central simple. We summarize the foregoing discussion as follows:

Theorem 27.3.2 *Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$ and $\mathcal{V}$ a vector space over $\mathbb{F}$.*

(a) *If* $\dim\mathcal{V}$ *is even, then* $\mathcal{C}_V(\mathbb{F})$ *is central simple.*
(b) *If* $\dim\mathcal{V}$ *is odd, then* $\mathcal{C}_V(\mathbb{C})$ *is the direct sum of two isomorphic central simple Clifford algebras.* $\mathcal{C}_V(\mathbb{R})$ *is the direct sum of two isomorphic central simple Clifford algebras if* $\mathbf{e}_\Delta^2 = +\mathbf{1}$.

We are now ready to classify all complex Clifford algebras. All we have to do is to use Theorem 3.5.29:

Theorem 27.3.3 *A complex Clifford algebra* $\mathcal{C}_V(\mathbb{C})$ *is isomorphic to either a total complex matrix algebra or a direct sum of two such algebras*:

(a) $\mathcal{C}_V(\mathbb{C}) \cong \mathcal{M}_r(\mathbb{C})$ *for some positive integer* r, *if* $\dim\mathcal{V}$ *is even.*
(b) $\mathcal{C}_V(\mathbb{C}) \cong \mathcal{M}_s(\mathbb{C}) \oplus \mathcal{M}_s(\mathbb{C})$ *for some positive integer* s, *if* $\dim\mathcal{V}$ *is odd.*

Although the real Clifford algebras are classified in the next section in much more detail, it is instructive to give a classification of $\mathcal{C}_V(\mathbb{R})$ based on what we know from our study of algebras in general. If $\mathcal{V}$ is even-dimensional, $\mathcal{C}_V(\mathbb{R})$ is central simple, and by Theorem 3.5.30, it is of the form $\mathcal{D} \otimes \mathcal{M}_n$ where $\mathcal{D}$ is $\mathbb{R}$ or $\mathbb{H}$.

If $\mathcal{V}$ is odd-dimensional, then we have to consider two cases. If $\mathbf{e}_\Delta^2 = +\mathbf{1}$, then $\mathcal{C}_V(\mathbb{R})$ is the direct sum of two central algebras and thus isomorphic to

$$\mathbb{R} \otimes \mathcal{M}_r \cong \mathcal{M}_r(\mathbb{R}) \quad \text{or to} \quad \mathbb{H} \otimes \mathcal{M}_s \cong \mathcal{M}_s(\mathbb{H}) \cong \mathbb{H} \otimes \mathcal{M}_s(\mathbb{R}),$$

for some nonnegative integers r and s. If $\mathbf{e}_\Delta^2 = -\mathbf{1}$, then the center of $\mathcal{C}_V(\mathbb{R})$, which is $\text{Span}\{\mathbf{1}, \mathbf{e}_\Delta\}$, is isomorphic to $\mathbb{C}$, and again by Theorem 3.5.30, $\mathcal{C}_V(\mathbb{R})$ is isomorphic to

$$\mathbb{C} \otimes \mathcal{M}_p \cong \mathcal{M}_p(\mathbb{C}) \quad \text{or to} \quad \mathbb{C} \otimes \mathbb{H} \otimes \mathcal{M}_q \cong \mathbb{H} \otimes \mathcal{M}_q(\mathbb{C}),$$

for some nonnegative integers p and q.

We summarize this discussion in

Theorem 27.3.4 *A real Clifford algebra $\mathcal{C}_V(\mathbb{R})$ is classified as follows:*

(a) *If $\mathcal{V}$ is even-dimensional, then $\mathcal{C}_V(\mathbb{R}) \cong \mathcal{D} \otimes \mathcal{M}_r \cong \mathcal{M}_r(\mathcal{D})$, where r is a positive integer and $\mathcal{D} = \mathbb{R}$ or $\mathbb{H}$, i.e., $\mathcal{C}_V(\mathbb{R})$ is a total matrix algebra over reals or quaternions.*

(b) *If $\mathcal{V}$ is odd-dimensional, then we have to consider two cases:*
1. *If $\mathbf{e}_\Delta^2 = -\mathbf{1}$, then $\mathcal{C}_V(\mathbb{R}) \cong \mathcal{M}_s(\mathcal{D})$, where s is a positive integer and $\mathcal{D}$ is either $\mathbb{C}$ or $\mathbb{C} \otimes \mathbb{H}$.*
2. *If $\mathbf{e}_\Delta^2 = \mathbf{1}$, then $\mathcal{C}_V(\mathbb{R}) \cong \mathcal{M}_p(\mathcal{D}) \oplus \mathcal{M}_p(\mathcal{D})$, where p is a positive integer and $\mathcal{D}$ is either $\mathbb{R}$ or $\mathbb{H}$.*

27.4 The Clifford Algebras $\mathbf{C}_\mu^\nu(\mathbb{R})$

Our discussion of inner products in Sect. 26.5 showed that orthonormal bases are especially convenient. In such bases, the metric matrix $g_{ij} \equiv \eta_{ij}$ is diagonal, with $\eta_{ii} = \pm 1$. In fact, if ν is the index of $\mathcal{V}$ (see Theorem 26.5.21), then, introducing $\mu \equiv N - \nu$, we have

$$g_{ij} = \eta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ +1 & \text{if } 1 \le i \le \mu, \\ -1 & \text{if } \mu + 1 \le i \le N, \end{cases} \tag{27.40}$$

and Eq. (27.15) becomes

$$\begin{aligned} \mathbf{e}_i \vee \mathbf{e}_j &= -\mathbf{e}_j \vee \mathbf{e}_i && \text{if } i \neq j, \\ \mathbf{e}_i \vee \mathbf{e}_i &= +1 && \text{if } 1 \le i \le \mu, \\ \mathbf{e}_i \vee \mathbf{e}_i &= -1 && \text{if } \mu + 1 \le i \le N. \end{aligned} \tag{27.41}$$

The Clifford algebra $\mathbf{C}_\mu^\nu(\mathbb{R})$

The Clifford algebra determined by (27.41) is denoted by $\mathbf{C}_\mu^\nu(\mathbb{R})$.[3] It is the Clifford algebra of the vector space $\mathbb{R}_\nu^n$ introduced on page 815.

Example 27.4.1 The simplest $\mathbf{C}_\mu^\nu(\mathbb{R})$ is when one of the subscripts is 0 and the other 1. First let $\mu = 0$ and $\nu = 1$. In this case, $\mathcal{V}$ is one-dimensional. Let $\mathbf{e}$ be the basis vector of $\mathcal{V}$. Then a basis of the Clifford algebra $\mathbf{C}_0^1(\mathbb{R})$ is $\{\mathbf{1}, \mathbf{e}\}$, and an arbitrary element of $\mathbf{C}_0^1(\mathbb{R})$ can be written as $\alpha\mathbf{1} + \beta\mathbf{e}$. The multiplication of any two such elements is completely determined by the multiplication of the basis vectors:

$$\mathbf{1} \vee \mathbf{1} = \mathbf{1}, \qquad \mathbf{1} \vee \mathbf{e} = \mathbf{e} \vee \mathbf{1} = \mathbf{e}, \qquad \mathbf{e} \vee \mathbf{e} = -\mathbf{1}.$$

[3] Many other notations are also used to denote this algebra. Among them are $\mathbf{C}_{\mu,\nu}(\mathbb{R})$, $\mathbf{C}(\mu,\nu)$, $C\ell_{\mu,\nu}(\mathbb{R})$, $C\ell_{p,q}(\mathbb{R})$, and $C(p,q)$ where $q = \nu$ and $p = \mu$. Occasionally, we'll use one of these notations as well.

If we identify $\mathbf{e}$ with $i = \sqrt{-1}$ and $\vee$ with ordinary multiplication, then $\mathbf{C}_0^1(\mathbb{R})$ becomes identical with the (algebra of) complex numbers. Thus, $\mathbf{C}_0^1(\mathbb{R}) \cong \mathbb{C}$.

$\mathbf{C}_0^1(\mathbb{R}) \cong \mathbb{C}$

Now let $\mu = 1$ and $\nu = 0$. Again, $\mathcal{V}$ is one-dimensional with $\mathbf{e}$ as its basis vector. The basis of the Clifford algebra $\mathbf{C}_1^0(\mathbb{R})$ is again $\{\mathbf{1}, \mathbf{e}\}$. The multiplication of the basis vectors gives

$$\mathbf{1} \vee \mathbf{1} = \mathbf{1}, \qquad \mathbf{1} \vee \mathbf{e} = \mathbf{e} \vee \mathbf{1} = \mathbf{e}, \qquad \mathbf{e} \vee \mathbf{e} = \mathbf{1}.$$

This shows that $\mathbf{1}$ and $\mathbf{e}$ have identical properties, and since $\mathbf{1}$ is a basis of $\mathbb{R}$, so must be $\mathbf{e}$. We conclude that $\mathbf{C}_1^0(\mathbb{R}) \cong \mathbb{R} \oplus \mathbb{R}$. As a direct sum of two algebras, $\mathbb{R} \oplus \mathbb{R}$ has the product rule,

$\mathbf{C}_1^0(\mathbb{R}) \cong \mathbb{R} \oplus \mathbb{R}$

$$(\alpha_1 \oplus \alpha_2)(\beta_1 \oplus \beta_2) = (\alpha_1\beta_1 \oplus \alpha_2\beta_2).$$

In analogy with the ordinary complex numbers, $\mathbb{R} \oplus \mathbb{R}$ with this multiplication rule is called **split complex numbers**. Problem 27.6 establishes a concrete algebra isomorphism between $\mathbf{C}_1^0$ and $\mathbb{R} \oplus \mathbb{R}$.

split complex numbers

Example 27.4.2 In this example, we consider a slightly more complicated Clifford algebra than Example 27.4.1, namely $\mathbf{C}_0^2(\mathbb{R})$. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the two orthonormal basis vectors of the two-dimensional vector space $\mathcal{V}$ on which the Clifford algebra $\mathbf{C}_0^2(\mathbb{R})$ is defined. This algebra is 4-dimensional with a basis $\{\mathbf{1}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 \vee \mathbf{e}_2\}$. To make the multiplication of the basis vectors more transparent, let's set $\mathbf{e}_1 = \mathbf{a}$, $\mathbf{e}_2 = \mathbf{b}$, $\mathbf{e}_1 \vee \mathbf{e}_2 = \mathbf{c}$. Then it is clear that

$$\begin{aligned}
&\mathbf{a} \vee \mathbf{a} = -\mathbf{1}, \qquad \mathbf{a} \vee \mathbf{b} = \mathbf{c}, \qquad \mathbf{a} \vee \mathbf{c} = -\mathbf{b} \\
&\mathbf{b} \vee \mathbf{a} = -\mathbf{c}, \qquad \mathbf{b} \vee \mathbf{b} = -\mathbf{1}, \qquad \mathbf{b} \vee \mathbf{c} = \mathbf{a} \\
&\mathbf{c} \vee \mathbf{a} = \mathbf{b}, \qquad \mathbf{c} \vee \mathbf{b} = -\mathbf{a}, \qquad \mathbf{c} \vee \mathbf{c} = -\mathbf{1}.
\end{aligned} \tag{27.42}$$

Most of these are self-evident. The less obvious ones can be easily shown using Eq. (27.41). For example,

$$\mathbf{c} \vee \mathbf{c} = \mathbf{e}_1 \vee \underbrace{\mathbf{e}_2 \vee \mathbf{e}_1}_{=-\mathbf{e}_1 \vee \mathbf{e}_2} \vee \mathbf{e}_2 = -\underbrace{\mathbf{e}_1 \vee \mathbf{e}_1}_{=-\mathbf{1}} \vee \underbrace{\mathbf{e}_2 \vee \mathbf{e}_2}_{=-\mathbf{1}} = -\mathbf{1}.$$

Comparison of Eq. (27.42) with Example 3.1.16 reveals that $\mathbf{C}_0^2(\mathbb{R})$ is the algebra of quaternions: $\mathbf{C}_0^2(\mathbb{R}) \cong \mathbb{H}$.

$\mathbf{C}_0^2(\mathbb{R}) \cong \mathbb{H}$

The two examples above identified some low-dimensional Clifford algebras of the type $\mathbf{C}_\mu^\nu(\mathbb{R})$ with certain familiar algebras. It is possible to identify all $\mathbf{C}_\mu^\nu(\mathbb{R})$ with more familiar algebras as we proceed to show in the following. We first need to establish some isomorphisms among the algebras $\mathbf{C}_\mu^\nu(\mathbb{R})$ themselves.

Set $N = 2$ in Eq. (27.38) to get $\mathbf{e}_\Delta^2 = (-1)^{1+\nu}\mathbf{1}$. In particular, for $\nu = 0$ and $\nu = 2$, we get $\mathbf{e}_\Delta^2 = -\mathbf{1}$ for both $\mathbf{C}_2^0(\mathbb{R})$ and $\mathbf{C}_0^2(\mathbb{R})$. Now in Theorem 27.2.16, let $\mathcal{U} = \mathbb{R}_{n-\nu}^n$ and $\mathcal{V} = \mathbb{R}_2^2$ (recall that $\mathcal{V}$ has to be even-dimensional) and note that $\tilde{\mathcal{U}} = \mathbb{R}_\nu^n$. The second identity of that theorem then gives

$$C_{\mathbb{R}^2_2 \oplus \mathbb{R}^n_\nu} \cong C_{\mathbb{R}^2_2} \otimes C_{\mathbb{R}^n_{n-\nu}} \equiv \mathbf{C}^2_0(\mathbb{R}) \otimes \mathbf{C}^\mu_\nu(\mathbb{R}). \tag{27.43}$$

Note the position of μ and ν in the last term! Also note that since $\mathbb{R}^2_2 \oplus \mathbb{R}^n_\nu = \mathbb{R}^n_\nu \oplus \mathbb{R}^2_2$, we must have $\mathbf{C}^2_0(\mathbb{R}) \otimes \mathbf{C}^\mu_\nu(\mathbb{R}) = \mathbf{C}^\mu_\nu(\mathbb{R}) \otimes \mathbf{C}^2_0(\mathbb{R})$. But $\mathbb{R}^2_2 \oplus \mathbb{R}^n_\nu = \mathbb{R}^{n+2}_{\nu+2}$ by Proposition 26.5.23. Hence, the left-hand side of the equation above is simply $\mathbf{C}^{\nu+2}_\mu(\mathbb{R})$. By choosing $\mathcal{U} = \mathbb{R}^2_0$, and going through the same procedure we obtain a similar result. The following theorem, in which we have restored μ and ν to their normal position on the right-hand side of (27.43) (now written on the left in the following theorem), summarizes these results.

Theorem 27.4.3 *There exist the following Clifford algebra isomorphisms*:

$$\mathbf{C}^\nu_\mu(\mathbb{R}) \otimes \mathbf{C}^2_0(\mathbb{R}) \cong \mathbf{C}^{\mu+2}_\nu(\mathbb{R})$$

$$\mathbf{C}^\nu_\mu(\mathbb{R}) \otimes \mathbf{C}^0_2(\mathbb{R}) \cong \mathbf{C}^\mu_{\nu+2}(\mathbb{R})$$

Theorem 27.4.4 *Suppose that $\mu = \nu + 4k$ for some integer k. Then*

$$\mathbf{C}^\nu_\mu(\mathbb{R}) \cong \mathbf{C}^\mu_\nu(\mathbb{R})$$

Proof We note that $N = \mu + \nu = 2\nu + 4k = 2(\nu + 2k) \equiv 2m$. Therefore, if $\boldsymbol{\Delta}$ is the normed determinant function, then

$$\lambda_\Delta = (-1)^\nu = (-1)^{m-2k} = (-1)^m.$$

Now apply Theorem 27.2.15, noting that $\widetilde{\mathbb{R}^n_\nu} = \mathbb{R}^n_\mu$. □

For the special case of $\nu = 0$, we obtain

$$\mathbf{C}^0_\mu(\mathbb{R}) \cong \mathbf{C}^\mu_0(\mathbb{R}) \quad \text{if } N = 4k. \tag{27.44}$$

The case of $\mu = \nu$ is important in the classification of the Clifford algebras. In this case, we can write

$$\mathbb{R}^n_\nu = \mathbb{R}^{2\nu}_\nu = \mathbb{R}^{2\mu}_\mu = \mathbb{R}^\mu_0 \oplus \mathbb{R}^\mu_\mu, \tag{27.45}$$

where the inner product is positive definite in the first subspace and negative definite in the second. Let $\{\hat{\mathbf{e}}_i\}_{i=1}^\mu$ and $\{\hat{\mathbf{f}}_i\}_{i=1}^\mu$ be orthonormal bases of $\mathbb{R}^\mu_0$ and $\mathbb{R}^\mu_\mu$, respectively, so that

$$\langle \hat{\mathbf{e}}_i, \hat{\mathbf{e}}_j \rangle = \delta_{ij}, \qquad \langle \hat{\mathbf{f}}_i, \hat{\mathbf{f}}_j \rangle = -\delta_{ij}, \quad i, j = 1, 2, \dots, \mu.$$

Now define an involution ω on $\mathbb{R}^n_\nu$ as follows:

$$\omega(\hat{\mathbf{e}}_i) = \hat{\mathbf{f}}_i, \quad \text{and} \quad \omega(\hat{\mathbf{f}}_i) = \hat{\mathbf{e}}_i, \ i = 1, 2, \dots, \mu. \tag{27.46}$$

Then it can easily be shown that $\omega^t = -\omega$ (see Problem 27.12). Hence, by Theorem 27.2.17 and Eq. (27.36), $\mathbf{C}_\mu^\mu(\mathbb{R}) = \mathcal{L}(\mathbb{R}^{2^{\dim \mathcal{V}_1}})$, where $\mathcal{V}_1 = \ker(\omega - \iota)$. But

$$\mathbf{u} = \sum_{i=1}^{\mu}(\alpha_i \hat{\mathbf{e}}_i + \beta_i \hat{\mathbf{f}}_i) \in \ker(\omega - \iota) \quad \Longleftrightarrow \quad \alpha_i = \beta_i$$

as can be readily shown. This yields $\mathcal{V}_1 = \operatorname{Span}\{\hat{\mathbf{e}}_i + \hat{\mathbf{f}}_i\}_{i=1}^{\mu}$. Therefore, $\dim \mathcal{V}_1 = \mu$, and (27.36) gives the isomorphism

$$\mathbf{C}_\mu^\mu(\mathbb{R}) \cong \mathcal{L}\big(\mathbb{R}^{2^\mu}\big). \tag{27.47}$$

Example 27.4.5 For the simplest case of $\mu = 1$, we have $\mathbf{C}_1^1(\mathbb{R}) \cong \mathcal{L}(\mathbb{R}^2) \cong \mathcal{M}^{2\times 2}$. The isomorphism can be established directly by a procedure similar to Example 27.2.6.

For the case of $\mu = 2$, we have $\mathbf{C}_2^2(\mathbb{R}) \cong \mathcal{L}(\mathbb{R}^4) \cong \mathcal{M}^{4\times 4}$. Furthermore, setting $\mu = 2$ and $\nu = 0$ in the second equation of Theorem 27.4.3, we obtain

$$\mathbf{C}_2^2(\mathbb{R}) \cong \mathbf{C}_2^0(\mathbb{R}) \otimes \mathbf{C}_2^0(\mathbb{R}) \cong \mathbb{H} \otimes \mathbb{H}$$

where we used the result of Example 27.4.2. Hence, we have the isomorphisms

$$\mathbf{C}_2^2(\mathbb{R}) \cong \mathcal{L}\big(\mathbb{R}^4\big) \cong \mathcal{M}^{4\times 4} \cong \mathbb{H} \otimes \mathbb{H}. \tag{27.48}$$

Problem 27.14 gives a direct isomorphic map from $\mathbb{H} \otimes \mathbb{H}$ to $\mathcal{L}(\mathbb{R}^4)$.

27.4.1 Classification of $\mathbf{C}_n^0(\mathbb{R})$ and $\mathbf{C}_0^n(\mathbb{R})$

From the structure of $\mathbf{C}_n^0(\mathbb{R})$ and $\mathbf{C}_0^n(\mathbb{R})$ for low values of n, we can construct all of these algebras by using Theorem 27.4.3. First, let us collect the results of Examples 27.2.6, 27.4.1, and 27.4.2:

$$\mathbf{C}_0^1(\mathbb{R}) \cong \mathbb{C}, \qquad \mathbf{C}_1^0(\mathbb{R}) \cong \mathbb{R} \oplus \mathbb{R}, \qquad \mathbf{C}_2^0(\mathbb{R}) \cong \mathcal{L}\big(\mathbb{R}^2\big), \qquad \mathbf{C}_0^2(\mathbb{R}) \cong \mathbb{H}. \tag{27.49}$$

Next let $\mu = 1$ and $\nu = 0$ in the first equation of Theorem 27.4.3 to obtain

$$\mathbf{C}_0^3(\mathbb{R}) \cong \mathbf{C}_1^0(\mathbb{R}) \otimes \mathbf{C}_0^2(\mathbb{R}) \cong (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H} \cong (\mathbb{R} \otimes \mathbb{H}) \oplus (\mathbb{R} \otimes \mathbb{H}) \cong \mathbb{H} \oplus \mathbb{H},$$

where we used Eqs. (2.16) and (2.18). Similarly, with $\mu = 0$ and $\nu = 1$, the second equation of Theorem 27.4.3 yields

$$\mathbf{C}_3^0(\mathbb{R}) \cong \mathbf{C}_0^1(\mathbb{R}) \otimes \mathbf{C}_2^0(\mathbb{R}) \cong \mathbb{C} \otimes \mathcal{L}\big(\mathbb{R}^2\big).$$

Setting $\mu = 2$, $\nu = 0$ in the first equation of Theorem 27.4.3, we obtain

$$\mathbf{C}_0^4(\mathbb{R}) \cong \mathbf{C}_4^0(\mathbb{R}) \cong \mathbf{C}_2^0(\mathbb{R}) \otimes \mathbf{C}_0^2(\mathbb{R}) \cong \mathcal{L}\big(\mathbb{R}^2\big) \otimes \mathbb{H} \cong \mathbb{H} \otimes \mathcal{L}\big(\mathbb{R}^2\big)$$

Table 27.1 Classification of $\mathbf{C}_n^0(\mathbb{R})$ and $\mathbf{C}_0^n(\mathbb{R})$ for $n \le 8$

n	$\mathbf{C}_n^0(\mathbb{R})$	$\mathbf{C}_0^n(\mathbb{R})$
1	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{C}$
2	$\mathcal{L}(\mathbb{R}^2)$	$\mathbb{H}$
3	$\mathbb{C} \otimes \mathcal{L}(\mathbb{R}^2)$	$\mathbb{H} \oplus \mathbb{H}$
4	$\mathbb{H} \otimes \mathcal{L}(\mathbb{R}^2)$	$\mathbb{H} \otimes \mathcal{L}(\mathbb{R}^2)$
5	$(\mathbb{H} \otimes \mathcal{L}(\mathbb{R}^2)) \oplus (\mathbb{H} \otimes \mathcal{L}(\mathbb{R}^2))$	$\mathbb{C} \otimes \mathcal{L}(\mathbb{R}^2) \otimes \mathbb{H}$
6	$\mathbb{H} \otimes \mathcal{L}(\mathbb{R}^4)$	$\mathcal{L}(\mathbb{R}^8)$
7	$\mathbb{C} \otimes \mathbb{H} \otimes \mathcal{L}(\mathbb{R}^4)$	$\mathcal{L}(\mathbb{R}^8) \oplus \mathcal{L}(\mathbb{R}^8)$
8	$\mathcal{L}(\mathbb{R}^{16})$	$\mathcal{L}(\mathbb{R}^{16})$

where the first isomorphism comes from Eq. (27.44) and the last from Eq. (2.17). We can continue the construction of the rest of $\mathbf{C}_n^0(\mathbb{R})$ and $\mathbf{C}_0^n(\mathbb{R})$ for $n \le 8$. The results are tabulated in Table 27.1. The reader is urged to verify that the entries of the table are consistent with Theorem 27.3.4, keeping in mind that $\mathcal{L}(\mathbb{R}^n)$ can be identified as the total matrix algebra $\mathbb{R} \otimes \mathcal{M}_n$.

Let $\mathcal{V} = \mathbb{R}_0^8$ and $\mathcal{U} = \mathbb{R}_0^n$, noting that $\mathbf{e}_\Delta^2 = \mathbf{1}$ for $\mathcal{C}_V = \mathbf{C}_8^0(\mathbb{R})$. Now use Theorem 27.2.16 to obtain

$$\mathcal{C}_{\mathbb{R}_0^8 \oplus \mathbb{R}_0^n} \cong \mathcal{C}_{\mathbb{R}_0^8} \otimes \mathcal{C}_{\mathbb{R}_0^n} \quad \text{or} \quad \mathcal{C}_{\mathbb{R}_0^{n+8}} \cong \mathcal{C}_{\mathbb{R}_0^8} \otimes \mathcal{C}_{\mathbb{R}_0^n}$$

or

$$\mathbf{C}_{n+8}^0(\mathbb{R}) \cong \mathbf{C}_n^0(\mathbb{R}) \otimes \mathbf{C}_8^0(\mathbb{R}) \cong \mathbf{C}_n^0(\mathbb{R}) \otimes \mathcal{L}\big(\mathbb{R}^{16}\big). \tag{27.50}$$

Similarly, using $\mathcal{V} = \mathbb{R}_8^8$ and $\mathcal{U} = \mathbb{R}_n^n$ in Theorem 27.2.16 yields

$$\mathbf{C}_0^{n+8}(\mathbb{R}) \cong \mathbf{C}_0^n(\mathbb{R}) \otimes \mathcal{L}\big(\mathbb{R}^{16}\big). \tag{27.51}$$

It is clear that these two equations plus Table 27.1 generate $\mathbf{C}_0^n(\mathbb{R})$ and $\mathbf{C}_n^0(\mathbb{R})$ for all n.[4]

Matrices are much more convenient and intuitive to use than linear transformations. It is therefore instructive to rewrite Table 27.1 in terms of matrices keeping in mind that with $\mathbb{F}$ being $\mathbb{C}$ or $\mathbb{H}$,

$$\mathcal{L}\big(\mathbb{R}^n\big) \cong \mathcal{M}_n(\mathbb{R}) \quad \text{and} \quad \mathbb{F} \otimes \mathcal{L}\big(\mathbb{R}^n\big) \cong \mathbb{F} \otimes \mathcal{M}_n(\mathbb{R}) \cong \mathcal{M}_n(\mathbb{F}), \tag{27.52}$$

where $\mathcal{M}_n(\mathbb{F})$ denotes an $n \times n$ matrix with entries in $\mathbb{F}$. The results are given in Table 27.2.

We also write the periodicity relations (27.50) and (27.51) in terms of matrices:

$$\begin{aligned} \mathbf{C}_{n+8}^0(\mathbb{R}) &\cong \mathbf{C}_n^0(\mathbb{R}) \otimes \mathcal{M}_{16}(\mathbb{R}) \\ \mathbf{C}_0^{n+8}(\mathbb{R}) &\cong \mathbf{C}_0^n(\mathbb{R}) \otimes \mathcal{M}_{16}(\mathbb{R}). \end{aligned} \tag{27.53}$$

[4]It is worth noting that, by using $\mathcal{V} = \mathbb{R}_0^4$ or $\mathcal{V} = \mathbb{R}_4^4$, we could obtain formulas for $\mathbf{C}_{n+4}^0(\mathbb{R})$ and $\mathbf{C}_0^{n+4}(\mathbb{R})$ analogous to (27.50) and (27.51). However, as entries 5, 6, and 7 of Table 27.1 can testify, they would not be as appealing as the formulas obtained above. This is primarily because $\mathbf{C}_0^8(\mathbb{R}) \cong \mathbf{C}_8^0(\mathbb{R}) \cong \mathcal{L}(\mathbb{R}^{16})$.

Table 27.2 Classification of $\mathbf{C}_n^0(\mathbb{R})$ and $\mathbf{C}_0^n(\mathbb{R})$ for $n \le 8$ in terms of matrices

n	$\mathbf{C}_n^0(\mathbb{R})$	$\mathbf{C}_0^n(\mathbb{R})$
1	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{C}$
2	$\mathcal{M}_2(\mathbb{R})$	$\mathbb{H}$
3	$\mathbb{C} \otimes \mathcal{M}_2(\mathbb{R}) \cong \mathcal{M}_2(\mathbb{C})$	$\mathbb{H} \oplus \mathbb{H}$
4	$\mathbb{H} \otimes \mathcal{M}_2(\mathbb{R}) \cong \mathcal{M}_2(\mathbb{H})$	$\mathbb{H} \otimes \mathcal{M}_2(\mathbb{R}) \cong \mathcal{M}_2(\mathbb{H})$
5	$\mathcal{M}_2(\mathbb{H}) \oplus \mathcal{M}_2(\mathbb{H})$	$\mathbb{H} \otimes \mathcal{M}_2(\mathbb{C}) \cong \mathbb{C} \otimes \mathcal{M}_2(\mathbb{H})$
6	$\mathbb{H} \otimes \mathcal{M}_4(\mathbb{R}) \cong \mathcal{M}_4(\mathbb{H})$	$\mathcal{M}_8(\mathbb{R})$
7	$\mathbb{H} \otimes \mathcal{M}_4(\mathbb{C}) \cong \mathbb{C} \otimes \mathcal{M}_4(\mathbb{H})$	$\mathcal{M}_8(\mathbb{R}) \oplus \mathcal{M}_8(\mathbb{R})$
8	$\mathcal{M}_{16}(\mathbb{R})$	$\mathcal{M}_{16}(\mathbb{R})$

27.4.2 Classification of $\mathbf{C}_\mu^\nu(\mathbb{R})$

We can now complete the task of classifying all of the algebras $\mathbf{C}_\mu^\nu(\mathbb{R})$. The case of $\mu = \nu$ is given by Eq. (27.47). For the case of $\mu > \nu$, let $\mu = \nu + \sigma$ and note that

$$\mathbb{R}_\nu^n = \mathbb{R}_0^\mu \oplus \mathbb{R}_\nu^\nu = \left(\mathbb{R}_0^\sigma \oplus \mathbb{R}_0^\nu\right) \oplus \mathbb{R}_\nu^\nu = \mathbb{R}_0^\sigma \oplus \left(\mathbb{R}_0^\nu \oplus \mathbb{R}_\nu^\nu\right) = \mathbb{R}_0^\sigma \oplus \mathbb{R}_\nu^{2\nu}.$$

Now let $\mathcal{V} = \mathbb{R}_\nu^{2\nu}$ and note that $\mathcal{C}_V = \mathbf{C}_\nu^\nu(\mathbb{R})$. Furthermore, Eq. (27.38), with $N = 2\nu$ gives $\mathbf{e}_\Delta^2 = \mathbf{1}$. Hence, with $\mathcal{U} = \mathbb{R}_0^\sigma$, the first equation of Theorem 27.2.16 yields

$$\mathbf{C}_\mu^\nu(\mathbb{R}) \cong \mathbf{C}_\nu^\nu(\mathbb{R}) \otimes \mathbf{C}_\sigma^0(\mathbb{R}) \cong \mathcal{L}\left(\mathbb{R}^{2^\nu}\right) \otimes \mathbf{C}_\sigma^0(\mathbb{R}), \tag{27.54}$$

where we used (27.47).

If $\mu < \nu$, let $\nu = \mu + \rho$ and note that $\mathbb{R}_\nu^n = \mathbb{R}_\rho^\rho \oplus \mathbb{R}_\mu^{2\mu}$. With $\mathcal{V} = \mathbb{R}_\mu^{2\mu}$ and $\mathcal{U} = \mathbb{R}_\rho^\rho$, the first equation of Theorem 27.2.16 yields

$$\mathbf{C}_\mu^\nu(\mathbb{R}) \cong \mathbf{C}_\mu^\mu(\mathbb{R}) \otimes \mathbf{C}_0^\rho(\mathbb{R}) \cong \mathcal{L}\left(\mathbb{R}^{2^\mu}\right) \otimes \mathbf{C}_0^\rho(\mathbb{R}). \tag{27.55}$$

It is worthwhile to collect these results in a theorem. Noting that $\mathbf{C}_0^0 \cong \mathbb{R}$ and that $\mathcal{A} \otimes \mathbb{R} = \mathcal{A}$ for any real algebra, we can combine the three cases of $\mu = \nu$, $\mu > \nu$, and $\mu < \nu$ into two cases:

Theorem 27.4.6 *The following isomorphisms hold*:

$$\mathbf{C}_\mu^\nu(\mathbb{R}) \cong \mathcal{L}\left(\mathbb{R}^{2^\nu}\right) \otimes \mathbf{C}_{\mu-\nu}^0(\mathbb{R}) \cong \mathcal{M}_{2^\nu}(\mathbb{R}) \otimes \mathbf{C}_{\mu-\nu}^0(\mathbb{R}), \quad \text{if } \mu \ge \nu,$$
$$\mathbf{C}_\mu^\nu(\mathbb{R}) \cong \mathcal{L}\left(\mathbb{R}^{2^\mu}\right) \otimes \mathbf{C}_0^{\nu-\mu}(\mathbb{R}) \cong \mathcal{M}_{2^\mu}(\mathbb{R}) \otimes \mathbf{C}_0^{\nu-\mu}(\mathbb{R}), \quad \text{if } \mu \le \nu.$$

This theorem together with Table (27.2) and the periodicity relations

$$\begin{aligned} \mathbf{C}_{\mu+8}^\nu(\mathbb{R}) &\cong \mathbf{C}_\mu^\nu(\mathbb{R}) \otimes \mathcal{M}_{16}(\mathbb{R}) \\ \mathbf{C}_\mu^{\nu+8}(\mathbb{R}) &\cong \mathbf{C}_\mu^\nu(\mathbb{R}) \otimes \mathcal{M}_{16}(\mathbb{R}), \end{aligned} \tag{27.56}$$

which come from Eq. (27.53), determine all the algebras $\mathbf{C}_\mu^\nu(\mathbb{R})$.

From Theorem 27.4.6, the periodicity relation (27.53), and Table 27.2, we get the following:

Theorem 27.4.7 *All Clifford algebras* $\mathbf{C}_\mu^\nu(\mathbb{R})$ *with* $\mu - \nu \neq 1 \mod 4$ *are simple. Those with* $\mu - \nu = 1 \mod 4$ *are direct sums of two identical simple algebras.*

27.4.3 The Algebra $\mathbf{C}_3^1(\mathbb{R})$

For the Minkowski n-space, $\mathbb{R}_1^n$, Theorem 27.4.6 gives

$$\mathbf{C}_{n-1}^1(\mathbb{R}) \cong \mathcal{M}_2(\mathbb{R}) \otimes \mathbf{C}_{n-2}^0(\mathbb{R}).$$

When $n = 4$, this reduces to

$$\mathbf{C}_3^1(\mathbb{R}) \cong \mathcal{M}_2(\mathbb{R}) \otimes \mathbf{C}_2^0(\mathbb{R}) \cong \mathcal{M}_2(\mathbb{R}) \otimes \mathcal{M}_2(\mathbb{R}) \cong \mathcal{M}_4(\mathbb{R}).$$

In the language of Chap. 3, $\mathbf{C}_3^1(\mathbb{R})$ is a total matrix algebra, which, by either Theorem 3.5.27 (and the remarks after it) or Theorem 3.3.2, is simple. We now find a basis $\{\mathbf{e}_{ij}\}$ of this algebra.

First we find the diagonals $\{\mathbf{e}_{ii}\}_{i=1}^4$, which are obviously primitive, orthogonal to each other, and

$$\mathbf{1} = \mathbf{e}_{11} + \mathbf{e}_{22} + \mathbf{e}_{33} + \mathbf{e}_{44}.$$

Thus, by Theorem 3.5.32, the identity has rank 4. Next, we construct four primitive orthogonal idempotents $\{\mathbf{P}_i\}_{i=1}^4$ out of the basis vectors[5] $\{\hat{\mathbf{e}}_\eta\}_{\eta=0}^3$ of $\mathbf{C}_3^1(\mathbb{R})$ and their products and identify them with $\{\mathbf{e}_{ii}\}_{i=1}^4$. The easiest way to construct these idempotents is to find $\mathbf{x}$ and $\mathbf{y}$ such that

$$\mathbf{x}^2 = \mathbf{1} = \mathbf{y}^2, \qquad \mathbf{xy} = \mathbf{yx}.$$

Then the product of $\frac{1}{2}(\mathbf{1} \pm \mathbf{x})$ and $\frac{1}{2}(\mathbf{1} \pm \mathbf{y})$ for all sign choices yields four primitive orthogonal idempotents, as the reader may verify. There are many choices for $\mathbf{x}$ and $\mathbf{y}$. We choose $\mathbf{x} = \hat{\mathbf{e}}_1$ and $\mathbf{y} = \hat{\mathbf{e}}_{02}$, where we use the common abbreviation $\hat{\mathbf{e}}_{\eta_1 \dots \eta_p} \equiv \hat{\mathbf{e}}_{\eta_1} \vee \cdots \vee \hat{\mathbf{e}}_{\eta_p}$, and set

$$\begin{aligned}
\mathbf{P}_1 &= \frac{1}{4}(\mathbf{1} + \hat{\mathbf{e}}_1)(\mathbf{1} + \hat{\mathbf{e}}_{02}) \equiv \mathbf{e}_{11}, \\
\mathbf{P}_2 &= \frac{1}{4}(\mathbf{1} + \hat{\mathbf{e}}_1)(\mathbf{1} - \hat{\mathbf{e}}_{02}) \equiv \mathbf{e}_{22}, \\
\mathbf{P}_3 &= \frac{1}{4}(\mathbf{1} - \hat{\mathbf{e}}_1)(\mathbf{1} + \hat{\mathbf{e}}_{02}) \equiv \mathbf{e}_{33}, \\
\mathbf{P}_4 &= \frac{1}{4}(\mathbf{1} - \hat{\mathbf{e}}_1)(\mathbf{1} - \hat{\mathbf{e}}_{02}) \equiv \mathbf{e}_{44}.
\end{aligned} \tag{27.57}$$

[5]Here we are using the physicists' convention of numbering the basis vectors from 0 to 3 with $\hat{\mathbf{e}}_0 = \hat{\mathbf{e}}_4$ and using Greek letters for indices.

Since the P_is are all primitive (thus, of rank 1), by Theorem 3.5.32, they are similar. Indeed, one can easily show that

$$\begin{aligned}\hat{\mathbf{e}}_{03}\mathsf{P}_1\hat{\mathbf{e}}_{03}^{-1} &= \hat{\mathbf{e}}_{03}\mathsf{P}_1\hat{\mathbf{e}}_{03} = \mathsf{P}_2,\\ \hat{\mathbf{e}}_3\mathsf{P}_1\hat{\mathbf{e}}_3^{-1} &= \hat{\mathbf{e}}_3\mathsf{P}_1\hat{\mathbf{e}}_3 = \mathsf{P}_3,\\ \hat{\mathbf{e}}_0\mathsf{P}_1\hat{\mathbf{e}}_0^{-1} &= -\hat{\mathbf{e}}_0\mathsf{P}_1\hat{\mathbf{e}}_0 = \mathsf{P}_4.\end{aligned} \tag{27.58}$$

Equations (27.57) and (27.58) determine all $\mathbf{e}_{ij}$s, as we now demonstrate. Write the first relation of (27.58) as $\hat{\mathbf{e}}_{03}\mathsf{P}_1 = \mathsf{P}_2\hat{\mathbf{e}}_{03}$ or $\hat{\mathbf{e}}_{03}\mathbf{e}_{11} = \mathbf{e}_{22}\hat{\mathbf{e}}_{03}$. Since $\hat{\mathbf{e}}_{03} \in \mathbf{C}_3^1(\mathbb{R})$, it can be written as a linear combination of $\{\mathbf{e}_{ij}\}$. With $\hat{\mathbf{e}}_{03} = \sum_{i,j=1}^4 \alpha_{ij}\mathbf{e}_{ij}$, we have

$$\sum_{i,j=1}^4 \alpha_{ij}\mathbf{e}_{ij}\mathbf{e}_{11} = \mathbf{e}_{22}\sum_{i,j=1}^4 \alpha_{ij}\mathbf{e}_{ij} \quad \text{or} \quad \sum_{i=1}^4 \alpha_{i1}\mathbf{e}_{i1} = \sum_{j=1}^4 \alpha_{2j}\mathbf{e}_{2j}.$$

Linear independence of $\{\mathbf{e}_{ij}\}$ implies that $\alpha_{i1} = 0$ for $i \neq 2$ and $\alpha_{2j} = 0$ for $j \neq 1$. Therefore, the left-hand side (or the right-hand side) of the equation reduces to $\alpha_{21}\mathbf{e}_{21}$. Hence, we have

$$\hat{\mathbf{e}}_{03}\mathsf{P}_1 = \alpha_{21}\mathbf{e}_{21}. \tag{27.59}$$

We can also write the first relation of (27.58) as $\mathsf{P}_1\hat{\mathbf{e}}_{03} = \hat{\mathbf{e}}_{03}\mathsf{P}_2$ or $\mathbf{e}_{11}\hat{\mathbf{e}}_{03} = \hat{\mathbf{e}}_{03}\mathbf{e}_{22}$, which yields

$$\mathbf{e}_{11}\sum_{i,j=1}^4 \alpha_{ij}\mathbf{e}_{ij} = \sum_{i,j=1}^4 \alpha_{ij}\mathbf{e}_{ij}\mathbf{e}_{22} \quad \text{or} \quad \sum_{j=1}^4 \alpha_{1j}\mathbf{e}_{1j} = \sum_{i=1}^4 \alpha_{i2}\mathbf{e}_{i2}.$$

Again, linear independence of $\{\mathbf{e}_{ij}\}$ implies that $\alpha_{1j} = 0$ for $j \neq 2$ and $\alpha_{i2} = 0$ for $i \neq 1$. Therefore, the left-hand side (or the right-hand side) of the equation reduces to $\alpha_{12}\mathbf{e}_{12}$, and we get

$$\mathsf{P}_1\hat{\mathbf{e}}_{03} = \alpha_{12}\mathbf{e}_{12}. \tag{27.60}$$

Multiply Eqs. (27.59) and (27.60) to get

$$(\hat{\mathbf{e}}_{03}\mathsf{P}_1)(\mathsf{P}_1\hat{\mathbf{e}}_{03}) = (\alpha_{21}\mathbf{e}_{21})(\alpha_{12}\mathbf{e}_{12})$$

or

$$\hat{\mathbf{e}}_{03}\mathsf{P}_1\hat{\mathbf{e}}_{03} = \alpha_{21}\alpha_{12}\mathbf{e}_{21}\mathbf{e}_{12} = \alpha_{21}\alpha_{12}\mathbf{e}_{22} = \alpha_{21}\alpha_{12}\mathsf{P}_2.$$

Comparing this with the first equation in (27.58), we conclude that $\alpha_{21}\alpha_{12} = 1$, which is also a consistency condition for $\hat{\mathbf{e}}_{03} \vee \hat{\mathbf{e}}_{03} = \mathbf{1}$. There are several choices for α_{ij}, all of which satisfy this as well as other conditions obtained above. Therefore, we are at liberty to set $\alpha_{21} = 1 = \alpha_{12}$ and write

$$\hat{\mathbf{e}}_{03}\mathsf{P}_1 = \mathbf{e}_{21}, \qquad \mathsf{P}_1\hat{\mathbf{e}}_{03} = \hat{\mathbf{e}}_{03}\mathsf{P}_2 = \mathbf{e}_{12}. \tag{27.61}$$

Table 27.3 The basis $\mathbf{e}_{ij}$ for the total matrix algebra $\mathbf{C}_3^1(\mathbb{R})$

$\mathbf{e}_{ij}$	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	$\mathbf{P}_1$	$\hat{\mathbf{e}}_{03}\mathbf{P}_2$	$\hat{\mathbf{e}}_3\mathbf{P}_3$	$-\hat{\mathbf{e}}_0\mathbf{P}_4$
$i=2$	$\hat{\mathbf{e}}_{03}\mathbf{P}_1$	$\mathbf{P}_2$	$\hat{\mathbf{e}}_0\mathbf{P}_3$	$-\hat{\mathbf{e}}_3\mathbf{P}_4$
$i=3$	$\hat{\mathbf{e}}_3\mathbf{P}_1$	$-\hat{\mathbf{e}}_0\mathbf{P}_2$	$\mathbf{P}_3$	$\hat{\mathbf{e}}_{03}\mathbf{P}_4$
$i=4$	$\hat{\mathbf{e}}_0\mathbf{P}_1$	$-\hat{\mathbf{e}}_3\mathbf{P}_2$	$\hat{\mathbf{e}}_{03}\mathbf{P}_3$	$\mathbf{P}_4$

Going through the same procedure using the second and third relations of (27.58), we obtain

$$\begin{aligned} \hat{\mathbf{e}}_3\mathbf{P}_1 &= \mathbf{e}_{31}, & \mathbf{P}_1\hat{\mathbf{e}}_3 &= \hat{\mathbf{e}}_3\mathbf{P}_3 = \mathbf{e}_{13}, \\ \hat{\mathbf{e}}_0\mathbf{P}_1 &= \mathbf{e}_{41}, & \mathbf{P}_1\hat{\mathbf{e}}_0 &= \hat{\mathbf{e}}_0\mathbf{P}_4 = -\mathbf{e}_{14}. \end{aligned} \tag{27.62}$$

Having found $\mathbf{e}_{i1}$ and $\mathbf{e}_{1j}$, we can find all the $\mathbf{e}_{ij}$ because $\mathbf{e}_{ij} = \mathbf{e}_{i1}\mathbf{e}_{1j}$. The result is summarized in Table 27.3.

Now that we have the basis we were after, we can express the basis vectors $\{\hat{\mathbf{e}}_\eta\}_{\eta=0}^3$ of the underlying vector space in terms of the new basis. Writing

$$\hat{\mathbf{e}}_\eta = \sum_{i,j=1}^4 \gamma_{\eta,ij}\mathbf{e}_{ij},$$

multiplying it on the right by $\mathbf{e}_{kl}$ and on the left by $\mathbf{e}_{mn}$, we obtain

$$\mathbf{e}_{mn}\hat{\mathbf{e}}_\eta\mathbf{e}_{kl} = \gamma_{\eta,nk}\mathbf{e}_{ml},$$

which, for $m = 1 = l$ yields

$$\mathbf{e}_{1n}\hat{\mathbf{e}}_\eta\mathbf{e}_{k1} = \gamma_{\eta,nk}\mathbf{e}_{11}. \tag{27.63}$$

Thus, to find $\gamma_{\eta,nk}$, multiply $\hat{\mathbf{e}}_\eta$ on the left by $\mathbf{e}_{1n}$ and on the right by $\mathbf{e}_{k1}$ and read the coefficient of $\mathbf{e}_{11}$ in the expression obtained. As an example, we find $\gamma_{3,12}$. We have

$$\mathbf{e}_{11}\hat{\mathbf{e}}_3\mathbf{e}_{21} = \gamma_{3,12}\mathbf{e}_{11}.$$

The left-hand side can be evaluated from the table:

$$\begin{aligned} \mathbf{e}_{11}\hat{\mathbf{e}}_3\mathbf{e}_{21} = \mathbf{e}_{11}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_{03}\mathbf{P}_1 &= \mathbf{e}_{11}\hat{\mathbf{e}}_3 \vee \hat{\mathbf{e}}_0 \vee \hat{\mathbf{e}}_3\mathbf{P}_1 \\ &= -\mathbf{e}_{11}\hat{\mathbf{e}}_3 \vee \hat{\mathbf{e}}_3 \vee \hat{\mathbf{e}}_0\mathbf{P}_1 \\ &= -\mathbf{e}_{11}\hat{\mathbf{e}}_0\mathbf{P}_1 = -\mathbf{e}_{11}\mathbf{e}_{41} = \mathbf{0}. \end{aligned}$$

Thus, $\gamma_{3,12} = 0$. Similarly, we find $\gamma_{3,13}$:

$$\mathbf{e}_{11}\hat{\mathbf{e}}_3\mathbf{e}_{31} = \gamma_{3,13}\mathbf{e}_{11}.$$

Using Table 27.3, we get

$$\mathbf{e}_{11}\hat{\mathbf{e}}_3\mathbf{e}_{31} = \mathbf{e}_{11}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_3\mathbf{P}_1 = \mathbf{e}_{11}\hat{\mathbf{e}}_3 \vee \hat{\mathbf{e}}_3\mathbf{P}_1 = \mathbf{e}_{11}\mathbf{1}\mathbf{e}_{11} = \mathbf{e}_{11}.$$

Thus, $\gamma_{3,13} = 1$.

We can continue this way and obtain all coefficients $\gamma_{\eta,ij}$. However, an easier way is to solve for $\hat{\mathbf{e}}_\eta$ from Eq. (27.57). Thus

$$\hat{\mathbf{e}}_1 = \mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_3 - \mathbf{P}_4 = \mathbf{e}_{11} + \mathbf{e}_{22} - \mathbf{e}_{33} - \mathbf{e}_{44},$$

giving the matrix

$$\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Similarly,

$$\hat{\mathbf{e}}_{02} = \mathbf{P}_1 + \mathbf{P}_3 - \mathbf{P}_2 - \mathbf{P}_4, \qquad (27.64)$$

from which we can get $\hat{\mathbf{e}}_2$ by multiplying on the left by $\hat{\mathbf{e}}_0$ and noting that

$$\hat{\mathbf{e}}_0\hat{\mathbf{e}}_{02} = \hat{\mathbf{e}}_0\hat{\mathbf{e}}_0\hat{\mathbf{e}}_2 = -\hat{\mathbf{e}}_2.$$

Thus,

$$\hat{\mathbf{e}}_2 = -\hat{\mathbf{e}}_0\mathbf{P}_1 - \hat{\mathbf{e}}_0\mathbf{P}_3 + \hat{\mathbf{e}}_0\mathbf{P}_2 + \hat{\mathbf{e}}_0\mathbf{P}_4 = -\mathbf{e}_{41} - \mathbf{e}_{23} - \mathbf{e}_{32} - \mathbf{e}_{14} \qquad (27.65)$$

where use was made of Table 27.3 in the last step. It follows that

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

The remaining two matrices can be obtained similarly. The details are left as Problem 27.22. The result is

$$\gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \gamma_0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since $\hat{\mathbf{e}}_\mu \vee \hat{\mathbf{e}}_\nu + \hat{\mathbf{e}}_\nu \vee \hat{\mathbf{e}}_\mu = 2\eta_{\mu\nu}$ by (27.15) and (27.40), we have

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\mathbf{1}, \qquad (27.66)$$

which can also be verified directly by matrix multiplication. Equation (27.66) is identical to (27.11) obeyed by the Dirac gamma matrices. The matrices that Dirac used in his equation had complex entries. The matrices constructed above are all real. They are called the **Majorana representation** of the Dirac matrices.

Majorana representation

27.5 Problems

27.1 Starting with Eq. (27.8), write $\mathbf{e}_{i_{p-1}} \vee \mathbf{e}_{i_p}$ in terms of the wedge product using Eq. (27.3). Then use the more general Clifford product (27.2) repeatedly until you have turned all the $\vee$'s to $\wedge$'s.

27.2 Find the coefficients of $\mathbf{1}$, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_1 \vee \mathbf{e}_2$ for the Clifford product $\mathbf{u} \vee \mathbf{v}$ of Example 27.2.2.

27.3 Show that Eqs. (27.24) and (27.25) are equivalent.

27.4 Show that because of (27.24), φ can be extended to an algebra homomorphism only if it is an *injective* linear map.

27.5 Show that the conjugation involution of Definition 27.2.5 coincides with the usual complex and quaternion conjugation. Show that $\overline{\mathbf{a} \vee \mathbf{b}} = \bar{\mathbf{b}} \vee \bar{\mathbf{a}}$.

27.6 Let $\varphi : \mathbb{R} \to \mathbb{R} \oplus \mathbb{R}$ be a linear map. Assume a completely general form for φ, i.e., assume $\varphi(\alpha) = (\beta \oplus \gamma)$. Extend this linear map to a homomorphism $\phi : \mathbf{C}_1^0 \to \mathbb{R} \oplus \mathbb{R}$. Imposing the consistency condition (27.25), deduce that $\beta^2 = \alpha^2 = \gamma^2$. Now show that a non-trivial homomorphism sends $\mathbf{1}$ to $1 \oplus 1$ and $\mathbf{e}$ to $1 \oplus -1$, and therefore is an isomorphism. Finally for a general element of $\mathbf{C}_1^0$, show that

$$\phi(\alpha 1 + \beta \mathbf{e}) = (\alpha + \beta, \alpha - \beta), \quad \alpha, \beta \in \mathbb{R}.$$

27.7 Show that the four matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are linearly independent.

27.8 Derive Eq. (27.33).

27.9 Show that the center $\mathcal{Z}_V$ is a subalgebra of $\mathcal{C}_V$.

27.10 Show that both $\mathcal{Z}_V$ and $\overline{\mathcal{Z}}_V$ are invariant under the degree involution ω_V.

27.11 Let $\mathbf{Q}_\nu : \mathcal{V} \to \mathbb{F}$ be a quadratic form defined in terms of the basis $\{\mathbf{e}_i\}_{i=1}^n$ and $\mathbf{g}$ the inner product derived from $\mathbf{Q}_\nu$. Show that $\mathbf{g}(\mathbf{e}_i, \mathbf{e}_j) = \pm\delta_{ij}$.

27.12 Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}_\mu^{2\mu}$ of Eq. (27.45). Write $\mathbf{u}$ and $\mathbf{v}$ in terms of the basis vectors $\{\hat{\mathbf{e}}_i\}_{i=1}^\mu$ and $\{\hat{\mathbf{f}}_i\}_{i=1}^\mu$ and show that the ω of Eq. (27.46) satisfies $\langle \mathbf{u}, \omega \mathbf{v} \rangle = \langle -\omega \mathbf{u}, \mathbf{v} \rangle$, implying that $\omega^t = -\omega$. With $\mathbf{u} = \sum_{i=1}^\mu (\alpha_i \hat{\mathbf{e}}_i + \beta_i \hat{\mathbf{f}}_i)$, show that $\mathbf{u} \in \ker(\omega - \iota)$ iff $\alpha_i = \beta_i$.

27.13 Following Example 27.2.6, show directly that $\mathbf{C}_1^1(\mathbb{R}) \cong \mathcal{L}(\mathbb{R}^2) \cong \mathcal{M}^{2\times 2}$.

27.14 Let $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathcal{L}(\mathbb{R}^4)$. Define $\phi : \mathbb{H} \otimes \mathbb{H} \to \mathcal{L}(\mathbb{R}^4)$ by

$$\phi(\mathbf{p} \otimes \mathbf{q})\mathbf{x} = \mathbf{p} \cdot \mathbf{x} \cdot \mathbf{q}^*, \qquad \mathbf{p}, \mathbf{q} \in \mathbb{H}, \ \mathbf{x} \in \mathcal{L}(\mathbb{R}^4)$$

where on the right-hand side, $\mathbf{x} = x_1 + x_2 i + x_3 j + x_4 k$ is a quaternion. Show that ϕ is an algebra homomorphism, whose kernel is zero. Now invoke the dimension theorem and the fact that $\mathbb{H} \otimes \mathbb{H}$ and $\mathcal{L}(\mathbb{R}^4)$ have the same dimension to show that ϕ is an isomorphism.

27.15 Let $\mathcal{V} = \mathbb{R}_0^4$ or $\mathcal{V} = \mathbb{R}_4^4$ and note that $\mathbf{e}_\Delta^2 = \mathbf{1}$ for $\mathcal{C}_{\mathcal{V}}$. Now use Theorem 27.2.17 to show that

$$\mathbf{C}_{8 \text{ or } 0}^{0 \text{ or } 8}(\mathbb{R}) \cong \mathbf{C}_{4 \text{ or } 0}^{0 \text{ or } 4}(\mathbb{R}) \otimes \mathbf{C}_{4 \text{ or } 0}^{0 \text{ or } 4}(\mathbb{R}).$$

27.16 Complete the remainder of Table 27.1.

27.17 Using $\mathcal{V} = \mathbb{R}_0^4$ or $\mathcal{V} = \mathbb{R}_4^4$, derive formulas for $\mathbf{C}_{n+4}^0(\mathbb{R})$ and $\mathbf{C}_0^{n+4}(\mathbb{R})$ analogous to (27.50) and (27.51).

27.18 Show that

$$\mathbf{C}_0^9(\mathbb{R}) \cong \mathbb{C} \otimes \mathcal{L}(\mathbb{R}^{16}), \qquad \mathbf{C}_9^0(\mathbb{R}) \cong \mathcal{L}(\mathbb{R}^{16}) \oplus \mathcal{L}(\mathbb{R}^{16}),$$
$$\mathbf{C}_0^{10}(\mathbb{R}) \cong \mathbb{H} \otimes \mathcal{L}(\mathbb{R}^{16}), \qquad \mathbf{C}_{10}^0(\mathbb{R}) \cong \mathcal{L}(\mathbb{R}^{32}).$$

27.19 Show that if $\mathbf{x}^2 = \mathbf{1} = \mathbf{y}^2$ and $\mathbf{xy} = \mathbf{yx}$, then the four quantities $\frac{1}{4}(\mathbf{1} \pm \mathbf{x})(\mathbf{1} \pm \mathbf{y})$ are orthogonal idempotents.

27.20 Verify all of the relations in Eq. (27.58).

27.21 Derive Eq. (27.62).

27.22 Note that

$$\hat{\mathbf{e}}_0 = \hat{\mathbf{e}}_0 \mathbf{1} = \hat{\mathbf{e}}_0 \mathbf{P}_1 + \hat{\mathbf{e}}_0 \mathbf{P}_2 + \hat{\mathbf{e}}_0 \mathbf{P}_3 + \hat{\mathbf{e}}_0 \mathbf{P}_4$$
$$\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_3 \mathbf{1} = \hat{\mathbf{e}}_3 \mathbf{P}_1 + \hat{\mathbf{e}}_3 \mathbf{P}_2 + \hat{\mathbf{e}}_3 \mathbf{P}_3 + \hat{\mathbf{e}}_3 \mathbf{P}_4.$$

Now use Table 27.3 to express each term on the right in terms of $\mathbf{e}_{ij}$.