Complex Calculus

10

Complex analysis, just like real analysis, deals with questions of continuity, convergence of series, differentiation, integration, and so forth. The reader is assumed to have been exposed to the *algebra* of complex numbers.

10.1 Complex Functions

A complex function is a map $f : \mathbb{C} \to \mathbb{C}$, and we write $f(z) = w$, where both z and w are complex numbers.[1] The map f can be geometrically thought of as a correspondence between two complex planes, the z-plane and the w-plane. The w-plane has a real axis and an imaginary axis, which we can call u and v, respectively. Both u and v are real functions of the coordinates of z, i.e., x and y. Therefore, we may write

$$f(z) = u(x, y) + iv(x, y). \tag{10.1}$$

This equation gives a unique point (u, v) in the w-plane for each point (x, y) in the z-plane (see Fig. 10.1). Under f, regions of the z-plane are mapped onto regions of the w-plane. For instance, a curve in the z-plane may be mapped into a curve in the w-plane. The following example illustrates this point.

Example 10.1.1 Let us investigate the behavior of a couple of elementary complex functions. In particular, we shall look at the way a line $y = mx$ in the z-plane is mapped into curves in the w-plane.

(a) For $w = f(z) = z^2$, we have

$$w = (x + iy)^2 = x^2 - y^2 + 2ixy,$$

[1] Strictly speaking, we should write $f : S \to \mathbb{C}$ where S is a *subset* of the complex plane. The reason is that most functions are not defined for the entire set of complex numbers, so that the domain of such functions is not necessarily $\mathbb{C}$. We shall specify the domain only when it is absolutely necessary. Otherwise, we use the generic notation $f : \mathbb{C} \to \mathbb{C}$, even though f is defined only on a subset of $\mathbb{C}$.

S. Hassani, *Mathematical Physics*, DOI 10.1007/978-3-319-01195-0_10,

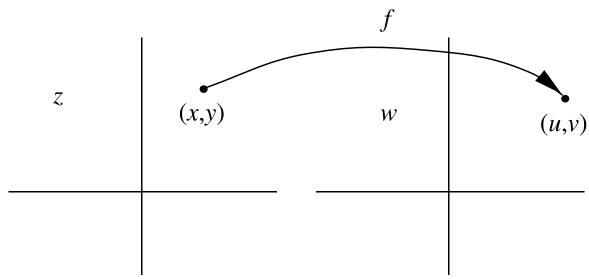

Fig. 10.1 A map from the z-plane to the w-plane

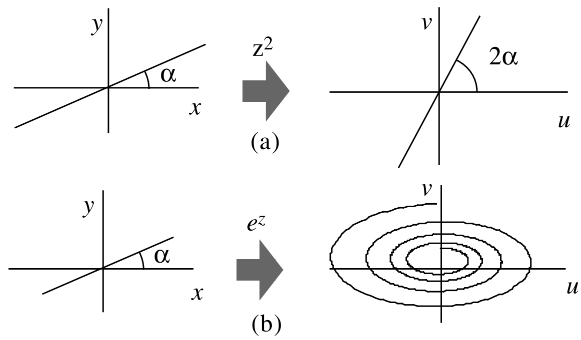

Fig. 10.2 (**a**) The map z^2 takes a line with slope angle α and maps it to a line with twice the angle in the w-plane. (**b**) The map e^z takes the same line and maps it to a spiral in the w-plane

with $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. How does the region of $\mathbb{C}$ consisting of all points of a line get mapped into $\mathbb{C}$? For $y = mx$, i.e., for a line in the z-plane with slope m, these equations yield $u = (1 - m^2)x^2$ and $v = 2mx^2$. Eliminating x in these equations, we find $v = [2m/(1 - m^2)]u$. This is a line passing through the origin of the w-plane [see Fig. 10.2(a)]. Note that the angle the image line makes with the real axis of the w-plane is twice the angle the original line makes with the x-axis. (Show this!).

(b) The function $w = f(z) = e^z = e^{x+iy}$ gives $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. What is the image of the line $y = mx$ under this map? Substituting $y = mx$, we obtain $u = e^x \cos mx$ and $v = e^x \sin mx$. Unlike part (a), we cannot eliminate x to find v as an explicit function of u. Nevertheless, the last pair of equations are *parametric equations* of a curve, which we can plot in a uv-plane as shown in Fig. 10.2(b).

Limits of complex functions are defined in terms of absolute values. Thus, $\lim_{z \to a} f(z) = w_0$ means that given any real number $\epsilon > 0$, we can find a corresponding real number $\delta > 0$ such that $|f(z) - w_0| < \epsilon$ whenever $|z - a| < \delta$. Similarly, we say that a function f is **continuous** at $z = a$ if $\lim_{z \to a} f(z) = f(a)$.

10.2 Analytic Functions

The derivative of a complex function is defined as usual:

Definition 10.2.1 Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function. The **derivative** of f at z_0 is

$$\left.\frac{df}{dz}\right|_{z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$

provided that the limit exists and is independent of Δz.

In this definition "independent of Δz" means independent of Δx and Δy (the components of Δz) and, therefore, independent of the direction of approach to z_0. The restrictions of this definition apply to the real case as well. For instance, the derivative of $f(x) = |x|$ at $x = 0$ does not exist[2] because it approaches $+1$ from the right and -1 from the left.

It can easily be shown that all the formal rules of differentiation that apply to the real case also apply to the complex case. For example, if f and g are differentiable, then $f \pm g$, fg, and—as long as g is not zero at the point of interest—f/g are also differentiable, and their derivatives are given by the usual rules of differentiation.

Example 10.2.2 Let us examine the derivative of $f(z) = x^2 + 2iy^2$ at $z = 1 + i$:

Example illustrating path dependence of derivative

$$\begin{aligned}
\left.\frac{df}{dz}\right|_{z=1+i} &= \lim_{\Delta z \to 0} \frac{f(1 + i + \Delta z) - f(1 + i)}{\Delta z} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(1 + \Delta x)^2 + 2i(1 + \Delta y)^2 - 1 - 2i}{\Delta x + i\Delta y} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{2\Delta x + 4i\Delta y + (\Delta x)^2 + 2i(\Delta y)^2}{\Delta x + i\Delta y}.
\end{aligned}$$

Let us approach $z = 1 + i$ along the line $y - 1 = m(x - 1)$. Then $\Delta y = m\Delta x$, and the limit yields

$$\left.\frac{df}{dz}\right|_{z=1+i} = \lim_{\Delta x \to 0} \frac{2\Delta x + 4im\Delta x + (\Delta x)^2 + 2im^2(\Delta x)^2}{\Delta x + im\Delta x} = \frac{2 + 4im}{1 + im}.$$

It follows that we get infinitely many values for the derivative depending on the value we assign to m, i.e., depending on the direction along which we approach $1 + i$. Thus, the derivative does not exist at $z = 1 + i$.

It is clear from the definition that differentiability puts a severe restriction on $f(z)$ because it requires the limit to be the same for *all paths* going through z_0. Furthermore, differentiability is a *local* property: To test

[2] One can rephrase this and say that the derivative exists, but not in terms of ordinary functions, rather, in terms of *generalized* functions—in this case $\theta(x)$—discussed in Sect. 7.3.

whether or not a function $f(z)$ is differentiable at z_0, we move away from z_0 only by a small amount Δz and check the existence of the limit in Definition 10.2.1.

What are the conditions under which a complex function is differentiable? For $f(z) = u(x, y) + iv(x, y)$, Definition 10.2.1 yields

$$\left.\frac{df}{dz}\right|_{z_0} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left\{ \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} + i\frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y} \right\}.$$

If this limit is to exist for all paths, it must exist for the two particular paths on which $\Delta y = 0$ (parallel to the x-axis) and $\Delta x = 0$ (parallel to the y-axis). For the first path we get

$$\left.\frac{df}{dz}\right|_{z_0} = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} = \left.\frac{\partial u}{\partial x}\right|_{(x_0, y_0)} + i \left.\frac{\partial v}{\partial x}\right|_{(x_0, y_0)}.$$

For the second path ($\Delta x = 0$), we obtain

$$\left.\frac{df}{dz}\right|_{z_0} = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} = -i \left.\frac{\partial u}{\partial y}\right|_{(x_0, y_0)} + \left.\frac{\partial v}{\partial y}\right|_{(x_0, y_0)}.$$

If f is to be differentiable at z_0, the derivatives along the two paths must be equal. Equating the real and imaginary parts of both sides of this equation and ignoring the subscript z_0 (x_0, y_0, or z_0 is arbitrary), we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{10.2}$$

Cauchy-Riemann conditions

These two conditions, which are necessary for the differentiability of f, are called the **Cauchy-Riemann conditions**.

An alternative way of writing the Cauchy-Riemann (C-R) conditions is obtained by making the substitution[3] $x = \frac{1}{2}(z + z^*)$ and $y = \frac{1}{2i}(z - z^*)$ in $u(x, y)$ and $v(x, y)$, using the chain rule to write Eq. (10.2) in terms of z and z^*, substituting the results in $\frac{\partial f}{\partial z^*} = \frac{\partial u}{\partial z^*} + i\frac{\partial v}{\partial z^*}$ and showing that Eq. (10.2) is equivalent to the single equation $\partial f/\partial z^* = 0$. This equation says that

Box 10.2.3 *If f is to be differentiable, it must be independent of z^*.*

[3]We use z^* to indicate the complex conjugate of z. Occasionally we may use $\bar{z}$.

If the derivative of f exists, the arguments leading to Eq. (10.2) imply that the derivative can be expressed as

Expression for the derivative of a differentiable complex function

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{10.3}$$

The C-R conditions assure us that these two equations are equivalent.

The following example illustrates the differentiability of complex functions.

Example 10.2.4 Let us determine whether or not the following functions are differentiable:

(a) We have already established that $f(z) = x^2 + 2iy^2$ is not differentiable at $z = 1 + i$. We can now show that it has no derivative at any point in the complex plane (except at the origin). This is easily seen by noting that $u = x^2$ and $v = 2y^2$, and that $\partial u/\partial x = 2x \neq \partial v/\partial y = 4y$, and the first Cauchy-Riemann condition is not satisfied. The second C-R condition is satisfied, but that is not enough.

We can also write $f(z)$ in terms of z and z^*:

$$\begin{aligned} f(z) &= \left[\frac{1}{2}(z + z^*)\right]^2 + 2i\left[\frac{1}{2i}(z - z^*)\right]^2 \\ &= \frac{1}{4}(1 - 2i)\left(z^2 + z^{*2}\right) + \frac{1}{2}(1 + 2i)zz^*. \end{aligned}$$

$f(z)$ has an explicit dependence on z^*. Therefore, it is not differentiable.

(b) Now consider $f(z) = x^2 - y^2 + 2ixy$, for which $u = x^2 - y^2$ and $v = 2xy$. The C-R conditions become $\partial u/\partial x = 2x = \partial v/\partial y$ and $\partial u/\partial y = -2y = -\partial v/\partial x$. Thus, $f(z)$ may be differentiable. Recall that the C-R conditions are only *necessary* conditions; we have not shown (but we will, shortly) that they are also sufficient.

To check the dependence of f on z^*, substitute $x = (z + z^*)/2$ and $y = (z - z^*)/(2i)$ in u and v to show that $f(z) = z^2$, and thus there is no z^* dependence.

(c) Let $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. Then $\partial u/\partial x = e^x \cos y = \partial v/\partial y$ and $\partial u/\partial y = -e^x \sin y = -\partial v/\partial x$, and the C-R conditions are satisfied. Also,

$$f(z) = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = e^x e^{iy} = e^{x+iy} = e^z,$$

and there is no z^* dependence.

The requirement of differentiability is very restrictive: The derivative must exist along infinitely many paths. On the other hand, the C-R conditions seem deceptively mild: They are derived for only two paths. Nevertheless, the two paths are, in fact, true representatives of all paths; that is, the C-R conditions are not only necessary, but also sufficient:

Theorem 10.2.5 *The function $f(z) = u(x, y) + iv(x, y)$ is differentiable in a region of the complex plane if and only if the Cauchy-Riemann conditions,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(or, equivalently, $\partial f/\partial z^* = 0$), are satisfied and all first partial derivatives of u and v are continuous in that region. In that case

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Proof We have already shown the "only if" part. To show the "if" part, note that if the derivative exists at all, it must equal (10.3). Thus, we have to show that

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

or, equivalently, that

$$\left| \frac{f(z + \Delta z) - f(z)}{\Delta z} - \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \right) \right| < \epsilon \quad \text{whenever } |\Delta z| < \delta.$$

By definition,

$$f(z + \Delta z) - f(z)$$
$$= u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y).$$

Since u and v have continuous first partial derivatives, we can write

$$u(x + \Delta x, y + \Delta y) = u(x, y) + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1 \Delta x + \delta_1 \Delta y,$$

$$v(x + \Delta x, y + \Delta y) = v(x, y) + \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \epsilon_2 \Delta x + \delta_2 \Delta y,$$

where $\epsilon_1, \epsilon_2, \delta_1$, and δ_2 are real numbers that approach zero as Δx and Δy approach zero. Using these expressions, we can write

$$\begin{aligned} f(z + \Delta z) - f(z) &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \right)\Delta x + i\left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right)\Delta y \\ &\quad + (\epsilon_1 + i\epsilon_2)\Delta x + (\delta_1 + i\delta_2)\Delta y \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \right)(\Delta x + i\Delta y) + \epsilon \Delta x + \delta \Delta y, \end{aligned}$$

where $\epsilon \equiv \epsilon_1 + i\epsilon_2$, $\delta \equiv \delta_1 + i\delta_2$, and we used the C-R conditions in the last step. Dividing both sides by $\Delta z = \Delta x + i\Delta y$, we get

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} - \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \right) = \epsilon \frac{\Delta x}{\Delta z} + \delta \frac{\Delta y}{\Delta z}.$$

By the triangle inequality, $|\text{RHS}| \le |\epsilon_1 + i\epsilon_2| + |\delta_1 + i\delta_2|$. This follows from the fact that $|\Delta x|/|\Delta z|$ and $|\Delta y|/|\Delta z|$ are both equal to at most 1. The ϵ and δ terms can be made as small as desired by making Δz small enough. We have thus established that when the C-R conditions hold, the function f is differentiable. □

Historical Notes

Augustin-Louis Cauchy
1789–1857

Augustin-Louis Cauchy (1789–1857) was one of the most influential French mathematicians of the nineteenth century. He began his career as a military engineer, but when his health broke down in 1813 he followed his natural inclination and devoted himself wholly to mathematics.

In mathematical productivity Cauchy was surpassed only by Euler, and his collected works fill 27 fat volumes. He made substantial contributions to number theory and determinants; is considered to be the originator of the theory of finite groups; and did extensive work in astronomy, mechanics, optics, and the theory of elasticity.

His greatest achievements, however, lay in the field of analysis. Together with his contemporaries Gauss and Abel, he was a pioneer in the rigorous treatment of limits, continuous functions, derivatives, integrals, and infinite series. Several of the basic tests for the convergence of series are associated with his name. He also provided the first existence proof for solutions of differential equations, gave the first proof of the convergence of a Taylor series, and was the first to feel the need for a careful study of the convergence behavior of Fourier series (see Chap. 9). However, his most important work was in the theory of functions of a complex variable, which in essence he created and which has continued to be one of the dominant branches of both pure and applied mathematics. In this field, Cauchy's integral theorem and Cauchy's integral formula are fundamental tools without which modern analysis could hardly exist (see Chap. 10).

Unfortunately, his personality did not harmonize with the fruitful power of his mind. He was an arrogant royalist in politics and a self-righteous, preaching, pious believer in religion—all this in an age of republican skepticism—and most of his fellow scientists disliked him and considered him a smug hypocrite. It might be fairer to put first things first and describe him as a great mathematician who happened also to be a sincere but narrow-minded bigot.

analyticity and singularity; regular and singular points; entire functions

Definition 10.2.6 A function $f : \mathbb{C} \to \mathbb{C}$ is called **analytic** at z_0 if it is differentiable at z_0 and at all other points in some neighborhood of z_0. A point at which f is analytic is called a **regular point** of f. A point at which f is not analytic is called a **singular point** or a **singularity** of f. A function for which all points in $\mathbb{C}$ are regular is called an **entire** function.

Example 10.2.7 (Derivatives of some functions)

(a) $f(z) = z$. Here $u = x$ and $v = y$; the C-R conditions are easily shown to hold, and for any z, we have $df/dz = \partial u/\partial x + i\partial v/\partial x = 1$. Therefore, the derivative exists at all points of the complex plane.

(b) $f(z) = z^2$. Here $u = x^2 - y^2$ and $v = 2xy$; the C-R conditions hold, and for all points z of the complex plane, we have $df/dz = \partial u/\partial x + i\partial v/\partial x = 2x + i2y = 2z$. Therefore, $f(z)$ is differentiable at all points.

(c) $f(z) = z^n$ for $n \ge 1$. We can use mathematical induction and the fact that the product of two entire functions is an entire function to show that $\frac{d}{dz}(z^n) = nz^{n-1}$.

(d) $f(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n$, where a_i are arbitrary constants. That $f(z)$ is entire follows directly from (c) and the fact that the sum of two entire functions is entire.

(e) $f(z) = 1/z$. The derivative can be found to be $f'(z) = -1/z^2$, which does not exist for $z = 0$. Thus, $z = 0$ is a singularity of $f(z)$. However, any other point of the complex plane is a regular point of f.

(f) $f(z) = |z|^2$. Using the definition of the derivative, we obtain

$$\frac{\Delta f}{\Delta z} = \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \frac{(z+\Delta z)(z^* + \Delta z^*) - zz^*}{\Delta z}$$
$$= z^* + \Delta z^* + z\frac{\Delta z^*}{\Delta z}.$$

For $z = 0$, $\Delta f/\Delta z = \Delta z^*$, which goes to zero as $\Delta z \to 0$. Therefore, $df/dz = 0$ at $z = 0$.[4] However, if $z \neq 0$, the limit of $\Delta f/\Delta z$ will depend on how z is approached. Thus, df/dz does not exist if $z \neq 0$. This shows that $|z|^2$ is differentiable only at $z = 0$ and nowhere else in its neighborhood. It also shows that even if the real (here, $u = x^2 + y^2$) and imaginary (here, $v = 0$) parts of a complex function have continuous partial derivatives of all orders at a point, the function may not be differentiable there.

(g) $f(z) = 1/\sin z$: This gives $df/dz = -\cos z/\sin^2 z$. Thus, f has infinitely many (isolated) singular points at $z = \pm n\pi$ for $n = 0, 1, 2, \ldots$.

complex exponential function

Example 10.2.8 (The complex exponential function) In this example, we find the (unique) function $f : \mathbb{C} \to \mathbb{C}$ that has the following three properties:

(a) f is single-valued and analytic for all z,
(b) $df/dz = f(z)$, and
(c) $f(z_1 + z_2) = f(z_1)f(z_2)$.

Property (b) shows that if $f(z)$ is well behaved, then df/dz is also well behaved. In particular, if $f(z)$ is defined for all values of z, then f must be entire.

For $z_1 = 0 = z_2$, property (c) yields $f(0) = [f(0)]^2 \Rightarrow f(0) = 1$, or $f(0) = 0$. On the other hand,

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{f(z)f(\Delta z) - f(z)}{\Delta z} = f(z) \lim_{\Delta z \to 0} \frac{f(\Delta z) - 1}{\Delta z}.$$

Property (b) now implies that

$$\lim_{\Delta z \to 0} \frac{f(\Delta z) - 1}{\Delta z} = 1 \quad \Rightarrow \quad f'(0) = 1 \quad \text{and} \quad f(0) = 1.$$

The first implication follows from the definition of derivative, and the second from the fact that the only other choice, namely $f(0) = 0$, would yield $-\infty$ for the limit.

[4] Although the derivative of $|z|^2$ exists at $z = 0$, it is not analytic there (or anywhere else). To be analytic at a point, a function must have derivatives at *all points* in some neighborhood of the given point.

Now, we write $f(z) = u(x, y) + iv(x, y)$, for which property (b) becomes

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u + iv \quad \Rightarrow \quad \frac{\partial u}{\partial x} = u, \qquad \frac{\partial v}{\partial x} = v.$$

These equations have the most general solution $u(x, y) = a(y)e^x$ and $v(x, y) = b(y)e^x$, where $a(y)$ and $b(y)$ are the "constants" of integration. The Cauchy-Riemann conditions now yield $a(y) = db/dy$ and $da/dy = -b(y)$, whose most general solution is $a(y) = A\cos y + B\sin y$, $b(y) = A\sin y - B\cos y$. On the other hand, $f(0) = 1$ yields $u(0, 0) = 1$ and $v(0, 0) = 0$, implying that $a(0) = 1$, $b(0) = 0$ or $A = 1$, $B = 0$. We therefore conclude that

$$f(z) = a(y)e^x + ib(y)e^x = e^x(\cos y + i\sin y) = e^x e^{iy} = e^z.$$

Both e^x and e^{iy} are well-defined in the entire complex plane. Hence, e^z is defined and differentiable over all $\mathbb{C}$; therefore, it is entire.

Example 10.2.7 shows that any polynomial in z is entire. Example 10.2.8 shows that the exponential function e^z is also entire. Therefore, any product and/or sum of polynomials and e^z will also be entire. We can build other entire functions. For instance, e^{iz} and e^{-iz} are entire functions; therefore, the trigonometric functions, defined as

trigonometric functions

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \tag{10.4}$$

are also entire functions. Problem 10.5 shows that $\sin z$ and $\cos z$ have only *real* zeros. The hyperbolic functions can be defined similarly:

hyperbolic functions

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^{z} + e^{-z}}{2}. \tag{10.5}$$

Although the sum and the product of entire functions are entire, the ratio, in general, is not. For instance, if $f(z)$ and $g(z)$ are polynomials of degrees m and n, respectively, then for $n > 0$, the ratio $f(z)/g(z)$ is not entire, because at the zeros of $g(z)$—which always exist and we assume that it is not a zero of $f(z)$—the derivative is not defined.

The functions $u(x, y)$ and $v(x, y)$ of an analytic function have an interesting property that the following example investigates.

Example 10.2.9 The family of curves $u(x, y) =$ constant is perpendicular to the family of curves $v(x, y) =$ constant at each point of the complex plane where $f(z) = u + iv$ is analytic.

This can easily be seen by looking at the normal to the curves. The normal to the curve $u(x, y) =$ constant is simply $\nabla u = (\partial u/\partial x, \partial u/\partial y)$. Similarly, the normal to the curve $v(x, y) =$ constant is $\nabla v = (\partial v/\partial x, \partial v/\partial y)$. Taking the dot product of these two normals, we obtain

$$(\nabla u) \cdot (\nabla v) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 0$$

by the C-R conditions.

10.3 Conformal Maps

The real and imaginary parts of an analytic function separately satisfy the two-dimensional Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \qquad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0. \tag{10.6}$$

This can easily be verified from the C-R conditions.

Laplace's equation in three dimensions,

$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}+\frac{\partial^2 \Phi}{\partial z^2}=0,$$

describes the electrostatic potential Φ in a charge-free region of space. In a typical electrostatic problem the potential Φ is given at certain boundaries (usually conducting surfaces), and its value at every point in space is sought. There are numerous techniques for solving such problems, and some of them will be discussed later in the book. However, some of these problems have a certain degree of symmetry that reduces them to two-dimensional problems. In such cases, the theory of analytic functions can be extremely helpful.

The symmetry mentioned above is cylindrical symmetry, where the potential is known a priori to be independent of the z-coordinate (the axis of symmetry). This situation occurs when conductors are cylinders and—if there are charge distributions in certain regions of space—the densities are z-independent. In such cases, $\partial\Phi/\partial z=0$, and the problem reduces to a two-dimensional one.

harmonic functions

Functions satisfying Laplace's equation are called **harmonic functions**. Thus, the electrostatic potential is a three-dimensional harmonic function, and the potential for a cylindrically symmetric charge distribution and boundary condition is a two-dimensional harmonic function. Since the real and the imaginary parts of a complex analytic function are also harmonic, techniques of complex analysis are sometimes useful in solving electrostatic problems with cylindrical symmetry.[5]

To illustrate the connection between electrostatics and complex analysis, consider a long straight filament with a constant linear charge density λ. It is shown in introductory electromagnetism that the potential Φ (disregarding the arbitrary constant that determines the reference potential) is given, in cylindrical coordinates, by

$$\Phi=2\lambda\ln\rho=2\lambda\ln\left[\left(x^2+y^2\right)^{1/2}\right]=2\lambda\ln|z|.$$

complex potential

Since Φ satisfies Laplace's equation, we conclude that Φ *could* be the real part of an analytic function $w(z)$, which we call the **complex potential**.

[5] We use electrostatics because it is more familiar to physics students. Engineering students are familiar with steady state heat transfer as well, which also involves Laplace's equation, and therefore is amenable to this technique.

Example 10.2.9, plus the fact that the curves $u = \Phi =$ constant are circles, imply that the constant-v curves are rays, i.e., $v \propto \varphi$. Choosing the constant of proportionality as 2λ, we obtain

$$w(z) = 2\lambda \ln \rho + i2\lambda\varphi = 2\lambda \ln\bigl(\rho e^{i\varphi}\bigr) = 2\lambda \ln z.$$

It is useful to know the complex potential of more than one filament of charge. To find such a potential we must first find $w(z)$ for a line charge when it is displaced from the origin. If the line is located at $z_0 = x_0 + iy_0$, then it is easy to show that $w(z) = 2\lambda \ln(z - z_0)$. If there are n line charges located at $z_1, z_2, \ldots, z_n$, then

$$w(z) = 2\sum_{k=1}^{n} \lambda_k \ln(z - z_k). \tag{10.7}$$

The function $w(z)$ can be used directly to solve a number of electrostatic problems involving simple charge distributions and conductor arrangements. Some of these are illustrated in problems at the end of this chapter. Instead of treating $w(z)$ as a complex potential, let us look at it as a map from the z-plane (or xy-plane) to the w-plane (or uv-plane). In particular, the equipotential curves (circles for a single line of charge) are mapped onto *lines parallel to the v-axis* in the w-plane. This is so because equipotential curves are defined by $u =$ constant. Similarly, the constant-v curves are mapped onto horizontal lines in the w-plane.

This is an enormous simplification of the geometry. Straight lines, especially when they are parallel to axes, are by far simpler geometrical objects than circles,[6] especially if the circles are not centered at the origin. So let us consider two complex "worlds". One is represented by the xy-plane and denoted by z. The other, the "prime world", is represented[7] by z', and its real and imaginary parts by x' and y'. We start in z, where we need to find a physical quantity such as the electrostatic potential $\Phi(x, y)$. If the problem is too complicated in the z-world, we transfer it to the z'-world, in which it may be easily solvable; we solve the problem there (in terms of x' and y') and then transfer back to the z-world (x and y). The mapping that relates z and z' must be cleverly chosen. Otherwise, there is no guarantee that the problem will simplify.

Two conditions are necessary for the above strategy to work. First, the differential equation describing the physics must not get more complicated with the transfer to z'. Since Laplace's equation is already of the simplest type, the z'-world must also respect Laplace's equation. Second, and more importantly, the mapping must preserve the angles between curves. This is necessary because we want the equipotential curves and the field lines to be perpendicular in both worlds. A mapping that preserves the angle between two curves at a given point is called a **conformal mapping**. We already have such mappings at our disposal, as the following proposition shows. conformal mapping

[6] This statement is valid only in Cartesian coordinates. But these are precisely the coordinates we are using in this discussion.

[7] We are using z' instead of w, and (x', y') instead of (u, v).

Proposition 10.3.1 *Let γ_1 and γ_2 be curves in the complex z-plane that intersect at a point z_0 at an angle α. Let $f:\mathbb{C}\to\mathbb{C}$ be a mapping given by $f(z)=z'=x'+iy'$ that is analytic at z_0. Let γ_1' and γ_2' be the images of γ_1 and γ_2 under this mapping, which intersect at an angle α'. Then,*

(a) $\alpha'=\alpha$, *that is, the mapping f is* conformal, *if* $(dz'/dz)_{z_0}\neq 0$.
(b) *If f is harmonic in (x,y), it is also harmonic in (x',y').*

Proof See Problem 10.21. □

The following are some examples of conformal mappings.

(a) $z'=z+a$, where a is an arbitrary complex constant. This is simply a **translation** of the z-plane.

translation

dilation

(b) $z'=bz$, where b is an arbitrary complex constant. This is a **dilation** whereby distances are dilated by a factor $|b|$. A graph in the z-plane is mapped onto a *similar* (congruent) graph in the z'-plane that will be reduced ($|b|<1$) or enlarged ($|b|>1$) by a factor of $|b|$.

inversion

(c) $z'=1/z$. This is called an **inversion**. Example 10.3.2 will show that under such a mapping, circles are mapped onto circles or straight lines.
(d) Combining the preceding three transformations yields the general mapping

$$z'=\frac{az+b}{cz+d}, \tag{10.8}$$

which is conformal if $cz+d\neq 0\neq dz'/dz$. These conditions are equivalent to $ad-bc\neq 0$.

Example 10.3.2 A circle of radius r whose center is at a in the z-plane is described by the equation $|z-a|=r$. When transforming to the z'-plane under inversion, this equation becomes $|1/z'-a|=r$, or $|1-az'|=r|z'|$. Squaring both sides and simplifying yields $(r^2-|a|^2)|z'|^2+2\,\mathrm{Re}(az')-1=0$. In terms of Cartesian coordinates, this becomes

$$\left(r^2-|a|^2\right)\left(x'^2+y'^2\right)+2\left(a_r x'-a_i y'\right)-1=0, \tag{10.9}$$

where $a\equiv a_r+ia_i$. We now consider two cases:

1. $r\neq|a|$: Divide by $r^2-|a|^2$ and complete the squares to get

$$\left(x'+\frac{a_r}{r^2-|a|^2}\right)^2+\left(y'-\frac{a_i}{r^2-|a|^2}\right)^2-\frac{a_r^2+a_i^2}{(r^2-|a|^2)^2}-\frac{1}{r^2-|a|^2}=0$$

or defining

$$a_r'\equiv -a_r/\left(r^2-|a|^2\right),$$
$$a_i'\equiv a_i/\left(r^2-|a|^2\right) \quad\text{and}\quad r'\equiv r/\left|r^2-|a|^2\right|,$$

we have $(x'-a_r')^2+(y'-a_i')^2=r'^2$, which can also be written as

$$\left|z'-a'\right|=r',\quad a'=a_r'+ia_i'=\frac{a^*}{|a|^2-r^2}.$$

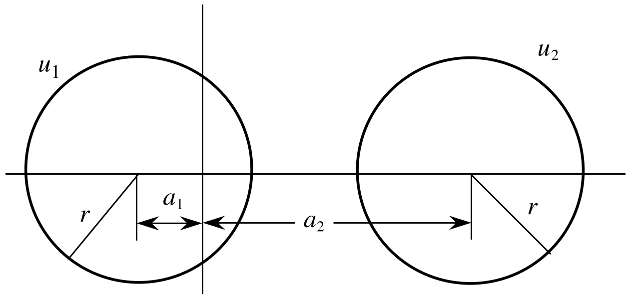

Fig. 10.3 In the z-plane, we see two equal cylinders whose centers are separated

This is a circle in the z'-plane with center at a' and radius of r'.

2. $r = a$: Then Eq. (10.9) reduces to $a_r x' - a_i y' = \frac{1}{2}$, which is the equation of a line.

If we use the transformation $z' = 1/(z - c)$ instead of $z' = 1/z$, then $|z - a| = r$ becomes $|1/z' - (a - c)| = r$, and all the above analysis will go through exactly as before, except that a is replaced by $a - c$.

Mappings of the form given in Eq. (10.8) are called **homographic transformations**. A useful property of such transformations is that they can map an infinite region of the z-plane onto a finite region of the z'-plane. In fact, points with very large values of z are mapped onto a neighborhood of the point $z' = a/c$. Of course, this argument goes both ways: Eq. (10.8) also maps a neighborhood of $-d/c$ in the z-plane onto large regions of the z'-plane. The usefulness of homographic transformations is illustrated in the following example.

homographic transformations

Example 10.3.3 Consider two cylindrical conductors of equal radius r, held at potentials u_1 and u_2, respectively, whose centers are D units of length apart. Choose the x-and the y-axes such that the centers of the cylinders are located on the x-axis at distances a_1 and a_2 from the origin, as shown in Fig. 10.3. Let us find the electrostatic potential produced by such a configuration in the xy-plane.

electrostatic potential of two charged cylinder

We know from elementary electrostatics that the problem becomes very simple if the two cylinders are concentric (and, of course, of different radii). Thus, we try to map the two circles onto two concentric circles in the z'-plane such that the infinite region outside the two circles in the z-plane gets mapped onto the finite annular region between the two concentric circles in the z'-plane. We then (easily) find the potential in the z'-plane, and transfer it back to the z-plane.

The most general mapping that may be able to do the job is that given by Eq. (10.8). However, it turns out that we do not have to be this general. In fact, the special case $z' = 1/(z - c)$ in which c is a *real* constant will be sufficient. So, $z = (1/z') + c$, and the circles $|z - a_k| = r$ for $k = 1, 2$ will be mapped onto the circles $|z' - a'_k| = r'_k$, where (by Example 10.3.2) $a'_k = (a_k - c)/[(a_k - c)^2 - r^2]$ and $r'_k = r/|(a_k - c)^2 - r^2|$.

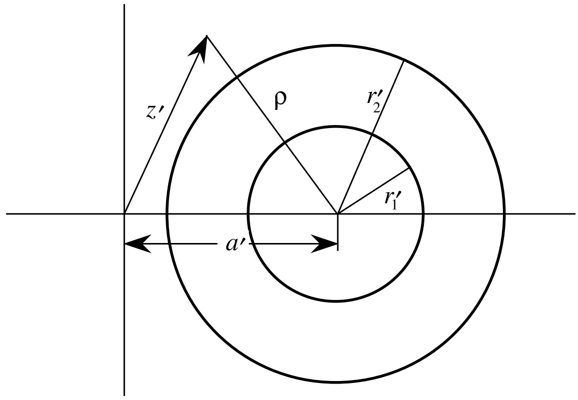

Fig. 10.4 In the z'-plane, we see two concentric unequal cylinders

Can we arrange the parameters so that the circles in the z'-plane are concentric, i.e., that $a'_1 = a'_2$? The answer is yes. We set $a'_1 = a'_2$ and solve for a_2 in terms of a_1. The result is either the trivial solution $a_2 = a_1$, or $a_2 = c - r^2/(a_1 - c)$. If we place the origin of the z-plane at the center of the first cylinder, then $a_1 = 0$ and $a_2 = D = c + r^2/c$. We can also find a'_1 and a'_2: $a'_1 = a'_2 \equiv a' = -c/(c^2 - r^2)$, and the geometry of the problem is as shown in Fig. 10.4.

For such a geometry the potential at a point in the annular region is given by

$$\Phi' = A \ln \rho + B = A \ln\left|z' - a'\right| + B,$$

where A and B are real constants determined by the conditions $\Phi'(r'_1) = u_1$ and $\Phi'(r'_2) = u_2$, which yields

$$A = \frac{u_1 - u_2}{\ln(r'_1/r'_2)} \quad \text{and} \quad B = \frac{u_2 \ln r'_1 - u_1 \ln r'_2}{\ln(r'_1/r'_2)}.$$

The potential Φ' is the real part of the complex function[8]

$$F\left(z'\right) = A \ln\left(z' - a'\right) + B,$$

which is analytic except at $z' = a'$, a point lying outside the region of interest. We can now go back to the z-plane by substituting $z' = 1/(z - c)$ to obtain

$$G(z) = A \ln\left(\frac{1}{z - c} - a'\right) + B,$$

whose real part is the potential in the z-plane:

$$\Phi(x, y) = \text{Re}\left[G(z)\right] = A \ln\left|\frac{1 - a'z + a'c}{z - c}\right| + B$$

[8] Writing $z = |z|e^{i\theta}$, we note that $\ln z = \ln|z| + i\theta$, so that the real part of a complex log function is the log of the absolute value.

$$= A \ln\left|\frac{(1+a'c-a'x)-ia'y}{(x-c)+iy}\right| + B$$

$$= \frac{A}{2}\ln\left[\frac{(1+a'c-a'x)^2+a'^2y^2}{(x-c)^2+y^2}\right] + B.$$

This is the potential we want.

10.4 Integration of Complex Functions

The derivative of a complex function is an important concept and, as the previous section demonstrated, provides a powerful tool in physical applications. The concept of integration is even more important. In fact, we will see in the next section that derivatives can be written in terms of integrals. We will study integrals of complex functions in detail in this section.

The definite integral of a complex function is defined in analogy to that of a real function:

$$\int_{\alpha_1}^{\alpha_2} f(z)\,dz = \lim_{\substack{N\to\infty \\ \Delta z_i \to 0}} \sum_{i=1}^{N} f(z_i)\Delta z_i,$$

where Δz_i is a small segment, situated at z_i, of the curve that connects the complex number α_1 to the complex number α_2 in the z-plane. Since there are infinitely many ways of connecting α_1 to α_2, it is possible to obtain different values for the integral for different paths. Before discussing the integral itself, let us first consider the various kinds of path encountered in complex analysis.

curve, simple arc, path, and smooth arc defined

1. A **curve** is a map $\gamma : [a,b] \to \mathbb{C}$ from the real interval into the complex plane given by $\gamma(t) = \gamma_r(t) + i\gamma_i(t)$, where $a \le t \le b$, and γ_r and γ_i are the real and imaginary parts of γ; $\gamma(a)$ is called the **initial point** of the curve and $\gamma(b)$ its **final point**.
2. A **simple arc**, or a Jordan arc, is a curve that does not cross itself, i.e., γ is injective (or one to one), so that $\gamma(t_1) \ne \gamma(t_2)$ when $t_1 \ne t_2$.
3. A **path** is a finite collection $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ of simple arcs such that the initial point of γ_{k+1} coincides with the final point of γ_k.
4. A **smooth** arc is a curve for which $d\gamma/dt = d\gamma_r/dt + i\,d\gamma_i/dt$ exists and is nonzero for $t \in [a,b]$.
5. A **contour** is a path whose arcs are smooth. When the initial point of γ_1 coincides with the final point of γ_n, the contour is said to be a **simple closed contour**.

contour defined

The path dependence of a complex integral is analogous to the line integral of a vector field encountered in vector analysis. In fact, we can turn the integral of a complex function into a line integral as follows. We substitute $f(z) = u + iv$ and $dz = dx + i\,dy$ in the integral to obtain

$$\int_{\alpha_1}^{\alpha_2} f(z)\,dz = \int_{\alpha_1}^{\alpha_2} (u\,dx - v\,dy) + i\int_{\alpha_1}^{\alpha_2} (v\,dx + u\,dy).$$

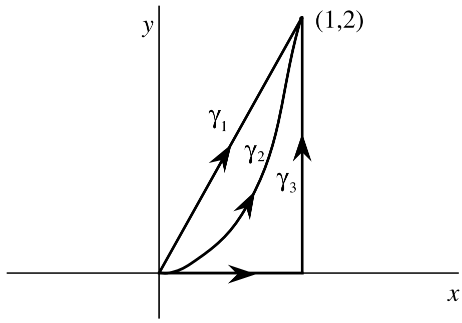

Fig. 10.5 The three different paths of integration corresponding to the integrals I_1, I_1', I_2, and I_2'

If we define the two-dimensional vectors $\mathbf{A}_1 \equiv (u, -v)$ and $\mathbf{A}_2 \equiv (v, u)$, we get

$$\int_{\alpha_1}^{\alpha_2} f(z)\,dz = \int_{\alpha_1}^{\alpha_2} \mathbf{A}_1 \cdot d\mathbf{r} + i \int_{\alpha_1}^{\alpha_2} \mathbf{A}_2 \cdot d\mathbf{r}.$$

It follows from Stokes' theorem (or Green's theorem, since the vectors lie in a plane) that the integral of f is path-independent only if both $\mathbf{A}_1$ and $\mathbf{A}_2$ have vanishing curls. This in turn follows if and only if u and v satisfy the C-R conditions, and this is exactly what is needed for $f(z)$ to be analytic.

Path-independence of a line integral of a vector $\mathbf{A}$ is equivalent to the vanishing of the integral along a closed path, and the latter is equivalent to the vanishing of $\nabla \times \mathbf{A} = 0$ at *every* point of the region bordered by the closed path. In the case of complex integrals, this result is stated as

Cauchy-Goursat theorem

Theorem 10.4.1 (Cauchy-Goursat theorem) *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic on a simple closed contour C and at all points inside C. Then*

$$\oint_C f(z)\,dz = 0.$$

Example 10.4.2 (Examples of definite integrals)

(a) Let us evaluate the integral $I_1 = \int_{\gamma_1} z\,dz$ where γ_1 is the straight line drawn from the origin to the point $(1, 2)$ (see Fig. 10.5). Along such a line $y = 2x$ and, using t for x, $\gamma_1(t) = t + 2it$ where $0 \le t \le 1$; so

$$I_1 = \int_{\gamma_1} z\,dz = \int_0^1 (t + 2it)(dt + 2i\,dt)$$
$$= \int_0^1 (-3t\,dt + 4it\,dt) = -\frac{3}{2} + 2i.$$

For a different path γ_2, along which $y = 2x^2$, we get $\gamma_2(t) = t + 2it^2$ where $0 \le t \le 1$, and

$$I_1' = \int_{\gamma_2} z\,dz = \int_0^1 \big(t + 2it^2\big)(dt + 4it\,dt) = -\frac{3}{2} + 2i.$$

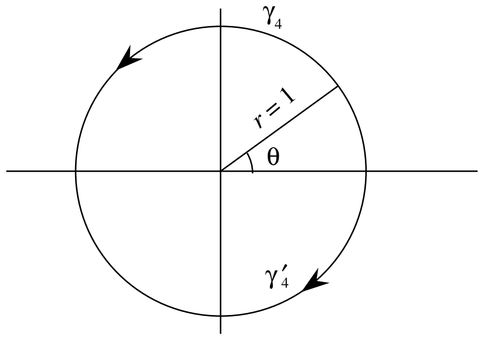

Fig. 10.6 The two semicircular paths for calculating I_3 and I_3'

Therefore, $I_1 = I_1'$. This is what is expected from the Cauchy-Goursat theorem because the function $f(z) = z$ is analytic on the two paths and in the region bounded by them.

(b) To find $I_2 \equiv \int_{\gamma_1} z^2 dz$ with γ_1 as in part (a), substitute for z in terms of t:

$$I_2 = \int_{\gamma_1} (t + 2it)^2 (dt + 2idt) = (1 + 2i)^3 \int_0^1 t^2 dt = -\frac{11}{3} - \frac{2}{3}i.$$

Next we compare I_2 with $I_2' = \int_{\gamma_3} z^2 dz$ where γ_3 is as shown in Fig. 10.5. This path can be described by

$$\gamma_3(t) = \begin{cases} t & \text{for } 0 \le t \le 1, \\ 1 + i(t-1) & \text{for } 1 \le t \le 3. \end{cases}$$

Therefore,

$$I_2' = \int_0^1 t^2 dt + \int_1^3 \left[1 + i(t-1)\right]^2 (idt) = \frac{1}{3} - 4 - \frac{2}{3}i = -\frac{11}{3} - \frac{2}{3}i,$$

which is identical to I_2, once again because the function is analytic on γ_1 and γ_3 as well as in the region bounded by them.

(c) Now consider $I_3 \equiv \int_{\gamma_4} dz/z$ where γ_4 is the upper semicircle of unit radius, as shown in Fig. 10.6. A parametric equation for γ_4 can be given in terms of θ:

$$\gamma_4(\theta) = \cos\theta + i\sin\theta = e^{i\theta} \quad \Rightarrow \quad dz = ie^{i\theta} d\theta, \quad 0 \le \theta \le \pi.$$

Thus, we obtain

$$I_3 = \int_0^\pi \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = i\pi.$$

On the other hand, for γ_4', the lower semicircle of unit radius, we get

$$I_3' = \int_{\gamma_4'} \frac{1}{z} dz = \int_{2\pi}^{\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = -i\pi.$$

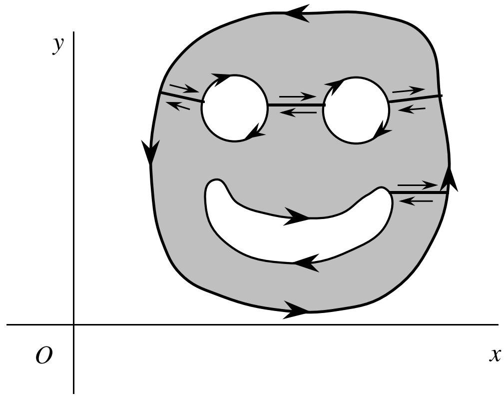

Fig. 10.7 A complicated contour can be broken up into simpler ones. Note that the boundaries of the "eyes" and the "mouth" are forced to be traversed in the (negative) clockwise direction

Here the two integrals are not equal. From γ_4 and γ_4' we can construct a counterclockwise simple closed contour C, along which the integral of $f(z) = 1/z$ becomes $\oint_C dz/z = I_3 - I_3' = 2i\pi$. That the integral is not zero is a consequence of the fact that $1/z$ is *not* analytic at all points of the region bounded by the closed contour C.

The Cauchy-Goursat theorem applies to more complicated regions. When a region contains points at which $f(z)$ is not analytic, those points can be avoided by redefining the region and the contour. Such a procedure requires an agreement on the direction we will take.

convention for positive sense of integration around a closed contour

Convention *When integrating along a closed contour, we agree to move along the contour in such a way that the enclosed region lies to our left. An integration that follows this convention is called integration in the* ***positive sense****. Integration performed in the opposite direction acquires a minus sign.*

For a simple closed contour, movement in the counterclockwise direction yields integration in the positive sense. However, as the contour becomes more complicated, this conclusion breaks down. Figure 10.7 shows a complicated path enclosing a region (shaded) in which the integrand is analytic. Note that it is possible to traverse a portion of the region twice in opposite directions without affecting the integral, which may be a sum of integrals for different pieces of the contour. Also note that the "eyes" and the "mouth" are traversed clockwise! This is necessary because of the convention above. A region such as that shown in Fig. 10.7, in which holes are "punched out", is called **multiply connected**. In contrast, a **simply connected** region is one in which every simple closed contour encloses only points of the region.

simply and multiply connected regions

Cauchy Integral Formula (CIF)

One important consequence of the Cauchy-Goursat theorem is the following:

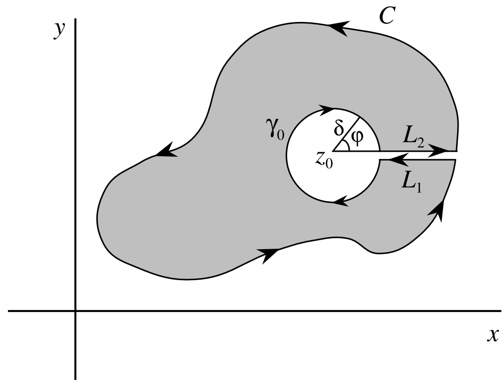

Fig. 10.8 The integrand is analytic within and on the boundary of the *shaded region*. It is always possible to construct contours that exclude all singular points

Theorem 10.4.3 (Cauchy integral formula) *Let f be analytic on and within a simple closed contour C integrated in the positive sense. Let z_0 be any interior point to C. Then*

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

To prove the Cauchy integral formula (CIF), we need the following lemma.

Lemma 10.4.4 (Darboux inequality) *Suppose $f : \mathbb{C} \to \mathbb{C}$ is continuous and bounded on a path γ, i.e., there exists a positive number M such that $|f(z)| \le M$ for all values $z \in \gamma$. Then*

$$\left| \int_\gamma f(z)\, dz \right| \le M L_\gamma,$$

where L_γ is the length of the path of integration.

Proof See Problem 10.27. □

Now we are ready to prove the Cauchy integral formula.

Proof of CIF Consider the shaded region in Fig. 10.8, which is bounded by C, by γ_0 (a circle of arbitrarily small radius δ centered at z_0), and by L_1 and L_2, two straight line segments infinitesimally close to one another (we can, in fact, assume that L_1 and L_2 are right on top of one another; however, they are separated in the figure for clarity). Let $C' = C \cup \gamma_0 \cup L_1 \cup L_2$.

Since $f(z)/(z - z_0)$ is analytic everywhere on the contour C' and inside the shaded region, we can write

$$0 = \oint_{C'} \frac{f(z)}{z - z_0} dz = \oint_C \frac{f(z)}{z - z_0} dz + \oint_{\gamma_0} \frac{f(z)}{z - z_0} dz \qquad (10.10)$$

because the contributions from L_1 and L_2 cancel. Let us evaluate the contribution from the infinitesimal circle γ_0. First we note that because $f(z)$ is continuous (differentiability implies continuity), we can write

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| = \frac{|f(z)-f(z_0)|}{|z-z_0|} = \frac{|f(z)-f(z_0)|}{\delta} < \frac{\epsilon}{\delta}$$

for $z \in \gamma_0$, where ϵ is a small positive number. We now apply the Darboux inequality and write

$$\left|\oint_{\gamma_0} \frac{f(z)-f(z_0)}{z-z_0}dz\right| < \frac{\epsilon}{\delta}2\pi\delta = 2\pi\epsilon.$$

This means that the integral goes to zero as $\delta \to 0$, or

$$\oint_{\gamma_0} \frac{f(z)}{z-z_0}dz = \oint_{\gamma_0} \frac{f(z_0)}{z-z_0}dz = f(z_0)\oint_{\gamma_0} \frac{dz}{z-z_0}.$$

We can easily calculate the integral on the RHS by noting that $z - z_0 = \delta e^{i\varphi}$ and that γ_0 has a *clockwise* direction:

$$\oint_{\gamma_0} \frac{dz}{z-z_0} = -\int_0^{2\pi} \frac{i\delta e^{i\varphi}d\varphi}{\delta e^{i\varphi}} = -2\pi i \quad \Rightarrow \quad \oint_{\gamma_0} \frac{f(z)}{z-z_0}dz = -2\pi i f(z_0).$$

Substituting this in (10.10) yields the desired result. □

Example 10.4.5 We can use the CIF to evaluate the integrals

$$I_1 = \oint_{C_1} \frac{z^2dz}{(z^2+3)^2(z-i)}, \qquad I_2 = \oint_{C_2} \frac{(z^2-1)dz}{(z-\frac{1}{2})(z^2-4)^3},$$

$$I_3 = \oint_{C_3} \frac{e^{z/2}dz}{(z-i\pi)(z^2-20)^4},$$

where C_1, C_2, and C_3 are circles centered at the origin with radii $r_1 = 3/2$, $r_2 = 1$, and $r_3 = 4$.

For I_1 we note that $f(z) = z^2/(z^2+3)^2$ is analytic within and on C_1, and $z_0 = i$ lies in the interior of C_1. Thus,

$$I_1 = \oint_{C_1} \frac{f(z)dz}{z-i} = 2\pi i f(i) = 2\pi i \frac{i^2}{(i^2+3)^2} = -i\frac{\pi}{2}.$$

Similarly, $f(z) = (z^2-1)/(z^2-4)^3$ for the integral I_2 is analytic on and within C_2, and $z_0 = 1/2$ is an interior point of C_2. Thus, the CIF gives

$$I_2 = \oint_{C_2} \frac{f(z)dz}{z-\frac{1}{2}} = 2\pi i f(1/2) = \frac{32\pi}{1125}i.$$

For the last integral, $f(z) = e^{z/2}/(z^2-20)^4$, and the interior point is $z_0 = i\pi$:

$$I_3 = \oint_{C_3} \frac{f(z)dz}{z-i\pi} = 2\pi i f(i\pi) = -\frac{2\pi}{(\pi^2+20)^4}.$$

The Cauchy integral formula gives the value of an analytic function at every point inside a simple closed contour when it is given the value of the function only at points on the contour. It seems as though an analytic function is not free to change inside a region once its value is fixed on the contour enclosing that region.

There is an analogous situation in electrostatics: The specification of the potential at the boundaries, such as the surfaces of conductors, automatically determines the potential at any other point in the region of space bounded by the conductors. This is the content of the uniqueness theorem used in electrostatic boundary value problems. However, the electrostatic potential Φ is bound by another condition, Laplace's equation; and the *combination* of Laplace's equation and the boundary conditions furnishes the uniqueness of Φ. Similarly, the real and imaginary parts of an analytic function separately satisfy Laplace's equation in two dimensions! Thus, it should come as no surprise that the value of an analytic function on a boundary (contour) determines the function at all points inside the boundary.

Explanation of why the Cauchy integral formula works!

10.5 Derivatives as Integrals

The Cauchy Integral Formula is a powerful tool for working with analytic functions. One of the applications of this formula is in evaluating the derivatives of such functions. It is convenient to change the dummy integration variable to ξ and write the CIF as

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)\, d\xi}{\xi - z}, \tag{10.11}$$

where C is a simple closed contour in the ξ-plane and z is a point within C. As preparation for defining the derivative of an analytic function, we need the following result.

Proposition 10.5.1 *Let γ be any path—a contour, for example—and g a continuous function on that path. The function $f(z)$ defined by*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{g(\xi)\, d\xi}{\xi - z}$$

is analytic at every point $z \notin \gamma$.

Proof The proof follows immediately from differentiation of the integral:

$$\begin{aligned}\frac{df}{dz} &= \frac{1}{2\pi i} \frac{d}{dz} \int_\gamma \frac{g(\xi)\, d\xi}{\xi - z} \\ &= \frac{1}{2\pi i} \int_\gamma g(\xi)\, d\xi \frac{d}{dz}\left(\frac{1}{\xi - z}\right) = \frac{1}{2\pi i} \int_\gamma \frac{g(\xi)\, d\xi}{(\xi - z)^2}.\end{aligned}$$

This is defined for all values of z not on γ.[9] Thus, $f(z)$ is analytic there. □

We can generalize the formula above to the nth derivative, and obtain

$$\frac{d^n f}{dz^n} = \frac{n!}{2\pi i}\int_\gamma \frac{g(\xi)\,d\xi}{(\xi - z)^{n+1}}.$$

Applying this result to an analytic function expressed by Eq. (10.11), we obtain the following important theorem.

derivative of an analytic function given in terms of an integral

Theorem 10.5.2 *The derivatives of all orders of an analytic function $f(z)$ exist in the domain of analyticity of the function and are themselves analytic in that domain. The nth derivative of $f(z)$ is given by*

$$f^{(n)}(z) = \frac{d^n f}{dz^n} = \frac{n!}{2\pi i}\oint_C \frac{f(\xi)\,d\xi}{(\xi - z)^{n+1}}. \tag{10.12}$$

Example 10.5.3 Let us apply Eq. (10.12) directly to some simple functions. In all cases, we will assume that the contour is a circle of radius r centered at z.

(a) Let $f(z) = K$, a constant. Then, for $n = 1$ we have

$$\frac{df}{dz} = \frac{1}{2\pi i}\oint_C \frac{K\,d\xi}{(\xi - z)^2}.$$

Since ξ is on the circle C centered at z, $\xi - z = re^{i\theta}$ and $d\xi = rie^{i\theta}d\theta$. So we have

$$\frac{df}{dz} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{Kire^{i\theta}d\theta}{(re^{i\theta})^2} = \frac{K}{2\pi r}\int_0^{2\pi} e^{-i\theta}d\theta = 0.$$

(b) Given $f(z) = z$, its first derivative will be

$$\begin{aligned}\frac{df}{dz} &= \frac{1}{2\pi i}\oint_C \frac{\xi\,d\xi}{(\xi - z)^2} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(z + re^{i\theta})ire^{i\theta}d\theta}{(re^{i\theta})^2}\\ &= \frac{1}{2\pi}\left(\frac{z}{r}\int_0^{2\pi} e^{-i\theta}d\theta + \int_0^{2\pi} d\theta\right) = \frac{1}{2\pi}(0 + 2\pi) = 1.\end{aligned}$$

(c) Given $f(z) = z^2$, for the first derivative, Eq. (10.12) yields

$$\begin{aligned}\frac{df}{dz} &= \frac{1}{2\pi i}\oint_C \frac{\xi^2\,d\xi}{(\xi - z)^2} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(z + re^{i\theta})^2 ire^{i\theta}d\theta}{(re^{i\theta})^2}\\ &= \frac{1}{2\pi}\int_0^{2\pi}\left[z^2 + \left(re^{i\theta}\right)^2 + 2zre^{i\theta}\right]\left(re^{i\theta}\right)^{-1}d\theta\end{aligned}$$

[9]The interchange of differentiation and integration requires justification. Such an interchange can be done if the integral has some restrictive properties. We shall not concern ourselves with such details. In fact, one can achieve the same result by using the definition of derivatives and the usual properties of integrals.

$$= \frac{1}{2\pi}\left(\frac{z^2}{r}\int_0^{2\pi} e^{-i\theta}d\theta + r\int_0^{2\pi} e^{i\theta}d\theta + 2z\int_0^{2\pi} d\theta\right) = 2z.$$

It can be shown that, in general, $(d/dz)z^m = mz^{m-1}$. The proof is left as Problem 10.30.

The CIF is a central formula in complex analysis, and we shall see its significance in much of the later development of complex analysis. For now, let us demonstrate its usefulness in proving a couple of important properties of analytic functions.

Proposition 10.5.4 *The absolute value of an analytic function $f(z)$ cannot have a local maximum within the region of analyticity of the function.*

Proof Let $S \subset \mathbb{C}$ be the region of analyticity of f and z_0 a point in S. Let γ_0 be a circle of radius δ in S, centered at z_0. Using the CIF, and noting that $z - z_0 = \delta e^{i\theta}$, we have

$$\begin{aligned} \left|f(z_0)\right| &= \left|\frac{1}{2\pi i}\oint_{\gamma_0}\frac{f(z)}{z-z_0}dz\right| = \frac{1}{2\pi}\left|\int_0^{2\pi}\frac{f(z)}{\delta e^{i\theta}}i\delta e^{i\theta}d\theta\right| \\ &\le \frac{1}{2\pi}\int_0^{2\pi}\left|f(z)\right|d\theta \le \frac{1}{2\pi}\int_0^{2\pi}\left|f(z_{\max})\right|d\theta = \left|f(z_{\max})\right|, \end{aligned}$$

where $z_{\max}$ is where the maximum value of $|f(z)|$ occurs on γ_0. This inequality says that for any point z_0 that one picks, there is always another point which produces a larger absolute value for f. Therefore, there can be no local maximum within S. □

Proposition 10.5.5 *A bounded entire function is necessarily a constant.*

Proof We show that the derivative of such a function is zero. Consider

$$\frac{df}{dz} = \frac{1}{2\pi i}\oint_C \frac{f(\xi)\,d\xi}{(\xi - z)^2}.$$

Since f is an entire function, the closed contour C can be chosen to be a very large circle of radius R with center at z. Taking the absolute value of both sides yields

$$\begin{aligned} \left|\frac{df}{dz}\right| &= \frac{1}{2\pi}\left|\int_0^{2\pi}\frac{f(z)}{(Re^{i\theta})^2}iRe^{i\theta}d\theta\right| \\ &\le \frac{1}{2\pi}\int_0^{2\pi}\frac{|f(z)|}{R}d\theta \le \frac{1}{2\pi}\int_0^{2\pi}\frac{M}{R}d\theta = \frac{M}{R}, \end{aligned}$$

where M is the maximum of the function in the complex plane. Now, as $R \to \infty$, the derivative goes to zero, and the function must be a constant. □

Proposition 10.5.5 is a very powerful statement about analytic functions. There are many interesting and nontrivial *real* functions that are bounded and have derivatives of all orders on the entire real line. For instance, e^{-x^2}

is such a function. No such freedom exists for *complex* analytic functions. Any nontrivial analytic function is either not bounded (goes to infinity somewhere on the complex plane) or not entire (it is not analytic at some point(s) of the complex plane).

A consequence of Proposition 10.5.5 is the following

fundamental theorem of algebra

Theorem 10.5.6 (Fundamental theorem of algebra) *Any polynomial*

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad a_n \neq 0$$

can be factored completely as

$$p(x) = a_n(x - z_1)(x - z_2) \cdots (x - z_n),$$

where the z_i are complex numbers.

Proof Let $f(z) = 1/p(z)$ and assume the contrary, i.e., that $p(z)$ is never zero for any (finite) $z \in \mathbb{C}$. Then $f(z)$ is bounded and analytic for all $z \in \mathbb{C}$, and Proposition 10.5.5 says that $f(z)$ is a constant. This is obviously wrong if $n > 0$. Thus, there must be at least one z, say $z = z_1$, for which $p(z)$ is zero. So, we can factor out $(z - z_1)$ from $p(z)$ and write $p(z) = (z - z_1)q(z)$ where $q(z)$ is of degree $n - 1$. Applying the above argument to $q(z)$, we have $p(z) = (z - z_1)(z - z_2)r(z)$ where $r(z)$ is of degree $n - 2$. Continuing in this way, we can factor $p(z)$ into linear factors. The last polynomial will be a constant (a polynomial of degree zero) which has to be equal to a_n to make the coefficient of z^n equal to the original polynomial. □

The primitive (indefinite integral) of an analytic function can be defined using definite integrals just as in the real case. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic in a region S of the complex plane. Let z_0 and z be two points in S, and define[10] $F(z) \equiv \int_{z_0}^{z} f(\xi)\, d\xi$. We can show that $F(z)$ is the primitive of $f(z)$ by showing that

$$\lim_{\Delta z \to 0} \left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| = 0.$$

We leave the details as a problem for the reader.

Proposition 10.5.7 *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic in a region S of $\mathbb{C}$. Then at every point $z \in S$, there exists an analytic function $F : \mathbb{C} \to \mathbb{C}$ such that*

$$\frac{dF}{dz} = f(z).$$

[10] Note that the integral is path-independent due to the analyticity of f. Thus, F is well-defined.

In the sketch of the proof of Proposition 10.5.7, we used only the continuity of f and the fact that the integral was well-defined. These two conditions are sufficient to establish the analyticity of F *and* f, since the latter is the derivative of the former. The following theorem, due to Morera, states this fact and is the converse of the Cauchy-Goursat theorem.

Theorem 10.5.8 (Morera's theorem) *Let a function $f : \mathbb{C} \to \mathbb{C}$ be continuous in a simply connected region S. If for each simple closed contour C in S we have $\oint_C f(\xi)\, d\xi = 0$, then f is analytic throughout S.* Morera's theorem

10.6 Infinite Complex Series

The expansion of functions in terms of polynomials or monomials is important in calculus and was emphasized in Chaps. 7 and 8. We now apply this concept to analytic functions.

10.6.1 Properties of Series

Complex series are very similar to real series with which the reader is assumed to have some familiarity. Therefore, we state (without proof) the most important properties of complex series before discussing the quintessential Taylor and Laurent series.

A complex series is said to **converge absolutely** if the *real* series absolute convergence

$$\sum_{k=0}^{\infty} |z_k| = \sum_{k=0}^{\infty} \sqrt{x_k^2 + y_k^2}$$

converges. Clearly, absolute convergence implies convergence.

Proposition 10.6.1 *If the* ***power series*** $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ *converges for* $z_1 \neq z_0$, *then it converges* absolutely *for every value of z such that* $|z - z_0| < |z_1 - z_0|$. *Similarly if the power series* $\sum_{k=0}^{\infty} b_k/(z - z_0)^k$ *converges for* $z_2 \neq z_0$, *then it converges* absolutely *for every value of z such that* $|z - z_0| > |z_2 - z_0|$. power series

A geometric interpretation of this proposition is that if a power series—with positive powers—converges for a point at a distance r_1 from z_0, then it converges for *all interior* points of the circle whose center is z_0, and whose radius is r_1. Similarly, if a power series—with negative powers—converges for a point at a distance r_2 from z_0, then it converges for *all exterior* points of the circle whose center is z_0 and whose radius is r_2 (see Fig. 10.9). Generally speaking, positive powers are used for points inside a circle and negative powers for points outside it.

The largest circle about z_0 such that the first power series of Proposition 10.6.1 converges is called the **circle of convergence** of the power series. circle of convergence

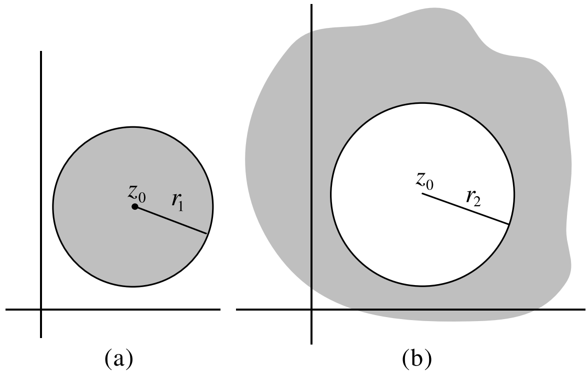

Fig. 10.9 (**a**) Power series with positive exponents converge for the interior points of a circle. (**b**) Power series with negative exponents converge for the exterior points of a circle

The proposition implies that the series cannot converge at *any* point outside the circle of convergence.

In determining the convergence of a power series

$$S(z) \equiv \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{10.13}$$

we look at the behavior of the sequence of partial sums

$$S_N(z) \equiv \sum_{n=0}^{N} a_n (z - z_0)^n.$$

Convergence of (10.13) implies that for any $\varepsilon > 0$, there exists an integer N_ε such that

$$\left|S(z) - S_N(z)\right| < \varepsilon \quad \text{whenever } N > N_\varepsilon.$$

uniform convergence explained

In general, the integer N_ε may be dependent on z; that is, for different values of z, we may be forced to pick different N_ε's. When N_ε is independent of z, we say that the convergence is **uniform**.

power series are uniformly convergent and analytic

Theorem 10.6.2 *The power series $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is* ***uniformly*** *convergent for all points within its circle of convergence and represents a function that is analytic there.*

By substituting the reciprocal of $(z - z_0)$ in the power series, we can show that if $\sum_{k=0}^{\infty} b_k/(z - z_0)^k$ is convergent in the annulus $r_2 < |z - z_0| < r_1$, then it is uniformly convergent for all z in that annulus.

power series can be differentiated and integrated term by term

Theorem 10.6.3 *A convergent power series can be differentiated and integrated term by term; that is, if $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, then*

$$\frac{dS(z)}{dz} = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}, \qquad \int_\gamma S(z)\,dz = \sum_{n=0}^{\infty} a_n \int_\gamma (z-z_0)^n\,dz$$

for any path γ lying in the circle of convergence of the power series.

10.6.2 Taylor and Laurent Series

We now state and prove the two main theorems of this section. A Taylor series consists of terms with only positive powers. A Laurent series allows for negative powers as well.

Taylor series

Theorem 10.6.4 (Taylor series) *Let f be analytic throughout the interior of a circle C_0 having radius r_0 and centered at z_0. Then at each point z inside C_0,*

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n. \qquad (10.14)$$

Proof From the CIF and the fact that z is inside C_0, we have

$$f(z) = \frac{1}{2\pi i}\oint_{C_0} \frac{f(\xi)}{\xi - z}\,d\xi.$$

On the other hand,

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0 + z_0 - z} = \frac{1}{(\xi - z_0)(1 - \frac{z-z_0}{\xi-z_0})}$$
$$= \frac{1}{\xi - z_0}\frac{1}{1 - \frac{z-z_0}{\xi-z_0}} = \frac{1}{\xi - z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\xi-z_0}\right)^n.$$

The last equality follows from the fact that $|(z-z_0)/(\xi-z_0)| < 1$—because z is inside the circle C_0 and ξ is on it—and from the sum of a geometric series. Substituting in the CIF and using Theorem 10.5.2, we obtain the result. □

Maclaurin series

For $z_0 = 0$ we obtain the **Maclaurin series**:

$$f(z) = f(0) + f'(0)z + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

The Taylor expansion requires analyticity of the function at all points interior to the circle C_0. On many occasions there may be a point inside C_0 at which the function is not analytic. The Laurent series accommodates such cases.

Laurent series

Theorem 10.6.5 (Laurent series) *Let C_1 and C_2 be circles of radii r_1 and r_2, both centered at z_0 in the z-plane with $r_1 > r_2$. Let $f : \mathbb{C} \to \mathbb{C}$*

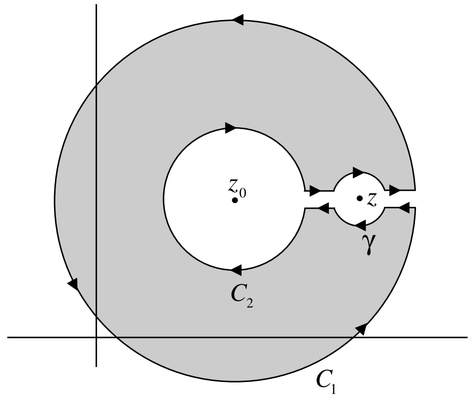

Fig. 10.10 The annular region within and on whose contour the expanded function is analytic

be analytic on C_1 and C_2 and throughout S, the annular region between the two circles. Then, at each point $z \in S$, $f(z)$ is given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \textit{where} \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$

and C is any contour within S that encircles z_0.

Proof Let γ be a small closed contour in S enclosing z, as shown in Fig. 10.10. For the composite contour $C' = C_1 \cup C_2 \cup \gamma$, the Cauchy-Goursat theorem gives

$$0 = \oint_{C'} \frac{f(\xi)}{\xi - z} d\xi = \oint_{C_1} \frac{f(\xi)}{\xi - z} d\xi - \oint_{C_2} \frac{f(\xi)}{\xi - z} d\xi - \oint_{\gamma} \frac{f(\xi)}{\xi - z} d\xi,$$

where the γ and C_2 integrations are negative because their interior lies to our right as we traverse them. The γ integral is simply $2\pi i f(z)$ by the CIF. Thus, we obtain

$$2\pi i f(z) = \oint_{C_1} \frac{f(\xi)}{\xi - z} d\xi - \oint_{C_2} \frac{f(\xi)}{\xi - z} d\xi. \tag{10.15}$$

Now we use the same trick we used in deriving the Taylor expansion. Since z is located in the annular region, $r_2 < |z - z_0| < r_1$. We have to keep this in mind when expanding the fractions. In particular, for $\xi \in C_1$ we want the ξ term to be in the denominator, and for $\xi \in C_2$ we want it to be in the numerator. Substituting such expansions in Eq. (10.15) yields

$$\begin{aligned} 2\pi i f(z) = &\sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}} \\ &+ \sum_{n=0}^{\infty} \frac{1}{(z - z_0)^{n+1}} \oint_{C_2} f(\xi)(\xi - z_0)^n \, d\xi. \end{aligned} \tag{10.16}$$

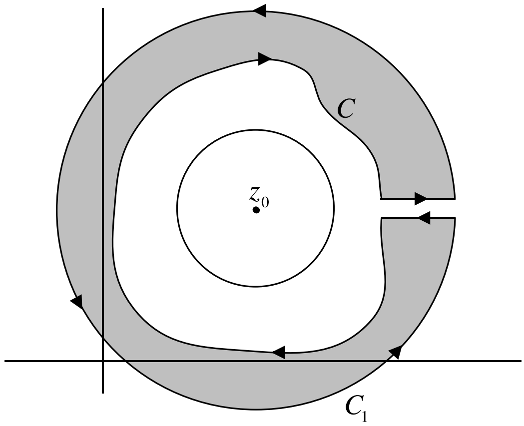

Fig. 10.11 The arbitrary contour in the annular region used in the Laurent expansion. The break in C and the gap in the shaded region are magnified for visual clarity

Now we consider an arbitrary contour C in S that encircles z_0. Figure 10.11 shows a region bounded by a contour composed of C_1 and C. In this region $f(\xi)/(\xi - z_0)^{n+1}$ is analytic (because ξ can never equal z_0). Thus, the integral over the composite contour must vanish by the Cauchy-Goursat theorem. It follows that the integral over C_1 is equal to that over C. A similar argument shows that the C_2 integral can also be replaced by an integral over C. We let $n + 1 = -m$ in the second sum of Eq. (10.16) to transform it into

$$\sum_{m=-1}^{-\infty} \frac{1}{(z - z_0)^{-m}} \oint_C f(\xi)(\xi - z_0)^{-m-1}\, d\xi$$
$$= \sum_{m=-\infty}^{-1} (z - z_0)^m \oint_C \frac{f(\xi)\, d\xi}{(\xi - z_0)^{m+1}}.$$

Changing the dummy index back to n and substituting the result in Eq. (10.16) yields

$$2\pi i f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi + \sum_{-\infty}^{n=-1} (z - z_0)^n \oint_C \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi.$$

We can now combine the sums and divide both sides by $2\pi i$ to get the desired expansion. □

The Laurent expansion is convergent as long as $r_2 < |z - z_0| < r_1$. In particular, if $r_2 = 0$, and if the function is analytic throughout the interior of the larger circle, then a_n will be zero for $n = -1, -2, \ldots$ because $f(\xi)/(\xi - z_0)^{n+1}$ will be analytic for negative n, and the integral will be zero by the

Cauchy-Goursat theorem. Thus, only positive powers of $(z - z_0)$ will be present in the series, and we recover the Taylor series, as we should.

It is clear that we can expand C_1 and shrink C_2 until we encounter a point at which f is no longer analytic. This is obvious from the construction of the proof, in which only the analyticity in the annular region is important, not its size. Thus, we can include all the possible analytic points by expanding C_1 and shrinking C_2.

Example 10.6.6 Let us expand some functions in terms of series. For an entire function there is no point in the entire complex plane at which it is not analytic. Thus, only positive powers of $(z - z_0)$ will be present, and we will have a Taylor expansion that is valid for all values of z.

(a) Let us expand e^z around $z_0 = 0$. The nth derivative of e^z is e^z. Thus, $f^{(n)}(0) = 1$, and Taylor (Maclaurin) expansion gives

$$e^z = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

(b) The Maclaurin series for $\sin z$ is obtained by noting that

$$\frac{d^n}{dz^n} \sin z \bigg|_{z=0} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{(n-1)/2} & \text{if } n \text{ is odd} \end{cases}$$

and substituting this in the Maclaurin expansion:

$$\sin z = \sum_{n \text{ odd}} (-1)^{(n-1)/2} \frac{z^n}{n!} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}.$$

Similarly, we can obtain

$$\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}, \qquad \sinh z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!},$$

$$\cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}.$$

(c) The function $1/(1 + z)$ is not entire, so the region of its convergence is limited. Let us find the Maclaurin expansion of this function. The function is analytic within all circles of radii $r < 1$. At $r = 1$ we encounter a singularity, the point $z = -1$. Thus, the series converges for all points[11] z for which $|z| < 1$. For such points we have

$$f^{(n)}(0) = \frac{d^n}{dz^n}\left[(1+z)^{-1}\right]\bigg|_{z=0} = (-1)^n n!.$$

[11] As remarked before, the series diverges for *all* points outside the circle $|z| = 1$. This does not mean that the function cannot be represented by a series for points outside the circle. On the contrary, we shall see shortly that Laurent series, with *negative* powers of $z - z_0$ are designed precisely for such a purpose.

Thus,

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} (-1)^n z^n.$$

Taylor and Laurent series allow us to express an analytic function as a power series. For a Taylor series of $f(z)$, the expansion is routine because the coefficient of its nth term is simply $f^{(n)}(z_0)/n!$, where z_0 is the center of the circle of convergence. When a Laurent series is applicable, however, the nth coefficient is not, in general, easy to evaluate. Usually it can be found by inspection and certain manipulations of other known series. But if we use such an intuitive approach to determine the coefficients, can we be sure that we have obtained the correct Laurent series? The following theorem answers this question.

Laurent series is unique

Theorem 10.6.7 *If the series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ converges to $f(z)$ at all points in some annular region about z_0, then it is* the unique *Laurent series expansion of $f(z)$ in that region.*

Proof Multiply both sides of $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ by

$$\frac{1}{2\pi i(z-z_0)^{k+1}},$$

integrate the result along a contour C in the annular region, and use the easily verifiable fact that

$$\frac{1}{2\pi i} \oint_C \frac{dz}{(z-z_0)^{k-n+1}} = \delta_{kn}$$

to obtain

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{k+1}}\, dz = a_k.$$

Thus, the coefficient in the power series of f is precisely the coefficient in the Laurent series, and the two must be identical. □

We will look at some examples that illustrate the abstract ideas developed in the preceding collection of theorems and propositions. However, we can consider a much broader range of examples if we know the arithmetic of power series. The following theorem about arithmetical manipulations with power series is not difficult to prove (see [Chur 74]).

Theorem 10.6.8 *Let the two power series*

You can add, subtract, and multiply convergent power series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \quad \textit{and} \quad g(z) = \sum_{n=-\infty}^{\infty} b_n(z-z_0)^n$$

be convergent within some annular region $r_2 < |z - z_0| < r_1$. Then

$$f(z) + g(z) = \sum_{n=-\infty}^{\infty} (a_n + b_n)(z - z_0)^n$$

and

$$f(z)g(z) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_n b_m (z - z_0)^{m+n} \equiv \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k$$

for z interior to the annular region. Furthermore, if $g(z) \neq 0$ for some neighborhood of z_0, then the series obtained by long division of the first series by the second converges to $f(z)/g(z)$ in that neighborhood.

This theorem, in essence, says that converging power series can be manipulated as though they were finite sums (polynomials). Such manipulations are extremely useful when dealing with Taylor and Laurent expansions in which the straightforward calculation of coefficients may be tedious. The following examples illustrate the power of infinite-series arithmetic.

Example 10.6.9 To expand the function $f(z) = \frac{2+3z}{z^2+z^3}$ in a Laurent series about $z = 0$, rewrite it as

$$f(z) = \frac{1}{z^2}\left(\frac{2+3z}{1+z}\right) = \frac{1}{z^2}\left(3 - \frac{1}{1+z}\right) = \frac{1}{z^2}\left(3 - \sum_{n=0}^{\infty}(-1)^n z^n\right)$$
$$= \frac{1}{z^2}(3 - 1 + z - z^2 + z^3 - \cdots) = \frac{2}{z^2} + \frac{1}{z} - 1 + z - z^2 + \cdots .$$

This series converges for $0 < |z| < 1$. We note that negative powers of z are also present.[12] Using the notation of Theorem 10.6.5, we have $a_n = 0$ for $n \le -3$, $a_{-2} = 2$, $a_{-1} = 1$, and $a_n = (-1)^{n+1}$ for $n \ge 0$.

Example 10.6.10 The function $f(z) = 1/(4z - z^2)$ is the ratio of two entire functions. Therefore, by Theorem 10.6.8, it is analytic everywhere except at the zeros of its denominator, $z = 0$ and $z = 4$. For the annular region (here r_2 of Theorem 10.6.5 is zero) $0 < |z| < 4$, we expand $f(z)$ in the Laurent series around $z = 0$. Instead of actually calculating a_n, we first note that

$$f(z) = \frac{1}{4z}\left(\frac{1}{1 - z/4}\right).$$

The second factor can be expanded in a geometric series because $|z/4| < 1$:

$$\frac{1}{1 - z/4} = \sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n = \sum_{n=0}^{\infty} 4^{-n} z^n.$$

[12]This is a reflection of the fact that the function is not analytic inside the entire circle $|z| = 1$; it blows up at $z = 0$.

Dividing this by $4z$, and noting that $z = 0$ is the only zero of $4z$ and is excluded from the annular region, we obtain the expansion

$$f(z) = \sum_{n=0}^{\infty} 4^{-n} \frac{z^n}{4z} = \sum_{n=0}^{\infty} 4^{-n-1} z^{n-1}.$$

Although we derived this series using manipulations of other series, the uniqueness of series representations assures us that this is *the* Laurent series for the indicated region.

How can we represent $f(z)$ in the region for which $|z| > 4$? This region is exterior to the circle $|z| = 4$, so we expect negative powers of z. To find the Laurent expansion we write

$$f(z) = -\frac{1}{z^2}\left(\frac{1}{1 - 4/z}\right)$$

and note that $|4/z| < 1$ for points exterior to the larger circle. The second factor can be written as a geometric series:

$$\frac{1}{1 - 4/z} = \sum_{n=0}^{\infty} \left(\frac{4}{z}\right)^n = \sum_{n=0}^{\infty} 4^n z^{-n}.$$

Dividing by $-z^2$, which is nonzero in the region exterior to the larger circle, yields

$$f(z) = -\sum_{n=0}^{\infty} 4^n z^{-n-2}.$$

Example 10.6.11 The function $f(z) = z/[(z-1)(z-2)]$ has a Taylor expansion around the origin for $|z| < 1$. To find this expansion, we write[13]

$$f(z) = -\frac{1}{z-1} + \frac{2}{z-2} = \frac{1}{1-z} - \frac{1}{1-z/2}.$$

Expanding both fractions in geometric series (both $|z|$ and $|z/2|$ are less than 1), we obtain $f(z) = \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} (z/2)^n$. Adding the two series—using Theorem 10.6.8—yields

$$f(z) = \sum_{n=0}^{\infty} \left(1 - 2^{-n}\right) z^n \quad \text{for } |z| < 1.$$

This is the unique Taylor expansion of $f(z)$ within the circle $|z| = 1$.

For $1 < |z| < 2$ we have a Laurent series. To obtain this series, write

$$f(z) = \frac{1/z}{1/z - 1} - \frac{1}{1 - z/2} = -\frac{1}{z}\left(\frac{1}{1 - 1/z}\right) - \frac{1}{1 - z/2}.$$

[13] We could, of course, evaluate the derivatives of all orders of the function at $z = 0$ and use Maclaurin's formula. However, the present method gives the same result much more quickly.

Since both fractions on the RHS converge in the annular region ($|1/z| < 1$, $|z/2| < 1$), we get

$$f(z) = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty} z^{-n-1} - \sum_{n=0}^{\infty} 2^{-n} z^n$$

$$= -\sum_{n=-1}^{-\infty} z^n - \sum_{n=0}^{\infty} 2^{-n} z^n = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where $a_n = -1$ for $n < 0$ and $a_n = -2^{-n}$ for $n \geq 0$. This is the unique Laurent expansion of $f(z)$ in the given region.

Finally, for $|z| > 2$ we have only negative powers of z. We obtain the expansion in this region by rewriting $f(z)$ as follows:

$$f(z) = -\frac{1/z}{1 - 1/z} + \frac{2/z}{1 - 2/z}.$$

Expanding the fractions yields

$$f(z) = -\sum_{n=0}^{\infty} z^{-n-1} + \sum_{n=0}^{\infty} 2^{n+1} z^{-n-1} = \sum_{n=0}^{\infty} \left(2^{n+1} - 1\right) z^{-n-1}.$$

This is again the unique expansion of $f(z)$ in the region $|z| > 2$.

Example 10.6.12 Define $f(z)$ as

$$f(z) = \begin{cases} (1 - \cos z)/z^2 & \text{for } z \neq 0, \\ \frac{1}{2} & \text{for } z = 0. \end{cases}$$

We can show that $f(z)$ is an entire function.

Since $1 - \cos z$ and z^2 are entire functions, their ratio is analytic everywhere except at the zeros of its denominator. The only such zero is $z = 0$. Thus, Theorem 10.6.8 implies that $f(z)$ is analytic everywhere except possibly at $z = 0$. To see the behavior of $f(z)$ at $z = 0$, we look at its Maclaurin series:

$$1 - \cos z = 1 - \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$

which implies that

$$\frac{1 - \cos z}{z^2} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^{2n-2}}{(2n)!} = \frac{1}{2} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots.$$

The expansion on the RHS shows that the value of the series at $z = 0$ is $\frac{1}{2}$, which, by definition, is $f(0)$. Thus, the series converges for all z, and Theorem 10.6.2 says that $f(z)$ is entire.

A Laurent series can give information about the integral of a function around a closed contour in whose interior the function may not be analytic. In fact, the coefficient of the first negative power in a Laurent series is given by

$$a_{-1} = \frac{1}{2\pi i}\oint_C f(\xi)\,d\xi. \qquad (10.17)$$

Thus, to find the integral of a (nonanalytic) function around a closed contour surrounding z_0, we write the Laurent series for the function and read off the coefficient of the $1/(z-z_0)$ term.

Example 10.6.13 As an illustration of this idea, let us evaluate the integral $I = \oint_C dz/[z^2(z-2)]$, where C is the circle of radius 1 centered at the origin. The function is analytic in the annular region $0 < |z| < 2$. We can therefore expand it as a Laurent series about $z = 0$ in that region:

$$\frac{1}{z^2(z-2)} = -\frac{1}{2z^2}\left(\frac{1}{1-z/2}\right) = -\frac{1}{2z^2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$
$$= -\frac{1}{2}\left(\frac{1}{z^2}\right) - \frac{1}{4}\left(\frac{1}{z}\right) - \frac{1}{8} - \cdots.$$

Thus, $a_{-1} = -\frac{1}{4}$, and $\oint_C dz/[z^2(z-2)] = 2\pi i a_{-1} = -i\pi/2$. A direct evaluation of the integral is nontrivial. In fact, we will see later that to find certain integrals, it is advantageous to cast them in the form of a contour integral and use either Eq. (10.17) or a related equation.

Let $f : \mathbb{C} \to \mathbb{C}$ be analytic *at* z_0. Then by definition, there exists a *neighborhood* of z_0 in which f is analytic. In particular, we can find a circle $|z - z_0| = r > 0$ in whose interior f has a Taylor expansion.

Definition 10.6.14 Let

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \equiv \sum_{n=0}^{\infty}a_n(z-z_0)^n.$$

Then f is said to have a **zero of order** k at z_0 if $f^{(n)}(z_0) = 0$ for $n = 0, 1, \dots, k-1$ but $f^{(k)}(z_0) \neq 0$. zero of order k

In that case $f(z) = (z-z_0)^k \sum_{n=0}^{\infty} a_{k+n}(z-z_0)^n$, where $a_k \neq 0$ and $|z - z_0| < r$. We define $g(z)$ as

$$g(z) = \sum_{n=0}^{\infty} a_{k+n}(z-z_0)^n \quad \text{where } |z-z_0| < r$$

and note that $g(z_0) = a_k \neq 0$. Convergence of the series on the RHS implies that $g(z)$ is continuous at z_0. Consequently, for each $\epsilon > 0$, there exists δ such that $|g(z) - a_k| < \epsilon$ whenever $|z - z_0| < \delta$. If we choose $\epsilon = |a_k|/2$, then, for some $\delta_0 > 0$, $|g(z) - a_k| < |a_k|/2$ whenever $|z - z_0| < \delta_0$. Thus, as long as z is inside the circle $|z - z_0| < \delta_0$, $g(z)$ cannot vanish (because if it did the first inequality would imply that $|a_k| < |a_k|/2$). We therefore have the following result.

the zeros of an analytic function are isolated

Theorem 10.6.15 *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic at z_0 and $f(z_0) = 0$. Then there exists a neighborhood of z_0 throughout which f has no other zeros unless f is identically zero there. Thus, the zeros of an analytic function are* ***isolated***.

simple zero

When $k = 1$, we say that z_0 is a **simple zero** of f. To find the order of the zero of a function at a point, we differentiate the function, evaluate the derivative at that point, and continue the process until we obtain a nonzero value for the derivative.

Example 10.6.16 Here are some functions with their zeros:

(a) The zeros of $\cos z$, which are $z = (2k+1)\pi/2$, are all simple, because

$$\frac{d}{dz}\cos z\bigg|_{z=(2k+1)\pi/2} = -\sin\left[(2k+1)\frac{\pi}{2}\right] \neq 0.$$

(b) To find the order of the zero of $f(z) = e^z - 1 - z - z^2/2$ at $z = 0$, we differentiate $f(z)$ and evaluate $f'(0)$:

$$f'(0) = \left(e^z - 1 - z\right)_{z=0} = 0.$$

Differentiating again gives $f''(0) = (e^z - 1)_{z=0} = 0$. Differentiating once more yields $f'''(0) = (e^z)_{z=0} = 1$. Thus, the zero is of order 3.

10.7 Problems

10.1 Show that the function $w = 1/z$ maps the straight line $y = a$ in the z-plane onto a circle in the w-plane with radius $1/(2|a|)$ and center $(0, 1/(2a))$.

10.2 Treating x and y as functions of z and z^*,

(a) use the chain rule to find $\partial f/\partial z^*$ and $\partial f/\partial z$ in terms of partial derivatives with respect to x and y.
(b) Evaluate $\partial f/\partial z^*$ and $\partial f/\partial z$ assuming that the Cauchy-Riemann conditions hold.

10.3 Show that when z is represented by polar coordinates, the derivative of a function $f(z)$ can be written as

$$\frac{df}{dz} = e^{-i\theta}\left(\frac{\partial U}{\partial r} + i\frac{\partial V}{\partial r}\right),$$

where U and V are the real and imaginary parts of $f(z)$ written in polar coordinates. What are the C-R conditions in polar coordinates? Hint: Start with the C-R conditions in Cartesian coordinates and apply the chain rule to them using $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x) = \cos^{-1}(x/\sqrt{x^2 + y^2})$.

10.4 Show that $\frac{d}{dz}(\ln z) = \frac{1}{z}$. Hint: Find $u(x, y)$ and $v(x, y)$ for $\ln z$ and differentiate them.

10.5 Show that $\sin z$ and $\cos z$ have only real roots.

10.6 Show that

(a) the sum and the product of two entire functions are entire, and
(b) the ratio of two entire functions is analytic everywhere except at the zeros of the denominator.

10.7 Given that $u = 2\lambda \ln[(x^2 + y^2)^{1/2}]$, show that $v = 2\lambda \tan^{-1}(y/x)$, where u and v are the real and imaginary parts of an analytic function $w(z)$.

10.8 If $w(z)$ is any complex potential, show that its (complex) derivative gives the components of the electric field.

10.9 Show that

(a) the flux through an element of area da of the lateral surface of a cylinder (with arbitrary cross section) is $d\phi = dz(|\mathbf{E}|\, ds)$ where ds is an arc length along the equipotential surface.
(b) Prove that $|\mathbf{E}| = |dw/dz| = \partial v/\partial s$ where v is the imaginary part of the complex potential, and s is the parameter describing the length along the equipotential curves.
(c) Combine (a) and (b) to get

$$\text{flux per unit } z\text{-length} = \frac{\phi}{z_2 - z_1} = v(P_2) - v(P_1)$$

for any two points P_1 and P_2 on the cross-sectional curve of the lateral surface. Conclude that the total flux per unit z-length through a cylinder (with arbitrary cross section) is $[v]$, the total change in v as one goes around the curve.
(d) Using Gauss's law, show that the capacitance per unit length for the capacitor consisting of the two conductors with potentials u_1 and u_2 is

$$c \equiv \frac{\text{charge per unit length}}{\text{potential difference}} = \frac{[v]/4\pi}{|u_2 - u_1|}.$$

10.10 Using Eq. (10.7)

(a) find the equipotential curves (curves of constant u) and curves of constant v for two line charges of equal magnitude and opposite signs located at $y = a$ and $y = -a$ in the xy-plane.
(b) Show that

$$z = a\left(\sin\frac{v}{2\lambda} + i\sinh\frac{u}{2\lambda}\right)\Big/\left(\cosh\frac{u}{2\lambda} - \cos\frac{v}{2\lambda}\right)$$

by solving Eq. (10.7) for z and simplifying.

(c) Show that the equipotential curves are circles in the xy-plane of radii $a/\sinh(u/2\lambda)$ with centers at $(0, a\coth(u/2\lambda))$, and that the curves of constant v are circles of radii $a/\sin(v/2\lambda)$ with centers at $(a\cot(v/2\lambda), 0)$.

10.11 In this problem, you will find the capacitance per unit length of two cylindrical conductors of radii R_1 and R_2 the distance between whose centers is D by looking for two line charge densities $+\lambda$ and $-\lambda$ such that the two cylinders are two of the equipotential surfaces. From Problem 10.10, we have

$$R_i = \frac{a}{\sinh(u_i/2\lambda)}, \qquad y_i = a\coth(u_i/2\lambda), \quad i = 1, 2,$$

where y_1 and y_2 are the locations of the centers of the two conductors on the y-axis (which we assume to connect the two centers).

(a) Show that $D = |y_1 - y_2| = |R_1 \cosh \frac{u_1}{2\lambda} - R_2 \cosh \frac{u_2}{2\lambda}|$.
(b) Square both sides and use $\cosh(a-b) = \cosh a \cosh b - \sinh a \sinh b$ and the expressions for the R's and the y's given above to obtain

$$\cosh\left(\frac{u_1 - u_2}{2\lambda}\right) = \left|\frac{R_1^2 + R_2^2 - D^2}{2R_1R_2}\right|.$$

(c) Now find the capacitance per unit length. Consider the special case of two concentric cylinders.
(d) Find the capacitance per unit length of a cylinder and a plane, by letting one of the radii, say R_1, go to infinity while $h \equiv R_1 - D$ remains fixed.

10.12 Use Equations (10.4) and (10.5) to establish the following identities.

(a) $\mathrm{Re}(\sin z) = \sin x \cosh y$, (b) $\mathrm{Im}(\sin z) = \cos x \sinh y$,

(c) $\mathrm{Re}(\cos z) = \cos x \cosh y$, (d) $\mathrm{Im}(\cos z) = -\sin x \sinh y$,

(e) $\mathrm{Re}(\sinh z) = \sinh x \cos y$, (f) $\mathrm{Im}(\sinh z) = \cosh x \sin y$,

(g) $\mathrm{Re}(\cosh z) = \cosh x \cos y$, (h) $\mathrm{Im}(\cosh z) = \sinh x \sin y$,

(i) $|\sin z|^2 = \sin^2 x + \sinh^2 y$, (j) $|\cos z|^2 = \cos^2 x + \sinh^2 y$,

(k) $|\sinh z|^2 = \sinh^2 x + \sin^2 y$, (l) $|\cosh z|^2 = \sinh^2 x + \cos^2 y$.

10.13 Find all the zeros of $\sinh z$ and $\cosh z$.

10.14 Verify the following hyperbolic identities.

(a) $\cosh^2 z - \sinh^2 z = 1$.

(b) $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$.

(c) $\sinh(z_1 + z_2) = \sin z_1 \cosh z_2 + \cosh z_1 \sinh z_2$.

(d) $\cosh 2z = \cosh^2 z + \sinh^2 z, \qquad \sinh 2z = 2\sinh z \cosh z$.

(e) $\tanh(z_1 + z_2) = \dfrac{\tanh z_1 + \tanh z_2}{1 + \tanh z_1 \tanh z_2}$.

10.15 Show that

(a) $\tanh\left(\frac{z}{2}\right) = \frac{\sinh x + i \sin y}{\cosh x + \cos y}$, (b) $\coth\left(\frac{z}{2}\right) = \frac{\sinh x - i \sin y}{\cosh x - \cos y}$.

10.16 Find all values of z such that

(a) $e^z = -3$, (b) $e^z = 1 + i\sqrt{3}$, (c) $e^{2z-1} = 1$.

10.17 Show that $|e^{-z}| < 1$ if and only if $\mathrm{Re}(z) > 0$.

10.18 Show that each of the following functions—call each one $u(x, y)$—is harmonic, and find the function's harmonic partner, $v(x, y)$, such that $u(x, y) + iv(x, y)$ is analytic.

(a) $x^3 - 3xy^2$; (b) $e^x \cos y$;

(c) $\frac{x}{x^2 + y^2}$, where $x^2 + y^2 \neq 0$;

(d) $e^{-2y} \cos 2x$; (e) $e^{y^2 - x^2} \cos 2xy$;

(f) $e^x (x \cos y - y \sin y) + 2 \sinh y \sin x + x^3 - 3xy^2 + y$.

10.19 Prove the following identities.

(a) $\cos^{-1} z = -i \ln\left(z \pm \sqrt{z^2 - 1}\right)$,

(b) $\sin^{-1} z = -i \ln\left[iz \pm \sqrt{1 - z^2}\right]$,

(c) $\tan^{-1} z = \frac{1}{2i} \ln\left(\frac{i - z}{i + z}\right)$,

(d) $\cosh^{-1} z = \ln\left(z \pm \sqrt{z^2 - 1}\right)$,

(e) $\sinh^{-1} z = \ln\left(z \pm \sqrt{z^2 + 1}\right)$,

(f) $\tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1 + z}{1 - z}\right)$.

10.20 Find the curve defined by each of the following equations.

(a) $z = 1 - it$, $0 \leq t \leq 2$,

(b) $z = t + it^2$, $-\infty < t < \infty$,

(c) $z = a(\cos t + i \sin t)$, $\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$,

(d) $z = t + \frac{i}{t}$, $-\infty < t < 0$.

10.21 Prove part (a) of Proposition 10.3.1. Hint: A small displacement along γ_i can be written as $\hat{\mathbf{e}}_x \Delta x_i + \hat{\mathbf{e}}_y \Delta y_i$ for $i = 1, 2$. Find a unit vector along each displacement and take the dot product of the two. Do the same for γ_i'; use the C-R condition to show that the two are equal. Prove part (b) by showing that if $f(z) = z' = x' + iy'$ is analytic and $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$, then $\frac{\partial^2 \Phi}{\partial x'^2} + \frac{\partial^2 \Phi}{\partial y'^2} = 0$.

10.22 Let $f(t) = u(t) + iv(t)$ be a (piecewise) continuous complex-valued function of a real variable t defined in the interval $a \le t \le b$. Show that if $F(t) = U(t) + iV(t)$ is a function such that $dF/dt = f(t)$, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

This is the fundamental theorem of calculus for complex variables.

10.23 Find the value of the integral $\int_C [(z+2)/z]dz$, where C is

(a) the semicircle $z = 2e^{i\theta}$, for $0 \le \theta \le \pi$,
(b) the semicircle $z = 2e^{i\theta}$, for $\pi \le \theta \le 2\pi$, and
(c) the circle $z = 2e^{i\theta}$, for $-\pi \le \theta \le \pi$.

10.24 Evaluate the integral $\int_\gamma dz/(z - 1 - i)$ where γ is

(a) the line joining $z_1 = 2i$ and $z_2 = 3$, and
(b) the broken path from z_1 to the origin and from there to z_2.

10.25 Evaluate the integral $\int_C z^m (z^*)^n dz$, where m and n are integers and C is the circle $|z| = 1$ taken counterclockwise.

10.26 Let C be the boundary of the square with vertices at the points $z = 0$, $z = 1$, $z = 1 + i$, and $z = i$ with counterclockwise direction. Evaluate

$$\oint_C (5z + 2)\,dz \quad \text{and} \quad \oint_C e^{\pi z^*} dz.$$

10.27 Use the definition of an integral as the limit of a sum and the fact that absolute value of a sum is less than or equal to the sum of absolute values to prove the Darboux inequality.

10.28 Let C_1 be a simple closed contour. Deform C_1 into a new contour C_2 in such a way that C_1 does not encounter any singularity of an analytic function f in the process. Show that

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz.$$

That is, the contour can always be deformed into simpler shapes (such as a circle) and the integral evaluated.

10.29 Use the result of the previous problem to show that

$$\oint_C \frac{dz}{z - 1 - i} = 2\pi i \quad \text{and} \quad \oint_C (z - 1 - i)^{m-1} dz = 0 \quad \text{for } m = \pm 1, \pm 2, \ldots$$

when C is the boundary of a square with vertices at $z = 0$, $z = 2$, $z = 2 + 2i$, and $z = 2i$, taken counterclockwise.

10.30 Use Eq. (10.12) and the binomial expansion to show that $\frac{d}{dz}(z^m) = mz^{m-1}$.

10.31 Evaluate $\oint_C dz/(z^2 - 1)$ where C is the circle $|z| = 3$ integrated in the positive sense. Hint: Deform C into a contour C' that avoids the singularities of the integrand. Then use Cauchy Goursat theorem.

10.32 Show that when f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\oint_C \frac{f'(z)\,dz}{z - z_0} = \oint_C \frac{f(z)\,dz}{(z - z_0)^2}.$$

10.33 Let C be the boundary of a square whose sides lie along the lines $x = \pm 3$ and $y = \pm 3$. For the positive sense of integration, evaluate each of the following integrals.

(a) $\oint_C \frac{e^{-z}dz}{z - i\pi/2}$, (b) $\oint_C \frac{e^{z}dz}{z(z^2 + 10)}$,

(c) $\oint_C \frac{\cos z dz}{(z - \frac{\pi}{4})(z^2 - 10)}$, (d) $\oint_C \frac{\sinh z dz}{z^4}$,

(e) $\oint_C \frac{\cosh z dz}{z^4}$, (f) $\oint_C \frac{\cos z dz}{z^3}$,

(g) $\oint_C \frac{\cos z dz}{(z - i\pi/2)^2}$, (h) $\oint_C \frac{e^{z}dz}{(z - i\pi)^2}$,

(i) $\oint_C \frac{\cos z dz}{z + i\pi}$, (j) $\oint_C \frac{e^{z}dz}{z^2 - 5z + 4}$,

(k) $\oint_C \frac{\sinh z dz}{(z - i\pi/2)^2}$, (l) $\oint_C \frac{\cosh z dz}{(z - i\pi/2)^2}$,

(m) $\oint_C \frac{\tan z dz}{(z - \alpha)^2}$, for $-3 < \alpha < 3$, (n) $\oint_C \frac{z^2 dz}{(z - 2)(z^2 - 10)}$.

10.34 Let C be the circle $|z - i| = 3$ integrated in the positive sense. Find the value of each of the following integrals.

(a) $\oint_C \frac{e^{z}}{z^2 + \pi^2}dz$, (b) $\oint_C \frac{\sinh z}{(z^2 + \pi^2)^2}dz$,

(c) $\oint_C \frac{dz}{z^2 + 9}$, (d) $\oint_C \frac{dz}{(z^2 + 9)^2}$,

(e) $\oint_C \frac{\cosh z}{(z^2 + \pi^2)^3}dz$, (f) $\oint_C \frac{z^2 - 3z + 4}{z^2 - 4z + 3}dz$.

10.35 Show that Legendre polynomials (for $|x| < 1$) can be represented as

$$P_n(x) = \frac{(-1)^n}{2^n(2\pi i)} \oint_C \frac{(1-z^2)^n}{(z-x)^{n+1}}\, dz,$$

where C is the unit circle around the origin.

Cauchy's inequality

10.36 Let f be analytic within and on the circle γ_0 given by $|z - z_0| = r_0$ and integrated in the positive sense. Show that **Cauchy's inequality** holds:

$$\left| f^{(n)}(z_0) \right| \le \frac{n! M}{r_0^n},$$

where M is the maximum value of $|f(z)|$ on γ_0.

10.37 Expand $\sinh z$ in a Taylor series about the point $z = i\pi$.

10.38 What is the largest circle within which the Maclaurin series for $\tanh z$ converges to $\tanh z$?

10.39 Find the (unique) Laurent expansion of each of the following functions about the origin for its entire region of analyticity.

(a) $\dfrac{1}{(z-2)(z-3)}$; (b) $z\cos(z^2)$; (c) $\dfrac{1}{z^2(1-z)}$;

(d) $\dfrac{\sinh z - z}{z^4}$; (e) $\dfrac{1}{(1-z)^3}$; (f) $\dfrac{1}{z^2-1}$;

(g) $\dfrac{z^2-4}{z^2-9}$; (h) $\dfrac{1}{(z^2-1)^2}$; (i) $\dfrac{z}{z-1}$.

10.40 Show that the following functions are entire.

$$\text{(a)} \quad f(z) = \begin{cases} \frac{e^{2z}-1}{z^2} - \frac{2}{z} & \text{for } z \ne 0, \\ 2 & \text{for } z = 0. \end{cases}$$

$$\text{(b)} \quad f(z) = \begin{cases} \frac{\sin z}{z} & \text{for } z \ne 0, \\ 1 & \text{for } z = 0. \end{cases}$$

$$\text{(c)} \quad f(z) = \begin{cases} \frac{\cos z}{z^2-\pi^2/4} & \text{for } z \ne \pm\pi/2, \\ -1/\pi & \text{for } z = \pm\pi/2. \end{cases}$$

10.41 Let f be analytic at z_0 and $f(z_0) = f'(z_0) = \cdots = f^{(k)}(z_0) = 0$. Show that the following function is analytic at z_0:

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^{k+1}} & \text{for } z \ne z_0, \\ \frac{f^{(k+1)}(z_0)}{(k+1)!} & \text{for } z = z_0. \end{cases}$$

10.42 Obtain the first few nonzero terms of the Laurent series expansion of each of the following functions about the origin. Also find the integral of the function along a small simple closed contour encircling the origin.

(a) $\dfrac{1}{\sin z}$; (b) $\dfrac{1}{1-\cos z}$; (c) $\dfrac{z}{1-\cosh z}$;

(d) $\dfrac{z^2}{z-\sin z}$; (e) $\dfrac{z^4}{6z+z^3-6\sinh z}$; (f) $\dfrac{1}{z^2 \sin z}$;

(g) $\dfrac{1}{e^z-1}$.