15 Complex Analysis of SOLDEs

We have familiarized ourselves with some useful techniques for finding solutions to differential equations. One powerful method that leads to formal solutions is power series. We also stated Theorem 14.6.4 which guarantees the convergence of the solution of the power series within a circle whose size is at least as large as the smallest of the circles of convergence of the coefficient functions. Thus, the convergence of the solution is related to the convergence of the coefficient functions. What about the nature of the convergence, or the analyticity of the solution? Is it related to the analyticity of the coefficient functions? If so, how? Are the singular points of the coefficients also singular points of the solution? Is the nature of the singularities the same? This chapter answers some of these questions.

Analyticity is best handled in the complex plane. An important reason for this is the property of analytic continuation discussed in Chap. 12. The differential equation $du/dx = u^2$ has a solution $u = -1/x$ for all x except $x = 0$. Thus, we have to "puncture" the real line by removing $x = 0$ from it. Then we have two solutions, because the domain of definition of $u = -1/x$ is not connected on the real line (technically, the definition of a function includes its domain as well as the rule for going from the domain to the range). In addition, if we confine ourselves to the real line, there is no way that we can connect the $x > 0$ region to the $x < 0$ region. However, in the complex plane the same equation, $dw/dz = w^2$, has the complex solution $w = -1/z$, which is analytic everywhere except at $z = 0$. Puncturing the complex plane does not destroy the connectivity of the region of definition of w. Thus, the solution in the $x > 0$ region can be analytically continued to the solution in the $x < 0$ region by going around the origin.

The aim of this chapter is to investigate the analytic properties of the solutions of some well known SOLDEs in mathematical physics. We begin with a result from differential equation theory (for a proof, see [Birk 78, p. 223]).

Proposition 15.0.1 (Continuation principle) *The function obtained by analytic continuation of any solution of an analytic differential equation along any path in the complex plane is a solution of the analytic continuation of the differential equation along the same path.*

continuation principle

S. Hassani, *Mathematical Physics*, DOI 10.1007/978-3-319-01195-0_15,

An analytic differential equation is one with analytic coefficient functions. This proposition makes it possible to find a solution in one region of the complex plane and then continue it analytically. The following example shows how the singularities of the coefficient functions affect the behavior of the solution.

Example 15.0.2 Let us consider the FODE $w' - (\gamma/z)w = 0$ for $\gamma \in \mathbb{R}$. The coefficient function $p(z) = -\gamma/z$ has a simple pole at $z = 0$. The solution to the FODE is easily found to be $w = z^\gamma$. Thus, depending on whether γ is a nonnegative integer, a negative integer $-m$, or a noninteger, the solution has a regular point, a pole of order m, or a branch point at $z = 0$, respectively.

This example shows that the singularities of the solution depend on the parameters of the differential equation.

15.1 Analytic Properties of Complex DEs

To prepare for discussing the analytic properties of the solutions of SOLDEs, let us consider some general properties of differential equations from a complex analytical point of view.

15.1.1 Complex FOLDEs

In the homogeneous FOLDE

$$\frac{dw}{dz} + p(z)w = 0, \tag{15.1}$$

$p(z)$ is assumed to have only isolated singular points. It follows that $p(z)$ can be expanded about a point z_0—which may be a singularity of $p(z)$—as a Laurent series in some annular region $r_1 < |z - z_0| < r_2$:

$$p(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \text{where} \quad r_1 < |z - z_0| < r_2.$$

The solution to Eq. (15.1), as given in Theorem 14.2.1 with $q = 0$, is

$$\begin{aligned} w(z) &= \exp\left[-\int p(z)\,dz\right] \\ &= C\exp\left[-a_{-1}\int \frac{dz}{z - z_0} - \sum_{n=0}^{\infty} a_n \int (z - z_0)^n dz \right. \\ &\quad \left. - \sum_{n=2}^{\infty} a_{-n} \int (z - z_0)^{-n} dz\right] \end{aligned}$$

$$= C \exp\left[-a_{-1} \ln(z - z_0) - \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1} + \sum_{n=1}^{\infty} \frac{a_{-n-1}}{n} (z - z_0)^{-n} \right]$$

We can write this solution as

$$w(z) = C(z - z_0)^{\alpha} g(z), \tag{15.2}$$

where $\alpha \equiv -a_{-1}$ and $g(z)$ is an analytic *single-valued* function in the annular region $r_1 < |z - z_0| < r_2$ because $g(z)$ is the exponential of an analytic function.

For the special case in which p has a simple pole, i.e., when $a_{-n} = 0$ for all $n \geq 2$, the second sum in the exponent will be absent, and g will be analytic even *at* z_0. In fact, $g(z_0) = 1$, and choosing $C = 1$, we can write

$$w(z) = (z - z_0)^{\alpha} \left[1 + \sum_{k=1}^{\infty} b_k (z - z_0)^k \right]. \tag{15.3}$$

Depending on the nature of the singularity of $p(z)$ at z_0, the solutions given by Eq. (15.2) have different classifications. For instance, if $p(z)$ has a removable singularity (if $a_{-n} = 0\ \forall n \geq 1$), the solution is $Cg(z)$, which is analytic. In this case, we say that the FOLDE [Eq. (15.1)] has *a removable singularity at* z_0. If $p(z)$ has a simple pole at z_0 (if $a_{-1} \neq 0$ and $a_{-n} = 0\ \forall n \geq 2$), then in general, the solution has a branch point at z_0. In this case we say that the FOLDE *has a regular singular point*. Finally, if $p(z)$ has a pole of order $m > 1$, then the solution will have an essential singularity (see Problem 15.1). In this case the FOLDE is said to have an *irregular singular point*.

The singularity of the coefficient functions of an FOLDE determines the singularity of the solution.

To arrive at the solution given by Eq. (15.2), we had to solve the FOLDE. Since higher-order differential equations are not as easily solved, it is desirable to obtain such a solution through other considerations. The following example sets the stage for this endeavor.

Example 15.1.1 A FOLDE has a unique solution, to within a multiplicative constant, given by Theorem 14.2.1. Thus, given a solution $w(z)$, any other solution must be of the form $Cw(z)$. Let z_0 be a singularity of $p(z)$, and let $z - z_0 = re^{i\theta}$. Start at a point z and circle z_0 so that $\theta \to \theta + 2\pi$. Even though $p(z)$ may have a simple pole at z_0, the solution may have a branch point there. This is clear from the general solution, where α may be a noninteger. Thus, $\tilde{w}(z) \equiv w(z_0 + re^{i(\theta+2\pi)})$ may be different from $w(z)$. To discover this branch point—without solving the DE—invoke Proposition 15.0.1 and conclude that $\tilde{w}(z)$ is also a solution to the FOLDE. Thus, $\tilde{w}(z)$ can be different from $w(z)$ by at most a multiplicative constant: $\tilde{w}(z) = Cw(z)$. Define the *complex* number α by $C = e^{2\pi i\alpha}$. Then the function $g(z) \equiv (z - z_0)^{-\alpha} w(z)$ is single-valued around z_0. In fact,

$$g\big(z_0 + re^{i(\theta+2\pi)}\big) = \big[re^{i(\theta+2\pi)}\big]^{-\alpha} w\big(z_0 + re^{i(\theta+2\pi)}\big)$$
$$= (z-z_0)^{-\alpha} e^{-2\pi i\alpha} e^{2\pi i\alpha} w(z) = (z-z_0)^{-\alpha} w(z) = g(z).$$

This argument shows that a solution $w(z)$ of the FOLDE of Eq. (15.1) can be written as $w(z) = (z-z_0)^{\alpha} g(z)$, where $g(z)$ is single-valued.

15.1.2 The Circuit Matrix

The method used in Example 15.1.1 can be generalized to obtain a similar result for the NOLDE

$$\mathsf{L}[w] = \frac{d^n w}{dz^n} + p_{n-1}(z)\frac{d^{n-1}w}{dz^{n-1}} + \cdots + p_1(z)\frac{dw}{dz} + p_0(z)w = 0 \quad (15.4)$$

where all the $p_i(z)$ are analytic in $r_1 < |z-z_0| < r_2$.

Let $\{w_j(z)\}_{j=1}^n$ be a basis of solutions of Eq. (15.4), and let $z - z_0 = re^{i\theta}$. Start at z and analytically continue the functions $w_j(z)$ one complete turn to $\theta + 2\pi$. Let $\tilde{w}_j(z) \equiv \tilde{w}_j(z_0 + re^{i\theta}) \equiv w_j(z_0 + re^{i(\theta+2\pi)})$. Then, by a generalization of Proposition 15.0.1, $\{\tilde{w}_j(z)\}_{j=1}^n$ are not only solutions, but they are linearly independent (because they are w_j's evaluated at a different point). Therefore, they also form a basis of solutions. On the other hand, $\tilde{w}_j(z)$ can be expressed as a linear combination of the $w_j(z)$. Thus,

$$\tilde{w}_j(z) = w_j\big(z_0 + re^{i(\theta+2\pi)}\big) = \sum_{k=1}^n a_{jk} w_k(z).$$

circuit matrix

The matrix $\mathsf{A} = (a_{jk})$, called the **circuit matrix** of the NOLDE, is invertible, because it transforms one basis into another. Therefore, it has only nonzero eigenvalues. We let λ be one such eigenvalue, and choose the column vector $\mathbf{C}$, with entries $\{c_i\}_{i=1}^n$, to be the corresponding eigenvector of the transpose of A (note that A and A^t, have the same set of eigenvalues). At least one such eigenvector always exists, because the characteristic polynomial of A^t has at least one root. Now we let $w(z) = \sum_{j=1}^n c_j w_j(z)$. Clearly, this $w(z)$ is a solution of (15.4), and

$$\tilde{w}(z) \equiv w\big(z_0 + re^{i(\theta+2\pi)}\big) = \sum_{j=1}^n c_j w_j\big(z_0 + re^{i(\theta+2\pi)}\big)$$
$$= \sum_{j=1}^n c_j \sum_{k=1}^n a_{jk} w_k(z) = \sum_{j,k} \big(\mathsf{A}^t\big)_{kj} c_j w_k(z) = \sum_{k=1}^n \lambda c_k w_k(z) = \lambda w(z).$$

If we define α by $\lambda = e^{2\pi i\alpha}$, then $w(z_0 + re^{i(\theta+2\pi)}) = e^{2\pi i\alpha} w(z)$. Now we write $f(z) \equiv (z-z_0)^{-\alpha} w(z)$. Following the argument used in Example 15.1.1, we get $f(z_0 + re^{i(\theta+2\pi)}) = f(z)$; that is, $f(z)$ is single-valued around z_0. We thus have the following theorem.

Theorem 15.1.2 *Any homogeneous NOLDE with analytic coefficient functions in $r_1 < |z - z_0| < r_2$ admits a solution of the form*

$$w(z) = (z - z_0)^\alpha f(z)$$

where $f(z)$ is single-valued around z_0 in $r_1 < |z - z_0| < r_2$.

An isolated singular point z_0 near which an analytic function $w(z)$ can be written as $w(z) = (z - z_0)^\alpha f(z)$, where $f(z)$ is single-valued and analytic in the punctured neighborhood of z_0, is called a **simple branch point** of $w(z)$. The arguments leading to Theorem 15.1.2 imply that a solution with a simple branch point exists if and only if the vector $\mathbf{C}$ whose components appear in $w(z)$ is an eigenvector of A^t, the transpose of the circuit matrix. Thus, there are as many solutions with simple branch points as there are linearly independent eigenvectors of A^t.

simple branch point

15.2 Complex SOLDEs

Let us now consider the SOLDE $w'' + p(z)w' + q(z)w = 0$. Given two linearly independent solutions $w_1(z)$ and $w_2(z)$, we form the 2×2 matrix A and try to diagonalize it. There are three possible outcomes:

1. The matrix A is diagonalizable, and we can find two eigenvectors, $F(z)$ and $G(z)$, corresponding, respectively, to two distinct eigenvalues, λ_1 and λ_2. This means that

$$F\big(z_0 + re^{i(\theta+2\pi)}\big) = \lambda_1 F(z), \quad \text{and} \quad G\big(z_0 + re^{i(\theta+2\pi)}\big) = \lambda_2 G(z).$$

   Defining $\lambda_1 = e^{2\pi i\alpha}$ and $\lambda_2 = e^{2\pi i\beta}$, we get

$$F(z) = (z - z_0)^\alpha f(z) \quad \text{and} \quad G(z) = (z - z_0)^\beta g(z),$$

   as Theorem 15.1.2 suggests. The set $\{F(z), G(z)\}$ is called a **canonical basis** of the SOLDE.

canonical basis of the SOLDE

2. The matrix A is diagonalizable, and the two eigenvalues are the same. In this case both $F(z)$ and $G(z)$ have the same constant α:

$$F(z) = (z - z_0)^\alpha f(z) \quad \text{and} \quad G(z) = (z - z_0)^\alpha g(z).$$

3. We cannot find two eigenvectors. This corresponds to the case where A is not diagonalizable. However, we can always find one eigenvector, so A has only one eigenvalue, λ. We let $w_1(z)$ be the solution of the form $(z - z_0)^\alpha f(z)$, where $f(z)$ is single-valued and $\lambda = e^{2\pi i\alpha}$. The existence of such a solution is guaranteed by Theorem 15.1.2. Let $w_2(z)$ be any other linearly independent solution (Theorem 14.3.5 ensures the existence of such a second solution). Then

$$w_2\big(z_0 + re^{i(\theta+2\pi)}\big) = aw_1(z) + bw_2(z),$$

and the circuit matrix will be $\mathsf{A} = \left(\begin{smallmatrix} \lambda & 0 \\ a & b \end{smallmatrix}\right)$, which has eigenvalues λ and b. Since A is assumed to have only one eigenvalue (otherwise we would have the first outcome again), we must have $b = \lambda$. This reduces A to $\mathsf{A} = \left(\begin{smallmatrix} \lambda & 0 \\ a & \lambda \end{smallmatrix}\right)$, where $a \neq 0$. The condition $a \neq 0$ is necessary to distinguish this case from the second outcome. Now we analytically continue $h(z) \equiv w_2(z)/w_1(z)$ one whole turn around z_0, obtaining

$$\begin{aligned} h\bigl(z_0 + re^{i(\theta+2\pi)}\bigr) &= \frac{w_2(z_0 + re^{i(\theta+2\pi)})}{w_1(z_0 + re^{i(\theta+2\pi)})} = \frac{aw_1(z) + \lambda w_2(z)}{\lambda w_1(z)} \\ &= \frac{a}{\lambda} + \frac{w_2(z)}{w_1(z)} = \frac{a}{\lambda} + h(z). \end{aligned}$$

It then follows that the function[1]

$$g_1(z) \equiv h(z) - \frac{a}{2\pi i \lambda} \ln(z - z_0)$$

is single-valued in $r_1 < |z - z_0| < r_2$. If we redefine $g_1(z)$ and $w_2(z)$ as $(2\pi i \lambda/a) g_1(z)$ and $(2\pi i \lambda/a) w_2(z)$, respectively, we have the following:

Theorem 15.2.1 *If $p(z)$ and $q(z)$ are analytic for $r_1 < |z - z_0| < r_2$, then the SOLDE $w'' + p(z)w' + q(z)w = 0$ admits a basis of solutions $\{w_1, w_2\}$ in the neighborhood of the singular point z_0, where either*

$$w_1(z) = (z - z_0)^\alpha f(z), \qquad w_2(z) = (z - z_0)^\beta g(z)$$

or, in exceptional cases (*when the circuit matrix is not diagonalizable*),

$$w_1(z) = (z - z_0)^\alpha f(z), \qquad w_2(z) = w_1(z)\bigl[g_1(z) + \ln(z - z_0)\bigr].$$

The functions $f(z)$, $g(z)$, and $g_1(z)$ are analytic and single-valued in the annular region.

This theorem allows us to factor out the *branch point* z_0 from the rest of the solutions. However, even though $f(z)$, $g(z)$, and $g_1(z)$ are analytic in the annular region $r_1 < |z - z_0| < r_2$, they may very well have poles of arbitrary orders at z_0. Can we also factor out the *poles*? In general, we cannot; however, under special circumstances, described in the following definition, we can.

regular singular point of a SOLDE defined

Definition 15.2.2 A SOLDE $w'' + p(z)w' + q(z)w = 0$ that is analytic in $0 < |z - z_0| < r$ is said to have a **regular singular point** at z_0 if $p(z)$ has at worst a simple pole and $q(z)$ has at worst a pole of order 2 there.

[1]Recall that $\ln(z - z_0)$ increases by $2\pi i$ for each turn around z_0.

In a neighborhood of a regular singular point z_0, the coefficient functions $p(z)$ and $q(z)$ have the power-series expansions

$$p(z) = \frac{a_{-1}}{z - z_0} + \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

$$q(z) = \frac{b_{-2}}{(z - z_0)^2} + \frac{b_{-1}}{z - z_0} + \sum_{k=0}^{\infty} b_k (z - z_0)^k.$$

Multiplying both sides of the first equation by $z - z_0$ and the second by $(z - z_0)^2$ and introducing $P(z) \equiv (z - z_0)p(z)$, $Q(z) \equiv (z - z_0)^2 q(z)$, we obtain

$$P(z) = \sum_{k=0}^{\infty} a_{k-1}(z - z_0)^k, \qquad Q(z) = \sum_{k=0}^{\infty} b_{k-2}(z - z_0)^k.$$

It is also convenient to multiply the SOLDE by $(z - z_0)^2$ and write it as

$$(z - z_0)^2 w'' + (z - z_0)P(z)w' + Q(z)w = 0. \tag{15.5}$$

Inspired by the discussion leading to Theorem 15.2.1, we write

$$w(z) = (z - z_0)^{\nu} \sum_{k=0}^{\infty} C_k (z - z_0)^k, \qquad C_0 = 1, \tag{15.6}$$

where we have chosen the arbitrary multiplicative constant in such a way that $C_0 = 1$. Substitute this in Eq. (15.5), and change the dummy variable—so that all sums start at 0—to obtain

$$\sum_{n=0}^{\infty} \left\{ (n + \nu)(n + \nu - 1)C_n + \sum_{k=0}^{n} \left[(k + \nu)a_{n-k-1} + b_{n-k-2}\right] C_k \right\} \times (z - z_0)^{n+\nu} = 0,$$

which results in the recursion relation

$$(n + \nu)(n + \nu - 1)C_n = -\sum_{k=0}^{n} \left[(k + \nu)a_{n-k-1} + b_{n-k-2}\right] C_k. \tag{15.7}$$

For $n = 0$, this leads to what is known as the **indicial equation** for the exponent ν:

$$I(\nu) \equiv \nu(\nu - 1) + a_{-1}\nu + b_{-2} = 0. \tag{15.8}$$

indicial equation, indicial polynomial, characteristic exponents

The roots of this equation are called the **characteristic exponents** of z_0, and $I(\nu)$ is called its **indicial polynomial**. In terms of this polynomial, (15.7) can be expressed as

$$I(n + \nu)C_n = -\sum_{k=0}^{n-1} \left[(k + \nu)a_{n-k-1} + b_{n-k-2}\right] C_k \quad \text{for } n = 1, 2, \ldots. \tag{15.9}$$

Equation (15.8) determines what values of ν are possible, and Eq. (15.9) gives $C_1, C_2, C_3, \ldots$, which in turn determine $w(z)$. Special care must be taken if the indicial polynomial vanishes at $n+\nu$ for some positive integer n, that is, if $n+\nu$, in addition to ν, is a root of the indicial polynomial: $I(n+\nu)=0=I(\nu)$.

If ν_1 and ν_2 are characteristic exponents of the indicial equation and $\mathrm{Re}(\nu_1) > \mathrm{Re}(\nu_2)$, then a solution for ν_1 always exists. A solution for ν_2 also exists if $\nu_1 - \nu_2 \neq n$ for any (positive) integer n. In particular, if z_0 is an ordinary point [a point at which both $p(z)$ and $q(z)$ are analytic], then only one solution is determined by (15.9). (Why?) The foregoing discussion is summarized in the following:

Theorem 15.2.3 *If the differential equation $w'' + p(z)w' + q(z)w = 0$ has a regular singular point at $z = z_0$, then at least one power series of the form of* (15.6) *formally solves the equation. If ν_1 and ν_2 are the characteristic exponents of z_0, then there are two linearly independent formal solutions unless $\nu_1 - \nu_2$ is an integer.*

Example 15.2.4 Let us consider some familiar differential equations.

(a) The Bessel equation is

$$w'' + \frac{1}{z}w' + \left(1 - \frac{\alpha^2}{z^2}\right)w = 0.$$

In this case, the origin is a regular singular point, $a_{-1} = 1$, and $b_{-2} = -\alpha^2$. Thus, the indicial equation is $\nu(\nu-1) + \nu - \alpha^2 = 0$, and its solutions are $\nu_1 = \alpha$ and $\nu_2 = -\alpha$. Therefore, there are two linearly independent solutions to the Bessel equation unless $\nu_1 - \nu_2 = 2\alpha$ is an integer, i.e., unless α is either an integer or a half-integer.

(b) For the Coulomb potential $f(r) = \beta/r$, the radial equation (13.14) reduces to

$$w'' + \frac{2}{z}w' + \left(\frac{\beta}{z} - \frac{\alpha}{z^2}\right)w = 0.$$

The point $z = 0$ is a regular singular point at which $a_{-1} = 2$ and $b_{-2} = -\alpha$. The indicial polynomial is $I(\nu) = \nu^2 + \nu - \alpha$ with characteristic exponents

$$\nu_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{1+4\alpha} \quad \text{and} \quad \nu_2 = -\frac{1}{2} - \frac{1}{2}\sqrt{1+4\alpha}.$$

There are two independent solutions unless $\nu_1 - \nu_2 = \sqrt{1+4\alpha}$ is an integer. In practice, $\alpha = l(l+1)$, where l is some integer; so $\nu_1 - \nu_2 = 2l + 1$, and only one solution is obtained.

hypergeometric differential equation

(c) The **hypergeometric** differential equation is

$$w'' + \frac{\gamma - (\alpha+\beta+1)z}{z(1-z)}w' - \frac{\alpha\beta}{z(1-z)}w = 0.$$

A substantial number of functions in mathematical physics are solutions of this remarkable equation, with appropriate values for α, β, and γ. The regular singular points[2] are $z=0$ and $z=1$. At $z=0$, $a_{-1}=\gamma$ and $b_{-2}=0$. The indicial polynomial is $I(\nu)=\nu(\nu+\gamma-1)$, whose roots are $\nu_1=0$ and $\nu_2=1-\gamma$. Unless γ is an integer, we have two formal solutions.

It is shown in differential equation theory [Birk 78, pp. 40–242] that as long as $\nu_1-\nu_2$ is not an integer, the series solution of Theorem 15.2.3 is convergent for a neighborhood of z_0. What happens when $\nu_1-\nu_2$ is an integer? First, as a convenience, we translate the coordinate axes so that the point z_0 coincides with the origin. This will save us some writing, because instead of powers of $z-z_0$, we will have powers of z. Next we let $\nu_1=\nu_2+n$ with n a *positive* integer. Then, since it is impossible to encounter any new zero of the indicial polynomial beyond ν_1, the recursion relation, Eq. (15.9), will be valid for all values of n, and we obtain a solution:

$$w_1(z)=z^{\nu_1}f(z)=z^{\nu_1}\left(1+\sum_{k=1}^{\infty}C_kz^k\right),$$

which is convergent in the region $0<|z|<r$ for some $r>0$.

To investigate the nature and the possibility of the second solution, write the recursion relations of Eq. (15.9) for the smaller characteristic root ν_2:

$$\begin{aligned}
&I(\nu_2+1)C_1=\overbrace{-(\nu_2a_0+b_{-1})}^{\equiv\rho_1I(\nu_2+1)}C_0 \quad\Rightarrow\quad C_1=\rho_1,\\
&I(\nu_2+2)C_2=-(\nu_2a_1+b_0)C_0-\bigl[(\nu_2+1)a_0+b_{-1}\bigr]C_1 \quad\Rightarrow\quad C_2\equiv\rho_2,\\
&\quad\vdots\\
&I(\nu_2+n-1)C_{n-1}\equiv\rho_{n-1}I(\nu_2+n-1)C_0 \quad\Rightarrow\quad C_{n-1}=\rho_{n-1},\\
&I(\nu_2+n)C_n=I(\nu_1)C_n=\rho_nC_0 \quad\Rightarrow\quad 0=\rho_n,
\end{aligned}\tag{15.10}$$

where in each step, we have used the result of the previous step in which C_k is given as a multiple of $C_0=1$. Here, the ρ's are constants depending (possibly in a very complicated way) on the a_k's and b_k's.

Theorem 15.2.3 guarantees two power series solutions only when $\nu_1-\nu_2$ is not an integer. When $\nu_1-\nu_2$ is an integer, Eq. (15.10) shows that a necessary condition for a second *power series* solution to exist is that $\rho_n=0$. Therefore, when $\rho_n\neq 0$, we have to resort to other means of obtaining the second solution.

Let us define the second solution as

$$w_2(z)\equiv w_1(z)h(z)=\overbrace{z^{\nu_1}f(z)}^{\equiv w_1(z)}h(z)\tag{15.11}$$

[2]The coefficient of w need not have a pole of order 2. Its pole can be of order one as well.

and substitute in the SOLDE to obtain a FOLDE in h', namely,

$$h'' + \bigl(p + 2w_1'/w_1\bigr)h' = 0,$$

or, by substituting $w_1'/w_1 = \nu_1/z + f'/f$, the equivalent FOLDE

$$h'' + \left(\frac{2\nu_1}{z} + \frac{2f'}{f} + p\right)h' = 0. \tag{15.12}$$

Lemma 15.2.5 *The coefficient of h' in Eq. (15.12) has a residue of $n+1$.*

Proof Recall that the residue of a function is the coefficient of z^{-1} in the Laurent expansion of the function (about $z=0$). Let us denote this residue for the coefficient of h' by A_{-1}. Since $f(0)=1$, the ratio f'/f is analytic at $z=0$. Thus, the simple pole at $z=0$ comes from the other two terms. Substituting the Laurent expansion of $p(z)$ gives

$$\frac{2\nu_1}{z} + p = \frac{2\nu_1}{z} + \frac{a_{-1}}{z} + a_0 + a_1 z + \cdots .$$

This shows that $A_{-1} = 2\nu_1 + a_{-1}$. On the other hand, comparing the two versions of the indicial polynomial

$$\nu^2 + (a_{-1} - 1)\nu + b_{-2} \quad \text{and} \quad (\nu - \nu_1)(\nu - \nu_2) = \nu^2 - (\nu_1 + \nu_2)\nu + \nu_1\nu_2$$

gives

$$\nu_1 + \nu_2 = -(a_{-1} - 1), \quad \text{or} \quad 2\nu_1 - n = -(a_{-1} - 1).$$

Therefore, $A_{-1} = 2\nu_1 + a_{-1} = n + 1$. □

Theorem 15.2.6 *Suppose that the characteristic exponents of a SOLDE with a regular singular point at $z=0$ are ν_1 and ν_2. Consider three cases:*

1. *$\nu_1 - \nu_2$ is not an integer.*
2. *$\nu_2 = \nu_1 - n$ where n is a nonnegative integer, and ρ_n, as defined in Eq. (15.10), vanishes.*
3. *$\nu_2 = \nu_1 - n$ where n is a nonnegative integer, and ρ_n, as defined in Eq. (15.10), does not vanish.*

Then, in the first two cases, there exists a basis of solutions $\{w_1, w_2\}$ of the form

$$w_i(z) = z^{\nu_i}\left(1 + \sum_{k=1}^{\infty} C_k^{(i)} z^k\right), \quad i = 1, 2,$$

and in the third case, the basis of solutions takes the form

$$w_1(z) = z^{\nu_1}\left(1 + \sum_{k=1}^{\infty} a_k z^k\right), \qquad w_2(z) = z^{\nu_2}\left(1 + \sum_{k=1}^{\infty} b_k z^k\right) + C w_1(z)\ln z,$$

where the power series are convergent in a neighborhood of $z=0$.

Proof The first two cases have been shown before. For the third case, we use Lemma 15.2.5 and write

$$\frac{2\nu_1}{z} + \frac{2f'}{f} + p = \frac{n+1}{z} + \sum_{k=0}^{\infty} c_k z^k,$$

and the solution for the FOLDE in h' will be [see Eq. (15.3) and the discussion preceding it]

$$h'(z) = z^{-n-1}\left(1 + \sum_{k=1}^{\infty} b_k z^k\right).$$

For $n = 0$, i.e., when the indicial polynomial has a double root, this yields

$$h'(z) = 1/z + \sum_{k=1}^{\infty} b_k z^{k-1} \quad \Rightarrow \quad h(z) = \ln z + g_1(z),$$

where g_1 is analytic in a neighborhood of $z = 0$. For $n \neq 0$, we have $h'(z) = b_n/z + \sum_{k\neq n}^{\infty} b_k z^{k-n-1}$ and, by integration,

$$\begin{aligned} h(z) &= b_n \ln z + \sum_{k\neq n}^{\infty} \frac{b_k}{k-n} z^{k-n} \\ &= b_n \ln z + z^{-n} \sum_{k\neq n}^{\infty} \frac{b_k}{k-n} z^k = b_n \ln z + z^{-n} g_2(z), \end{aligned}$$

where g_2 is analytic in a neighborhood of $z = 0$. Substituting h in Eq. (15.11) and recalling that $\nu_2 = \nu_1 - n$, we obtain the desired results of the theorem. □

15.3 Fuchsian Differential Equations

In many cases of physical interest, the behavior of the solution of a SOLDE at infinity is important. For instance, bound state solutions of the Schrödinger equation describing the probability amplitudes of particles in quantum mechanics must tend to zero as the distance from the center of the binding force increases.

We have seen that the behavior of a solution is determined by the behavior of the coefficient functions. To determine the behavior at infinity, we substitute $z = 1/t$ in the SOLDE

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0 \tag{15.13}$$

and obtain

$$\frac{d^2v}{dt^2} + \left[\frac{2}{t} - \frac{1}{t^2}r(t)\right]\frac{dv}{dt} + \frac{1}{t^4}s(t)v = 0, \tag{15.14}$$

where $v(t) = w(1/t)$, $r(t) = p(1/t)$, and $s(t) = q(1/t)$.

Clearly, as $z \to \infty$, $t \to 0$. Thus, we are interested in the behavior of (15.14) at $t = 0$. We assume that both $r(t)$ and $s(t)$ are analytic at $t = 0$. Equation (15.14) shows, however, that the solution $v(t)$ may still have singularities at $t = 0$ because of the extra terms appearing in the coefficient functions.

We assume that infinity is a regular singular point of (15.13), by which we mean that $t = 0$ is a regular singular point of (15.14). Therefore, in the Taylor expansions of $r(t)$ and $s(t)$, the first (constant) term of $r(t)$ and the first two terms of $s(t)$ must be zero. Thus, we write

$$r(t) = a_1 t + a_2 t^2 + \cdots = \sum_{k=1}^{\infty} a_k t^k,$$

$$s(t) = b_2 t^2 + b_3 t^3 + \cdots = \sum_{k=2}^{\infty} b_k t^k.$$

By their definitions, these two equations imply that for $p(z)$ and $q(z)$, and for large values of $|z|$, we must have expressions of the form

$$\begin{aligned} p(z) &= \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots = \sum_{k=1}^{\infty} \frac{a_k}{z^k}, \\ q(z) &= \frac{b_2}{z^2} + \frac{b_3}{z^3} + \cdots = \sum_{k=2}^{\infty} \frac{b_k}{z^k}. \end{aligned} \tag{15.15}$$

When infinity is a regular singular point of Eq. (15.13), or, equivalently, when the origin is a regular singular point of (15.14), it follows from Theorem 15.2.6 that there exists at least one solution of the form $v_1(t) = t^{\alpha}(1 + \sum_{k=1}^{\infty} C_k t^k)$ or, in terms of z,

$$w_1(z) = z^{-\alpha}\left(1 + \sum_{k=1}^{\infty} \frac{C_k}{z^k}\right). \tag{15.16}$$

Here α is a characteristic exponents at $t = 0$ of (15.14), whose indicial polynomial is easily found to be $\alpha(\alpha - 1) + (2 - a_1)\alpha + b_2 = 0$.

Fuchsian DE (or FDE)

Definition 15.3.1 A homogeneous differential equation with single-valued analytic coefficient functions is called a **Fuchsian** differential equation (FDE) if it has *only* regular singular points in the extended complex plane, i.e., the complex plane including the point at infinity.

It turns out that a particular kind of FDE describes a large class of nonelementary functions encountered in mathematical physics. Therefore, it is instructive to classify various kinds of FDEs. A fact that is used in such a classification is that complex functions whose only singularities in the *extended* complex plane are poles are rational functions, i.e., ratios of polynomials (see Proposition 11.2.2). We thus expect FDEs to have only rational functions as coefficients.

Consider the case where the equation has at most two regular singular points at z_1 and z_2. We introduce a new variable $\xi(z) = \frac{z-z_1}{z-z_2}$. The regular singular points at z_1 and z_2 are mapped onto the points $\xi_1 = \xi(z_1) = 0$ and $\xi_2 = \xi(z_2) = \infty$, respectively, in the extended ξ-plane. Equation (15.13) becomes

A second-order Fuchsian DE with two regular singular points leads to uninteresting solutions!

$$\frac{d^2u}{d\xi^2} + \Phi(\xi)\frac{du}{d\xi} + \Theta(\xi)u = 0, \qquad (15.17)$$

where u, Φ, and Θ are functions of ξ obtained when z is expressed in terms of ξ in $w(z)$, $p(z)$, and $q(z)$, respectively. From Eq. (15.15) and the fact that $\xi = 0$ is at most a simple pole of $\Phi(\xi)$, we obtain $\Phi(\xi) = a_1/\xi$. Similarly, $\Theta(\xi) = b_2/\xi^2$. Thus, a SOFDE with two regular singular points is equivalent to the DE $w'' + (a_1/z)w' + (b_2/z^2)w = 0$. Multiplying both sides by z^2, we obtain $z^2w'' + a_1zw' + b_2w = 0$, which is the second-order Euler differential equation. A general nth-order Euler differential equation is equivalent to a NOLDE with constant coefficients (see Problem 14.30). Thus, a second order Fuchsian DE (SOFDE) with two regular singular points is equivalent to a SOLDE with constant coefficients and produces nothing new.

The simplest SOFDE whose solutions may include nonelementary functions is therefore one having three regular singular points, at say z_1, z_2, and z_3. By the transformation

A second-order Fuchsian DE with three regular singular points leads to interesting solutions!

$$\xi(z) = \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}$$

we can map z_1, z_2, and z_3 onto $\xi_1 = 0$, $\xi_2 = \infty$, and $\xi_3 = 1$. Thus, we assume that the three regular singular points are at $z = 0$, $z = 1$, and $z = \infty$. It can be shown [see Problem (15.8)] that the most general $p(z)$ and $q(z)$ are

$$p(z) = \frac{A_1}{z} + \frac{B_1}{z-1} \quad \text{and} \quad q(z) = \frac{A_2}{z^2} + \frac{B_2}{(z-1)^2} - \frac{A_3}{z(z-1)}.$$

We thus have the following:

Theorem 15.3.2 *The most general second order Fuchsian DE with three regular singular points can be transformed into the form*

Riemann differential equation

$$w'' + \left(\frac{A_1}{z} + \frac{B_1}{z-1}\right)w' + \left[\frac{A_2}{z^2} + \frac{B_2}{(z-1)^2} - \frac{A_3}{z(z-1)}\right]w = 0, \qquad (15.18)$$

where A_1, A_2, A_3, B_1, and B_2 are constants. This equation is called the ***Riemann differential equation***.

We can write the Riemann DE in terms of pairs of characteristic exponents, (λ_1, λ_2), (μ_1, μ_2), and (ν_1, ν_2), belonging to the singular points 0, 1, and ∞, respectively. The indicial equations are easily found to be

$$\lambda^2 + (A_1 - 1)\lambda + A_2 = 0,$$

$$\mu^2 + (B_1 - 1)\mu + B_2 = 0,$$

$$\nu^2 + (1 - A_1 - B_1)\nu + A_2 + B_2 - A_3 = 0.$$

By writing the indicial equations as $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$, and so forth and comparing coefficients, we can find the following relations:

$$\begin{aligned} A_1 &= 1 - \lambda_1 - \lambda_2, & A_2 &= \lambda_1 \lambda_2, \\ B_1 &= 1 - \mu_1 - \mu_2, & B_2 &= \mu_1 \mu_2, \\ A_1 + B_1 &= \nu_1 + \nu_2 + 1, & A_2 + B_2 - A_3 &= \nu_1 \nu_2. \end{aligned}$$

Riemann identity

These equations lead easily to the **Riemann identity**

$$\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \nu_1 + \nu_2 = 1. \tag{15.19}$$

Substituting these results in (15.18) gives the following result.

Theorem 15.3.3 *A second order Fuchsian DE with three regular singular points in the extended complex plane is equivalent to the Riemann DE,*

$$\begin{aligned} & w'' + \left(\frac{1 - \lambda_1 - \lambda_2}{z} + \frac{1 - \mu_1 - \mu_2}{z - 1} \right) w' \\ & \quad + \left[\frac{\lambda_1 \lambda_2}{z^2} + \frac{\mu_1 \mu_2}{(z-1)^2} + \frac{\nu_1 \nu_2 - \lambda_1 \lambda_2 - \mu_1 \mu_2}{z(z-1)} \right] w = 0, \end{aligned} \tag{15.20}$$

which is uniquely *determined by the pairs of characteristic exponents at each singular point. The characteristic exponents satisfy the Riemann identity, Eq.* (15.19).

The uniqueness of the Riemann DE allows us to derive identities for solutions and reduce the independent parameters of Eq. (15.20) from five to three. We first note that if $w(z)$ is a solution of the Riemann DE corresponding to (λ_1, λ_2), (μ_1, μ_2), and (ν_1, ν_2), then the function

$$v(z) = z^{\lambda}(z-1)^{\mu} w(z)$$

has branch points at $z = 0, 1, \infty$ [because $w(z)$ does]; therefore, it is a solution of the Riemann DE. Its pairs of characteristic exponents are (see Problem 15.10)

$$(\lambda_1 + \lambda, \lambda_2 + \lambda), \quad (\mu_1 + \mu, \mu_2 + \mu), \quad (\nu_1 - \lambda - \mu, \nu_2 - \lambda - \mu).$$

In particular, if we let $\lambda = -\lambda_1$ and $\mu = -\mu_1$, then the pairs reduce to

$$(0, \lambda_2 - \lambda_1), \quad (0, \mu_2 - \mu_1), \quad (\nu_1 + \lambda_1 + \mu_1, \nu_2 + \lambda_1 + \mu_1).$$

Defining $\alpha \equiv \nu_1 + \lambda_1 + \mu_1$, $\beta \equiv \nu_2 + \lambda_1 + \mu_1$, and $\gamma \equiv 1 - \lambda_2 + \lambda_1$, and using (15.19), we can write the pairs as

$$(0, 1 - \gamma), \quad (0, \gamma - \alpha - \beta), \quad (\alpha, \beta),$$

which yield the third version of the Riemann DE

$$w'' + \left(\frac{\gamma}{z} + \frac{1 - \gamma + \alpha + \beta}{z - 1} \right) w' + \frac{\alpha\beta}{z(z-1)} w = 0.$$

This important equation is commonly written in the equivalent form

$$z(1-z)w'' + \big[\gamma - (1+\alpha+\beta)z\big]w' - \alpha\beta w = 0 \tag{15.21}$$

and is called the **hypergeometric** differential equation (HGDE). We will study this equation next.

hypergeometric DE

15.4 The Hypergeometric Function

The two characteristic exponents of Eq. (15.21) at $z=0$ are 0 and $1-\gamma$. It follows from Theorem 15.2.6 that there exists an *analytic* solution (corresponding to the characteristic exponent 0) at $z=0$. Let us denote this solution, the **hypergeometric function**, by $F(\alpha,\beta;\gamma;z)$ and write

$$F(\alpha,\beta;\gamma;z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{where } a_0 = 1.$$

Substituting in the DE, we obtain the recurrence relation

$$a_{k+1} = \frac{(\alpha+k)(\beta+k)}{(k+1)(\gamma+k)} a_k \quad \text{for} \quad k \ge 0.$$

These coefficients can be determined successively if γ is neither zero nor a negative integer:

hypergeometric series

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(k+1)\Gamma(\gamma+k)} z^k. \tag{15.22}$$

The series in (15.22) is called the **hypergeometric series**, because it is the generalization of $F(1,\beta;\beta;z)$, which is simply the geometric series.

We note immediately from (15.22) that

Box 15.4.1 *The hypergeometric series becomes a polynomial if either α or β is a negative integer.*

This is because for $k < |\alpha|$ (or $k < |\beta|$) both $\Gamma(\alpha+k)$ [or $\Gamma(\beta+k)$] and $\Gamma(\alpha)$ [or $\Gamma(\beta)$] have poles that cancel each other. However, $\Gamma(\alpha+k)$ [or $\Gamma(\beta+k)$] becomes finite for $k > |\alpha|$ (or $k > |\beta|$), and the pole in $\Gamma(\alpha)$ [or $\Gamma(\beta)$] makes the denominator infinite. Therefore, all terms of the series (15.22) beyond $k = |\alpha|$ (or $k = |\beta|$) will be zero.

Many of the properties of the hypergeometric function can be obtained directly from the HGDE (15.21). For instance, differentiating the HGDE and letting $v = w'$, we obtain

$$z(1-z)v'' + \big[\gamma + 1 - (\alpha+\beta+3)z\big]v' - (\alpha+1)(\beta+1)v = 0,$$

which shows that $F'(\alpha, \beta; \gamma; z) = CF(\alpha+1, \beta+1; \gamma+1; z)$. The constant C can be determined by differentiating Eq. (15.22), setting $z = 0$ in the result, and noting that $F(\alpha+1, \beta+1; \gamma+1; 0) = 1$. Then we obtain

$$F'(\alpha, \beta; \gamma; z) = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1; \gamma+1; z). \tag{15.23}$$

Now assume that $\gamma \neq 1$, and make the substitution $w = z^{1-\gamma}u$ in the HGDE to obtain[3]

$$z(1-z)u'' + \big[\gamma_1 - (\alpha_1 + \beta_1 + 1)z\big]u' - \alpha_1\beta_1 u = 0,$$

where $\alpha_1 = \alpha - \gamma + 1$, $\beta_1 = \beta - \gamma + 1$, and $\gamma_1 = 2 - \gamma$. Thus,

$$u = F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z),$$

and u is therefore analytic at $z = 0$. This leads to an interesting result. Provided that γ is not an integer, the two functions

$$w_1(z) \equiv F(\alpha, \beta; \gamma; z), \qquad w_2(z) \equiv z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; z) \tag{15.24}$$

form a canonical basis of solutions to the HGDE at $z = 0$. This follows from Theorem 15.2.6 and the fact that $(0, 1-\gamma)$ are a pair of (different) characteristic exponents at $z = 0$.

Johann Carl Friedrich Gauss 1777–1855

Historical Notes

Johann Carl Friedrich Gauss (1777–1855) was the greatest of all mathematicians and perhaps the most richly gifted genius of whom there is any record. He was born in the city of Brunswick in northern Germany. His exceptional skill with numbers was clear at a very early age, and in later life he joked that he knew how to count before he could talk. It is said that Goethe wrote and directed little plays for a puppet theater when he was 6 and that Mozart composed his first childish minuets when he was 5, but Gauss corrected an error in his father's payroll accounts at the age of 3. At the age of seven, when he started elementary school, his teacher was amazed when Gauss summed the integers from 1 to 100 instantly by spotting that the sum was 50 pairs of numbers each pair summing to 101. His long professional life is so filled with accomplishments that it is impossible to give a full account of them in the short space available here. All we can do is simply give a chronology of his almost uncountable discoveries.

1792–1794: Gauss reads the works of Newton, Euler, and Lagrange; discovers the prime number theorem (at the age of 14 or 15); invents the method of least squares; conceives the Gaussian law of distribution in the theory of probability.

1795: (only 18 years old!) Proves that a regular polygon with n sides is constructible (by ruler and compass) if and only if n is the product of a power of 2 and distinct prime numbers of the form $p_k = 2^{2^k} + 1$, and completely solves the 2000-year old problem of ruler-and-compass construction of regular polygons. He also discovers the law of quadratic reciprocity.

1799: Proves the **fundamental theorem of algebra** in his doctoral dissertation using the then-mysterious complex numbers with complete confidence.

1801: Gauss publishes his *Disquisitiones Arithmeticae* in which he creates the modern rigorous approach to mathematics; predicts the exact location of the asteroid Ceres.

[3] In the following discussion, α_1, β_1, and γ_1 will represent the parameters of the new DE satisfied by the new function defined in terms of the old.

1807: Becomes professor of astronomy and the director of the new observatory at Göttingen.

1809: Publishes his second book, *Theoria motus corporum coelestium*, a major two-volume treatise on the motion of celestial bodies and the bible of planetary astronomers for the next 100 years.

1812: Publishes *Disquisitiones generales circa seriem infinitam*, a rigorous treatment of infinite series, and introduces the **hypergeometric function** for the first time, for which he uses the notation $F(\alpha, \beta; \gamma; z)$; an essay on approximate integration.

1820–1830: Publishes over 70 papers, including *Disquisitiones generales circa superficies curvas*, in which he creates the intrinsic **differential geometry** of general curved surfaces, the forerunner of Riemannian geometry and the general theory of relativity. From the 1830s on, Gauss was increasingly occupied with physics, and he enriched every branch of the subject he touched. In the theory of **surface tension**, he developed the fundamental idea of conservation of energy and solved the earliest problem in the **calculus of variations**. In **optics**, he introduced the concept of the focal length of a system of lenses. He virtually created the science of **geomagnetism**, and in collaboration with his friend and colleague Wilhelm Weber he invented the electromagnetic telegraph. In 1839 Gauss published his fundamental paper on the general theory of inverse square forces, which established **potential theory** as a coherent branch of mathematics and in which he established the **divergence theorem**.

Gauss had many opportunities to leave Göttingen, but he refused all offers and remained there for the rest of his life, living quietly and simply, traveling rarely, and working with immense energy on a wide variety of problems in mathematics and its applications. Apart from science and his family—he married twice and had six children, two of whom emigrated to America—his main interests were history and world literature, international politics, and public finance. He owned a large library of about 6000 volumes in many languages, including Greek, Latin, English, French, Russian, Danish, and of course German. His acuteness in handling his own financial affairs is shown by the fact that although he started with virtually nothing, he left an estate over a hundred times as great as his average annual income during the last half of his life.

The foregoing list is the published portion of Gauss's total achievement; the unpublished and private part is almost equally impressive. His scientific diary, a little booklet of 19 pages, discovered in 1898, extends from 1796 to 1814 and consists of 146 very concise statements of the results of his investigations, which often occupied him for weeks or months. These ideas were so abundant and so frequent that he physically did not have time to publish them. Some of the ideas recorded in this diary:

1. **Cauchy Integral Formula**: Gauss discovers it in 1811, 16 years before Cauchy.
2. **Non-Euclidean Geometry**: After failing to prove Euclid's fifth postulate at the age of 15, Gauss came to the conclusion that the Euclidean form of geometry cannot be the only one possible.
3. **Elliptic Functions**: Gauss had found many of the results of Abel and Jacobi (the two main contributors to the subject) before these men were born. The facts became known partly through Jacobi himself. His attention was caught by a cryptic passage in the *Disquisitiones*, whose meaning can only be understood if one knows something about elliptic functions. He visited Gauss on several occasions to verify his suspicions and tell him about his own most recent discoveries, and each time Gauss pulled 30-year-old manuscripts out of his desk and showed Jacobi what Jacobi had just shown him. After a week's visit with Gauss in 1840, Jacobi wrote to his brother, "Mathematics would be in a very different position if practical astronomy had not diverted this colossal genius from his glorious career."

A possible explanation for not publishing such important ideas is suggested by his comments in a letter to Bolyai: "It is not knowledge but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it in order to go into darkness again." His was the temperament of an explorer who is reluctant to take the time to write an account of his last expedition when he could be starting another. As it was, Gauss wrote a great deal, but to have published every fundamental discovery he made in a form satisfactory to himself would have required several long lifetimes.

A third relation can be obtained by making the substitution $w=(1-z)^{\gamma-\alpha-\beta}u$. This leads to a hypergeometric equation for u with $\alpha_1=\gamma-\alpha$, $\beta_1=\gamma-\beta$, and $\gamma_1=\gamma$. Furthermore, w is analytic at $z=0$, and $w(0)=1$. We conclude that $w=F(\alpha,\beta;\gamma;z)$. We therefore have the identity

$$F(\alpha,\beta;\gamma;z)=(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma;z). \tag{15.25}$$

To obtain the canonical basis at $z=1$, we make the substitution $t=1-z$, and note that the result is again the HGDE, with $\alpha_1=\alpha$, $\beta_1=\beta$, and $\gamma_1=\alpha+\beta-\gamma+1$. It follows from Eq. (15.24) that

$$\begin{aligned}
w_3(z)&\equiv F(\alpha,\beta;\alpha+\beta-\gamma+1;1-z),\\
w_4(z)&\equiv(1-z)^{\gamma-\alpha-\beta}F(\gamma-\beta,\gamma-\alpha;\gamma-\alpha-\beta+1;1-z)
\end{aligned} \tag{15.26}$$

form a canonical basis of solutions to the HGDE at $z=1$.

A symmetry of the hypergeometric function that is easily obtained from the HGDE is

$$F(\alpha,\beta;\gamma;z)=F(\beta,\alpha;\gamma;z). \tag{15.27}$$

The six functions

$$F(\alpha\pm 1,\beta;\gamma;z),\qquad F(\alpha,\beta\pm 1;\gamma;z),\qquad F(\alpha,\beta;\gamma\pm 1;z)$$

are called hypergeometric functions **contiguous to** $F(\alpha,\beta;\gamma;z)$. The discussion above showed how to obtain the basis of solutions at $z=1$ from the regular solution to the HDE $z=0$, $F(\alpha,\beta;\gamma;z)$. We can show that the basis of solutions at $z=\infty$ can also be obtained from the hypergeometric function.

Equation (15.16) suggests a function of the form

$$v(z)=z^rF\left(\alpha_1,\beta_1;\gamma_1;\frac{1}{z}\right)\equiv z^rw\left(\frac{1}{z}\right)\quad\Rightarrow\quad w(z)=z^rv\left(\frac{1}{z}\right), \tag{15.28}$$

where r, α_1, β_1, and γ_1 are to be determined. Since $w(z)$ is a solution of the HGDE, v will satisfy the following DE (see Problem 15.15):

$$\begin{aligned}
&z(1-z)v''+\bigl[1-\alpha-\beta-2r-(2-\gamma-2r)z\bigr]v'\\
&\quad-\left[r^2-r+r\gamma-\frac{1}{z}(r+\alpha)(r+\beta)\right]v=0.
\end{aligned} \tag{15.29}$$

This reduces to the HGDE if $r=-\alpha$ or $r=-\beta$. For $r=-\alpha$, the parameters become $\alpha_1=\alpha$, $\beta_1=1+\alpha-\gamma$, and $\gamma_1=\alpha-\beta+1$. For $r=-\beta$, the parameters are $\alpha_1=\beta$, $\beta_1=1+\beta-\gamma$, and $\gamma_1=\beta-\alpha+1$. Thus,

$$\begin{aligned}
v_1(z)&=z^{-\alpha}F\left(\alpha,1+\alpha-\gamma;\alpha-\beta+1;\frac{1}{z}\right),\\
v_2(z)&=z^{-\beta}F\left(\beta,1+\beta-\gamma;\beta-\alpha+1;\frac{1}{z}\right)
\end{aligned} \tag{15.30}$$

form a canonical basis of solutions for the HGDE that are valid about $z = \infty$.

As the preceding discussion suggests, it is possible to obtain many relations among the hypergeometric functions with different parameters and independent variables. In fact, the nineteenth-century mathematician Kummer showed that there are 24 different (but linearly dependent, of course) solutions to the HGDE. These are collectively known as **Kummer's solutions**, and six of them were derived above. Another important relation (shown in Problem 15.16) is that

Kummer's solutions

$$z^{\alpha-\gamma}(1-z)^{\gamma-\alpha-\beta} F\left(\gamma-\alpha, 1-\alpha; 1-\alpha+\beta; \frac{1}{z}\right) \tag{15.31}$$

also solves the HGDE.

Many of the functions that occur in mathematical physics are related to the hypergeometric function. Even some of the common elementary functions can be expressed in terms of the hypergeometric function with appropriate parameters. For example, when $\beta = \gamma$, we obtain

$$F(\alpha, \beta; \beta; z) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(k+1)} z^k = (1-z)^{-\alpha}.$$

Similarly,

$$F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) = \frac{\sin^{-1} z}{z}, \quad \text{and} \quad F(1, 1; 2; -z) = \frac{\ln(1+z)}{z}.$$

However, the real power of the hypergeometric function is that it encompasses almost all of the nonelementary functions encountered in physics. Let us look briefly at a few of these.

Jacobi functions are solutions of the DE

Jacobi functions

$$\begin{aligned}(1-x^2)\frac{d^2u}{dx^2} &+ [\beta-\alpha-(\alpha+\beta+2)x]\frac{du}{dx} \\ &+ \lambda(\lambda+\alpha+\beta+1)u = 0.\end{aligned} \tag{15.32}$$

Defining $x = 1-2z$ changes this DE into the HGDE with parameters $\alpha_1 = \lambda$, $\beta_1 = \lambda+\alpha+\beta+1$, and $\gamma_1 = 1+\alpha$. The solutions of Eq. (15.32), called the **Jacobi functions of the first kind**, are, with appropriate normalization,

$$P_\lambda^{(\alpha,\beta)}(z) = \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)\Gamma(\alpha+1)} F\left(-\lambda, \lambda+\alpha+\beta+1; 1+\alpha; \frac{1-z}{2}\right).$$

When $\lambda = n$, a nonnegative integer, the Jacobi function turns into a polynomial of degree n with the following expansion:

$$\begin{aligned}P_n^{(\alpha,\beta)}(z) &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} \\ &\quad \times \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(\alpha+k+1)} \left(\frac{z-1}{2}\right)^k.\end{aligned}$$

These are the Jacobi polynomials discussed in Chap. 8. In fact, the DE satisfied by $P_n^{(\alpha,\beta)}(x)$ of Chap. 8 is identical to Eq. (15.32). Note that the transformation $x = 1 - 2z$ translates the points $z = 0$ and $z = 1$ to the points $x = 1$ and $x = -1$, respectively. Thus the regular singular points of the Jacobi functions of the first kind are at ± 1 and ∞.

A second, linearly independent, solution of Eq. (15.32) is obtained by using (15.31). These are called the **Jacobi functions of the second kind**:

$$Q_\lambda^{(\alpha,\beta)}(z) = \frac{2^{\lambda+\alpha+\beta}\Gamma(\lambda+\alpha+1)\Gamma(\lambda+\beta+1)}{\Gamma(2\lambda+\alpha+\beta+2)(z-1)^{\lambda+\alpha+1}(z+1)^\beta} \times F\left(\lambda+\alpha+1, \lambda+1; 2\lambda+\alpha+\beta+2; \frac{2}{1-z}\right). \tag{15.33}$$

Gegenbauer functions

Gegenbauer functions, or ultraspherical functions, are special cases of Jacobi functions for which $\alpha = \beta = \mu - \frac{1}{2}$. They are defined by

$$C_\lambda^\mu(z) = \frac{\Gamma(\lambda+2\mu)}{\Gamma(\lambda+1)\Gamma(2\mu)} F\left(-\lambda, \lambda+2\mu; \mu+\frac{1}{2}; \frac{1-z}{2}\right). \tag{15.34}$$

Note the change in the normalization constant. Linearly independent Gegenbauer functions "of the second kind" can be obtained from the Jacobi functions of the second kind by the substitution $\alpha = \beta = \mu - \frac{1}{2}$.

Another special case of the Jacobi functions is obtained when $\alpha = \beta = 0$. Those obtained from the Jacobi functions of the first kind are called **Legendre functions of the first kind**:

Legendre functions

$$P_\lambda(z) \equiv P_\lambda^{(0,0)}(z) = C_\lambda^{1/2} = F\left(-\lambda, \lambda+1; 1; \frac{1-z}{2}\right). \tag{15.35}$$

Legendre functions of the second kind are obtained from the Jacobi functions of the second kind in a similar way:

$$Q_\lambda(z) = \frac{2^\lambda \Gamma^2(\lambda+1)}{\Gamma(2\lambda+2)(z-1)^{\lambda+1}} F\left(\lambda+1, \lambda+1; 2\lambda+2; \frac{2}{1-z}\right).$$

Other functions derived from the Jacobi functions are obtained similarly (see Chap. 8).

15.5 Confluent Hypergeometric Functions

The transformation $x = 1 - 2z$ translates the regular singular points of the HGDE by a finite amount. Consequently, the new functions still have two regular singular points, $z = \pm 1$, in the complex plane. In some physical cases of importance, only the origin, corresponding to $r = 0$ in spherical coordinates (typically the location of the source of a central force), is the singular point. If we want to obtain a differential equation consistent with such a case, we have to "push" the singular point $z = 1$ to infinity. This can be achieved by making the substitution $t = rz$ in the HGDE and taking the limit $r \to \infty$. The substitution yields

$$\frac{d^2w}{dt^2} + \left(\frac{\gamma}{t} + \frac{1-\gamma+\alpha+\beta}{t-r}\right)\frac{dw}{dt} + \frac{\alpha\beta}{t(t-r)}w = 0. \qquad (15.36)$$

If we blindly take the limit $r \to \infty$ with α, β, and γ remaining finite, Eq. (15.36) reduces to $\ddot{w} + (\gamma/t)\dot{w} = 0$, an elementary FODE in $\dot{w}$. To obtain a nonelementary DE, we need to manipulate the parameters, to let some of them tend to infinity. We want γ to remain finite, because otherwise the coefficient of dw/dt will blow up. We therefore let β or α tend to infinity. The result will be the same either way because α and β appear symmetrically in the equation. It is customary to let $\beta = r \to \infty$. In that case, Eq. (15.36) becomes

$$\frac{d^2w}{dt^2} + \left(\frac{\gamma}{t} - 1\right)\frac{dw}{dt} - \frac{\alpha}{t}w = 0.$$

Multiplying by t and changing the independent variable back to z yields

confluent hypergeometric DE

$$zw''(z) + (\gamma - z)w'(z) - \alpha w(z) = 0. \qquad (15.37)$$

This is called the **confluent hypergeometric** DE (CHGDE).

Since $z = 0$ is still a regular singular point of the CHGDE, we can obtain expansions about that point. The characteristic exponents are 0 and $1-\gamma$, as before. Thus, there is an analytic solution (corresponding to the characteristic exponent 0) to the CHGDE at the origin, which is called the **confluent hypergeometric function** and denoted by $\Phi(\alpha;\gamma;z)$. Since $z = 0$ is the only possible (finite) singularity of the CHGDE, $\Phi(\alpha;\gamma;z)$ is an entire function.

We can obtain the series expansion of $\Phi(\alpha;\gamma;z)$ directly from Eq. (15.22) and the fact that $\Phi(\alpha;\gamma;z) = \lim_{\beta\to 0} F(\alpha,\beta;\gamma;z/\beta)$. The result is

confluent hypergeometric function and series

$$\Phi(\alpha;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)}{\Gamma(k+1)\Gamma(\gamma+k)}z^k. \qquad (15.38)$$

This is called the **confluent hypergeometric series**. An argument similar to the one given in the case of the hypergeometric function shows that

Box 15.5.1 *The confluent hypergeometric function $\Phi(\alpha;\gamma;z)$ reduces to a polynomial when α is a negative integer.*

A second solution of the CHGDE can be obtained, as for the HGDE. If $1-\gamma$ is not an integer, then by taking the limit $\beta \to \infty$ of Eq. (15.24), we obtain the second solution $z^{1-\gamma}\Phi(\alpha-\gamma+1, 2-\gamma; z)$. Thus,

Proposition 15.5.2 *Any solution of the CHGDE can be written as a linear combination of $\Phi(\alpha;\gamma;z)$ and $z^{1-\gamma}\Phi(\alpha-\gamma+1, 2-\gamma; z)$.*

15.5.1 Hydrogen-Like Atoms

The time-independent Schrödinger equation for a central potential, in units in which $\hbar = m = 1$, is $-\frac{1}{2}\nabla^2\Psi + V(r)\Psi = E\Psi$. For the case of hydrogen-like atoms, $V(r) = -Ze^2/r$, where Z is the atomic number, and the equation reduces to

$$\nabla^2\Psi + \left(2E + \frac{2Ze^2}{r}\right)\Psi = 0.$$

The radial part of this equation is given by Eq. (13.14) with $f(r) = 2E + 2Ze^2/r$. Defining $u = rR(r)$, we may write

$$\frac{d^2u}{dr^2} + \left(\lambda + \frac{a}{r} - \frac{b}{r^2}\right)u = 0, \tag{15.39}$$

where $\lambda = 2E$, $a = 2Ze^2$, and $b = l(l+1)$. This equation can be further simplified by defining $r \equiv kz$ (k is an arbitrary constant to be determined later):

$$\frac{d^2u}{dz^2} + \left(\lambda k^2 + \frac{ak}{z} - \frac{b}{z^2}\right)u = 0.$$

Choosing $\lambda k^2 = -\frac{1}{4}$ and introducing $\alpha \equiv a/(2\sqrt{-\lambda})$ yields

$$\frac{d^2u}{dz^2} + \left(-\frac{1}{4} + \frac{\alpha}{z} - \frac{b}{z^2}\right)u = 0.$$

Equations of this form can be transformed into the CHGDE by making the substitution $u(z) = z^{\mu}e^{-\nu z}f(z)$. It then follows that

$$\frac{d^2f}{dz^2} + \left(\frac{2\mu}{z} - 2\nu\right)\frac{df}{dz} + \left[-\frac{1}{4} + \frac{\mu(\mu-1)}{z^2} - \frac{2\mu\nu}{z} + \frac{\alpha}{z} - \frac{b}{z^2} + \nu^2\right]f = 0.$$

Choosing $\nu^2 = \frac{1}{4}$ and $\mu(\mu-1) = b$ reduces this equation to

$$f'' + \left(\frac{2\mu}{z} - 2\nu\right)f' - \frac{2\mu\nu - \alpha}{z}f = 0,$$

which is in the form of (15.37).

quantization of the energy of the hydrogen atom

On physical grounds, we expect $u(z) \to 0$ as $z \to \infty$.[4] Therefore, $\nu = \frac{1}{2}$. Similarly, with $\mu(\mu-1) = b = l(l+1)$, we obtain the two possibilities $\mu = -l$ and $\mu = l+1$. Again on physical grounds, we demand that $u(0)$ be finite (the wave function must not blow up at $r = 0$). This implies[5] that $\mu = l+1$. We thus obtain

$$f'' + \left[\frac{2(l+1)}{z} - 1\right]f' - \frac{l+1-\alpha}{z}f = 0.$$

[4] This is because the volume integral of $|\Psi|^2$ over all space must be finite. The radial part of this integral is simply the integral of $r^2R^2(r) = u^2(r)$. This latter integral will not be finite unless $u(\infty) = 0$.

[5] Recall that μ is the exponent of $z = r/k$.

Multiplying by z gives

$$zf'' + \left[2(l+1) - z\right]f' - (l+1-\alpha)f = 0.$$

Comparing this with Eq. (15.37) shows that f is proportional to $\Phi(l+1-\alpha, 2l+2; z)$. Thus, the solution of (15.39) can be written as

$$u(z) = Cz^{l+1}e^{-z/2}\Phi(l+1-\alpha, 2l+2; z).$$

An argument similar to that used in Problem 14.22 will reveal that the product $e^{-z/2}\Phi(l+1-\alpha, 2l+2; z)$ will be infinite unless the power series representing Φ terminates (becomes a polynomial). It follows from Box 15.5.1 that this will take place if

$$l+1-\alpha = -N \tag{15.40}$$

for some integer $N \geq 0$. In that case we obtain the Laguerre polynomials

$$L_N^j \equiv \frac{\Gamma(N+j+1)}{\Gamma(N+1)\Gamma(j+1)}\Phi(-N, j+1; z), \quad \text{where } j = 2l+1.$$

Condition (15.40) is the quantization rule for the energy levels of a hydrogen-like atom. Writing everything in terms of the original parameters and defining $n = N + l + 1$ yields—after restoring all the m's and the $\hbar$'s—the energy levels of a hydrogen-like atom:

$$E = -\frac{Z^2me^4}{2\hbar^2n^2} = -Z^2\left(\frac{mc^2}{2}\right)\alpha^2\frac{1}{n^2},$$

where $\alpha = e^2/(\hbar c) = 1/137$ is the **fine-structure constant**. fine-structure constant

The radial wave functions can now be written as

$$R_{n,l}(r) = \frac{u_{n,l}(r)}{r} = Cr^l e^{-Zr/(na_0)}\Phi\left(-n+l+1, 2l+2; \frac{2Zr}{na_0}\right),$$

where

$$a_0 = \hbar^2/\left(me^2\right) = 0.529 \times 10^{-8} \text{ cm}$$

is the **Bohr radius**. Bohr radius

Historical Notes

Friedrich Wilhelm Bessel (1784–1846) showed no signs of unusual academic ability in school, although he did show a liking for mathematics and physics. He left school intending to become a merchant's apprentice, a desire that soon materialized with a seven-year unpaid apprenticeship with a large mercantile firm in Bremen. The young Bessel proved so adept at accounting and calculation that he was granted a small salary, with raises, after only the first year. An interest in foreign trade led Bessel to study geography and languages at night, astonishingly learning to read and write English in only three months. He also studied navigation in order to qualify as a cargo officer aboard ship, but his innate curiosity soon compelled him to investigate astronomy at a more fundamental level. Still serving his apprenticeship, Bessel learned to observe the positions of stars with sufficient accuracy to determine the longitude of Bremen, checking his results against professional astronomical journals. He then tackled the more formidable problem of determining the orbit of Halley's comet from published observations. After seeing the close

Friedrich Wilhelm Bessel
1784–1846

agreement between Bessel's calculations and those of Halley, the German astronomer Olbers encouraged Bessel to improve his already impressive work with more observations. The improved calculations, an achievement tantamount to a modern doctoral dissertation, were published with Olbers's recommendation. Bessel later received appointments with increasing authority at observatories near Bremen and in Königsberg, the latter position being accompanied by a professorship. (The title of doctor, required for the professorship, was granted by the University of Göttingen on the recommendation of Gauss.)

Bessel proved himself an excellent observational astronomer. His careful measurements coupled with his mathematical aptitude allowed him to produce accurate positions for a number of previously mapped stars, taking account of instrumental effects, atmospheric refraction, and the position and motion of the observation site. In 1820 he determined the position of the vernal equinox accurate to 0.01 second, in agreement with modern values. His observation of the variation of the proper motion of the stars Sirius and Procyon led him to posit the existence of nearby, large, low-luminosity stars called dark companions. Between 1821 and 1833 he catalogued the positions of about 75,000 stars, publishing his measurements in detail. One of his most important contributions to astronomy was the determination of the distance to a star using parallax. This method uses triangulation, or the determination of the apparent positions of a distant object viewed from two points a known distance apart, in this case two diametrically opposed points of the Earth's orbit. The angle subtended by the baseline of Earth's orbit, viewed from the star's perspective, is known as the star's parallax. Before Bessel's measurement, stars were assumed to be so distant that their parallaxes were too small to measure, and it was further assumed that bright stars (thought to be nearer) would have the largest parallax. Bessel correctly reasoned that stars with large proper motions were more likely to be nearby ones and selected such a star, 61 Cygni, for his historic measurement. His measured parallax for that star differs by less than 8 % from the currently accepted value.

Given such an impressive record in astronomy, it seems only fitting that the famous functions that bear Bessel's name grew out of his investigations of perturbations in planetary systems. He showed that such perturbations could be divided into two effects and treated separately: the obvious direct attraction due to the perturbing planet and an indirect effect caused by the sun's response to the perturber's force. The so-called Bessel functions then appear as coefficients in the series treatment of the indirect perturbation. Although special cases of Bessel functions were discovered by Bernoulli, Euler, and Lagrange, the systematic treatment by Bessel clearly established his preeminence, a fitting tribute to the creator of the most famous functions in mathematical physics.

15.5.2 Bessel Functions

Bessel differential equation

The **Bessel differential equation** is usually written as

$$w'' + \frac{1}{z} w' + \left(1 - \frac{\nu^2}{z^2}\right) w = 0. \tag{15.41}$$

As in the example above, the substitution $w = z^\mu e^{-\eta z} f(z)$ transforms (15.41) into

$$\frac{d^2 f}{dz^2} + \left(\frac{2\mu + 1}{z} - 2\eta\right)\frac{df}{dz} + \left[\frac{\mu^2 - \nu^2}{z^2} - \frac{\eta(2\mu + 1)}{z} + \eta^2 + 1\right] f = 0,$$

which, if we set $\mu = \nu$ and $\eta = i$, reduces to

$$f'' + \left(\frac{2\nu + 1}{z} - 2i\right) f' - \frac{(2\nu + 1)i}{z} f = 0.$$

Making the further substitution $2iz = t$, and multiplying out by t, we obtain

$$t\frac{d^2 f}{dt^2} + (2\nu + 1 - t)\frac{df}{dt} - \left(\nu + \frac{1}{2}\right) f = 0,$$

which is in the form of (15.37) with $\alpha = \nu + \frac{1}{2}$ and $\gamma = 2\nu + 1$.

Thus, solutions of the Bessel equation, Eq. (15.41), can be written as constant multiples of $z^\nu e^{-iz}\Phi(\nu + \frac{1}{2}, 2\nu + 1; 2iz)$. With proper normalization, we define the **Bessel function of the first kind of order** ν as

Bessel function of the first kind

$$J_\nu(z) = \frac{1}{\Gamma(\nu + 1)}\left(\frac{z}{2}\right)^\nu e^{-iz}\Phi\left(\nu + \frac{1}{2}, 2\nu + 1; 2iz\right). \tag{15.42}$$

Using Eq. (15.38) and the expansion for e^{-iz}, we can show that

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu + k + 1)}\left(\frac{z}{2}\right)^{2k}. \tag{15.43}$$

The second linearly independent solution can be obtained as usual and is proportional to

$$\begin{aligned}
&z^{1-(2\nu+1)}\left(\frac{z}{2}\right)^\nu e^{-iz}\Phi\left(\nu + \frac{1}{2} - (2\nu + 1) + 1, 2 - (2\nu + 1); 2iz\right) \\
&= C\left(\frac{z}{2}\right)^{-\nu} e^{-iz}\Phi\left(-\nu + \frac{1}{2}, -2\nu + 1; 2iz\right) = CJ_{-\nu}(z),
\end{aligned}$$

provided that $1 - \gamma = 1 - (2\nu + 1) = -2\nu$ is not an integer. When ν is an integer, $J_{-n}(z) = (-1)^n J_n(z)$ (see Problem 15.25). Thus, when ν is a noninteger, the most general solution is of the form $AJ_\nu(z) + BJ_{-\nu}(z)$.

How do we find a second linearly independent solution when ν is an integer n? We first define

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}, \tag{15.44}$$

called the Bessel function of the second kind, or the **Neumann function**. For noninteger ν this is simply a linear combination of the two linearly independent solutions. For integer ν the function is indeterminate. Therefore, we use l'Hôpital's rule and define

Bessel function of the second kind, or Neumann function

$$Y_n(z) \equiv \lim_{\nu\to n} Y_\nu(z) = \frac{1}{\pi}\lim_{\nu\to n}\left[\frac{\partial J_\nu}{\partial\nu} - (-1)^n\frac{\partial J_{-\nu}}{\partial\nu}\right].$$

Equation (15.43) yields

$$\frac{\partial J_\nu}{\partial\nu} = J_\nu(z)\ln\left(\frac{z}{2}\right) - \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty}(-1)^k\frac{\Psi(\nu + k + 1)}{k!\Gamma(\nu + k + 1)}\left(\frac{z}{2}\right)^{2k},$$

where $\Psi(z) = (d/dz)\ln\Gamma(z)$. Similarly,

$$\frac{\partial J_{-\nu}}{\partial\nu} = -J_{-\nu}(z)\ln\left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^{-\nu}\sum_{k=0}^{\infty}\frac{\Psi(-\nu + k + 1)}{k!\Gamma(-\nu + k + 1)}\left(\frac{z}{2}\right)^{2k}.$$

Substituting these expressions in the definition of $Y_n(z)$ and using $J_{-n}(z) = (-1)^n J_n(z)$, we obtain

$$Y_n(z) = \frac{2}{\pi} J_n(z) \ln\left(\frac{z}{2}\right) - \frac{1}{\pi}\left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} (-1)^k \frac{\Psi(n+k+1)}{k!\Gamma(n+k+1)}\left(\frac{z}{2}\right)^{2k}$$
$$- \frac{1}{\pi}\left(\frac{z}{2}\right)^{-n} (-1)^n \sum_{k=0}^{\infty} (-1)^k \frac{\Psi(k-n+1)}{k!\Gamma(k-n+1)}\left(\frac{z}{2}\right)^{2k}. \qquad (15.45)$$

The natural log term is indicative of the solution suggested by Theorem 15.2.6. Since $Y_\nu(z)$ is linearly independent of $J_\nu(z)$ for any ν, integer or noninteger, it is convenient to consider $\{J_\nu(z), Y_\nu(z)\}$ as a basis of solutions for the Bessel equation.

Bessel function of the third kind or Hankel function

Another basis of solutions is defined as

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z), \qquad H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z), \qquad (15.46)$$

which are called Bessel functions of the third kind, or **Hankel functions**.

Replacing z by iz in the Bessel equation yields

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} - \left(1 + \frac{\nu^2}{z^2}\right)w = 0,$$

whose basis of solutions consists of multiples of $J_\nu(iz)$ and $J_{-\nu}(iz)$. Thus, the **modified Bessel functions of the first kind** are defined as

modified Bessel functions

$$I_\nu(z) \equiv e^{-i\pi\nu/2} J_\nu(iz) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k}.$$

Similarly, the **modified Bessel functions of the second kind** are defined as

$$K_\nu(z) = \frac{\pi}{2\sin\nu\pi}\bigl[I_{-\nu}(z) - I_\nu(z)\bigr].$$

When ν is an integer n, $I_n = I_{-n}$, and K_n is indeterminate. Thus, we define $K_n(z)$ as $\lim_{\nu\to n} K_\nu(z)$. This gives

$$K_n(z) = \frac{(-1)^n}{2} \lim_{\nu\to n}\left[\frac{\partial I_{-\nu}}{\partial\nu} - \frac{\partial I_\nu}{\partial\nu}\right],$$

which has the power-series representation

$$K_n(z) = (-1)^{n+1} I_n(z) \ln\left(\frac{z}{2}\right) + \frac{1}{2}(-1)^n\left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{\Psi(n+k+1)}{k!\Gamma(n+k+1)}\left(\frac{z}{2}\right)^{2k}$$
$$+ \frac{1}{2}(-1)^n\left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{\infty} \frac{\Psi(k-n+1)}{k!\Gamma(k-n+1)}\left(\frac{z}{2}\right)^{2k}.$$

We can obtain a recurrence relation for solutions of the Bessel equation as follows. If $Z_\nu(z)$ is a solution of order ν, then (see Problem 15.28)

recurrence relation for solutions of the Bessel DE

$$Z_{\nu+1} = C_1 z^\nu \frac{d}{dz}\big[z^{-\nu} Z_\nu(z)\big] \quad \text{and} \quad Z_{\nu-1} = C_2 z^{-\nu} \frac{d}{dz}\big[z^\nu Z_\nu(z)\big].$$

If the constants are chosen in such a way that Z_ν, $Z_{-\nu}$, $Z_{\nu+1}$, and $Z_{\nu-1}$ satisfy their appropriate series expansions, then $C_1 = -1$ and $C_2 = 1$. Carrying out the differentiation in the equations for $Z_{\nu+1}$ and $Z_{\nu-1}$, we obtain

$$Z_{\nu+1} = \frac{\nu}{z} Z_\nu - \frac{dZ_\nu}{dz}, \qquad Z_{\nu-1} = \frac{\nu}{z} Z_\nu + \frac{dZ_\nu}{dz}. \tag{15.47}$$

Adding these two equations yields the recursion relation

$$Z_{\nu-1}(z) + Z_{\nu+1}(z) = \frac{2\nu}{z} Z_\nu(z), \tag{15.48}$$

where $Z_\nu(z)$ can be any of the three kinds of Bessel functions.

15.6 Problems

15.1 Show that the solution of $w' + w/z^2 = 0$ has an essential singularity at $z = 0$.

15.2 Derive the recursion relation of Eq. (15.7) and express it in terms of the indicial polynomial, as in Eq. (15.9).

15.3 Find the characteristic exponent associated with the solution of

$$w'' + p(z)w' + q(z)w = 0$$

at an ordinary point [a point at which $p(z)$ and $q(z)$ have no poles]. How many solutions can you find?

15.4 The Laplace equation in electrostatics when separated in spherical coordinates yields a DE in the radial coordinate given by

$$\frac{d}{dx}\left(x^2 \frac{dy}{dx}\right) - n(n+1)y = 0 \quad \text{for } n \geq 0.$$

Starting with an infinite series of the form (15.6), show that the two independent solutions of this ODE are of the form x^n and x^{-n-1}.

15.5 Find the indicial polynomial, characteristic exponents, and recursion relation at both of the regular singular points of the Legendre equation,

$$w'' - \frac{2z}{1-z^2} w' + \frac{\alpha}{1-z^2} w = 0.$$

What is a_k, the coefficient of the Laurent expansion, for the point $z = +1$?

15.6 Show that the substitution $z = 1/t$ transforms Eq. (15.13) into Eq. (15.14).

15.7 Obtain the indicial polynomial of Eq. (15.14) for expansion about $t = 0$.

15.8 Show that Riemann DE represents the most general second order Fuchsian DE.

15.9 Derive the indicial equation for the Riemann DE.

15.10 Show that the transformation $v(z) = z^\lambda (z-1)^\mu w(z)$ changes the pairs of characteristic exponents (λ_1, λ_2), (μ_1, μ_2), and (ν_1, ν_2) for the Riemann DE to $(\lambda_1+\lambda, \lambda_2+\lambda)$, $(\mu_1+\mu, \mu_2+\mu)$, and $(\nu_1-\lambda-\mu, \nu_2-\lambda-\mu)$.

15.11 Go through the steps leading to Eqs. (15.24), (15.25), and (15.26).

15.12 Show that the elliptic function of the first kind, defined as

$$K(z) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - z^2 \sin^2\theta}},$$

can be expressed as $(\pi/2)F(\frac{1}{2}, \frac{1}{2}; 1; z^2)$.

15.13 By differentiating the hypergeometric series, show that

$$\frac{d^n}{dz^n} F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\alpha+n)\Gamma(\beta+n)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma+n)} F(\alpha+n, \beta+n; \gamma+n; z).$$

15.14 Use direct substitution in the hypergeometric series to show that

$$F(-\alpha, \beta; \beta; -z) = (1+z)^\alpha, \qquad F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) = \frac{1}{z} \sin^{-1} z,$$

$$F(1, 1; 2; -z) = \frac{1}{z} \ln(1+z).$$

15.15 Show that the substitution $v(z) = z^r w(1/z)$ [see Eq. (15.28)] transforms the HGDE into Eq. (15.29).

15.16 Consider the function $v(z) \equiv z^r (1-z)^s F(\alpha_1, \beta_1; \gamma_1; 1/z)$ and assume that it is a solution of HGDE. Find a relation among r, s, α_1, β_1, and γ_1 such that $v(z)$ is written in terms of three parameters rather than five. In particular, show that one possibility is

$$v(z) = z^{\alpha-\gamma}(1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, 1-\alpha; 1+\beta-\alpha; 1/z).$$

Find all such possibilities.

15.17 Show that the Jacobi functions are related to the hypergeometric functions.

15.18 Derive the expression for the Jacobi function of the second kind as given in Eq. (15.33).

15.19 Show that $z = \infty$ is not a regular singular point of the CHGDE.

15.20 Derive the confluent hypergeometric series from hypergeometric series.

15.21 Show that the **Weber-Hermite equation**, $u'' + (\nu + \frac{1}{2} - \frac{1}{4}z^2)u = 0$ can be transformed into the CHGDE. Hint: Make the substitution $u(z) = \exp(-\frac{1}{4}z^2)v(z)$.

Weber-Hermite equation

15.22 The linear combination

$$\begin{aligned}\Psi(\alpha, \gamma; z) \equiv & \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)}\Phi(\alpha, \gamma; z) \\ & + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)}z^{1-\gamma}\Phi(\alpha-\gamma+1, 2-\gamma; z)\end{aligned}$$

is also a solution of the CHGDE. Show that the Hermite polynomials can be written as

$$H_n\left(\frac{z}{\sqrt{2}}\right) = 2^n\Psi\left(-\frac{n}{2}, \frac{1}{2}; \frac{z^2}{2}\right).$$

15.23 Verify that the error function $\operatorname{erf}(z) = \int_0^z e^{-t^2}dt$ satisfies the relation $\operatorname{erf}(z) = z\Phi(\frac{1}{2}, \frac{3}{2}; -z^2)$.

15.24 Derive the series expansion of the Bessel function of the first kind from that of the confluent hypergeometric series and the expansion of the exponential. Check your answer by obtaining the same result by substituting the power series directly in the Bessel DE.

15.25 Show that $J_{-n}(z) = (-1)^n J_n(z)$. Hint: Let $\nu = -n$ in the expansion of $J_\nu(z)$ and use $\Gamma(m) = \infty$ for a nonpositive integer m.

15.26 In a potential-free region, the radial part of the Schrödinger equation reduces to

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[\lambda - \frac{\alpha}{r^2}\right]R = 0.$$

Write the solutions of this DE in terms of Bessel functions. Hint: Substitute $R = u/\sqrt{r}$. These solutions are called **spherical Bessel functions**.

spherical Bessel functions

15.27 Theorem 15.2.6 states that under certain conditions, linearly independent solutions of SOLDE at regular singular points exist even though the difference between the characteristic exponents is an integer. An example is the case of Bessel functions of half-odd-integer orders. Evaluate the Wronskian of the two linearly independent solutions, J_ν and $J_{-\nu}$, of the Bessel equation and show that it vanishes *only if ν is an integer*. This shows, in particular, that $J_{n+1/2}$ and $J_{-n-1/2}$ are linearly independent. Hint: Consider the value of the Wronskian at $z=0$, and use the formula $\Gamma(\nu)\Gamma(1-\nu)=\pi/\sin\nu\pi$.

15.28 Show that $z^{\pm\nu}(d/dz)[z^{\mp\nu}Z_\nu(z)]$ is a solution of the Bessel equation of order $\nu\pm 1$ if Z_ν is a solution of order ν.

15.29 Use the recursion relation of Eq. (15.47) to prove that

$$\left(\frac{1}{z}\frac{d}{dz}\right)^m\bigl[z^\nu Z_\nu(z)\bigr]=z^{\nu-m}Z_{\nu-m}(z),$$
$$\left(\frac{1}{z}\frac{d}{dz}\right)^m\bigl[z^{-\nu} Z_\nu(z)\bigr]=(-1)^m z^{-\nu-m}Z_{\nu+m}(z).$$

15.30 Using the series expansion of the Bessel function, write $J_{1/2}(z)$ and $J_{-1/2}(z)$ in terms of elementary functions. Hint: First show that

$$\Gamma\left(k+\frac{3}{2}\right)=\sqrt{\pi}\,(2k+1)!/\bigl(k!2^{2k+1}\bigr).$$

15.31 From the results of the previous two problems, derive the relations

$$J_{-n-1/2}(z)=\sqrt{\frac{2}{\pi}}z^{n+1/2}\left(\frac{1}{z}\frac{d}{dz}\right)^n\left(\frac{\cos z}{z}\right),$$
$$J_{n+1/2}(z)=\sqrt{\frac{2}{\pi}}z^{n+1/2}\left(-\frac{1}{z}\frac{d}{dz}\right)^n\left(\frac{\sin z}{z}\right).$$

15.32 Obtain the following integral identities:

(a) $$\int z^{\nu+1}J_\nu(z)\,dz=z^{\nu+1}J_{\nu+1}(z).$$

(b) $$\int z^{-\nu+1}J_\nu(z)\,dz=-z^{-\nu+1}J_{\nu-1}(z).$$

(c) $$\int z^{\mu+1}J_\nu(z)\,dz=z^{\mu+1}J_{\nu+1}(z)+(\mu-\nu)z^\mu J_\nu(z)-\bigl(\mu^2-\nu^2\bigr)\int z^{\mu-1}J_\nu(z)\,dz,$$

and evaluate

(d) $$\int z^3 J_0(z)\,dz.$$

Hint: For (c) write $z^{\mu+1}=z^{\mu-\nu}z^{\nu+1}$ and use integration by parts.

15.33 Use Theorem 15.2.6 and the fact that $J_n(z)$ is entire to show that for integer n, a second solution to the Bessel equation exists and can be written as $Y_n(z)=J_n(z)[f_n(z)+C_n\ln z]$, where $f_n(z)$ is analytic about $z=0$.

15.34 (a) Show that the Wronskian $W(J_\nu, Z; z)$ of J_ν and any other solution Z of the Bessel equation, satisfies the equation

$$\frac{d}{dz}\big[zW(J_\nu, Z; z)\big] = 0.$$

(b) For some constant A, show that

$$\frac{d}{dz}\left[\frac{Z}{J_\nu}\right] = \frac{W(z)}{J_\nu^2(z)} = \frac{A}{zJ_\nu^2(z)}.$$

(c) Show that the general second solution of the Bessel equation can be written as

$$Z_\nu(z) = J_\nu(z)\left[B + A\int \frac{dz}{zJ_\nu^2(z)}\right].$$

15.35 Spherical Bessel functions are defined by

$$f_l(z) \equiv \sqrt{\frac{\pi}{2}}\left(\frac{Z_{l+1/2}(z)}{\sqrt{z}}\right).$$

Let $f_l(z)$ denote a spherical Bessel function "of some kind." By direct differentiation and substitution in the Bessel equation, show that

$$\text{(a)}\quad \frac{d}{dz}\big[z^{l+1} f_l(z)\big] = z^{l+1} f_{l-1}(z),$$

$$\text{(b)}\quad \frac{d}{dz}\big[z^{-l} f_l(z)\big] = -z^{-l} f_{l+1}(z).$$

(c) Combine the results of parts (a) and (b) to derive the recursion relations

$$f_{l-1}(z) + f_{l+1}(z) = \frac{2l+1}{z} f_l(z),$$

$$l f_{l-1}(z) - (l+1) f_{l+1}(z) = (2l+1)\frac{df_l}{dz}.$$

15.36 Show that

$$W(J_\nu, Y_\nu; z) = \frac{2}{\pi z}, \qquad W\big(H_\nu^{(1)}, H_\nu^{(2)}; z\big) = \frac{4}{i\pi z}.$$

Hint: Use Problem 15.34.

15.37 Verify the following relations:

(a) $Y_{n+1/2}(z) = (-1)^{n+1} J_{-n-1/2}(z), \qquad Y_{-n-1/2}(z) = (-1)^n J_{n+1/2}(z),$

(b) $Y_{-\nu}(z) = \sin\nu\pi\, J_\nu(z) + \cos\nu\pi\, Y_\nu(z) = \dfrac{J_\nu(z) - \cos\nu\pi\, J_{-\nu}(z)}{\sin\nu\pi},$

(c) $Y_{-n}(z) = (-1)^n Y_n(z)$ in the limit $\nu \to n$ in part (b).

15.38 Use the recurrence relation for the Bessel function to show that $J_1(z) = -J_0'(z)$.

15.39 Let $u = J_\nu(\lambda z)$ and $v = J_\nu(\mu z)$. Multiply the Bessel DE for u by v/z and that of v by u/z. Subtract the two equations to obtain

$$(\lambda^2 - \mu^2)zuv = \frac{d}{dz}\left[z\left(u\frac{dv}{dz} - v\frac{du}{dz}\right)\right].$$

(a) Write the above equation in terms of $J_\nu(\lambda z)$ and $J_\nu(\mu z)$ and integrate both sides with respect to z.

(b) Now divide both sides by $\lambda^2 - \mu^2$ and take the limit as $\mu \to \lambda$. You will need to use L'Hôpital's rule.

(c) Substitute for $J_\nu''(\lambda z)$ from the Bessel DE and simplify to get

$$\int z\big[J_\nu(\lambda z)\big]^2 dz = \frac{z^2}{2}\left\{\big[J_\nu'(\lambda z)\big]^2 + \left(1 - \frac{\nu^2}{\lambda^2 z^2}\right)\big[J_\nu(\lambda z)\big]^2\right\}.$$

(d) Finally, let $\lambda = x_{\nu n}/a$, where $x_{\nu n}$ is the nth root of J_ν, and use Eq. (15.47) to arrive at

$$\int_0^a z J_\nu^2\left(\frac{x_{\nu n}}{a}z\right)dz = \frac{a^2}{2}J_{\nu+1}^2(x_{\nu n}).$$

15.40 The generating function $g(z,t)$ for Bessel functions of integer order is

$$g(z,t) = \exp\left[\frac{1}{2}z(t - 1/t)\right].$$

To see this, rewrite $g(z,t)$ as $e^{zt/2}e^{-z/2t}$, expand both factors, and write the product as powers of t^n. Now show that the coefficient of t^n is simply $J_n(z)$. Finally, use $J_{-n}(z) = (-1)^n J_n(z)$ to derive the formula

$$\exp\left[\frac{1}{2}z(t - 1/t)\right] = \sum_{n=-\infty}^{\infty} J_n(z)t^n.$$

15.41 Make the substitutions $z = \beta t^\gamma$ and $w = t^\alpha u$ to transform the Bessel DE into

$$t^2\frac{d^2u}{dt^2} + (2\alpha + 1)t\frac{du}{dt} + \left(\beta^2\gamma^2 t^{2\gamma} + \alpha^2 - \nu^2\gamma^2\right)u = 0.$$

Airy's differential equation

Now show that **Airy's DE**, $\ddot{u} - tu = 0$, has solutions of the form $J_{1/3}(\frac{2}{3}it^{3/2})$ and $J_{-1/3}(\frac{2}{3}it^{3/2})$.

15.42 Show that the general solution of $\frac{d^2w}{dt^2} + \frac{e^{2t} - \nu^2}{t^4}w = 0$ is $w = t[AJ_\nu(e^{1/t}) + BY_\nu(e^{1/t})]$.

15.43 Transform $dw/dz + w^2 + z^m = 0$ by making the substitution $w = (d/dz)\ln v$. Now make the further substitutions

$$v = u\sqrt{z} \quad \text{and} \quad t = \frac{2}{m+2}z^{1+(1/2)m}$$

to show that the new DE can be transformed into a Bessel equation of order $1/(m+2)$.

15.44 Starting with the relation

$$\exp\left[\frac{1}{2}x(t-1/t)\right]\exp\left[\frac{1}{2}y(t-1/t)\right]=\exp\left[\frac{1}{2}(x+y)(t-1/t)\right]$$

and the fact that the exponential function is the generating function for $J_n(z)$, prove the "addition theorem" for Bessel functions:

$$J_n(x+y)=\sum_{k=-\infty}^{\infty} J_k(x)J_{n-k}(y).$$